Chapter 7

Arithmetic and Asymptotic Properties for Some Functions Related to the Least r-Gaps in Partitions

7.1 Introduction

The arithmetic functions related with the notion of minimal excludants and the least r-gaps in partitions are introduced in introductory chapter of the thesis. We refer to Section 1.9 of Chapter 1 for the definitions of the functions relevant in this chapter and their generating functions.

Baruah, Bhoria, Eyyunni, and Maji [24] proved the following refinement result for the expression of $\sigma\text{mex}(n)$ of Andrews and Newman [6].

Theorem 7.1 (Baruah, Bhoria, Eyyunni and Maji [24])**.** *We have*

$$\sum_{n=0}^{\infty} \sigma_o\text{mex}(n)q^n = \frac{(-q;q)_\infty^2 + (q;q)_\infty^2}{2} \quad \text{and} \quad \sum_{n=0}^{\infty} \sigma_e\text{mex}(n)q^n = \frac{(-q;q)_\infty^2 - (q;q)_\infty^2}{2},$$

where $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$ are as defined in Section 1.9.

They also obtained congruences for $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$ and studied the k-th moments of minimal excludants.

The contents of this chapter have been submitted for publication [23].

In their paper, Ballantine and Merca [11] found new identities relating $p(n)$ and $S_r(n)$. They also found the generating function for $\sigma_r\text{mex}(n)$ as stated in the following theorem.

Theorem 7.2 (Ballantine and Merca [11])**.** *We have*

$$\sum_{n=0}^{\infty} \sigma_r\text{mex}(n)q^n = \frac{(q^{2r};q^{2r})_\infty}{(q;q)_\infty\,(q^r;q^{2r})_\infty} = \frac{f_{2r}^2}{f_1 f_r}. \tag{7.1}$$

For further works on mex related functions and minimal excludants for other restricted partitions, we refer the readers to [7, 11, 12, 17, 40, 49, 70, 71, 74, 75, 77, 101].

In this chapter, we present an alternative proof of Theorem 7.2.

In the following three theorems, we obtain the generating functions for $\sigma_{r,o}\text{mex}(n)$, $\sigma_{r,e}\text{mex}(n)$, $\sigma_r\text{moex}(n)$, and $a_r(n)$.

Theorem 7.3. *We have*

$$\sum_{n=0}^{\infty} \sigma_{r,o}\text{mex}(n)q^n = \frac{1}{2}(\mathcal{M}_r(q) + \mathcal{N}_r(q)), \tag{7.2}$$

$$\sum_{n=0}^{\infty} \sigma_{r,e}\text{mex}(n)q^n = \frac{1}{2}(\mathcal{M}_r(q) - \mathcal{N}_r(q)), \tag{7.3}$$

where

$$\mathcal{M}_r(q) = \sum_{n=0}^{\infty} \beta_r(n)q^n = \frac{f_{2r}^2}{f_1 f_r} \tag{7.4}$$

and

$$\mathcal{N}_r(q) = \sum_{n=0}^{\infty} \gamma_r(n)q^n = \frac{f_r^3}{f_1}. \tag{7.5}$$

Theorem 7.4. *We have*

$$\sum_{n=0}^{\infty} \sigma_r\text{moex}(n)q^n = \frac{f_{2r}^5}{f_1 f_r^2 f_{4r}^2}.$$

Theorem 7.5. *We have*

$$\sum_{n=0}^{\infty} a_r(n)q^n = \frac{1}{f_1}\sum_{m=0}^{\infty}(-1)^m q^{r\cdot\frac{m(m+1)}{2}}.$$

Note that Theorem 7.3 is analogous to Theorem 7.1 of Baruah, Bhoria, Eyyunni, and Maji [24].

An overpartition of n, as defined by Corteel and Lovejoy [47], is a non-increasing sequence of positive integers whose sum is n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined. In 2015, Andrews [4] introduced the combinatorial objects which he called singular overpartitions and showed that these singular overpartitions, which depend on two parameters k and i, can be enumerated by the function $\overline{C}_{k,i}(n)$ which gives the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined.

Andrews and Newman [7] studied a generalization $\mathrm{mex}_{A,a}(\pi)$ of $\mathrm{mex}(\pi)$ defined to be the smallest positive integer congruent to a modulo A that is not a part of the partition π. Let $p_{A,a}(n)$ and $\overline{p}_{A,a}(n)$ denote the number of partitions π with $\mathrm{mex}_{A,a}(\pi) \equiv a \pmod{2A}$ and $\mathrm{mex}_{A,a}(\pi) \equiv A + a \pmod{2A}$, respectively.

In the next theorem, we establish connections of $\sigma_r\mathrm{mex}(n)$ and $a_r(n)$ with $\overline{C}_{k,i}(n)$ and $p_{A,a}(n)$, respectively.

Theorem 7.6. *For all $n \geq 0$ and $r \geq 1$, we have*

(i) $\sigma_r\mathrm{mex}(n) = \overline{C}_{4r,r}(n)$,

(ii) $a_r(n) = p_{r,r}(n)$.

Asymptotic behaviours of various partition functions and related q-products have been studied extensively after Hardy and Ramanujan [66] proved an asymptotic formula satisfied by $p(n)$. Using the circle method, they proved the following result for the unrestricted partition function $p(n)$:

$$p(n) \sim \frac{1}{4\sqrt{3}n}\exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as} \quad n \to \infty.$$

Grabner and Knopfmacher [59] obtained a Hardy-Ramanujan-type asymptotic formula for the smallest gap of a partition. Their results can be stated as:

$$\sigma\mathrm{mex}(n) \sim \frac{1}{4\sqrt[4]{6n^3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as} \quad n \to \infty.$$

Kaur, Bhoria, Eyyunni, and Maji [77] studied the minimal excludant over partitions into distinct parts and derived an asymptotic formula for the corresponding function. Recently, Barman and Singh [18] found the following Hardy-Ramanujan-type asymptotic formulae for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$:

$$\sigma_o\mathrm{mex}(n) \sim \sigma_e\mathrm{mex}(n) \sim \frac{1}{8\sqrt[4]{6n^3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as} \quad n \to \infty.$$

The result of Grabner and Knopfmacher for $\sigma\mathrm{mex}(n)$ can be deduced as an easy consequence of this result.

We study the asymptotic behaviour of $\sigma_r\mathrm{mex}(n)$ and $\sigma_r\mathrm{moex}(n)$ using Ingham's Tauberian theorem (see [73]). More specifically, we prove the following theorems.

Theorem 7.7. *We have*

$$\sigma_r\mathrm{mex}(n) \sim \frac{1}{4\sqrt[4]{6n^3r^2}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

as $n \to \infty$.

Theorem 7.8. *We have*

$$\sigma_r\mathrm{moex}(n) \sim \frac{1}{2\sqrt[4]{24n^3r^2}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

as $n \to \infty$.

As a corollary of Theorems 7.6 and 7.7, we obtain the following asymptotic formula for Andrews' singular overpartition function $\overline{C}_{4r,r}(n)$.

Corollary 7.9. *We have*

$$\overline{C}_{4r,r}(n) \sim \frac{1}{4\sqrt[4]{6n^3r^2}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

as $n \to \infty$.

Next, we focus on the arithmetic properties of these functions. Ray [101] found infinite families of congruences for $\sigma\mathrm{mex}(n)$ modulo 4. Baruah, Bhoria, Eyyunni, and Maji [24] proved the following Ramanujan-type congruences for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$ modulo 4 and 8:

$$\begin{aligned}\sigma_o\mathrm{mex}(2n+1) &\equiv 0 \pmod{4},\\ \sigma_o\mathrm{mex}(4n+1) &\equiv 0 \pmod{8},\\ \sigma_e\mathrm{mex}(4n) &\equiv 0 \pmod{4}.\end{aligned}$$

They also posed the two conjectural congruences for $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$ modulo 16 and 8. Using q-series manipulations and the theory of modular forms, Du and Tang [53] settled these conjectures.

Barman and Singh [18] discovered some infinite families as well as individual Ramanujan-type congruences $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$. For example, they proved that

$$\sigma_e\text{mex}(10n+r) \equiv 0 \pmod 4, \text{ for } r \in \{6,8\},$$

$$\sigma_e\text{mex}\left(2p^2n+kp+\frac{p^2-1}{12}\right) \equiv 0 \pmod 4,$$

where p is a prime congruent to 5, 7 or 11 modulo 12 and k is odd integer satisfying $1 \leq k < p$.

In this chapter, we prove some arithmetic relations between $\sigma_{r,o}\text{mex}(n)$ and $\sigma_{r,e}\text{mex}(n)$ for $r = 2$ and 3. With the help of some identities of Ahlgren [1] and Cooper, Hirschhorn, and Lewis [46], we obtain the following families of identities satisfied by these functions.

Theorem 7.10. *Let $n \geq 0$, $k \geq 0$, and $1 \leq s \leq p-1$ be positive integers and $p \equiv 7, 11, 13, 17, 19, 23 \pmod{24}$ be a prime. We have*

$$\sigma_{2,o}\text{mex}\left(p^{2k+2}n+p^{2k+1}s+\frac{5(p^{2k+2}-1)}{24}\right) = \sigma_{2,e}\text{mex}\left(p^{2k+2}n+p^{2k+1}s+5\frac{5(p^{2k+2}-1)}{24}\right). \tag{7.6}$$

Theorem 7.11. *Let $n \geq 0$, $k \geq 0$, and $1 \leq s \leq p-1$ be positive integers and $p \equiv 5 \pmod 6$ be a prime. We have*

$$\sigma_{3,o}\text{mex}\left(p^{2k+2}n+p^{2k+1}s+\frac{p^{2k+2}-1}{3}\right) = \sigma_{3,e}\text{mex}\left(p^{2k+2}n+p^{2k+1}s+\frac{p^{2k+2}-1}{3}\right). \tag{7.7}$$

Again, using two identities of Newman [86] and the theory of Lucas sequences, we prove the following families of congruence properties relating $\sigma_{2,o}\text{mex}(n)$ with $\sigma_{2,e}\text{mex}(n)$ and $\sigma_{3,o}\text{mex}(n)$ with $\sigma_{3,e}\text{mex}(n)$ modulo any positive integer $M \geq 2$.

Theorem 7.12. *Let p be a prime with $p \equiv 1 \pmod{24}$ and $M \geq 2$ be an integer.*

Let $G_p(k)$ and $H_p(k)$ be defined by

$$G_p(k+2) = \gamma_2\left(\frac{5(p-1)}{24}\right) G_p(k+1) - \left(\frac{2}{p}\right)_L G_p(k), \tag{7.8}$$

$$H_p(k+2) = \gamma_2\left(\frac{5(p-1)}{24}\right) H_p(k+1) - \left(\frac{2}{p}\right)_L H_p(k), \tag{7.9}$$

with $G_p(0) = 0$, $G_p(1) = 1$, $H_p(0) = 1$, $H_p(1) = 0$, $\gamma_r(n)$ defined as in (7.5) and $R_{G_p}(M)$ be the rank of $G_p(n)$ modulo M.

(a) For $n, k \geq 0$ with $p \nmid (24n+5)$, we have

$$\sigma_{2,o}\text{mex}\left(p^{R_{G_p}(M)(k+1)-1}n + \frac{5\left(p^{R_{G_p}(M)(k+1)-1}-1\right)}{24}\right) \equiv \sigma_{2,e}\text{mex}\left(p^{R_{G_p}(M)(k+1)-1}n + \frac{5\left(p^{R_{G_p}(M)(k+1)-1}-1\right)}{24}\right) \pmod{M}. \tag{7.10}$$

(b) If $\gamma_2\left(\frac{5(p-1)}{24}\right) \equiv 0 \pmod{M}$, then we have

$$\sigma_{2,o}\text{mex}\left(p^{R_{G_p}(M)k+1}n + \frac{5\left(p^{R_{G_p}(M)k+1}-1\right)}{24}\right) \equiv \sigma_{2,e}\text{mex}\left(p^{R_{G_p}(M)k+1}n + \frac{5\left(p^{R_{G_p}(M)k+1}-1\right)}{24}\right) \pmod{M}. \tag{7.11}$$

Theorem 7.13. *Let p be a prime with $p \equiv 1 \pmod 6$ and $M \geq 2$ be an integer. Let $U_p(k)$ and $V_p(k)$ be defined by*

$$U_p(k+2) = \gamma_3\left(\frac{p-1}{3}\right) U_p(k+1) - \left(\frac{-3}{p}\right)_L U_p(k), \tag{7.12}$$

$$V_p(k+2) = \gamma_3\left(\frac{p-1}{3}\right) V_p(k+1) - \left(\frac{-3}{p}\right)_L V_p(k), \tag{7.13}$$

with $U_p(0) = 0$, $U_p(1) = 1$, $V_p(0) = 1$, $V_p(1) = 0$, $\gamma_r(n)$ defined as in (7.5) and $R_{U_p}(M)$ be the rank of $U_p(n)$ modulo M.

(a) For $n, k \geq 0$ with $p \nmid (3n+1)$, we have

$$\sigma_{3,o}\text{mex}\left(p^{R_{U_p}(M)(k+1)-1}n + \frac{p^{R_{U_p}(M)(k+1)-1}-1}{3}\right) \equiv \sigma_{3,e}\text{mex}\left(p^{R_{U_p}(M)(k+1)-1}n + \frac{p^{R_{U_p}(M)(k+1)-1}-1}{3}\right) \pmod{M}. \tag{7.14}$$

(b) If $\gamma_3\left(\frac{p-1}{3}\right) \equiv 0 \pmod{M}$*, then we have*

$$\begin{aligned}\sigma_{3,o}\mathrm{mex}&\left(p^{R_{U_p}(M)k+1}n+\frac{p^{R_{U_p}(M)k+1}-1}{3}\right)\\ &\equiv \sigma_{3,e}\mathrm{mex}\left(p^{R_{U_p}(M)k+1}n+\frac{p^{R_{U_p}(M)k+1}-1}{3}\right) \pmod{M}. \qquad (7.15)\end{aligned}$$

Employing a result of Ono and Taguchi [88] on the nilpotency of Hecke operators, Singh and Barman [17, 113] found certain infinite families of congruences for $\overline{C}_{4r,r}(n)$ modulo arbitrary powers of 2 (see [113, Theorems 1.3 and 1.4]) and $p_{r,r}$ modulo 2 (see [17, Theorems 1.3 and 1.4]). Those results along with Theorem 7.6 give new families of congruences for $\sigma_r\mathrm{mex}(n)$ and $p_{r,r}(n)$. We state them in the following.

Theorem 7.14. *Let* $r = 2^\alpha$ *with* $\alpha \geq 0$ *an integer. Then there exists an integer* $c \geq 0$ *such that for every* $d \geq 1$ *and distinct primes* $p_1, \ldots, p_{c+d}$ *coprime to 6, we have*

$$\sigma_r\mathrm{mex}\left(\frac{p_1\cdots p_{c+d}\cdot n + 1 - 3\cdot 2^\alpha}{24}\right) \equiv 0 \pmod{2^d}$$

whenever n *is coprime to* $p_1, \ldots, p_{c+d}$.

Theorem 7.15. *Let* $r = 2^\alpha$ *with* $\alpha \geq 0$ *an integer. Then there exists an integer* $k \geq 0$ *such that for every* $\ell \geq 1$ *and distinct primes* $s_1, \cdots, s_{k+\ell}$ *coprime to 6, we have*

$$a_r\left(\frac{s_1\cdots s_{k+\ell}\cdot n + 1 - 3\cdot 2^\alpha}{24}\right) \equiv 0 \pmod{2}$$

whenever n *is coprime to* $s_1, \cdots, s_{k+\ell}$.

Similar infinite families of congruences for $\overline{C}_{4r,r}(n)$ and $p_{r,r}(n)$ can be also deduced for $r = 3\cdot 2^\alpha$.

Chakraborty and Ray [40] proved that $\sigma_r\mathrm{mex}(n)$ is lacunary modulo 2^k for $r = 2$ and 3. Using a result of Singh and Barman [113] on the density of $\overline{C}_{4r,r}(n)$ for some general values of r (see [113, Theorem 1.1]), we find the arithmetic densities of $\sigma_{r,o}\mathrm{mex}(n)$ and $\sigma_{r,e}\mathrm{mex}(n)$ modulo arbitrary powers of 2. To be specific, we prove the following two theorems.

Theorem 7.16. *Let* k *be a fixed positive integer. Then for* $r = 2^\alpha m$*, where* α *is nonnegative integer and* m *is odd integer with* $2^\alpha \geq m$*, the series* $\sum_{n=0}^{\infty}\sigma_{r,o}\mathrm{mex}(n)q^n$

and $\sum_{n=0}^{\infty} \sigma_{r,e}\mathrm{mex}(n)q^n$ *are lacunary modulo* 2^k*. And, consequently,* $\sum_{n=0}^{\infty} \sigma_r\mathrm{mex}(n)q^n$ *is lacunary modulo* 2^k*.*

Barman and Singh [17] also studied the density properties of $p_{r,r}(n)$ for certain values of r (see [17, Theorems 1.1 and 1.2]). In view of their result and Theorem 7.6, we have the following result regarding the lacunarity of $a_r(n)$ modulo 2.

Theorem 7.17. *For* $r = 2^\alpha$ *and* $3 \cdot 2^\alpha$*, where* $\alpha \geq 1$*, the series* $\sum_{n=0}^{\infty} a_r(n)q^n$ *is lacunary modulo 2.*

Recently, Ray [101] studied the parity distribution of $\sigma\mathrm{moex}(n)$ (see [101, Theorems 1.4 and 1.5]). As a consequence of their result, we have the following result on the distribution of $\sigma_r\mathrm{moex}(n)$ modulo 2.

Theorem 7.18. *For every positive integer* n *and* r*, we have*

$$\{1 \leq n \leq X : \sigma_r\mathrm{moex}(n) \equiv t \pmod 2\} \geq \alpha_t \log\log X,$$

where $t \in \{0, 1\}$ *and* α_t *is a constant.*

We organize the chapter in the following way. In Section 7.2, we present an alternative proof of Theorem 7.2 and establish Theorems 7.3–7.6. We prove the asymptotic results in the Theorems 7.7 and 7.8 in Section 7.3 by using a Tauberian theorem of Ingham [73]. Employing certain identities of Ahlgren [1], Cooper, Hirschhorn, and Lewis [46] and Newman [86], we prove Theorems 7.10–7.11 and Theorems 7.12–7.13 in Sections 7.4 and 7.5, respectively. The remaining theorems are deduced in Section 7.6. We conclude the chapter by mentioning some possible directions for future study in Section 7.7.

7.2 Proofs of the Theorems 7.2–7.6

7.2.1 An alternative proof of Theorem 7.2

Proof. Let $p_{r\text{-mex}}(m,n)$ be the number of partitions π of n such that r-mex(π) i.e., least r-gap of π, is m. We have

$$\begin{aligned}
M(z,q) :&= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p_{r\text{-mex}}(m,n) z^m q^n \\
&= \sum_{m=1}^{\infty} z^m q^{r\cdot 1+r\cdot 2+\cdots+r(m-1)} \left(1+q^m+\cdots+q^{m(r-1)}\right) \frac{1}{\prod\limits_{\substack{n=1\\ n\neq m}}^{\infty}(1-q^n)} \\
&= \frac{1}{(q;q)_\infty} \sum_{m=1}^{\infty} z^m q^{r\cdot\frac{m(m-1)}{2}} \left(1+q^m+\cdots+q^{m(r-1)}\right)(1-q^m) \\
&= \frac{1}{(q;q)_\infty} \sum_{m=1}^{\infty} z^m q^{r\cdot\frac{m(m-1)}{2}} (1-q^{rm}).
\end{aligned}$$

Differentiating $M(z,q)$ with respect to z and putting $z=1$, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \sigma_r \mathrm{mex}(n) q^n &= \frac{\partial}{\partial z} M(z,q)\Big|_{z=1} \\
&= \frac{1}{(q;q)_\infty} \sum_{m=1}^{\infty} m q^{r\cdot\frac{m(m-1)}{2}} (1-q^{rm}) \\
&= \frac{1}{(q;q)_\infty} \left(\sum_{m=0}^{\infty} (m+1) q^{r\cdot\frac{m(m+1)}{2}} - \sum_{m=0}^{\infty} m q^{r\cdot\frac{m(m+1)}{2}} \right) \\
&= \frac{1}{(q;q)_\infty} \sum_{m=0}^{\infty} q^{r\cdot\frac{m(m+1)}{2}} \\
&= \frac{1}{(q;q)_\infty} \frac{(q^{2r};q^{2r})_\infty}{(q^r;q^{2r})_\infty} = \frac{f_{2r}^2}{f_1 f_r}.
\end{aligned}$$

□

7.2.2 Proof of Theorem 7.3

Proof. Let $p^{o}_{r\text{-mex}}(2m+1,n)$ be the number of partitions of n with odd r-mex $2m+1$. Then we have

$$\begin{aligned}
M_1(z,q) :&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p^{o}_{r\text{-mex}}(2m+1,n) z^{2m+1} q^{n} \\
&= \sum_{m=0}^{\infty} z^{2m+1} q^{r\cdot 1 + r\cdot 2 + \cdots + r\cdot 2m} \left(1+q^{2m+1}+\cdots+q^{(2m+1)(r-1)}\right) \frac{1}{\prod\limits_{\substack{n=1\\ n\neq 2m+1}}^{\infty}(1-q^n)} \\
&= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty} z^{2m+1} q^{r\cdot\frac{2m(2m+1)}{2}} \frac{1-q^{r(2m+1)}}{1-q^{2m+1}}\left(1-q^{2m+1}\right) \\
&= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty} z^{2m+1} q^{r\cdot\frac{2m(2m+1)}{2}} \left(1-q^{r(2m+1)}\right). \qquad (7.16)
\end{aligned}$$

Differentiating $M_1(z,q)$ with respect to z and putting $z=1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\sigma_{r,o}\text{mex}(n)q^n &= \frac{\partial}{\partial z}M_1(z,q)\Big|_{z=1} \\
&= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty}(2m+1)q^{r\cdot\frac{2m(2m+1)}{2}}\left(1-q^{r(2m+1)}\right) \\
&= \frac{1}{(q;q)_\infty}\left(\sum_{m=0}^{\infty}(2m+1)q^{r\cdot\frac{2m(2m+1)}{2}} - \sum_{m=0}^{\infty}(2m+1)q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\right) \\
&= \frac{1}{(q;q)_\infty}\left(\sum_{m=0}^{\infty}(2m+1)q^{r\cdot\frac{2m(2m+1)}{2}} - \sum_{m=0}^{\infty}(2m+2)q^{r\cdot\frac{(2m+1)(2m+2)}{2}} \right. \\
&\qquad \left. + \sum_{m=0}^{\infty} q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\right) \\
&= \frac{1}{(q;q)_\infty}\left(\sum_{m=0}^{\infty}(m+1)(-1)^m q^{r\cdot\frac{m(m+1)}{2}} + \sum_{m=0}^{\infty} q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\right) \\
&= \frac{1}{2(q;q)_\infty}\left(\sum_{m=0}^{\infty}(2m+2)(-1)^m q^{r\cdot\frac{m(m+1)}{2}} + \sum_{m=0}^{\infty} 2q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\right) \\
&= \frac{1}{2(q;q)_\infty}\left(\sum_{m=0}^{\infty}(2m+1)(-1)^m q^{r\cdot\frac{m(m+1)}{2}} + \sum_{m=0}^{\infty}(-1)^m q^{r\cdot\frac{m(m+1)}{2}} \right. \\
&\qquad \left. + 2\sum_{m=0}^{\infty} q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\right) \\
&= \frac{1}{2(q;q)_\infty}\left((q^r;q^r)_\infty^3 + \sum_{m=0}^{\infty} q^{r\cdot\frac{2m(2m+1)}{2}} - \sum_{m=0}^{\infty} q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\right.
\end{aligned}$$

$$+2\sum_{m=0}^{\infty} q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\Bigg)$$

$$= \frac{1}{2(q;q)_\infty}\left((q^r;q^r)_\infty^3 + \sum_{m=0}^{\infty} q^{r\cdot\frac{2m(2m+1)}{2}} + \sum_{m=0}^{\infty} q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\right)$$

$$= \frac{1}{2(q;q)_\infty}\left((q^r;q^r)_\infty^3 + \sum_{m=0}^{\infty} q^{r\cdot\frac{m(m+1)}{2}}\right)$$

$$= \frac{1}{2f_1}\left(f_r^3 + \frac{f_{2r}^2}{f_r}\right) = \frac{1}{2}\left(\frac{f_{2r}^2}{f_1 f_r} + \frac{f_r^3}{f_1}\right).$$

Again, using the expressions for $\sigma_r\mathrm{mex}(n)$ and $\sigma_{r,o}\mathrm{mex}(n)$ in the relation

$$\sigma_r\mathrm{mex}(n) = \sigma_{r,o}\mathrm{mex}(n) + \sigma_{r,e}\mathrm{mex}(n),$$

we find the expression for $\sigma_{r,e}\mathrm{mex}(n)$. □

7.2.3 Proof of Theorem 7.4

Proof. Let $p_{r\text{-moex}}(2m+1,n)$ be the number of partitions π of n with $r\text{-moex}(\pi) = 2m+1$. Then we have

$$M_2(z,q) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{r\text{-moex}}(2m+1,n) z^{2m+1} q^n$$

$$= \sum_{m=0}^{\infty} z^{2m+1} q^{r\cdot 1 + r\cdot 3 + \cdots + r(2m-1)} \left(1 + q^{2m+1} + \cdots + q^{(2m+1)(r-1)}\right) \frac{1}{\prod\limits_{\substack{n=1\\ n\neq 2m+1}}^{\infty} (1-q^n)}$$

$$= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty} z^{2m+1} q^{rm^2} \frac{1-q^{r(2m+1)}}{1-q^{2m+1}} \left(1-q^{2m+1}\right)$$

$$= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty} z^{2m+1} q^{rm^2} \left(1-q^{r(2m+1)}\right).$$

Differentiating $M_2(z,q)$ with respect to z and putting $z=1$, we obtain

$$\sum_{n=0}^{\infty} \sigma_r\mathrm{moex}(n) q^n = \frac{\partial}{\partial z} M_2(z,q)\Big|_{z=1}$$

$$= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty}(2m+1) q^{rm^2}\left(1-q^{r(2m+1)}\right)$$

$$= \frac{1}{(q;q)_\infty}\left(\sum_{m=0}^{\infty}(2m+1) q^{rm^2} - \sum_{m=1}^{\infty}(2m-1) q^{rm^2}\right)$$

$$= \frac{1}{(q;q)_\infty}\left(1+2\sum_{m=1}^{\infty} q^{rm^2}\right)$$
$$= \frac{1}{(q;q)_\infty}(q^{2r};q^{2r})_\infty(-q^r;q^{2r})_\infty^2 = \frac{f_{2r}^5}{f_1 f_r^2 f_{4r}^2}.$$

□

7.2.4 Proof of Theorem 7.5

Proof. Putting $z=1$ in $M_1(z,q)$ in (7.16), we have

$$\sum_{n=0}^{\infty} a_r(n)q^n = M_1(z,q)\big|_{z=1}$$
$$= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty} q^{r\cdot\frac{2m(2m+1)}{2}}\left(1-q^{r(2m+1)}\right)$$
$$= \frac{1}{(q;q)_\infty}\left(\sum_{m=0}^{\infty} q^{r\cdot\frac{2m(2m+1)}{2}} - \sum_{m=0}^{\infty} q^{r\cdot\frac{(2m+1)(2m+2)}{2}}\right)$$
$$= \frac{1}{(q;q)_\infty}\sum_{m=0}^{\infty}(-1)^m q^{r\cdot\frac{m(m+1)}{2}}.$$

□

7.2.5 Proof of Theorem 7.6

Proof. Putting $k=4r$ and $i=r$ in the generating function of $\overline{C}_{k,i}(n)$ (see [113, Eq. (1.1)]), we have

$$\sum_{n=0}^{\infty}\overline{C}_{4r,r}(n)q^n = \frac{(q^{4r};q^{4r})_\infty(-q^r;q^{4r})_\infty(-q^{3r};q^{4r})_\infty}{(q;q)_\infty}$$
$$= \frac{(q^{4r};q^{4r})_\infty(-q^r;q^{2r})_\infty}{(q;q)_\infty}$$
$$= \frac{(q^{4r};q^{4r})_\infty(q^{2r};q^{4r})_\infty}{(q;q)_\infty(q^r;q^{2r})_\infty}$$
$$= \frac{(q^{2r};q^{2r})_\infty^2}{(q;q)_\infty(q^r;q^r)_\infty} = \sum_{n=0}^{\infty}\sigma_r\text{mex}(n)q^n.$$

Comparing the coefficients of q^n on both sides of the above equation, we arrive at

$$\overline{C}_{4r,r}(n) = \sigma_r\text{mex}(n).$$

Again, let $p_{r,r}(n)$ denote the number of partitions π of n such that $\text{mex}_{r,r}(\pi) \equiv r \pmod{2r}$. Andrews and Newman [7, Lemma 9] proved that the generating function for $p_{r,r}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{r,r}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{r\cdot\frac{n(n+1)}{2}} = \sum_{n=0}^{\infty} a_r(n)q^n.$$

Comparing the coefficients of q^n on both sides of the above equation, we find that

$$p_{r,r}(n) = a_r(n).$$

Thus, we complete the proof of Theorem 7.6. $\square$

7.3 Proofs of Theorems 7.7 and 7.8

Proofs of Theorems 7.7 and 7.8 are based on the following result of Ingham [73] regarding the asymptotic behaviour of coefficients of a power series.

Theorem 7.19. *Let $C(q) := \sum_{n=0}^{\infty} c(n)q^n$ be a power series with radius of convergence 1. Assume that $\{c(n)\}$ is weakly increasing sequence of nonnegative real numbers. If there are constants $\mu, \nu \in \mathbb{R}$ and $\lambda > 0$ such that*

$$C\left(e^{-y}\right) \sim \mu y^{\nu} e^{\frac{\lambda}{y}}, \ \text{ as } y \to 0^+,$$

then we have

$$c(n) \sim \frac{\mu}{2\sqrt{\pi}}\frac{\lambda^{\frac{2\nu+1}{4}}}{n^{\frac{2\nu+3}{4}}}e^{2\sqrt{\lambda n}} \ \text{ as } n \to \infty.$$

First we prove the following lemma about the weakly increasing nature of $\sigma_r\text{mex}(n)$ using combinatorial arguments.

Lemma 7.20. *The sequences $\{\sigma_r\text{mex}(n)\}$ and $\{\sigma_r\text{moex}(n)\}$ are weakly increasing.*

Proof. Let us construct a map $\Psi : \mathcal{P}(n) \to \mathcal{P}(n+1)$ as the following:
(1) if $r\text{-mex}(\pi) \neq 1$, then $\Psi(\pi)$ is the partition of $n+1$ with 1 added as a part to π.
(2) if $r\text{-mex}(\pi) = 1$, then $\Psi(\pi)$ is the partition of $n+1$ with 1 added to the largest part of π, that is, the largest part of π is increased by 1.

Also, we note that for two distinct partitions π_1 and π_2 of $\mathcal{P}(n)$, we have $\Psi(\pi_1)$ and $\Psi(\pi_2)$ are also distinct, which implies that the map Ψ is injective. Also, for any

$\pi \in \mathcal{P}(n)$, r-mex(π) is invariant under the map Ψ. Hence for $n \geq 1$, we have

$$\sigma_r\text{mex}(n) \leq \sigma_r\text{mex}(n+1)$$

and the sequence $\{\sigma_r\text{mex}(n)\}$ is weakly increasing.

Next, we show that the sequence $\{\sigma_r\text{moex}(n)\}$ is also weakly increasing. We construct the map $\Phi : \mathcal{P}(n) \to \mathcal{P}(n+1)$ in a similar fashion to the previous map Ψ:
(1) if r-moex$(\pi) \neq 1$, then $\Phi(\pi)$ is the partition of $n+1$ with 1 added as a part to π.
(2) if r-moex$(\pi) = 1$, then $\Phi(\pi)$ is the partition of $n+1$ with 1 added to the largest part of π, that is, the largest part of π is increased by 1.

Then by proceeding via similar lines of arguments, we conclude that the sequence $\{\sigma_r\text{moex}(n)\}$ is weakly increasing.

□

Now, we are in a position to prove Theorems 7.7 and 7.8.

Proof of Theorem 7.7. We recall the generating function of $\sigma_r\text{mex}(n)$

$$X_r(q) := \sum_{n=0}^{\infty} \sigma_r\text{mex}(n)q^n = \frac{f_{2r}^2}{f_1 f_r}.$$

Using the transformation formula for the Dedekind's eta-function [79, p. 121], one can show that, for $t \to 0^+$,

$$\frac{1}{(e^{-t};e^{-t})_\infty} \sim \sqrt{\frac{t}{2\pi}}e^{\frac{\pi^2}{6t}}. \tag{7.17}$$

In view of (7.17), we have

$$X_r(e^{-t}) \sim \frac{1}{2\sqrt{r}}e^{\frac{\pi^2}{6t}} \qquad \text{as } t \to 0^+.$$

Also, from the above lemma, we know that the sequence $\{\sigma_r\text{mex}(n)\}$ is weakly increasing sequence of nonnegative real numbers. Therefore, we invoke Theorem 7.19 with $\mu = \frac{1}{2\sqrt{r}}$, $\nu = 0$, and $\lambda = \frac{\pi^2}{6}$ to obtain

$$\sigma_r\text{mex}(n) \sim \frac{1}{4\sqrt[4]{6n^3r^2}}e^{\pi\sqrt{\frac{2n}{3}}} \text{ as } n \to \infty.$$

Thus, we complete the proof. □

Proof of Theorem 7.8. Recall that the generating function for $\sigma_r\text{moex}(n)$ is given by

$$Y_r(q) := \sum_{n=0}^{\infty} \sigma_r\text{moex}(n)q^n = \frac{f_{2r}^5}{f_1 f_r^2 f_{4r}^2}.$$

Using (7.17), we obtain

$$Y_r(e^{-t}) \sim \frac{1}{\sqrt{2r}} e^{\frac{\pi^2}{6t}} \qquad \text{as} \quad t \to 0+.$$

From Lemma 7.20, we know that $\{\sigma_r\text{moex}(n)\}$ is weakly increasing sequence. Also, $Y_r(q)$ has real nonnegative coefficients. Hence, employing Theorem 7.19 with $\mu = \frac{1}{\sqrt{2r}}$, $\nu = 0$, and $\lambda = \frac{\pi^2}{6}$, we arrive at

$$\sigma_r\text{moex}(n) \sim \frac{1}{2\sqrt[4]{24n^3r^2}} e^{\pi\sqrt{\frac{2n}{3}}} \qquad \text{as} \quad n \to \infty,$$

which completes the proof. □

7.4 Proofs of Theorems 7.10 and 7.11

First, we proof some necessary lemmas.

Lemma 7.21. *Let $\gamma_r(n)$ is defined as in* (7.5)*. Then, for $n \geq 0$ and $r \geq 1$, we have*

$$\sigma_{r,o}\text{mex}(n) = \sigma_{r,e}\text{mex}(n) + \gamma_r(n).$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \sigma_{r,o}\text{mex}(n)q^n &= \frac{1}{2}\left(\frac{f_{2r}^2}{f_1 f_r} + \frac{f_r^3}{f_1}\right) \\ &= \frac{1}{2}\left(\frac{f_{2r}^2}{f_1 f_r} - \frac{f_r^3}{f_1}\right) + \frac{f_r^3}{f_1} \\ &= \sum_{n=0}^{\infty} \sigma_{r,e}\text{mex}(n)q^n + \sum_{n=0}^{\infty} \gamma_r(n)q^n. \end{aligned} \tag{7.18}$$

Comparing the coefficients of q^n on both sides of the above, we complete the proof. □

Lemma 7.22. *Let $n \geq 0$, $k \geq 0$, $1 \leq s \leq p-1$ be positive integers, p be a prime congruent to* $7, 11, 13, 17, 19, 23 \pmod{24}$ *and $\gamma_r(n)$ defined as in* (7.5)*. Then, we* *have*

$$\gamma_2\left(p^{2k+2}n+p^{2k+1}s+\frac{5(p^{2k+2}-1)}{24}\right)=0.$$

Proof. Firstly, we note the following two identities involving $\gamma_2(n)$ from [1, 46]:

$$\gamma_2\left(pn+\frac{5(p^2-1)}{24}\right)=\gamma_2\left(\frac{n}{p}\right),\quad \text{when } p\equiv 7,\ 13 \text{ or } 23 \pmod{24},$$
$$\gamma_2\left(pn+\frac{5(p^2-1)}{24}\right)=-\gamma_2\left(\frac{n}{p}\right),\quad \text{when } p\equiv 11,\ 17 \text{ or } 19 \pmod{24}.$$

Case 1: When $p\equiv 11,17,19 \pmod{24}$:

Let $\epsilon=-1$, Then for $p\equiv 11,17,19 \pmod{24}$, we have

$$\gamma_2\left(pn+\frac{5(p^2-1)}{24}\right)=\epsilon\ \gamma_2\left(\frac{n}{p}\right). \tag{7.19}$$

Replacing n by $pn+s$, $1\le s\le p-1$ in the above equation, we have

$$\gamma_2\left(p^2n+ps+\frac{5(p^2-1)}{24}\right)=0. \tag{7.20}$$

Also, replacing n by pn in (7.19), we find that

$$\gamma_2\left(p^2n+\frac{5(p^2-1)}{24}\right)=\epsilon\ \gamma_2\left(n\right). \tag{7.21}$$

Iterating the above identity, we obtain

$$\gamma_2\left(p^{2k}n+\frac{5(p^{2k}-1)}{24}\right)=\epsilon^k\ \gamma_2\left(n\right).$$

Replacing n by $p^2n+ps+\frac{5(p^2-1)}{24}$ in the above, we arrive at

$$\gamma_2\left(p^{2k+2}n+p^{2k+1}s+\frac{5(p^{2k+2}-1)}{24}\right)=0. \tag{7.22}$$

Case 2: When $p\equiv 7,13,23 \pmod{24}$:

For $p\equiv 7,13,23 \pmod{24}$, we have

$$\gamma_2\left(pn+\frac{5(p^2-1)}{24}\right)=\gamma_2\left(\frac{n}{p}\right). \tag{7.23}$$

Now, repeating the same steps as in the Case 1 with $\epsilon=1$, we complete the proof of Lemma 7.22. □

Proof of Theorem 7.10. Putting $r=2$ in Lemma 7.21, we have

$$\sigma_{2,o}\text{mex}(n)=\sigma_{2,e}\text{mex}(n)+\gamma_2(n).$$

Next, replacing n by $p^{2k+2}n+p^{2k+1}s+\frac{5(p^{2k+2}-1)}{24}$ in the above equation and employing Lemma 7.22, we complete the proof of Theorem 7.10. □

Proof of Theorem 7.11. The proof of Theorem 7.11 is similar to that of Theorem 7.10. Hence, we skip the details of the proof. We only mention an useful identity involving $\gamma_3(n)$ which was conjectured by Cooper, Hirschhorn, and Lewis [46], and proved by Ahlgren [1]:

$$\gamma_3\left(pn + \frac{p^2-1}{3}\right) = \gamma_3\left(\frac{n}{p}\right), \quad \text{when} \quad p \equiv 5 \pmod 6). \tag{7.24}$$

□

7.5 Proofs of Theorems 7.12 and 7.13

Firstly, we state some lemmas related to (a, b)-Lucas sequence $S(n)$ and the dual sequence $T(n)$ defined by

$$S(n) = aS(n-1) - bS(n-2) \tag{7.25}$$

and

$$T(n) = aT(n-1) - bT(n-2) \tag{7.26}$$

where $S(0) = 0$, $S(1) = 1$, $T(0) = 1$, and $T(1) = 0$.

The first two lemmas are based on relations between (a, b)-Lucas sequence $S(n)$ and their dual sequences $T(n)$. while the next two lemmas provide information regarding $R_S(M)$.

Lemma 7.23. *[126, Lemma 2.2] Let $S(n)$ and $T(n)$ be given by* (7.25) *and* (7.26)*, respectively. For $n, k \geq 0$, we have*

$$S(n+k) = S(k)S(n+1) + T(k)S(n) \tag{7.27}$$

and

$$T(n+k) = S(k)T(n+1) + T(k)T(n). \tag{7.28}$$

Lemma 7.24. *[126, Lemma 2.5] Let $S(n)$ and $T(n)$ be defined by* (7.25) *and* (7.26)*, respectively. For $n \geq 0$, we have*

$$aS(n) + T(n) = S(n+1). \tag{7.29}$$

Lemma 7.25. *[126, Lemma 2.1] Let $M \geq 2$ be an integer and let $S(n)$ be given by (7.25). Suppose that $M = \prod_{i=1}^{t} p_i^{k_i}$ ($k_i \geq 1$) is the prime factorization of M and* $\gcd(M, b) = 1$. *Then, $R_S(M)$ exists and*

$$R_S(M) \leq \prod_{i=1}^{t} p_i^{k_i - 1} (p_i + 1) . \tag{7.30}$$

Lemma 7.26. *[126, Lemma 2.3] Let $S(n)$ be defined by (7.25) and let $R_S(M)$ denote the rank of $S(n)$ modulo M. For $k \geq 0$, we have*

$$S(R_S(M)k) \equiv 0 \pmod{M}. \tag{7.31}$$

Next, we prove the following lemma involving $\gamma_2(n)$.

Lemma 7.27. *Let p be a prime with $p \equiv 1 \pmod{24}$ and $\gamma_r(n)$ is defined by (7.5). We have*

$$\gamma_2\left(p^k n + \frac{5(p^k-1)}{24}\right) = G_p(k)\gamma_2\left(pn + \frac{5(p-1)}{24}\right) + H_p(k)\gamma_2(n), \tag{7.32}$$

where $G_p(k)$ and $H_p(k)$ are defined by

$$G_p(k+2) = \gamma_2\left(\frac{5(p-1)}{24}\right) G_p(k+1) - \left(\frac{2}{p}\right)_L G_p(k) \tag{7.33}$$

and

$$H_p(k+2) = \gamma_2\left(\frac{5(p-1)}{24}\right) H_p(k+1) - \left(\frac{2}{p}\right)_L H_p(k), \tag{7.34}$$

with $G_p(0) = H_p(1) = 0$ and $G_p(1) = H_p(0) = 1$.

Proof. We prove the lemma by induction on k using the method of Xia [126] based on Newman's identities and Lucas sequences. Since $G_p(0) = H_p(1) = 0$ and $G_p(1) = H_p(0) = 1$, we have (7.32) is true for $k = 0$ and $k =1$. We now assume that (7.32) is true for $k = m$ and $k = m + 1$ for some $m \geq 0$, which gives

$$\gamma_2\left(p^m n + \frac{5(p^m-1)}{24}\right) = G_p(m)\gamma_2\left(pn + \frac{5(p-1)}{24}\right) + H_p(m)\gamma_2(n) \tag{7.35}$$

and

$$\gamma_2\left(p^{m+1} n + \frac{5(p^{m+1}-1)}{24}\right) = G_p(m+1)\gamma_2\left(pn + \frac{5(p-1)}{24}\right) + H_p(m+1)\gamma_2(n). \tag{7.36}$$

For primes $p \equiv 1 \pmod{24}$, Newman [86] proved the following relation involving

$\gamma_2(n)$

$$\gamma_2\left(pn+\frac{5(p-1)}{24}\right)=\gamma_2\left(\frac{5(p-1)}{24}\right)\gamma_2(n)-\left(\frac{2}{p}\right)_L\gamma_2\left(\frac{n-\frac{5(p-1)}{24}}{p}\right). \quad (7.37)$$

Replacing n by $pn+\frac{5(p-1)}{24}$ in (7.37), we have

$$\gamma_2\left(p^2n+\frac{5(p^2-1)}{24}\right)=\gamma_2\left(\frac{5(p-1)}{24}\right)\gamma_2\left(pn+\frac{5(p-1)}{24}\right)-\left(\frac{2}{p}\right)_L\gamma_2\left(n\right). \quad (7.38)$$

Again, replacing n by $p^mn+\frac{5(p^m-1)}{24}$ in (7.38) and then employing (7.35) and (7.36), we find that

$$\begin{aligned}
&\gamma_2\left(p^{m+2}n+\frac{5(p^{m+2}-1)}{24}\right)\\
&=\gamma_2\left(\frac{5(p-1)}{24}\right)\gamma_2\left(p^{m+1}n+\frac{5(p^{m+1}-1)}{24}\right)-\left(\frac{2}{p}\right)_L\gamma_2\left(p^mn+\frac{5(p^m-1)}{24}\right)\\
&=\gamma_2\left(\frac{5(p-1)}{24}\right)\left(G_p(m+1)\gamma_2\left(pn+\frac{5(p-1)}{24}\right)+H_p(m+1)\gamma_2(n)\right)\\
&\quad-\left(\frac{2}{p}\right)_L\left(G_p(m)\gamma_2\left(pn+\frac{5(p-1)}{24}\right)+H_p(m)\gamma_2(n)\right)\\
&=\gamma_2\left(pn+\frac{5(p-1)}{24}\right)\left(\gamma_2\left(\frac{5(p-1)}{24}\right)G_p(m+1)-\left(\frac{2}{p}\right)_L G_p(m)\right)\\
&\quad+\gamma_2(n)\left(\gamma_2\left(\frac{5(p-1)}{24}\right)H_p(m+1)-\left(\frac{2}{p}\right)_L H_p(m)\right)\\
&=\gamma_2\left(pn+\frac{5(p-1)}{24}\right)G_p(m+2)+\gamma_2(n)H_p(m+2),
\end{aligned}$$

which implies that (7.32) holds for $k=m+2$ also. Hence, by the principle of mathematical induction, we complete the proof of the lemma. □

Now, we are in a position to prove Theorem 7.12.

Proof of Theorem 7.12. First we substitute (7.37) in (7.32). Thus

$$\begin{aligned}
&\gamma_2\left(p^kn+\frac{5(p^k-1)}{24}\right)\\
&=G_p(k)\left(\gamma_2\left(\frac{5(p-1)}{24}\right)\gamma_2(n)-\left(\frac{2}{p}\right)_L\gamma_2\left(\frac{n-\frac{5(p-1)}{24}}{p}\right)\right)+H_p(k)\gamma_2(n)
\end{aligned}$$

$$= \left(\gamma_2\left(\frac{5(p-1)}{24}\right)G_p(k)+H_p(k)\right)\gamma_2(n)-\left(\frac{2}{p}\right)_L G_p(k)\gamma_2\left(\frac{n-\frac{5(p-1)}{24}}{p}\right). \tag{7.39}$$

Since $G_p(k)$ and $H_p(k)$ are Lucas sequences as defined by (7.33) and (7.34). Using Lemma 7.24, we find that

$$\gamma_2\left(\frac{5(p-1)}{24}\right)G_p(k)+H_p(k)=G_p(k+1). \tag{7.40}$$

Employing (7.40) in (7.39), we have

$$\gamma_2\left(p^k n+\frac{5(p^k-1)}{24}\right)=G_p(k+1)\gamma_2(n)-\left(\frac{2}{p}\right)_L G_p(k)\gamma_2\left(\frac{n-\frac{5(p-1)}{24}}{p}\right). \tag{7.41}$$

For any Lucas sequence $S(n)$, the existence of $R_S(M)$ is guaranteed by Lemma 7.25. Therefore, $R_{G_p}(M)$ also exists.

Replacing k by $R_{G_p}(M)(k+1)-1$ in (7.41), we obtain

$$\gamma_2\left(p^{R_{G_p}(M)(k+1)-1}n+\frac{5(p^{R_{G_p}(M)(k+1)-1}-1)}{24}\right)$$
$$=G_p(R_{G_p}(M)(k+1))\gamma_2(n)-\left(\frac{2}{p}\right)_L G_p(R_{G_p}(M)(k+1)-1)\gamma_2\left(\frac{n-\frac{5(p-1)}{24}}{p}\right). \tag{7.42}$$

Again, in view of Lemma 7.26, for $k\geq 0$, we have

$$G_p\left(R_{G_p}(M)k\right)\equiv 0 \pmod{M}. \tag{7.43}$$

Invoking (7.43) in (7.42), we find that

$$\gamma_2\left(p^{R_{G_p}(M)(k+1)-1}n+\frac{5(p^{R_{G_p}(M)(k+1)-1}-1)}{24}\right)$$
$$\equiv -\left(\frac{2}{p}\right)_L G_p(R_{G_p}(M)(k+1)-1)\gamma_2\left(\frac{n-\frac{5(p-1)}{24}}{p}\right) \pmod{M}. \tag{7.44}$$

Now, if $p \nmid (24n+5)$, then $\gamma_2\left(\dfrac{n-\dfrac{5(p-1)}{24}}{p}\right) = 0$ and from (7.44), we have

$$\gamma_2\left(p^{R_{G_p}(M)(k+1)-1}n + \frac{5(p^{R_{G_p}(M)(k+1)-1}-1)}{24}\right) \equiv 0 \pmod{M}. \tag{7.45}$$

Moreover, if $p \nmid (24n+5)$ and $\gamma_2\left(\frac{5(p-1)}{24}\right) \equiv 0 \pmod{M}$, then (7.37) implies that

$$\gamma_2\left(pn + \frac{5(p-1)}{24}\right) \equiv 0 \pmod{M}. \tag{7.46}$$

Replacing k by $R_{G_p}(M)k$ in (7.32) and then using (7.46) in the resulting identity, we find that

$$\gamma_2\left(p^{R_{G_p}(M)k}n + \frac{5(p^{R_{U_p}(M)k}-1)}{24}\right) \equiv H_p\left(R_{U_p}(M)k\right)\gamma_2(n) \pmod{M}.$$

Again, replacing n by $pn + \dfrac{5(p-1)}{24}$ in the above congruence and then employing (7.46), we have

$$\gamma_2\left(p^{R_{G_p}(M)k+1}n + \frac{5(p^{R_{G_p}(M)k+1}-1)}{24}\right) \equiv 0 \pmod{M}. \tag{7.47}$$

Next, we put $r=2$ in Lemma 7.21 to arrive at

$$\sigma_{2,o}\mathrm{mex}(n) = \sigma_{2,e}\mathrm{mex}(n) + \gamma_2(n). \tag{7.48}$$

We complete the proof of the theorem by employing (7.45) and (7.47) in (7.48). □

Proof of Theorem 7.13. The proof of Theorem 7.13 is similar to the proof of Theorem 7.12. Thus, we omit the details and mention only a lemma involving $\gamma_3(n)$ and an identity due to Newman [86].

Lemma 7.28. *Let p be a prime with $p \equiv 1 \pmod 6$ and $\gamma_r(n)$ is defined by* (7.5)*. We have*

$$\gamma_3\left(p^k n + \frac{p^k-1}{3}\right) = U_p(k)\gamma_3\left(pn + \frac{p-1}{3}\right) + V_p(k)\gamma_3(n), \tag{7.49}$$

where $U_p(k)$ and $V_p(k)$ are defined by

$$U_p(k+2) = \gamma_3\left(\frac{p-1}{3}\right)U_p(k+1) - \left(\frac{-3}{p}\right)_L U_p(k) \tag{7.50}$$

$$V_p(k+2) = \gamma_3\left(\frac{p-1}{3}\right)V_p(k+1) - \left(\frac{-3}{p}\right)_L V_p(k) \tag{7.51}$$

with $U_p(0) = V_p(1) = 0$ *and* $U_p(1) = V_p(0) = 1$.

We also need the following identity for $\gamma_3(n)$ from Newman [86]:

$$\gamma_3\left(pn + \frac{p-1}{3}\right) = \gamma_3\left(\frac{p-1}{3}\right)\gamma_3(n) - \left(\frac{-3}{p}\right)_L \gamma_3\left(\frac{n - \frac{p-1}{3}}{p}\right). \tag{7.52}$$

□

7.6 Proofs of the remaining theorems

7.6.1 Proof of Theorems 7.14–7.15

Proof. Singh and Barman [113, Theorem 1.3] proved the following infinite family of congruences for $\overline{C}_{k,i}(n)$:

Let $r = 2^\alpha$ with $\alpha \geq 0$ an integer. Then there exists an integer $c \geq 0$ such that for every $d \geq 1$ and distinct primes $p_1, \ldots, p_{c+d}$ coprime to 6, we have

$$\overline{C}_{4r,r}\left(\frac{p_1 \cdots p_{c+d} \cdot n + 1 - 3 \cdot 2^\alpha}{24}\right) \equiv 0 \pmod{2^d}$$

whenever n is coprime to $p_1, \ldots, p_{c+d}$.

The above result together with Theorem 7.6 gives Corollary 7.14.

Similarly, Corollary 7.15 can be deduced easily from Theorem 1.3 of [17] and Theorem 7.6.

□

7.6.2 Proof of Theorem 7.16

From (7.4) and the proof of Theorem 7.6, we have

$$\sum_{n=0}^{\infty} \beta_r(n)q^n = \frac{f_{2r}^2}{f_1 f_r} = \sum_{n=0}^{\infty} \overline{C}_{4r,r}(n)q^n.$$

Singh and Barman [113, Theorem 1.1] proved that the series $\sum_{n=0}^{\infty} \overline{C}_{4r,r}(n)q^n$ is lacunary modulo arbitrary powers of 2 whenever $r = 2^\alpha m$ with α is a nonnegative integer and m is positive odd satisfying $2^\alpha \geq m$. This implies that the series $\sum_{n=0}^{\infty} \beta_r(n)q^n$ is lacunary modulo arbitrary powers of 2 for these specific values of r.

Using a similar analysis, we can easily show that the series $\sum_{n=0}^{\infty} \gamma_r(n)q^n$ is also lacunary modulo arbitrary powers of 2 for these values of r. Then the theorem follows from the above observations and (7.2) and (7.3).

7.6.3 Proof of Theorem 7.17

Proof. Barman and Singh [17, Theorems 1.1 and 1.2] proved that for all $\alpha \geq 1$ and $r = 2^\alpha$ and $3 \cdot 2^\alpha$, the series $\sum_{n=0}^{\infty} p_{r,r}(n)q^n$ is lacunary modulo 2. This result together with Theorem 7.6 gives us Theorem 7.17. □

7.6.4 Proof of Theorem 7.18

Proof. From Theorem 7.4, we have

$$\sum_{n=0}^{\infty} \sigma_r\text{moex}(n)q^n = \frac{f_{2r}^5}{f_1 f_r^2 f_{4r}^2} \equiv \frac{1}{f_1} \equiv \sum_{n=0}^{\infty} \sigma\text{moex}(n)q^n \pmod 2,$$

which implies that

$$\sigma_r\text{moex}(n) \equiv \sigma\text{moex}(n) \pmod 2. \tag{7.53}$$

In view of the above equation and Theorems 1.4 and 1.5 of [101], we deduce the corollary.

□

7.7 Concluding remarks

(1) In this chapter, we have deduced some arithmetic properties relating $\sigma_{r,o}\text{mex}(n)$ and $\sigma_{r,e}\text{mex}(n)$ for $r = 2$ and 3. An interesting problem may be to find exact

congruences for these two functions. It is also highly desirable to explore more arithmetic properties of $a_r(n)$ and $\sigma_r\text{moex}(n)$ for general values of r.

(2) Chern [43] studied maex of a partition which is defined as the greatest part smaller than the largest part of the partition which is missing from that partition. It will be interesting to study its generalization analogous to the least r-gap or r-mex. This generalization may be called as the greatest r-gap or r-maex of an integer partition and be defined as the greatest part smaller than the largest part of the partition which occurs less than r times in that partition.

(3) Baruah, Bhoria, Eyyunni and Maji [24] studied the k-th moments of the minimal excludants. In a similar way, the k-th moments of least r-gaps may also be studied.