# Abstract


In this thesis, we find new modular relations for the Rogers-Ramanujan continued fraction and functions, arithmetic properties for some restricted partition functions as well as functions related to the least $r$-gaps in partitions.

We prove some new identities for the Rogers–Ramanujan continued fraction. For example, if $R(q)$ denotes the Rogers–Ramanujan continued fraction, then

$$R(q)R(q^4) = \frac{R(q^5) + R(q^{20}) - R(q^5)R(q^{20})}{1 + R(q^5) + R(q^{20})},$$

$$\frac{1}{R(q^2)R(q^3)} + R(q^2)R(q^3) = 1 + \frac{R(q)}{R(q^6)} + \frac{R(q^6)}{R(q)}.$$

In the process, we also find some new relations for the Rogers-Ramanujan functions by using dissections of theta functions and the quintuple product identity.

Let $d_k(n)$ count the partitions obtained by adding the links of the $k$-elongated plane partition diamonds of length $n$. Andrews and Paule [9] obtained several generating functions and congruences for $d_1(n)$, $d_2(n)$, and $d_3(n)$. Da Silva, Hirschhorn, and Sellers [50] found many congruences modulo 4, 5, 7, 8, 9, and 11 for $d_k(n)$ for certain values of $k$. We extend some individual congruences found by Andrews and Paule [8] and da Silva, Hirschhorn, and Sellers [50] to their respective families as well as find new families of congruences for $d_k(n)$. We also present a refinement in an existence result for congruences of $d_k(n)$ found by da Silva, Hirschhorn, and Sellers [50] , and prove some new families of congruences modulo 5, 7, 8, 11, 13, 16, 17, 19, 23, 25, 32, 49, 64, and 128.

Recently, two analogues, $\overline{a}_t(n)$ and $\overline{b}_t(n)$, of the $t$-core partition function, $c_t(n)$, have been introduced by Gireesh, Ray, and Shivashankar [57] and Bandyopadhyay and Baruah [14], respectively. Using the theory of modular forms, we find the arithmetic densities of $\overline{a}_t(n)$ and $\overline{b}_t(n)$ modulo arbitrary powers of 2 and 3 for $t =$

$3^\alpha m$ where gcd$(m,6)$=1 and $\overline{b}_t(n)$ modulo $p_i^k$ where $t = p_1^{a_1} \cdots p_m^{a_m}$ where $p_i \geq 5$ are primes. We further prove some infinite families of congruences for $\overline{a}_3(n)$ and $\overline{b}_3(n)$ modulo arbitrary powers of 2 and $\overline{b}_2(n)$ modulo 2.

Let $b_6(n)$ denote the number of 6-regular partitions of $n$. We establish some new infinite families of congruences modulo 3 and some individual congruences modulo 9 for $b_6(n)$ using 5-dissections of some $q$-products, two of the well-known forty identities for the Rogers-Ramanujan functions of Ramanujan [31], two identities of Newman [86] and a method of Radu [91]. In the process, we also deduce some Kolberg-type congruences.

We obtain several Ramanujan-type congruences modulo powers of 5 for partition $k$-tuples with 5-cores, for $k = 2, 3, 4$. We also discover some new infinite families of congruences modulo powers of primes for partition $k$-tuples with $p$-cores, where $p$ is a prime.

For $\ell \geq 2$, let $\text{pod}_\ell(n)$ and $\text{ped}_\ell(n)$ denote the number of $\ell$-regular partitions where odd parts are distinct (even parts are unrestricted) and even parts are distinct (odd parts are unrestricted) respectively. We find the arithmetic densities of $\text{pod}_\ell(n)$ for $\ell = 3$, 5, 7, 13, 17 and $\text{ped}_t(n)$ for $t = 13$, 17 modulo 2 and arbitrary powers of 2. We also prove some new multiplicative relations for $\text{pod}_5(n)$, $\text{pod}_9(n)$, $\text{ped}_5(n)$, and $\text{ped}_9(n)$ modulo small powers of 2 with the aid of the theory of Hecke eigenforms.

The minimal excludant of a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of $n$ is the smallest positive integer that is not present in $\pi$ and is denoted by mex$(\pi)$. The least $r$-gap of $\pi$ is the least positive integer that does not appear in the partition at least $r$ times. We study some arithmetic functions related to the sum of least $r$-gaps and establish their connections with certain known partition functions such as Andrews' singular overpartitions [4]. Using a Tauberian theorem due to Ingham [73], we obtain Hardy-Ramanujan-type asymptotic formula for two such functions. We also explore arithmetic properties for these functions.