Chapter 1

Introduction

The thesis is primarily focused on three topics, namely, the Rogers-Ramanujan continued fraction, some restricted partition functions, and the least r-gaps in partitions. We find some new modular identities involving the Rogers-Ramanujan continued fraction and corresponding Rogers-Ramanujan functions. We explore some divisibility properties of certain restricted partition functions and two analogues of the t-core partition function. We also obtain the generating functions for some functions related to the least r-gaps in partitions and deduce arithmetic and asymptotic properties of these functions.

The thesis consists of seven chapters including this introductory chapter. In the following six sections, we refer to a few useful definitions as well as some background material and in the remaining sections of this chapter, we present a brief outline of the work done in the thesis.

1.1 The q-Pochhammer symbol and Ramanujan's theta functions

For complex numbers a, q with $\mid q \mid< 1$, and integers $n \geq 1$, we define the usual q-products as

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad \text{and} \quad (a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

Furthermore, for brevity, we set

$$f_n := (q^n;q^n)_\infty, \quad (a_1,a_2,\ldots,a_n;q) := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty,$$

and $(q^{r\pm};q^s)_\infty := (q^r,q^{s-r};q^s)_\infty$, for positive integers r and s with $r<s$.

Ramanujan's general theta function $f(a,b)$ [29, Section 1.2] is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab|<1. \tag{1.1}$$

Jacobi's well-known triple product identity [26, p. 35, Entry 19] is given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2}$$

Certain special cases of (1.1) are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2f_4^2}, \tag{1.3}$$

$$\varphi(-q) := f(-q,-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{f_1^2}{f_2}, \tag{1.4}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}, \tag{1.5}$$

$$\psi(-q) := f(-q,-q^3) = \sum_{n=-\infty}^{\infty} (-q)^{n(n+1)/2} = \frac{f_1f_4}{f_2}, \tag{1.6}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1, \tag{1.7}$$

$$f(q) := f(q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = \frac{f_2^3}{f_1f_4}, \tag{1.8}$$

where the q-product representations arise from (1.2) and manipulation of the q-products.

After Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty = \frac{f_2^2}{f_1f_4}. \tag{1.9}$$

Replacing q by $-q$ in (1.9), we have

$$\chi(-q) = (q;q^2)_\infty = \frac{f_1}{f_2}. \tag{1.10}$$

Throughout the thesis, we will use these theta functions frequently.

1.2 The Rogers-Ramanujan continued fraction

The well-known Rogers-Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{q^{1/5}}{1}\,_{+}\frac{q}{1}\,_{+}\frac{q^2}{1}\,_{+}\frac{q^3}{1}\,_{+\cdots}, \quad |q| < 1. \tag{1.11}$$

The Rogers-Ramanujan identities, first proved by Rogers [104] and then rediscovered by Ramanujan [95], are given by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{1.12}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.13}$$

where $G(q)$ and $H(q)$ are known as the Rogers-Ramanujan functions.

Rogers [104] and Ramanujan [98] also proved that

$$R(q) = q^{1/5} \frac{H(q)}{G(q)}. \tag{1.14}$$

For occasional use in the sequel, we set

$$T(q) := \frac{H(q)}{G(q)} = q^{-1/5} R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.15}$$

1.3 Dissection of power series

For a positive integer $t \geq 2$, the t-dissection of a power series $P(q)$ in q is given by

$$P(q) = P_0(q^t) + qP_1(q^t) + q^2 P_2(q^t) + \cdots + q^{t-1} P_{t-1}(q^t),$$

where $P_j(q^t)$, $1 \leq j \leq t-1$ are power series in q^t.

For example, we have the following 5-dissections of f_1 and $\dfrac{1}{f_1}$ (see [29, pp. 161–165]):

$$f_1 = f_{25} \left(\frac{1}{T(q^5)} - q - q^2 T(q^5) \right), \tag{1.16}$$

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6} \left(\frac{1}{T^4(q^5)} + \frac{q}{T^3(q^5)} + \frac{2q^2}{T^2(q^5)} + \frac{3q^3}{T(q^5)} + 5q^4 - 3q^5 T(q^5) + 2q^6 T^2(q^5) \right.$$

$$- q^7T^3(q^5) + q^8T^4(q^5)\Bigg). \tag{1.17}$$

1.4 Partitions of positive integers

A partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of a positive integer n is a non-increasing sequence of positive integers such that $\sum_{j=1}^{k} \pi_j = n$. Each π_j is termed as a part of the partition. If $p(n)$ denotes the number of partitions of n, then its generating function due to Euler (1707-1783) is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where by convention $p(0) = 1$. For example, $p(4) = 5$, since there are five partitions of 4, namely, (4), (3,1), (2,2), (2,1,1), and (1,1,1,1).

Note that for an integer $\ell > 1$, we say a partition of n is an ℓ-regular partition if none of its parts are divisible by ℓ. The ℓ-regular partition function denoted by $b_\ell(n)$ counts the number of ℓ-regular partitions of n. We consider $b_\ell(0) = 1$ and $b_\ell(n) = 0$ when $n < 0$. The generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1} = \frac{f(-q^\ell)}{f(-q)}. \tag{1.18}$$

For example, $b_3(5) = 5$ where the relevant partitions are (5), (4,1), (2,2,1), (2,1,1,1), and (1,1,1,1 ,1).

A partition π of n can be represented graphically by a Ferrers-Young diagram. The Ferrers–Young diagram of $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is an array of nodes with π_i nodes in the ith row. The (i, j)th hook is the set of nodes directly to the right of (i, j)th node together with the set of nodes directly below it as well as the (i, j)th node itself. The hook number, $H(i, j)$, is the total number of nodes on the (i, j)th hook. For a positive integer $t \geq 2$, a partition of n is said to be t-core if none of the hook numbers are divisible by t. We illustrate the Ferrers-Young diagram of the partition $(4, 3, 1)$ of 8 with hook numbers as follows:

$$\begin{array}{llll} \bullet^{6} & \bullet^{4} & \bullet^{3} & \bullet^{1} \\ \bullet^{4} & \bullet^{2} & \bullet^{1} & \\ \bullet^{1} & & & \end{array}$$

It is a 5-core. Furthermore, it is clear that for $t \geq 7$, the partition $(4, 3, 1)$ of 8 is a t-core.

Suppose that $c_t(n)$ counts the number of t-cores of n, then the generating function of $c_t(n)$ is given by (see [55, Eq. 2.1])

$$\sum_{n=0}^{\infty} c_t(n) q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty} = \frac{f_t^t}{f_1} = \frac{f^t(-q^t)}{f(-q)}. \tag{1.19}$$

In an existence result, Granville and Ono [60] proved that if $t \geq 4$, then $c_t(n) > 0$ for every nonnegative integer n.

Partition theory has been a topic of interest for the mathematicians since a long time. Various properties of partitions, namely, arithmetic, asymptotic, combinatorial, etc., have been prominently studied in the literature by many authors. Ramanujan [94, 97], in his pioneering work from the aspect of arithmetic properties of partitions, proved the following congruences. For $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

Ramanujan [94] also conjectured more general congruences for $p(n)$ modulo arbitrary powers of 5, 7, and 11.

Apart from congruences, another interesting topic of study in theory of partitions is the study of distributions of various partition functions. To be precise, given an integral power series $F(q) := \sum_{n=0}^{\infty} a(n) q^n$ and $0 \leq r < M$, the arithmetic density $\delta_r(F, M; X)$ is defined as

$$\delta_r(F, M; X) := \frac{\#\left\{0 \leq n \leq X : a(n) \equiv r \pmod{M}\right\}}{X}.$$

A well-known conjecture of Parkin and Shanks [89] regarding the density of even and odd values of $p(n)$ states that

$$\lim_{X\to\infty} \frac{\#\{0 \le n \le X : p(n) \equiv c \pmod 2\}}{X} = \frac{1}{2},$$

where $c \in \{0, 1\}$. This conjecture is still open.

An integral power series F is called *lacunary modulo M* if

$$\lim_{X\to\infty} \delta_0(F, M; X) = 1,$$

that is, almost all of the coefficients of F are divisible by M. In one of the earliest results regarding the arithmetic densities of partition functions, Gordon and Ono [58] proved that $b_\ell(n)$ is lacunary modulo 2^k for any fixed positive integer k.

1.5 Lucas sequences

For integers a and b, the (a, b)-Lucas sequence $S(n)$ is defined by

$$S(n) = aS(n-1) - bS(n-2), \tag{1.20}$$

where $S(0) = 0$ and $S(1) = 1$. The *dual* (a, b)-Lucas sequence $T(n)$ of $S(n)$ is given by

$$T(n) = aT(n-1) - bT(n-2) \tag{1.21}$$

with $T(0) = 1$ and $T(1) = 0$.

Also, for an integer $M \ge 2$, the rank of (a, b)-Lucas sequence $S(n)$ modulo M is defined to be the least positive integer k such that $S(k) \equiv 0 \pmod M$. We denote the the rank of $S(n)$ modulo M by $R_S(M)$.

For instance, the well-known Fibonacci sequence, $F(n)$, is nothing but (1,-1) Lucas sequence. Again, we have $F(0) = 0$, $F(1) = 1$, $F(2) = 1$, $F(3) = 2$, $F(4) = 3$. Thus $R_F(2) = 3$ and $R_F(3) = 4$.

1.6 Modular forms and Hecke operators

The theory of modular forms has been instrumental in deducing a number of results of the thesis. In this section, we recall some basic facts and definitions on modular forms. For more details, one can see [79] and [87].

Firstly, we define the matrix groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\},$$

where N is a positive integer. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some N and the smallest N with this property is called the level of Γ. For instance, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N.

Let $\mathbb{H}$ denote the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \dfrac{az+b}{cz+d}$. We identify ∞ with $\dfrac{1}{0}$ and define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \dfrac{r}{s} = \dfrac{ar+bs}{cr+ds}$, where $\dfrac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, then define the slash operator $|_\ell$ by

$$(f|_\ell\gamma)(z) := (\det(\gamma))^{\ell/2}(cz+d)^{-\ell} f(\gamma z).$$

Definition 1.1. *Let Γ be a congruence subgroup of level N. A holomorphic function*

$f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if the following hold:

(1) We have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\ell} f(z)$$

for all $z \in \mathbb{H}$ and and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.

(2) If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_{\ell}\gamma)(z)$ has a Fourier expansion of the form

$$(f|_{\ell}\gamma)(z) = \sum_{n \geq 0} \mathcal{A}_{\gamma}(n) q_N^n,$$

where $q := e^{2\pi i z/N}$.

For a positive integer ℓ, the complex vector space of modular forms of weight ℓ with respect to a congruence subgroup Γ is denoted by $M_{\ell}(\Gamma)$.

Definition 1.2. *[87, Definition 1.15] If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_{\ell}(\Gamma_1(N))$ has Nebentypus character χ if*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^{\ell} f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_{\ell}(\Gamma_0(N), \chi)$.

The relevant modular forms for the results of this thesis arise from eta-quotients. Recall that the Dedekind eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_{\infty} = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_{\delta}},$$

where N is a positive integer and r_{δ} is an integer.

Finally, we recall the definition of Hecke operators. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_{\ell}(\Gamma_0(N), \chi)$. Then the action of Hecke operator T_m on

$f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty} \left(\sum_{d|\gcd(n,m)} \chi(d) d^{\ell-1} a\left(\frac{nm}{d^2}\right) \right) q^n.$$

In particular, if $m = p$ is prime, then we have

$$f(z)|T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p) p^{\ell-1} a\left(\frac{n}{p}\right) \right) q^n. \tag{1.22}$$

We note that $a(n) = 0$ unless n is a nonnegative integer.

Definition 1.3. *A modular form* $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_\ell(\Gamma_0(N), \chi)$ *is called a Hecke eigenform if for every* $m \geq 2$*, there exists a complex number* $\lambda(m)$ *such that*

$$f(z) \mid T_m = \lambda(m) f(z). \tag{1.23}$$

1.7 Identities for the Rogers-Ramanujan continued fraction

Ramanujan found numerous identities for the Rogers-Ramanujan continued fraction $R(q)$ and the corresponding Rogers-Ramanujan functions $G(q)$ and $H(q)$ and recorded them in his notebooks [98] and his lost notebooks [99]. After that, various other mathematicians also worked on these functions and proved such identities.

We devote Chapter 2 to finding new identities for the Rogers-Ramanujan continued fraction and functions. We also deduce some partition theoretic relations arising from some of those identities.

1.8 Arithmetic properties for some restricted partition functions

In the thesis, we study congruence properties and density results of certain restricted partitions. We introduce the other relevant restricted partitions in the following.

Andrews and Paule [8, 9] studied a combinatorial object called the k-elongated partition diamonds. Let $d_k(n)$ count the partitions obtained by adding the links of the k–elongated plane partition diamonds of length n. Then the generating function for $d_k(n)$ is given by

$$\sum_{n=0}^{\infty} d_k(n)q^n = \frac{f_2^k}{f_1^{3k+1}}. \tag{1.24}$$

They obtained several generating functions and congruences for $d_1(n)$, $d_2(n)$, and $d_3(n)$. Further divisibility properties of $d_k(n)$ are studied in [15, 50, 109, 115, 129].

In Chapter 3, we discover new infinite families of congruences as well as individual congruences for the k-elongated plane partition diamonds function $d_k(n)$ modulo 2, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 32, 49, 64, 81, 128, 243, and 729 for various values of k. We also refine an existence result of congruences for $d_k(n)$ modulo powers of primes due to da Silva, Hirschhorn, and Sellers [50].

Recently, Gireesh, Ray, and Shivashankar [57] considered a new function $\overline{a}_t(n)$ by substituting $\varphi(-q)$ in place of $f(-q)$ in the generating function (1.19) of $c_t(n)$, namely,

$$\sum_{n=0}^{\infty} \overline{a}_t(n)q^n = \frac{\varphi^t(-q^t)}{\varphi(-q)} = \frac{f_2 f_t^{2t}}{f_1^2 f_{2t}^t}. \tag{1.25}$$

Very recently, Bandyopadhyay and Baruah [14] introduced another analogue $\overline{b}_t(n)$ of $c_t(n)$, which is defined by

$$\sum_{n=0}^{\infty} \overline{b}_t(n)q^n = \frac{\psi^t(-q^t)}{\psi(-q)} = \frac{f_2 f_t^t f_{4t}^t}{f_1 f_4 f_{2t}^t}. \tag{1.26}$$

In Chapter 4, we use the theory of modular forms and Hecke operators to find the arithmetic densities and families of congruences for $\overline{a}_t(n)$ and $\overline{b}_t(n)$ modulo powers of primes for certain values of t.

We further define a partition k-tuple of n to be the k-tuple of partitions $(\Lambda_1, \Lambda_2, \ldots, \Lambda_k)$, such that the sum of all the parts equals n. In the thesis, we study a class of restricted partitions called as the partitions k-tuple with t-cores. For example, consider the tuple ((4,3,2), (5,1)). This is a 2-tuple whose hook numbers of the individual tuples are not divisible by any integer $t \geq 7$. So, this is a 2-tuple t-core partition of 15 for $t \geq 7$.

Let $\mathcal{A}_{t,k}(n)$ denote the number of partitions k-tuples of n with t-cores. Its generating function is given by

$$\sum_{n\geq 0}\mathcal{A}_{t,k}(n)q^n=\frac{(q^t;q^t)_\infty^{kt}}{(q;q)_\infty^k}=\frac{f_t^{kt}}{f_1^k}. \tag{1.27}$$

We study two restricted partition functions in Chapter 5. We employ dissections of certain q-products and two identities of Newman [86] to prove new infinite families of congruences as well as individual congruences for the 6-regular partition function $b_6(n)$ and partition k-tuples with 5-cores, $\mathcal{A}_{5,k}(n)$, for $k=2$, 3, and 4. We also prove some new infinite families of congruences modulo powers of primes for k-tuples with p-cores, where p is a prime.

By imposing certain restrictions on the parts of ℓ-regular partitions of n, we obtain the following partition functions. Let

(i) $\mathrm{pod}_\ell(n)$ denote the number of ℓ-regular partitions of n where the odd parts are distinct and the even parts are unrestricted,

(ii) $\mathrm{ped}_\ell(n)$ denote the number of ℓ-regular partitions of n where the even parts are distinct and the odd parts are unrestricted.

For example, $\mathrm{pod}_3(5)=3$ with the relevant partitions being (5), (4,1), (2,2,1); and $\mathrm{ped}_3(5)=4$ with the relevant partitions, namely, (5), (4,1), (2,1,1,1), (1,1,1,1,1).

The generating functions of $\mathrm{pod}_\ell(n)$ and $\mathrm{ped}_\ell(n)$ are given by

$$\sum_{n=0}^{\infty}\mathrm{pod}_\ell(n)q^n=\frac{f_2f_\ell f_{4\ell}}{f_1f_4f_{2\ell}} \tag{1.28}$$

and

$$\sum_{n=0}^{\infty}\mathrm{ped}_\ell(n)q^n=\frac{f_4f_\ell}{f_1f_{4\ell}}. \tag{1.29}$$

Arithmetic properties of $\mathrm{pod}_\ell(n)$ and $\mathrm{ped}_\ell(n)$ are studied in [52, 56, 67, 102, 107, 112, 122]. In Chapter 6, we find the arithmetic densities of $\mathrm{pod}_\ell(n)$ for $\ell=3$, 5, 7, 13, 17, and $\mathrm{ped}_t(n)$ and $t=13$, 17 modulo 2 and arbitrary powers of 2 with the aid of the theory of modular forms. We also prove new multiplicative relations for $\mathrm{pod}_5(n)$, $\mathrm{pod}_9(n)$, $\mathrm{ped}_5(n)$, and $\mathrm{ped}_9(n)$ modulo small powers of 2 using the theory of Hecke eigenforms.

1.9 Minimal excludant and least r-gaps in partitions

Let $\mathcal{P}(n)$ denote the set of all partitions of n. Andrews and Newman [6] introduced the concept of the minimal excludant or "mex" of a partition. The mex of π is the smallest positive integer which is not present in π. However, the term 'minimal excludant' was first appeared on a paper of Fraenkel and Peled [54] where they defined the minimal excludant for any set S of positive integers as the smallest positive integer missing from that set. Andrews and Newman [6] also studied another related concept called minimal odd excludant or moex of a partition π which is the smallest odd positive integer missing from π. They deduced some arithmetic properties of the mex related functions $\sigma\text{mex}(n)$, $\sigma\text{moex}(n)$, and $a(n)$, defined as follows:

(i) $\sigma\text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{mex}(\pi)$,

(ii) $\sigma\text{moex}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{moex}(\pi)$,

(iii) $a(n) := \sum_{\substack{\pi \in \mathcal{P}(n) \\ \text{mex}(\pi) \text{ is odd}}} 1.$

Andrews and Newman [6] also found the following generating functions:

(i) $\displaystyle\sum_{n=0}^{\infty} \sigma\text{mex}(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty (q;q^2)_\infty} = (-q;q)_\infty^2,$

(ii) $\displaystyle\sum_{n=0}^{\infty} \sigma\text{moex}(n) q^n = (-q;q)_\infty (-q;q^2)_\infty^2.$

For $n \geq 0$, Andrews and Newman [6] showed that $\sigma\text{mex}(n) \equiv a(n) \pmod 2$. Recently, Baruah, Bhoria, Eyyunni, and Maji [24] refined the expression for $\sigma\text{mex}(n)$ by introducing the following two functions:

(i) $\sigma_o\text{mex}(n) := \sum_{\substack{\pi \in \mathcal{P}(n) \\ 2 \nmid \text{mex}(\pi)}} \text{mex}(\pi)$,

(ii) $\sigma_e\text{mex}(n) := \sum_{\substack{\pi\in\mathcal{P}(n)\\ 2|\text{mex}(\pi)}} \text{mex}(\pi).$

Let the smallest positive integer that does not appear at least r times as a part of a partition π be defined as the least r-gap of π or $g_r(\pi)$. Ballantine and Merca [11] studied the topic of least r-gaps in partitions and introduced two new partition functions involving least r-gaps. One of these is $S_r(n)$ defined by $S_r(n) = \sum_{\pi\in\mathcal{P}(n)} g_r(\pi)$. In terms of "mex", we write $g_r(\pi) = r\text{-mex}(\pi)$ and $S_r(n) = \sigma_r\text{mex}(n)$. We also define r-moex(π) to be the smallest odd positive integer that does not appear at least r times in the partition π.

Apart from $\sigma_r\text{mex}(n)$, we also study the functions $\sigma_{r,o}\text{mex}(n)$, $\sigma_{r,e}\text{mex}(n)$, $\sigma_r\text{moex}(n)$, and $a_r(n)$, which are r-mex analogues of $\sigma_o\text{mex}(n)$, $\sigma_e\text{mex}(n)$, $\sigma\text{moex}(n)$, and $a(n)$, defined by

(i) $\sigma_{r,o}\text{mex}(n) := \sum_{\substack{\pi\in\mathcal{P}(n)\\ 2\nmid r\text{-mex}(\pi)}} r\text{-mex}(\pi),$

(ii) $\sigma_{r,e}\text{mex}(n) := \sum_{\substack{\pi\in\mathcal{P}(n)\\ 2| r\text{-mex}(\pi)}} r\text{-mex}(\pi),$

(iii) $\sigma_r\text{moex}(n) := \sum_{\pi\in\mathcal{P}(n)} r\text{-moex}(\pi),$

(iv) $a_r(n) := \sum_{\substack{\pi\in\mathcal{P}(n)\\ r\text{-mex}(\pi) \text{ is odd}}} 1.$

In Chapter 7, we find the generating functions of the arithmetic functions $\sigma_{r,o}\text{mex}(n)$, $\sigma_{r,e}\text{mex}(n)$, $\sigma_r\text{moex}(n)$, and $a_r(n)$, which are related to the least r-gaps or r-mex of partitions. We establish some of their connections with certain known partition functions. We also explore some arithmetic as well as asymptotic properties enjoyed by these functions.