# Chapter 2

# Identities for the Rogers-Ramanujan Continued Fraction

## 2.1 Introduction

The Rogers-Ramanujan continued fraction $R(q)$ and the corresponding Rogers-Ramanujan functions $G(q)$ and $H(q)$ are defined in Section 1.2 (see (1.11), (1.12), and (1.13)).

In his first letter to Hardy, written on January 16, 1913, Ramanujan astounded Hardy by an elegant identity writing $R^5(q)$ as a rational expression in $R(q^5)$, namely,

$$R^5(q) = R(q^5)\,\frac{1 - 2R(q^5) + 4R^2(q^5) - 3R^3(q^5) + R^4(q^5)}{1 + R(q^5) + 4R^2(q^5) + 2R^3(q^5) + R^4(q^5)}. \tag{2.1}$$

This identity was also recorded by Ramanujan on p. 289 of his second notebook [98] and p. 365 of his lost notebook [99]. To the best of our knowledge, this is the only identity of this sort. Proofs of (2.1) were given by Rogers [105] in 1921, Watson [124] in 1929, Ramanathan [93] in 1984, Yi [133] in 2001, and Gugg [62] in 2009.

In the literature, there are further modular identities relating $R(q)$ with $R(-q)$ and $R(q^n)$ for some positive integers $n$. Ramanujan recorded identities relating $R(q)$

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with $R(-q)$, $R(q^2)$, $R(q^3)$, and $R(q^4)$ in the scattered places of his notebooks [98] and the lost notebook [99] and proofs can be found in [105], [10], [28, Chapter 32], and [5, Chapter 1]. In 1921, Rogers [105] found a relation connecting $R(q)$ with $R(q^{11})$. Chan and Tan [42] found a relation connecting $R(q)$ with $R(q^{19})$ in 1999, and Yi [133] found another relation connecting $R(q)$ with $R(q^7)$ in 2001. We also refer to Trott [121] for other modular identities discovered "experimentally" by using Wolfram's *Mathematica*.

The main focus of this chapter is to prove some new modular identities for $R(q)$. In the process, we also find several new relations for $G(q)$ and $H(q)$.

In the next theorem and corollary, we present some new identities analogous to (2.1).

**Theorem 2.1.** *We have*

$$R(q)R(q^4) = \frac{R(q^5) + R(q^{20}) - R(q^5)R(q^{20})}{1 + R(q^5) + R(q^{20})}, \tag{2.2}$$

$$\frac{R^2(q)R(q^2)}{R(q^4)} = R(q^5)\ \frac{R(q^{10}) - R(q^5)R(q^{10}) + R(q^{20})}{R(q^5)R(q^{10}) + R(q^{20}) + R(q^5)R(q^{20})}, \tag{2.3}$$

$$\frac{1}{R(q)R^2(q^2)} - R(q)R^2(q^2) = \frac{N}{D}, \tag{2.4}$$

*where*

$$D = R(q^5)\left(1 + R(q^5)R^2(q^{10}) - R^2(q^5)R^4(q^{10})\right)$$

*and*

$$\begin{aligned} N = {} & 1 + R(q^5) + R^2(q^5) + 2R(q^5)R(q^{10}) + 2R^2(q^5)R(q^{10}) - 2R(q^5)R^2(q^{10}) \\ & + 2R^3(q^5)R^2(q^{10}) - 2R^2(q^5)R^3(q^{10}) + 2R^3(q^5)R^3(q^{10}) + R^2(q^5)R^4(q^{10}) \\ & - R^3(q^5)R^4(q^{10}) + R^4(q^5)R^4(q^{10}). \end{aligned}$$

**Corollary 2.2.** *We have*

$$\frac{R^2(q^4)}{R(q)R(q^2)} = R(q^{20})\ \frac{1 - R(q^5)R(q^{10}) - R(q^{20})}{R(q^5)R(q^{10}) - R(q^{20}) + R(q^5)R(q^{10})R(q^{20})}, \tag{2.5}$$

$$\frac{R(q)}{R(-q)} = \frac{1 - R(q^5) + R(q^5)R(-q^5)}{1 + R(-q^5) + R(q^5)R(-q^5)}, \tag{2.6}$$

$$\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} = 4\ \frac{D}{N}, \tag{2.7}$$

*where $N$ and $D$ are as stated in the previous theorem.*

In the following theorem, we present new addition to the list of modular identities for $R(q)$ available in the literature.

**Theorem 2.3.** *We have*

$$\frac{1}{R(q^2)R(q^3)} + R(q^2)R(q^3) = 1 + \frac{R(q)}{R(q^6)} + \frac{R(q^6)}{R(q)}, \tag{2.8}$$

$$R(q^2) = \frac{R^2(q^4)R(q^8) - 2R(q)R(q^4)R(q^{16}) + R(q^4)R^2(q^{16})}{R^2(q)R(q^4)R(q^8) - 2R(q)R(q^4)R(q^8)R(q^{16}) + R^2(q)R^2(q^{16})}, \tag{2.9}$$

$$R(q^6) = R(q)R^2(q^2)R(q^3) \times \frac{R(q)R(q^2) - R(q^3) + R^3(q)}{R^2(q)R(q^3) - R(q^2)R(q^3) + R^3(q)R(q^2)}, \tag{2.10}$$

$$R(q^2) = \frac{R(q)R(q^3)}{R(q^6)} \times \frac{R(q)R^2(q^3)R(q^6) + 2R(q^6)R(q^{12}) + R(q)R(q^3)R^2(q^{12})}{R(q^3)R(q^6) + 2R(q)R^2(q^3)R(q^{12}) + R^2(q^{12})}, \tag{2.11}$$

$$R(q^2) = \frac{1}{R(q)R(q^3)R(q^6)} \times \frac{2R(q)R(q^3)R(q^6)R(q^{24}) - R(q^4)R(q^6)R^2(q^{12}) - R(q^4)R(q^{12})R^2(q^{24})}{R(q)R(q^3)R(q^6)R(q^{12}) - 2R(q^4)R^2(q^{12})R(q^{24}) + R(q)R(q^3)R^2(q^{24})}. \tag{2.12}$$

We prove (2.2)–(2.9) by using dissection formulae for theta functions and certain relations for the Rogers-Ramanujan functions whereas we derive (2.10)–(2.12) from some new relations for the Rogers-Ramanujan functions arising from the so-called quintuple product identity (Eq. (2.49) in the next section).

In a brief communication, Ramanujan [96] gave two algebraic relations between $G(q)$ and $H(q)$, namely,

$$H(q)\{G(q)\}^{11} - q^2G(q)\{H(q)\}^{11} = 1 + 11q\{G(q)H(q)\}^6$$

and

$$H(q)G(q^{11}) - q^2G(q)H(q^{11}) = 1$$

and remarked that each of these formulae is the simplest of a large class. In fact, these two identities are from a set of forty identities for $G(q)$ and $H(q)$ that Ramanujan never published. Ramanujan shared some of the identities with Rogers, who proved ten of the identities in [105]. Watson had Ramanujan's identities and he proved eight identities (two of them already proven by Rogers) in the paper [125]. Watson prepared a handwritten list of the forty identities. In 1975, Birch found Watson's handwritten copy of them in the Oxford University Library and published it in [33]. Watson's handwritten list of Ramanujan's forty identities was finally published in [99]. Eventually, the forty identities were proved with a combined effort of Bressoud [34], Biagioli [32], Berndt, G. Choi, Y. -S. Choi, Hahn, Yeap, Yee, Yesilyurt and Yi [31], and Yesilyurt [131, 132]. We refer to [5, Chapter 8] for a more comprehensive detail.

Apart from Ramanujan's forty identities for the Rogers-Ramanujan functions, there are several other identities in the literature. In [30], Berndt and Yesilyurt established new representations for $G$ and $H$ as linear combinations of $G$ and $H$ at different arguments and used them in conjunction with some of the previously proved forty identities to prove new identities for the Rogers-Ramanujan functions. Work on further new identities for $G(q)$ and $H(q)$ can also be found in the papers by Robins [103], Chu [44], Gugg [61, 62, 63, 64, 65], Koike [80], Bringmann and Swisher [35, 36], Bulkhali and Dasappa [37], and Baruah and Das [20].

In the following two theorems, we present some new relations for $G(q)$ and $H(q)$.

**Theorem 2.4.** *We have*

$$\frac{G\left(q^2\right)G\left(q^3\right)}{H\left(q\right)G\left(q^6\right)}-\frac{H\left(q^2\right)H\left(q^3\right)}{G\left(q\right)H\left(q^6\right)}=q\frac{(q^5;q^5)_\infty^2(q^{30};q^{30})_\infty^2}{(q^{10};q^{10})_\infty^2(q^{15};q^{15})_\infty^2}, \tag{2.13}$$

$$\frac{H\left(q\right)G\left(q^6\right)}{H\left(q^2\right)H\left(q^3\right)}-q^2\frac{G\left(q\right)H\left(q^6\right)}{G\left(q^2\right)G\left(q^3\right)}=\frac{(q^{10};q^{10})_\infty^2(q^{15};q^{15})_\infty^2}{(q^5;q^5)_\infty^2(q^{30};q^{30})_\infty^2}. \tag{2.14}$$

**Theorem 2.5.** *Let*

$$\begin{aligned}
S &:= G^2(q)H(q^2)G(q^3)H^2(q^4)H(q^{12}),\\
T &:= H^2(q)G(q^2)H(q^3)G^2(q^4)G(q^{12}),\\
U &:= G^3(q)H^3(q)H(q^6)-H^3(q^2)G(q^3)H(q^3),\\
V &:= G^3(q)H^3(q)G(q^6)-G^3(q^2)G(q^3)H(q^3),\\
W &:= H^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)H(q^3),\\
X &:= G^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)G(q^3),\\
Y &:= G^3(q^2)H(q^3)G(q^{12})-H^3(q)G^3(q^4)G(q^6),\\
Z &:= H^3(q^2)G(q^3)H(q^{12})-G^3(q)H^3(q^4)H(q^6).
\end{aligned}$$

*We have*

$$\frac{H(q)H^2(q^3)G(q^{12})+qG(q)G^2(q^3)H(q^{12})}{G(q)G(q^3)H(q^6)G(q^{12})+q^2H(q)H(q^3)G(q^6)H(q^{12})}=\frac{\chi(q^5)\chi(-q^{10})}{\chi(-q^6)\chi(q^{15})}, \tag{2.15}$$

$$\frac{UG(q^2)H(q^3)G(q^{12}) - qVH(q^2)G(q^3)H(q^{12})}{VH(q^2)H(q^6)G(q^{12}) - q^2UG(q^2)G(q^6)H(q^{12})} = \frac{\chi(q^5)\chi(-q^{10})}{\chi(-q^6)\chi(q^{15})}, \tag{2.16}$$

$$\frac{WG(q)H(q^{12})G(q^{24}) - q^2XH(q)G(q^{12})H(q^{24})}{q^4WG(q)G(q^6)H(q^{24}) - XH(q)H(q^6)G(q^{24})} = q^3\frac{\chi(q^{10})\chi(-q^{12})\chi(-q^{20})\chi(-q^{30})}{\chi^3(-q^{60})}, \tag{2.17}$$

$$\frac{SYH(q^{12})G(q^{24}) - q^2TZG(q^{12})H(q^{24})}{TZH(q^6)G(q^{24}) - q^4SYG(q^6)H(q^{24})} = q^3\frac{\chi(q^{10})\chi(-q^{12})\chi(-q^{20})\chi(-q^{30})}{\chi^3(-q^{60})}, \tag{2.18}$$

$$\frac{V}{q^{6/5}U} = \frac{R(q)R(q^3)}{R^2(q^2)R(q^6)}, \tag{2.19}$$

$$\frac{q^{3/5}W}{X} = R^2(q)R(q^2)R(q^3)R(q^6), \tag{2.20}$$

$$\frac{Y}{q^3Z} = \frac{R(q)}{R(q^2)R(q^4)}, \tag{2.21}$$

*where* $\chi(q) := (-q; q^2)_\infty$.

We organize the rest of the chapter as follows. In Section 2.2, we present the preliminary results on Ramanujan's theta functions, the Rogers-Ramanujan functions, and some results arising from the quintuple product identity. In Section 2.3, we prove Theorem 2.4 by using dissection formulae for theta functions whereas Section 2.4 is devoted to proving Theorem 2.5 by using some theta function identities arising from the quintuple product identity. In Section 2.5, we prove Theorem 2.1 and Corollary 2.2. In Section 2.6, we prove Theorem 2.3. Partition-theoretic results may be derived from the identities in Theorems 2.4 and 2.5. As for examples, in Section 2.7, we present three such results arising from (2.13), (2.14), and (2.15). We offer some concluding remarks in the final section.

## 2.2 Preliminary results and lemmas

Ramanujan's general theta function $f(a,b)$ is defined as in (1.1). In the following two lemmas, we recall some preliminary identities of $f(a,b)$.

**Lemma 2.6.** (Berndt [26, pp. 45–46, Entries 29–30])

$$f(a,b) + f(-a,-b) = 2f(a^3b, ab^3), \tag{2.22}$$

$$f(a,b) - f(-a,-b) = 2af\left(\frac{b}{a}, a^5b^3\right), \tag{2.23}$$

*and if $ab = cd$, then*

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{2.24}$$

**Lemma 2.7.** (Berndt [26, p. 51 and p. 350]) *We have*

$$f(q,q^5) = \psi(-q^3)\chi(q), \tag{2.25}$$

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{2.26}$$

With the aid of the product representations (1.12) and (1.13) of $G(q)$ and $H(q)$, respectively, and (1.2), the following preliminary identities are apparent (see [31] and [64] for proofs).

**Lemma 2.8.** *We have*

$$f(-q^2,-q^3) = f(-q)G(q), \quad f(-q,-q^4) = f(-q)H(q), \tag{2.27}$$

$$G(q)H(q) = \frac{f(-q^5)}{f(-q)}, \tag{2.28}$$

$$f(q^2,q^3) = f(-q^5)\frac{H(q)}{H(q^2)} = f(-q)\frac{G(q)H^2(q)}{H(q^2)}, \tag{2.29}$$

$$f(q,q^4) = f(-q^5)\frac{G(q)}{G(q^2)}, \tag{2.30}$$

$$f(-q,-q^9) = f(-q^{10})\frac{H(q^2)}{G(q)}, \tag{2.31}$$

$$f(-q^3,-q^7) = f(-q^{10})\frac{G(q^2)}{H(q)}, \tag{2.32}$$

$$f(q,q^9) = f(-q^2)G(q)H(q^4), \quad f(q^3,q^7) = f(-q^2)H(q)G(q^4). \tag{2.33}$$

By manipulating $q$-products, it also follows easily that

$$G(-q) = \frac{G(q^2)H^2(q^2)}{G(q)H(q^4)}, \quad H(-q) = \frac{H(q^2)G^2(q^2)}{H(q)G(q^4)}, \tag{2.34}$$

$$T(-q) = \frac{T(q^4)}{T(q)T(q^2)}, \tag{2.35}$$

where $T(q)$ is as defined in (1.15).

In the following lemma, we recall two useful 5-dissection formulae from [26, p. 49 and p. 82].

**Lemma 2.9.** *We have*

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4f(q^5,q^{45}), \tag{2.36}$$

$$\psi(q) = f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}). \tag{2.37}$$

The two identities in the next lemma were recorded by Ramanujan on p. 56 and p. 53, respectively, of his lost notebook [99] and proved first by Kang [76].

**Lemma 2.10.** *If $u = R(q)$ and $v = R(q^2)$, then*

$$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1 + uv^2 - u^2v^4}{uv^2}, \tag{2.38}$$

$$\frac{\psi(q^{1/5})}{q^{3/5}\psi(q^5)} = \frac{1 - uv^2}{uv} + \frac{1 + uv^2}{v} + 1. \tag{2.39}$$

The identities in the next lemma were also recorded by Ramanujan on p. 56 of his lost notebook [99] and proved first by Kang [76] (Also see Son [116]).

**Lemma 2.11.** *We have*

$$\varphi(q) + \varphi(q^5) = 2q^{4/5} f(q, q^9) R^{-1}(q^4), \tag{2.40}$$

$$\varphi(q) - \varphi(q^5) = 2q^{1/5} f(q^3, q^7) R(q^4), \tag{2.41}$$

$$\psi(q^2) - q\psi(q^{10}) = q^{-1/5} f(q^4, q^6) R(q). \tag{2.42}$$

The identities in the next lemma, which were recorded by Ramanujan in Chapter 19 of his second notebook [98, Entries 9(iii), 9(vi), 10(iv), 10(v)], [26, p. 258 and p. 262], readily follow from the previous lemma and (1.2).

**Lemma 2.12.** *Let $\chi(q) := (-q; q^2)_\infty$. We have*

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q, q^9)f(q^3, q^7) = 4q\chi(q)f(-q^5)f(-q^{20}), \tag{2.43}$$

$$\psi^2(q) - q\psi^2(q^5) = f(q, q^4)f(q^2, q^3) = \frac{\varphi(-q^5)f(-q^5)}{\chi(-q)}. \tag{2.44}$$

The following two identities in the next lemma are from the list of forty identities of Ramanujan and they played important roles in proving several other identities in the list; see [125], [31], [5], and [132]. The identities can be easily proved by using Lemma 2.11 and some preliminary identities of Lemma 2.8.

**Lemma 2.13.** *The following identities hold:*

$$G(q)G(q^4) + qH(q)\,H\left(q^4\right) = \frac{\varphi(q)}{f(-q^2)}, \tag{2.45}$$

$$G(q)\,G\left(q^4\right) - qH(q)\,H\left(q^4\right) = \frac{\varphi(q^5)}{f(-q^2)}. \tag{2.46}$$

Next we recall two more relations on $G(q)$ and $H(q)$. The first identity is from the list of Ramanujan's forty identities proved first by Rogers [105] and the second one is from a recent paper by Baruah and Das [20].

**Lemma 2.14.** *We have*

$$G(q^{16})H(q) - q^3G(q)H(q^{16}) = \frac{\varphi(-q^4)}{f(-q^2)}, \tag{2.47}$$

$$G(q^{16})H(q) + q^3G(q)H(q^{16}) = \frac{\varphi(-q^{20})}{f(-q^2)} + 2q^3\frac{f(-q^8)G(q^{16})H(q^{16})}{f(-q^2)}. \tag{2.48}$$

In his lost notebook, Ramanujan [99, p. 207], recorded the quintuple product identity in the form

$$\frac{f(-\lambda x, -x^2)}{f(-x, -\lambda x^2)}f(-\lambda x^3) = f(-\lambda^2x^3, -\lambda x^6) + xf(-\lambda, -\lambda^2x^9),$$

which, by setting $\lambda x^3 = q^2$ and $x = -q/B$, may be transformed into (see Berndt [26, pp. 80–82, Theorem])

$$f(B^3q, q^5/B^3) - B^2f(q/B^3, B^3q^5) = f(-q^2)\ \frac{f(-B^2, -q^2/B^2)}{f(Bq, q/B)}. \tag{2.49}$$

For a comprehensive survey of the work on the quintuple product identity and a detailed analysis of various proofs, we refer to Cooper [45].

Now we state and prove some theta function identities arising from (2.49) which will be used in Section 2.4 to prove Theorem 2.5.

The following relation on $G(q)$ and $H(q)$ was found by Gugg [63, 64] by using some theta function identities arising from (2.49).

**Lemma 2.15.** *We have*

$$G^3(q)H(q^3) - G(q^3)H^3(q) = \frac{3qf^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)}. \tag{2.50}$$

The identities in the following lemma were recorded by Ramanujan in his notebooks [98] and proofs based on (2.49) can be found in [26, p. 379, Entry 10] and [27, p. 188, Entry 36].

**Lemma 2.16.** *We have*

$$f(-q^7, -q^8) + qf(-q^2, -q^{13}) = f(-q^5)\ \frac{f(-q^2, -q^3)}{f(-q, -q^4)}, \tag{2.51}$$

$$f(-q^4, -q^{11}) - qf(-q, -q^{14}) = f(-q^5)\ \frac{f(-q, -q^4)}{f(-q^2, -q^3)}, \tag{2.52}$$

$$f^3(-q^7,-q^8)+q^3f^3(-q^2,-q^{13})=f^3(-q^5)\,\frac{f(-q^6,-q^9)}{f(-q^3,-q^{12})}, \tag{2.53}$$

$$f^3(-q^4,-q^{11})-q^3f^3(-q,-q^{14})=f^3(-q^5)\,\frac{f(-q^3,-q^{12})}{f(-q^6,-q^9)}. \tag{2.54}$$

In the following three lemmas, we state and prove some more useful theta function identities arising from (2.49).

**Lemma 2.17.** *We have*

$$f(q^7,q^8)-qf(q^2,q^{13})=f(-q^5)\,\frac{f(-q^2,-q^3)}{f(q,q^4)}, \tag{2.55}$$

$$f(q^4,q^{11})-qf(q,q^{14})=f(-q^5)\,\frac{f(-q,-q^4)}{f(q^2,q^3)}, \tag{2.56}$$

$$f^3(q^7,q^8)-q^3f^3(q^2,q^{13})=f^3(-q^5)\,\frac{f(-q^6,-q^9)}{f(q^3,q^{12})}, \tag{2.57}$$

$$f^3(q^4,q^{11})-q^3f^3(q,q^{14})=f^3(-q^5)\,\frac{f(-q^3,-q^{12})}{f(q^6,q^9)}. \tag{2.58}$$

*Proof.* Setting $q=q^{5/2}$ and $B=q^{3/2}$ in (2.49), we readily arrive at

$$f(q^7,q^8)-qf(q^2,q^{13})=f(-q^5)\,\frac{f(-q^2,-q^3)}{f(q,q^4)},$$

which is (2.55).

Let $\omega=e^{2\pi i/3}$. Putting $q=q^{5/2}$ and $B=\omega q^{3/2},\ \omega^2q^{3/2}$, in turn, in (2.49), yields

$$f(q^7,q^8)-q\omega^2f(q^2,q^{13})=f(-q^5)\,\frac{f(-\omega q^2,-\omega^2q^3)}{f(\omega^2q,\omega q^4)},$$

$$f(q^7,q^8)-q\omega f(q^2,q^{13})=f(-q^5)\,\frac{f(-\omega^2q^2,-\omega q^3)}{f(\omega q,\omega^2q^4)},$$

where the elementary identity $f(a,b)=af\left(a^{-1},a^2b\right)$ has also been used. Multiplying the previous three identities and then using (1.2), we find that

$$\begin{aligned}
&f^3(q^7,q^8)-q^3f(q^2,q^{13})\\
&=f^3(-q^5)\,\frac{f(-q^2,-q^3)f(-\omega q^2,-\omega^2q^3)f(-\omega^2q^2,-\omega q^3)}{f(q,q^4)f(\omega^2q,\omega q^4)f(\omega q,\omega^2q^4)}\\
&=f^3(-q^5)\,\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(\omega^2q^3;q^5)_\infty(\omega q^2;q^5)_\infty(\omega q^3;q^5)_\infty(\omega^2q^2;q^5)_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty(-\omega q^4;q^5)_\infty(-\omega^2q;q^5)_\infty(-\omega^2q^4;q^5)_\infty(-\omega q;q^5)_\infty}\\
&=f^3(-q^5)\,\frac{(q^6;q^{15})_\infty(q^9;q^{15})_\infty}{(-q^3;q^{15})_\infty(-q^{12};q^{15})_\infty}\\
&=f^3(-q^5)\,\frac{f(-q^6,-q^9)}{f(q^3,q^{12})},
\end{aligned}$$

which is (2.57).

Similarly, putting $q = q^{5/2}$ and $B = q^{1/2}$ in (2.49), we arrive at (2.56). Setting $q = q^{5/2}$ and $B = \omega q^{1/2}$, $\omega^2 q^{1/2}$ in (2.49) and then multiplying the resulting identities and (2.56) together, we obtain (2.58). □

**Lemma 2.18.** *We have*

$$f(-q^{13}, -q^{17}) + qf(-q^7, -q^{23}) = f(-q^{10}) \frac{f(-q^2, -q^8)}{f(-q, -q^9)}, \tag{2.59}$$

$$f(-q^{11}, -q^{19}) + q^3 f(-q, -q^{29}) = f(-q^{10}) \frac{f(-q^4, -q^6)}{f(-q^3, -q^7)}, \tag{2.60}$$

$$f^3(-q^{13}, -q^{17}) + q^3 f^3(-q^7, -q^{23}) = f^3(-q^{10}) \frac{f(-q^6, -q^{24})}{f(-q^3, -q^{27})}, \tag{2.61}$$

$$f^3(-q^{11}, -q^{19}) + q^9 f^3(-q, -q^{29}) = f^3(-q^{10}) \frac{f(-q^{12}, -q^{18})}{f(-q^9, -q^{21})}. \tag{2.62}$$

*Proof.* Setting $q = q^5$ and $B = -q^4$ and $B = -q^2$, in turn, in (2.49), we obtain (2.59) and (2.60).

The identities (2.61) and (2.62) can be proved in a similar way as in the proof of the previous lemma. □

**Lemma 2.19.** *We have*

$$f(q^{13}, q^{17}) - qf(q^7, q^{23}) = f(-q^{10}) \frac{f(-q^2, -q^8)}{f(q, q^9)}, \tag{2.63}$$

$$f(q^{11}, q^{19}) - q^3 f(q, q^{29}) = f(-q^{10}) \frac{f(-q^4, -q^6)}{f(q^3, q^7)}, \tag{2.64}$$

$$f^3(q^{13}, q^{17}) - q^3 f^3(q^7, q^{23}) = f^3(-q^{10}) \frac{f(-q^6, -q^{24})}{f(q^3, q^{27})}, \tag{2.65}$$

$$f^3(q^{11}, q^{19}) - q^9 f^3(q, q^{29}) = f^3(-q^{10}) \frac{f(-q^{12}, -q^{18})}{f(q^9, q^{21})}. \tag{2.66}$$

*Proof.* Putting $q = q^5$ and $B = q^4$ and $B = q^2$, in turn, in (2.49) yields (2.63) and (2.64). The identities (2.65) and (2.66) can be proved by proceeding as in the proof of Lemma 2.17. □

## 2.3 Proof of Theorem 2.4

*Proof of* (2.13). Setting $a = q$ and $b = q^5$ in (1.1), we have

$$f(q, q^5) = \sum_{n=-\infty}^{\infty} q^{3n^2-2n}.$$

It is easily checked that $3n^2 - 2n \equiv 0, 1$, or 3 modulo 5. Therefore, in the series expansion of $f(q, q^5)$, the coefficients of the terms of the forms $q^{5n+2}$ and $q^{5n+4}$ vanish. To exploit this fact further, we try to find a 5-dissection of $f(q, q^5)$. To that end, first we see from Jacobi triple product identity, (1.2), that

$$f(q, q^5) = (-q; q^6)_\infty(-q^5; q^6)_\infty(q^6; q^6)_\infty = \chi(q)\psi(-q^3). \tag{2.67}$$

Next, from (2.43) and (2.36), we have

$$\begin{aligned}
&4q\chi(q) \\
&= \frac{1}{f(-q^5)f(-q^{20})}\left(\varphi^2(q) - \varphi^2(q^5)\right) \\
&= \frac{1}{f(-q^5)f(-q^{20})}\left(\left(\varphi(q^{25}) + 2qf(q^{15}, q^{35}) + 2q^4f(q^5, q^{45})\right)^2 - \varphi^2(q^5)\right).
\end{aligned}$$

Employing (2.43) again in the above, and then dividing both sides by $4q$, we obtain

$$\begin{aligned}
\chi(q) = \frac{1}{f(-q^5)f(-q^{20})}\Big(&q^4f(q^5, q^{45})f(q^{15}, q^{35}) + \varphi(q^{25})f(q^{15}, q^{35}) \\
&+ qf^2(q^{15}, q^{35}) + q^3\varphi(q^{25})f(q^5, q^{45}) + q^7f^2(q^5, q^{45})\Big).
\end{aligned} \tag{2.68}$$

Now, employing the above identity and (2.37) with $q$ replaced by $-q^3$, in (2.67), we obtain the following 5-dissection of $f(q, q^5)$:

$$\begin{aligned}
f(q, q^5) = \frac{1}{f(-q^5)f(-q^{20})}&\left(q^4f(q^5, q^{45})f(q^{15}, q^{35}) + \varphi(q^{25})f(q^{15}, q^{35})\right. \\
&\left.+ qf^2(q^{15}, q^{35}) + q^3\varphi(q^{25})f(q^5, q^{45}) + q^7f^2(q^5, q^{45})\right) \\
&\times \left(f(q^{30}, -q^{45}) - q^3f(-q^{15}, q^{60}) - q^9\psi(-q^{75})\right).
\end{aligned}$$

Extracting the terms of the form $q^{5n+2}$ (or $q^{5n+4}$) from both sides of the above after noting from our earlier observation that no such terms appear on the left-hand side, we find that

$$f(q^{15}, q^{35})f(-q^{15}, q^{60}) + q^5\varphi(q^{25})\psi(-q^{75}) - f(q^5, q^{45})f(q^{30}, -q^{45}) = 0,$$

which, by replacing $q^5$ by $-q$, reduces to

$$f(-q^3, -q^7)f(q^3, q^{12}) - f(-q, -q^9)f(q^6, q^9) = q\varphi(-q^5)\psi(q^{15}).$$

Employing (1.3), (1.5) and (2.29)–(2.32) in the above, we readily arrive at (2.13).

*Proof of* (2.14). Setting $a=q$ and $b=q^2$ in (1.1), we have

$$f(q,q^2)=\sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2}.$$

Observe that $(3n^2-n)/2 \equiv 0, 1$ or $2$ modulo 5. Thus, in the series expansion of $f(q,q^2)$ the terms of the forms $q^{5n+3}$ and $q^{5n+4}$ vanish.

We now find a 5-dissection of $f(q,q^2)$. By Jacobi triple product identity (1.2) and Euler's identity $(-q;q)_\infty = 1/(q;q^2)_\infty$, we have

$$\begin{aligned} f(q,q^2) &= (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty = \frac{(-q;q)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty} \\ &= \frac{(q^3;q^3)^2_\infty(q^6;q^6)_\infty}{(q;q^2)_\infty} = \frac{\varphi(-q^3)}{\chi(-q)}. \end{aligned} \tag{2.69}$$

Next, from (2.44) and (2.37), we have

$$\begin{aligned} \frac{1}{\chi(-q)} &= \frac{1}{f(-q^5)\varphi(-q^5)}\left(\psi^2(q)-q\psi^2(q^5)\right) \\ &= \frac{1}{f(-q^5)\varphi(-q^5)}\Big(\left(f(q^{10},q^{15})+qf(q^5,q^{20})+q^3\psi(q^{25})\right)^2 \\ &\quad - q\psi^2(q^5)\Big). \end{aligned}$$

Once again applying (2.44) in the above, we have

$$\begin{aligned} \frac{1}{\chi(-q)} = \frac{1}{f(-q^5)\varphi(-q^5)}\Big(&f^2(q^{10},q^{15})+qf(q^{10},q^{15})f(q^5,q^{20})+q^2f^2(q^5,q^{20}) \\ &+2q^3\psi(q^{25})f(q^{10},q^{15})+2q^4\psi(q^{25})f(q^5,q^{20})\Big). \end{aligned} \tag{2.70}$$

Employing the above identity and (2.36), with $q$ replaced by $-q^3$, in (2.69), we arrive at the following 5-dissection of $f(q,q^2)$:

$$\begin{aligned} f(q,q^2) = \frac{1}{f(-q^5)\varphi(-q^5)}\Big(&f^2(q^{10},q^{15})+qf(q^{10},q^{15})f(q^5,q^{20})+q^2f^2(q^5,q^{20}) \\ &+2q^3\psi(q^{25})f(q^{10},q^{15})+2q^4\psi(q^{25})f(q^5,q^{20})\Big) \\ &\times\Big(\varphi(-q^{75})-2q^3f(-q^{45},-q^{105})+2q^{12}f(-q^{15},-q^{135})\Big). \end{aligned}$$

Now, we recall that the series expansion of $f(q,q^2)$ does not contain terms of the forms $q^{5n+3}$ and $q^{5n+4}$. Therefore, extracting the terms of the form $q^{5n+3}$ (or $q^{5n+4}$) from both sides of the above, we find that

$$f(-q^{45},-q^{105})f(q^{10},q^{15})-q^{10}f(-q^{15},-q^{135})f(q^5,q^{20})-\varphi(-q^{75})\psi(q^{25})=0.$$

Replacing $q^5$ by $q$ in the above identity, we have

$$f(q^2,q^3)f(-q^9,-q^{21}) - q^2 f(q,q^4)f(-q^3,-q^{27}) = \psi(q^5)\varphi(-q^{15}).$$

Employing (1.3), (1.5), and (2.29)–(2.32) in the above, we arrive at (2.14) to finish the proof.

## 2.4 Proof of Theorem 2.5

*Proof of* (2.15). From (2.51) and (2.53), we have

$$\begin{aligned}
&f^3(-q^5)\frac{f^3(-q^2,-q^3)}{f^3(-q,-q^4)} - f^3(-q^5)\frac{f(-q^6,-q^9)}{f(-q^3,-q^{12})}\\
&= \left(f(-q^7,-q^8)+qf(-q^2,-q^{13})\right)^3 - \left(f^3(-q^7,-q^8)+q^3f^3(-q^2,-q^{13})\right)\\
&= 3qf(-q^7,-q^8)f(-q^2,-q^{13})\left(f(-q^7,-q^8)+qf(-q^2,-q^{13})\right)\\
&= 3qf(-q^7,-q^8)f(-q^2,-q^{13})\left(f(-q^5)\frac{f(-q^2,-q^3)}{f(-q,-q^4)}\right). \qquad (2.71)
\end{aligned}$$

Applying (2.27) in (2.71) yields

$$f^2(-q^5)\left(\frac{G^3(q)}{H^3(q)} - \frac{G(q^3)}{H(q^3)}\right) = 3q\frac{G(q)}{H(q)}f(-q^7,-q^8)f(-q^2,-q^{13}).$$

Therefore,

$$G^3(q)H(q^3) - G(q^3)H^3(q) = \frac{3q}{f^2(-q^5)}G(q)H^2(q)H(q^3)f(-q^7,-q^8)f(-q^2,-q^{13}). \qquad (2.72)$$

In a similar fashion, from (2.52), (2.54), and (2.27), we find that

$$G^3(q)H(q^3) - G(q^3)H^3(q) = \frac{3q}{f^2(-q^5)}G^2(q)H(q)G(q^3)f(-q^4,-q^{11})f(-q,-q^{14}). \qquad (2.73)$$

Comparing (2.72) and (2.73), and then applying (2.24), we arrive at

$$\begin{aligned}
&H(q)H(q^3)\left(f(q^9,q^{21})f(q^{10},q^{20}) - q^2f(q^6,q^{24})f(q^5,q^{25})\right)\\
&= G(q)G(q^3)\left(f(q^5,q^{25})f(q^{12},q^{18}) - qf(q^3,q^{27})f(q^{10},q^{20})\right).
\end{aligned}$$

Employing (2.25), (2.26), (2.29), (2.30), (2.33) in the above and then simplifying, we have

$$\frac{f(-q^6)\varphi(-q^{30})}{\chi(-q^{10})}\left(H(q)H^2(q^3)G(q^{12}) + qG(q)G^2(q^3)H(q^{12})\right)$$

$$= \frac{f(-q^{12})f(-q^{30})}{f(-q^{60})}\psi(-q^{15})\chi(q^5)$$
$$\times \left(G(q)G(q^3)H(q^6)G(q^{12}) + q^2H(q)H(q^3)G(q^6)H(q^{12})\right),$$

which, with the aid of (1.3)-(1.7), can be rearranged to (2.15).

*Proof of* (2.16). From (2.55) and (2.57), we have

$$f^3(-q^5)\left(\frac{f(-q^6,-q^9)}{f(q^3,q^{12})} - \frac{f^3(-q^2,-q^3)}{f^3(q,q^4)}\right)$$
$$= \left(f^3(q^7,q^8) - q^3f^3(q^2,q^{13})\right) - \left(f(q^7,q^8) - qf(q^2,q^{13})\right)^3$$
$$= 3qf(q^7,q^8)f(q^2,q^{13})\left(f(q^7,q^8) - qf(q^2,q^{13})\right)$$
$$= 3qf(-q^5)f(q^7,q^8)f(q^2,q^{13})\frac{f(-q^2,-q^3)}{f(q,q^4)}. \tag{2.74}$$

Applying (2.27) and (2.30) in the above yields

$$G^3(q)H^3(q)G(q^6) - G^3(q^2)G(q^3)H(q^3)$$
$$= \frac{3}{f^2(-q^5)}G^2(q)H^2(q)G(q^2)G(q^3)H(q^3)f(q^7,q^8)f(q^2,q^{13}). \tag{2.75}$$

In a similar way, from (2.56) and (2.58), we obtain

$$f^3(-q^5)\left(\frac{f(-q^3,-q^{12})}{f(q^6,q^9)} - \frac{f^3(-q,-q^4)}{f^3(q^2,q^3)}\right) = 3qf(-q^5)f(q^4,q^{11})f(q,q^{14})\frac{f(-q,-q^4)}{f(q^2,q^3)}. \tag{2.76}$$

Employing (2.27) and (2.29) in (2.76), we find that

$$G^3(q)H^3(q)H(q^6) - H^3(q^2)G(q^3)H(q^3)$$
$$= \frac{3}{f^2(-q^5)}G^2(q)H^2(q)H(q^2)G(q^3)H(q^3)f(q^4,q^{11})f(q,q^{14}). \tag{2.77}$$

Dividing (2.75) by (2.77), we have

$$\frac{G^3(q)H^3(q)G(q^6) - G^3(q^2)G(q^3)H(q^3)}{G^3(q)H^3(q)H(q^6) - H^3(q^2)G(q^3)H(q^3)} = \frac{G(q^2)f(q^7,q^8)f(q^2,q^{13})}{H(q^2)f(q^4,q^{11})f(q,q^{14})}.$$

With the aid of (2.24), the above identity may be rewritten as

$$H(q^2)\Big(f(q^5,q^{25})f(q^{12},q^{18}) + qf(q^{10},q^{20})f(q^3,q^{27})\Big)$$
$$\times\Big(G^3(q)H^3(q)G(q^6) - G^3(q^2)G(q^3)H(q^3)\Big)$$
$$= G(q^2)\Big(f(q^9,q^{21})f(q^{10},q^{20}) + q^2f(q^6,q^{24})f(q^5,q^{25})\Big)$$

$$\times\left(G^3(q)H^3(q)H(q^6)-H^3(q^2)G(q^3)H(q^3)\right).$$

Employing (2.25), (2.26), (2.29), (2.30), (2.33) in the above and then simplifying further, we arrive at (2.16).

*Proof of* (2.17). As the proof is similar to that of (2.16), we only mention two intermediate identities below.

From (2.59)–(2.62), we have

$$\begin{aligned}&G^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)G(q^3)\\&=\frac{3q}{f^2(-q^{10})}G(q)G^2(q^2)H^2(q^2)G(q^6)H(q^6)f(-q^{13},-q^{17})f(-q^7,-q^{23})\end{aligned}$$

and

$$\begin{aligned}&H^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)H(q^3)\\&=\frac{3q^3}{f^2(-q^{10})}H(q)G^2(q^2)H^2(q^2)G(q^6)H(q^6)f(-q^{11},-q^{19})f(-q,-q^{29}).\end{aligned}$$

*Proof of* (2.18). The proof is also similar to that of (2.16). So we state only two intermediate identities below.

From (2.63)–(2.66), it can be shown that

$$\begin{aligned}&G^3(q)H^3(q^4)H(q^6)-H^3(q^2)G(q^3)H(q^{12})\\&\quad=\frac{3q}{f^2(-q^{10})}G^2(q)H(q^2)G(q^3)H^2(q^4)H(q^{12})f(q^{13},q^{17})f(q^7,q^{23})\end{aligned}$$

and

$$\begin{aligned}&H^3(q)G^3(q^4)G(q^6)-G^3(q^2)H(q^3)G(q^{12})\\&\quad=\frac{3q^3}{f^2(-q^{10})}H^2(q)G(q^2)H(q^3)G^2(q^4)G(q^{12})f(q^{11},q^{19})f(q,q^{29}).\end{aligned}$$

*Proof of* (2.19). From (2.74), we have

$$\begin{aligned}&f(-q^6,-q^9)f^3(q,q^4)-f(q^3,q^{12})f^3(-q^2,-q^3)\\&\quad=\frac{3q}{f^2(-q^5)}f(-q^2,-q^3)f^2(q,q^4)f(q^3,q^{12})f(q^7,q^8)f(q^2,q^{13}).\end{aligned}$$

Employing the Jacobi triple product identity, (1.2), in the above, we find that

$$f(-q^6,-q^9)f^3(q,q^4)-f(q^3,q^{12})f^3(-q^2,-q^3)$$

$$= \frac{3q}{(q^5;q^5)_\infty^2} f(q,q^4)(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty(-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty$$
$$\times (-q^3;q^{15})_\infty(-q^{12};q^{15})_\infty(q^{15};q^{15})_\infty(-q^7;q^{15})_\infty(-q^8;q^{15})_\infty(q^{15};q^{15})_\infty$$
$$\times (-q^2;q^{15})_\infty(-q^{13};q^{15})_\infty(q^{15};q^{15})_\infty$$
$$= \frac{3qf(-q^2)f^3(-q^{15})}{f(-q)f(-q^{10})} f(q,q^4)f(-q^2,-q^3). \tag{2.78}$$

Applying (2.27) and (2.30) in (2.78), we have

$$\frac{G^3(q)H^3(q)G(q^3)}{G^3(q^2)} - \frac{G^2(q^3)H(q^3)}{G(q^6)} = \frac{3qf(-q^2)f^3(-q^{15})H(q)}{f^2(-q)f(-q^3)f(-q^{10})G(q^2)}. \tag{2.79}$$

Next, from (2.76), we have

$$f(-q^3,-q^{12})f^3(q^2,q^3) - f(q^6,q^9)f^3(-q,-q^4)$$
$$= \frac{3q}{f^2(-q^5)} f(q^4,q^{11})f(q,q^{14})f(-q,-q^4)f^2(q^2,q^3)f(q^6,q^9).$$

Applying (1.2) in the above, we obtain

$$f(-q^3,-q^{12})f^3(q^2,q^3) - f(q^6,q^9)f^3(-q,-q^4)$$
$$= \frac{3qf(-q^2)f^3(-q^{15})}{f(-q)f(-q^{10})} f(-q,-q^4)f(q^2,q^3),$$

which by (2.27) and (2.29), reduces to

$$\frac{G^3(q)H^3(q)H(q^3)}{H^3(q^2)} - \frac{G(q^3)H^2(q^3)}{H(q^6)} = \frac{3qf(-q^2)f^3(-q^{15})G(q)}{f^2(-q)f(-q^3)f(-q^{10})H(q^2)}. \tag{2.80}$$

Dividing (2.79) by (2.80) and then employing (1.14), we deduce (2.19).

*Proof of* (2.20). The proof is similar to that of (2.19) given above. Therefore, we mention only two theta function identities below.

From (2.59)–(2.62), we find that

$$f^3(-q^2,-q^8)f(-q^3,-q^{27}) - f^3(-q,-q^9)f(-q^6,-q^{24})$$
$$= 3q\frac{f(-q)f(-q^{10})f^3(-q^{30})}{f(-q^5)} \cdot \frac{f(-q,-q^9)}{f(-q^4,-q^6)}$$

and

$$f^3(-q^4,-q^6)f(-q^9,-q^{21}) - f^3(-q^3,-q^7)f(-q^{12},-q^{18})$$
$$= 3q^3\frac{f(-q)f(-q^{10})f^3(-q^{30})}{f(-q^5)} \cdot \frac{f(-q^3,-q^7)}{f(-q^2,-q^8)}.$$

*Proof of* (2.21). The proof of this identity is also similar to that of (2.19) given earlier. So we mention only two important theta function identities.

We obtain the following identities from (2.63)–(2.66) by proceeding as in the proof of (2.19).

$$\begin{aligned}&f^3(q,q^9)f(-q^6,-q^{24})-f^3(-q^2,-q^8)f(q^3,q^{27})\\&\quad=3q\frac{f(-q^2)f(-q^5)f^3(-q^{30})}{f(-q)}\cdot\frac{f(-q^2,-q^8)f(q,q^9)}{f(q^2,q^8)f(q^4,q^6)}\end{aligned}$$

and

$$\begin{aligned}&f^3(q^3,q^7)f(-q^{12},-q^{18})-f^3(-q^4,-q^6)f(q^9,q^{21})\\&\quad=3q^3\frac{f(-q^2)f(-q^5)f^3(-q^{30})}{f(-q)}\cdot\frac{f(-q^4,-q^6)f(q^3,q^7)}{f(q^2,q^8)f(q^4,q^6)}.\end{aligned}$$

## 2.5 Proofs of Theorem 2.1 and Corollary 2.2

*Proof of* (2.2). By (2.41), we have

$$\begin{aligned}\varphi(q)&=\varphi(q^5)+2q^{1/5}f(q^3,q^7)R(q^4)\\&=\varphi(q^{25})+2qf(q^{15},q^{35})R(q^{20})+2q^{1/5}f(q^3,q^7)R(q^4).\end{aligned}\tag{2.81}$$

But, from (2.36), we recall that

$$\varphi(q)=\varphi(q^{25})+2qf(q^{15},q^{35})+2q^4f(q^5,q^{45}).\tag{2.82}$$

Comparing (2.81) and (2.82), we find that

$$q^{1/5}f(q^3,q^7)R(q^4)=qf(q^{15},q^{35})+q^4f(q^5,q^{45})-qf(q^{15},q^{35})R(q^{20}).$$

Employing (1.15) and (2.33) in the above, we obtain

$$f(-q^2)H(q)H(q^4)=f(-q^{10})\left(H(q^5)G(q^{20})+q^3G(q^5)H(q^{20})-q^4H(q^5)H(q^{20})\right).\tag{2.83}$$

Again, by (2.40), we have

$$\begin{aligned}\varphi(q)&=-\varphi(q^5)+2q^{4/5}f(q,q^9)R^{-1}(q^4)\\&=\varphi(q^{25})-2q^4f(q^5,q^{45})R^{-1}(q^{20})+2q^{4/5}f(q,q^9)R^{-1}(q^4).\end{aligned}\tag{2.84}$$

Comparing (2.82) and (2.84), we find that

$$q^{4/5}f(q,q^9)R^{-1}(q^4)=q^4f(q^5,q^{45})R^{-1}(q^{20})+qf(q^{15},q^{35})+q^4f(q^5,q^{45}).$$

Using (1.15) and (2.33) in the above, we have

$$
\begin{aligned}
&f(-q^2)G(q)G(q^4)\\
&= f(-q^{10})\left(G(q^5)G(q^{20}) + qH(q^5)G(q^{20}) + q^4G(q^5)H(q^{20})\right).
\end{aligned}
$$

It follows from the above identity and (2.83) that

$$
\frac{H(q)H(q^4)}{G(q)G(q^4)} = \frac{H(q^5)G(q^{20}) + q^3G(q^5)H(q^{20}) - q^4H(q^5)H(q^{20})}{G(q^5)G(q^{20}) + qH(q^5)G(q^{20}) + q^4G(q^5)H(q^{20})},
$$

from which, by (1.14), we arrive at

$$
R(q)R(q^4) = \frac{R(q^5) + R(q^{20}) - R(q^5)R(q^{20})}{1 + R(q^5) + R(q^{20})}.
$$

Thus, we complete the proof of (2.2).

*Proof of* (2.3). By (2.42), we have

$$
\begin{aligned}
\psi(q^2) &= q\psi(q^{10}) + q^{-1/5}f(q^4, q^6)R(q)\\
&= q^6\psi(q^{50}) + f(q^{20}, q^{30})R(q^5) + q^{-1/5}f(q^4, q^6)R(q). \qquad (2.85)
\end{aligned}
$$

Comparing (2.85) with (2.37), we find that

$$
q^{-1/5}f(q^4, q^6)R(q) = f(q^{20}, q^{30}) - f(q^{20}, q^{30})R(q^5) + q^2f(q^{10}, q^{40}).
$$

With the aid of (1.15), (2.29), and (2.30), the above identity may be recast as

$$
\begin{aligned}
&f(-q^2)\frac{H(q)G(q^2)H^2(q^2)}{G(q)H(q^4)}\\
&\qquad = f(-q^{50})\left(\frac{H(q^{10})}{H(q^{20})} - q\frac{H(q^5)H(q^{10})}{G(q^5)H(q^{20})} + q^2\frac{G(q^{10})}{G(q^{20})}\right). \qquad (2.86)
\end{aligned}
$$

Replacing $q$ by $-q$ in the above and then employing (2.34), we find that

$$
\begin{aligned}
&f(-q^2)\frac{G(q)G^2(q^2)H(q^2)}{H(q)G(q^4)}\\
&\qquad = f(-q^{50})\left(\frac{H(q^{10})}{H(q^{20})} + q\frac{G(q^5)G(q^{10})}{H(q^5)G(q^{20})} + q^2\frac{G(q^{10})}{G(q^{20})}\right). \qquad (2.87)
\end{aligned}
$$

Dividing (2.86) by (2.87), we find that

$$
\frac{H^2(q)H(q^2)G(q^4)}{G^2(q)G(q^2)H(q^4)} = \frac{\dfrac{H(q^{10})}{H(q^{20})} - q\dfrac{H(q^5)H(q^{10})}{G(q^5)H(q^{20})} + q^2\dfrac{G(q^{10})}{G(q^{20})}}{\dfrac{H(q^{10})}{H(q^{20})} + q\dfrac{G(q^5)G(q^{10})}{H(q^5)G(q^{20})} + q^2\dfrac{G(q^{10})}{G(q^{20})}}.
$$

By (1.14), the above transforms into

$$\frac{R^2(q)R(q^2)}{R(q^4)} = \frac{\dfrac{R(q^{10})}{R(q^{20})} - \dfrac{R(q^5)R(q^{10})}{R(q^{20})} + 1}{\dfrac{R(q^{10})}{R(q^{20})} + \dfrac{1}{R(q^5)} + 1},$$

which is clearly equivalent to (2.3).

*Proof of* (2.4). Replacing $q$ by $q^5$ in (2.39) and then squaring, we have

$$\frac{\psi^2(q)}{q^6\psi^2(q^{25})} = \left(\frac{1 - R(q^5)R^2(q^{10})}{R(q^5)R(q^{10})} + \frac{1 + R(q^5)R^2(q^{10})}{R(q^{10})} + 1\right)^2. \qquad (2.88)$$

Again, replacing $q$ by $q^5$ in (2.38) and then multiplying the resulting identity with (2.38) again, we find that

$$\frac{\psi^2(q)}{q^6\psi^2(q^{25})} = \left(\frac{1 + R(q^5)R^2(q^{10}) - R^2(q^5)R^4(q^{10})}{R(q^5)R^2(q^{10})}\right)\left(1 + \frac{1}{uv^2} - uv^2\right), \qquad (2.89)$$

where $u = R(q)$ and $v = R(q^2)$. From (2.88) and (2.89), it follows that

$$\frac{1}{uv^2} - uv^2 = \left(\frac{R(q^5)R^2(q^{10})}{1 + R(q^5)R^2(q^{10}) - R^2(q^5)R^4(q^{10})}\right) \times \left(\frac{1 - R(q^5)R^2(q^{10})}{R(q^5)R(q^{10})} + \frac{1 + R(q^5)R^2(q^{10})}{R(q^{10})} + 1\right)^2 - 1,$$

which upon simplification gives (2.4). This completes the proof of Theorem 2.1.

Now we prove the identities in Corollary 2.2.

*Proofs of* (2.5) and (2.6). Replacing $q$ by $-q$ in (2.2) and then applying (2.35) and (1.15), we easily deduce (2.5). Similarly, one can deduce (2.6) by using (2.35) and (1.15) in (2.3).

*Proof of* (2.7). On p. 326 in his second notebook [98], [28, p. 12], Ramanujan recorded the following modular equation relating $R(q)$ and $R(q^2)$:

$$R(q)R(q^2)^2 = \frac{R(q^2) - R^2(q)}{R(q^2) + R^2(q)}.$$

This identity was proved first by Rogers [105].

The above identity may be recast in the form

$$\frac{1}{R(q)R(q^2)^2} - R(q)R(q^2)^2 = \frac{4}{\dfrac{R(q^2)}{R(q)^2} - \dfrac{R(q)^2}{R(q^2)}}.$$

From the above identity and (2.4), we readily deduce (2.7) to finish the proof of Corollary 2.2.

## 2.6 Proof of Theorem 2.3

*Proof of* (2.8). Multiplying (2.13) and (2.14), we have

$$\frac{G(q^2)G(q^3)}{H(q^2)H(q^3)}+q^2\frac{H(q^2)H(q^3)}{G(q^2)G(q^3)}-\frac{G(q^6)H(q)}{G(q)H(q^6)}-q^2\frac{G(q)H(q^6)}{G(q^6)H(q)}=q,$$

which, by (1.14), readily transforms into (2.8).

*Proof of* (2.9). Dividing (2.48) by (2.47), we have

$$\frac{G(q^{16})H(q)+q^3G(q)H(q^{16})}{G(q^{16})H(q)-q^3G(q)H(q^{16})}=\frac{\varphi(-q^{20})+2q^3f(-q^8)G(q^{16})H(q^{16})}{\varphi(-q^4)}.$$

Replacing $q$ by $-q^4$ in (2.45) and (2.46), and then using the resulting identities in the above identity, we find that

$$\begin{aligned}&\frac{G(q^{16})H(q)+q^3G(q)H(q^{16})}{G(q^{16})H(q)-q^3G(q)H(q^{16})}\\&=\frac{G(-q^4)G(q^{16})+q^4H(-q^4)H(q^{16})+2q^3G(q^{16})H(q^{16})}{G(-q^4)G(q^{16})-q^4H(-q^4)H(q^{16})},\end{aligned}$$

which may be rewritten with the aid of (1.15) as

$$\frac{T(q)+q^3T(q^{16})}{T(q)-q^3T(q^{16})}=\frac{G(-q^4)G(q^{16})+q^4H(-q^4)H(q^{16})+2q^3G(q^{16})H(q^{16})}{G(-q^4)G(q^{16})-q^4H(-q^4)H(q^{16})}. \quad (2.90)$$

Replacing $q$ by $-q$ in (2.90), we have

$$\frac{T(-q)-q^3T(q^{16})}{T(-q)+q^3T(q^{16})}=\frac{G(-q^4)G(q^{16})+q^4H(-q^4)H(q^{16})-2q^3G(q^{16})H(q^{16})}{G(-q^4)G(q^{16})-q^4H(-q^4)H(q^{16})}. \quad (2.91)$$

Adding (2.90) and (2.91), and then using (1.15), we find that

$$\begin{aligned}\frac{T(q)+q^3T(q^{16})}{T(q)-q^3T(q^{16})}+\frac{T(-q)-q^3T(q^{16})}{T(-q)+q^3T(q^{16})}&=2\frac{G(-q^4)G(q^{16})+q^4H(-q^4)H(q^{16})}{G(-q^4)G(q^{16})-q^4H(-q^4)H(q^{16})}\\&=2\frac{1+q^4T(-q^4)T(q^{16})}{1-q^4T(-q^4)T(q^{16})}.\end{aligned}$$

Employing (2.35) in the above and then simplifying, we obtain

$$T(q^2)=\frac{T^2(q^4)T(q^8)-2qT(q)T(q^4)T(q^{16})+q^4T(q^4)T^2(q^{16})}{T^2(q)T(q^4)T(q^8)-2q^3T(q)T(q^4)T(q^8)T(q^{16})+q^4T^2(q)T^2(q^{16})}.$$

Applying (1.15) in the above, we readily arrive at (2.9).

*Proof of* (2.10). From (2.79) and (2.50), we have

$$\begin{aligned}
&\frac{G^3(q)H^3(q)G(q^3)}{G^3(q^2)}-\frac{G^2(q^3)H(q^3)}{G(q^6)}\\
&=3q\frac{f(-q^2)f^3(-q^{15})}{f^2(-q)f(-q^3)f(-q^{10})}\frac{H(q)}{G(q^2)}\\
&=3q\frac{f^3(-q^{15})}{f(-q)f(-q^3)f(-q^5)}\frac{G(q)H^2(q)}{G^2(q^2)H(q^2)}\\
&=\frac{G(q)H^2(q)}{G^2(q^2)H(q^2)}\Big(G^3(q)H(q^3)-H^3(q)G(q^3)\Big),
\end{aligned}$$

which on simplification gives

$$\begin{aligned}
G^3(q)H^3(q)H(q^2)G(q^3)-G^4(q)H^2(q)G(q^2)H(q^3)+G(q)H^5(q)G(q^2)G(q^3)\\
=\frac{G^3(q^2)H(q^2)G^2(q^3)H(q^3)}{G(q^6)}.
\end{aligned} \tag{2.92}$$

Similarly, from (2.80) and (2.50), we find that

$$\begin{aligned}
G^3(q)H^3(q)G(q^2)H(q^3)+G^2(q)H^4(q)H(q^2)G(q^3)-G^5(q)H(q)H(q^2)H(q^3)\\
=\frac{G(q^2)H^3(q^2)G(q^3)H^2(q^3)}{H(q^6)}.
\end{aligned} \tag{2.93}$$

Dividing (2.92) by (2.93), we have

$$\begin{aligned}
&\frac{G^3(q)H^3(q)H(q^2)G(q^3)-G^4(q)H^2(q)G(q^2)H(q^3)+G(q)H^5(q)G(q^2)G(q^3)}{G^3(q)H^3(q)G(q^2)H(q^3)+G^2(q)H^4(q)H(q^2)G(q^3)-G^5(q)H(q)H(q^2)H(q^3)}\\
&=\frac{G^2(q^2)G(q^3)H(q^6)}{H^2(q^2)H(q^3)G(q^6)}.
\end{aligned} \tag{2.94}$$

Dividing the numerator and denominator of the left-hand side of the above identity by $G^4(q)H^2(q)G(q^2)H(q^3)$ and then employing (1.14), we obtain

$$\frac{\dfrac{R(q)R(q^2)}{R(q^3)}-1+\dfrac{R^3(q)}{R(q^3)}}{R(q)+\dfrac{R^2(q)R(q^2)}{R(q^3)}-\dfrac{R(q^2)}{R(q)}}=\frac{R(q^6)}{R^2(q^2)R(q^3)},$$

which can be easily reduced to (2.10).

*Proof of* (2.11). Comparing (2.15) and (2.16), we have

$$\begin{aligned}
&\frac{H(q)H^2(q^3)G(q^{12})+qG(q)G^2(q^3)H(q^{12})}{G(q)G(q^3)H(q^6)G(q^{12})+q^2H(q)H(q^3)G(q^6)H(q^{12})}\\
&=\frac{UG(q^2)H(q^3)G(q^{12})-qVH(q^2)G(q^3)H(q^{12})}{VH(q^2)H(q^6)G(q^{12})-q^2UG(q^2)G(q^6)H(q^{12})}.
\end{aligned} \tag{2.95}$$

Now, we rewrite the terms on the numerators and denominators of the above as follows:

$$H(q)H^2(q^3)G(q^{12})+qG(q)G^2(q^3)H(q^{12})=G(q)G^2(q^3)H(q^{12})\left(\frac{T(q)T^2(q^3)}{T(q^{12})}+q\right),$$

$$\begin{aligned}&G(q)G(q^3)H(q^6)G(q^{12})+q^2H(q)H(q^3)G(q^6)H(q^{12})\\&\quad=H(q)H(q^3)G(q^6)H(q^{12})\left(\frac{T(q^6)}{T(q)T(q^3)T(q^{12})}+q^2\right),\end{aligned}$$

$$\begin{aligned}&UG(q^2)H(q^3)G(q^{12})-qVH(q^2)G(q^3)H(q^{12})\\&\quad=\Big(H(q^2)G(q^3)H(q^{12})\Big)\Big(G^3(q)H^3(q)H(q^6)-H^3(q^2)G(q^3)H(q^3)\Big)\\&\qquad\times\left(\frac{T(q^3)}{T(q^2)T(q^{12})}-q\frac{G^3(q)H^3(q)G(q^6)-G^3(q^2)G(q^3)H(q^3)}{G^3(q)H^3(q)H(q^6)-H^3(q^2)G(q^3)H(q^3)}\right),\end{aligned}$$

$$\begin{aligned}&VH(q^2)H(q^6)G(q^{12})-q^2UG(q^2)G(q^6)H(q^{12})\\&\quad=\Big(G(q^2)G(q^6)H(q^{12})\Big)\Big(G^3(q)H^3(q)H(q^6)-H^3(q^2)G(q^3)H(q^3)\Big)\\&\qquad\times\left(\frac{T(q^2)T(q^6)}{T(q^{12})}\cdot\frac{G^3(q)H^3(q)G(q^6)-G^3(q^2)G(q^3)H(q^3)}{G^3(q)H^3(q)H(q^6)-H^3(q^2)G(q^3)H(q^3)}-q^2\right),\end{aligned}$$

where $T(q)$ is as defined in (1.15).

Using the above expressions in (2.95) and then employing (2.19), we find that

$$\begin{aligned}&\frac{G(q)G^2(q^3)}{H(q)H(q^3)G(q^6)}\times\frac{\dfrac{T(q)T^2(q^3)}{T(q^{12})}+q}{\dfrac{T(q^6)}{T(q)T(q^3)T(q^{12})}+q^2}\\&\qquad=\frac{H(q^2)G(q^3)}{G(q^2)G(q^6)}\times\frac{\dfrac{T(q^3)}{T(q^2)T(q^{12})}-q\dfrac{T(q)T(q^3)}{T^2(q^2)T(q^6)}}{\dfrac{T(q^2)T(q^6)}{T(q^{12})}\times\dfrac{T(q)T(q^3)}{T^2(q^2)T(q^6)}-q^2},\end{aligned}$$

which, by (1.15), yields

$$\frac{R(q)R^2(q^3)+R(q^{12})}{R(q^6)+R(q)R(q^3)R(q^{12})}=\frac{R(q^3)}{R(q^6)}\times\frac{R(q^2)R(q^6)-R(q)R(q^{12})}{R(q)R(q^3)-R(q^2)R(q^{12})}.$$

Simplifying the previous identity for $R(q^2)$, we arrive at (2.11).

*Proof of* (2.12). Comparing (2.17) and (2.18), we have

$$\begin{aligned}&\frac{WG(q)H(q^{12})G(q^{24})-q^2XH(q)G(q^{12})H(q^{24})}{q^4WG(q)G(q^6)H(q^{24})-XH(q)H(q^6)G(q^{24})}\\&\qquad=\frac{SYH(q^{12})G(q^{24})-q^2TZG(q^{12})H(q^{24})}{TZH(q^6)G(q^{24})-q^4SYG(q^6)H(q^{24})}.\end{aligned}\tag{2.96}$$

We rewrite the terms on the numerators and denominators of the above as follows:

$$\begin{aligned}
&WG(q)H(q^{12})G(q^{24})-q^2XH(q)G(q^{12})H(q^{24})\\
&=\Big(G(q)H(q^{12})G(q^{24})\Big)\Big(G^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)G(q^3)\Big)\\
&\quad\times\left(\frac{H^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)H(q^3)}{G^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)G(q^3)}-q^2\frac{T(q)T(q^{24})}{T(q^{12})}\right),\\
&q^4WG(q)G(q^6)H(q^{24})-XH(q)H(q^6)G(q^{24})\\
&=\Big(G(q)G(q^6)H(q^{24})\Big)\Big(G^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)G(q^3)\Big)\\
&\quad\times\left(q^4\frac{H^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)H(q^3)}{G^3(q)G(q^6)H(q^6)-G^3(q^2)H^3(q^2)G(q^3)}-\frac{T(q)T(q^6)}{T(q^{24})}\right),\\
&SYH(q^{12})G(q^{24})-q^2TZG(q^{12})H(q^{24})\\
&=\Big(G^2(q)H(q^2)G(q^3)H^2(q^4)H^2(q^{12})G(q^{24})\Big)\\
&\quad\times\Big(H^3(q^2)G(q^3)H(q^{12})-G^3(q)H^3(q^4)H(q^6)\Big)\\
&\quad\times\left(\frac{G^3(q^2)H(q^3)G(q^{12})-H^3(q)G^3(q^4)G(q^6)}{H^3(q^2)G(q^3)H(q^{12})-G^3(q)H^3(q^4)H(q^6)}-q^2\frac{T^2(q)T(q^3)T(q^{24})}{T(q^2)T^2(q^4)T^2(q^{12})}\right),
\end{aligned}$$

$$\begin{aligned}
&TZH(q^6)G(q^{24})-q^4SYG(q^6)H(q^{24})\\
&=\Big(H^2(q)G(q^2)H(q^3)G^2(q^4)H(q^6)G(q^{12})G(q^{24})\Big)\\
&\quad\times\Big(H^3(q^2)G(q^3)H(q^{12})-G^3(q)H^3(q^4)H(q^6)\Big)\\
&\quad\times\left(1-q^4\frac{T(q^2)T^2(q^4)T(q^{12})T(q^{24})}{T^2(q)T(q^3)T(q^6)}\cdot\frac{G^3(q^2)H(q^3)G(q^{12})-H^3(q)G^3(q^4)G(q^6)}{H^3(q^2)G(q^3)H(q^{12})-G^3(q)H^3(q^4)H(q^6)}\right).
\end{aligned}$$

Using the above expressions in (2.96) and then employing (2.20) and (2.21), we obtain

$$\begin{aligned}
&\frac{H(q^{12})G(q^{24})}{G(q^6)H(q^{24})}\cdot\frac{q^2T^2(q)T(q^2)T(q^3)T(q^6)-q^2\dfrac{T(q)T(q^{24})}{T(q^{12})}}{q^6T^2(q)T(q^2)T(q^3)T(q^6)-\dfrac{T(q)T(q^6)}{T(q^{24})}}\\
&\qquad=\frac{G^2(q)H(q^2)G(q^3)H^2(q^4)H^2(q^{12})}{H^2(q)G(q^2)H(q^3)G^2(q^4)H(q^6)G(q^{12})}\\
&\qquad\quad\times\frac{q^2\dfrac{T(q)}{T(q^2)T(q^4)}-q^2\dfrac{T^2(q)T(q^3)T(q^{24})}{T(q^2)T^2(q^4)T^2(q^{12})}}{1-q^6\dfrac{T(q^2)T^2(q^4)T(q^{12})T(q^{24})}{T^2(q)T(q^3)T(q^6)}\cdot\dfrac{T(q)}{T(q^2)T(q^4)}},
\end{aligned}$$

which, by (1.15), yields

$$\frac{R(q)R(q^2)R(q^3)R(q^6)R(q^{12})-R(q^{24})}{R(q)R(q^2)R(q^3)R(q^6)R(q^{24})-R(q^6)}=\frac{R(q^4)R^2(q^{12})-R(q)R(q^3)R(q^{24})}{R(q)R(q^3)R(q^6)-R(q^4)R(q^{12})R(q^{24})}.$$

Simplifying the above for $R(q^2)$, we readily arrive at (2.12).

## 2.7 Partition-theoretic identities

In this section, we present some simple partition-theoretic results arising from the identities in Theorems 2.4 and 2.5.

We first note that

$$\frac{1}{(q^r;q^s)_\infty^k}$$

is the generating function for the number of partitions of a positive integer into parts that are congruent to $r \pmod{s}$ and having $k$ colors.

The following theorem arises from (2.13).

**Theorem 2.20.** *Let $p_1(n)$ denote the number of partitions of $n$ into parts congruent to $\pm1, \pm5, \pm11, \pm12 \pmod{30}$, where the parts congruent to $\pm5, \pm12 \pmod{30}$ have two colors. Let $p_2(n)$ denote the number of partitions of $n$ into parts congruent to $\pm5, \pm6, \pm7, \pm13 \pmod{30}$, where the parts congruent to $\pm5, \pm6 \pmod{30}$ have two colors. Let $p_3(n)$ denote the number of partitions of $n$ into parts congruent to $\pm1, \pm6, \pm7, \pm11, \pm12, \pm13 \pmod{30}$. Then, for each positive integer $n$,*

$$p_1(n)-p_2(n)=p_3(n-1). \tag{2.97}$$

*Proof.* Applying (1.12) and (1.13) in (2.13), we find that

$$\frac{(q^{2\pm};q^5)_\infty(q^{6\pm};q^{30})_\infty}{(q^{2\pm};q^{10})_\infty(q^{3\pm};q^{15})_\infty}-\frac{(q^{1\pm};q^5)_\infty(q^{12\pm};q^{30})_\infty}{(q^{4\pm};q^{10})_\infty(q^{6\pm};q^{15})_\infty}=q(q^{5\pm};q^{30})_\infty^2.$$

Dividing both sides of the above by $(q^{1\pm},q^{2\pm};q^5)_\infty(q^{5\pm};q^{30})_\infty^2(q^{6\pm},q^{12\pm};q^{30})_\infty$, reducing all the products into base $q^{30}$, and then cancelling the common terms, we obtain

$$\frac{1}{(q^{1\pm},q^{5\pm},q^{5\pm},q^{11\pm},q^{12\pm},q^{12\pm};q^{30})_\infty}-\frac{1}{(q^{5\pm},q^{5\pm},q^{6\pm},q^{6\pm},q^{7\pm},q^{13\pm};q^{30})_\infty}$$
$$=\frac{q}{(q^{1\pm},q^{6\pm},q^{7\pm},q^{11\pm},q^{12\pm},q^{13\pm};q^{30})_\infty},$$

which is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n - \sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^{n+1}.$$

Equating the coefficients of $q^n$ for $n \geq 1$, we readily arrive at (2.97) to finish the proof. □

In a similar way, one can easily derive the following two theorems from (2.14) and (2.15), respectively.

**Theorem 2.21.** *Let $p_4(n)$ denote the number of partitions of $n$ into parts congruent to* $\pm 2, \pm 3, \pm 7, \pm 8, \pm 12, \pm 13 \pmod{30}$*, where the parts congruent to* $\pm 2, \pm 3, \pm 8, \pm 12 \pmod{30}$ *have two colors. Let $p_5(n)$ denote the number of partitions of $n$ into parts congruent to* $\pm 1, \pm 4, \pm 6, \pm 9, \pm 11, \pm 14 \pmod{30}$*, where the parts congruent to* $\pm 4, \pm 6, \pm 9, \pm 14 \pmod{30}$ *have two colors. Let $p_6(n)$ denote the number of partitions of $n$ into parts congruent to* $\pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 9, \pm 12, \pm 14 \pmod{30}$ *where the parts congruent to* $\pm 5 \pmod{30}$ *have two colors. Then, for each positive integer $n > 1$,*

$$p_4(n) - p_5(n-2) = p_6(n).$$

**Theorem 2.22.** *Let $p_7(n)$ denote the number of partitions of $n$ into parts congruent to* $\pm 2, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 12, \pm 13, \pm 17, \pm 21, \pm 22, \pm 23, \pm 24, \pm 28 \pmod{60}$*, where the parts congruent to* $\pm 10 \pmod{60}$ *have two colors. Let $p_8(n)$ denote the number of partitions of $n$ into parts congruent to* $\pm 1, \pm 3, \pm 4, \pm 10, \pm 11, \pm 12, \pm 14, \pm 16, \pm 18, \pm 19, \pm 24, \pm 26, \pm 27, \pm 29 \pmod{60}$*, where the parts congruent to* $\pm 10 \pmod{60}$ *have two colors. Let $p_9(n)$ denote the number of partitions of $n$ into parts congruent to* $\pm 1, \pm 4, \pm 5, \pm 6, \pm 11, \pm 12, \pm 14, \pm 16, \pm 18, \pm 19, \pm 25, \pm 26, \pm 29 \pmod{60}$*, where the parts congruent to* $\pm 12, \pm 18 \pmod{60}$ *have two colors. Let $p_{10}(n)$ denote the number of partitions of $n$ into parts congruent to* $\pm 2, \pm 5, \pm 6, \pm 7, \pm 8, \pm 13, \pm 17, \pm 18, \pm 22, \pm 23, \pm 24, \pm 25, \pm 28 \pmod{60}$*, where the parts congruent to* $\pm 6, \pm 24 \pmod{60}$ *have two colors. Then, for each positive integer $n > 1$,*

$$p_7(n) + p_8(n-1) = p_9(n) + p_{10}(n-2).$$

Similar partition-theoretic identities may also be derived from the other identities in Theorem 2.5. However, we omit those considerably lengthy statements.

## 2.8 Concluding remarks

In this chapter, we find some new modular identities for the Rogers-Ramanujan continued fraction, namely, Thereom 2.1 and Theorem 2.3 by using two approaches. The identities in Thereom 2.1 and (2.8)–(2.9) in Thereom 2.3 are derived by using dissections of theta functions. Identities (2.10)–(2.12) in Thereom 2.3 are derived from some identities for the Rogers-Ramanujan functions arising from the quintuple product identity (2.49). In Lemmas 2.16–2.19, some theta function identities derived from (2.49) are given. We then use those identities to deduce the identities in Theorem 2.5 involving the Rogers-Ramanujan functions, which then imply the identities (2.10)–(2.12) for the Rogers-Ramanujan continued fraction in Theorem 2.5. The next set of theta function identities analogous to those in Lemmas 2.16–2.19 may be explored. However, it would be challenging to derive identities for the Rogers-Ramanujan continued fraction analogous to (2.10)–(2.12) from them. In the following, we briefly outline a few steps in that direction.

First, setting $q = q^{10}$ and $B = q$, $q^3$, $q^7$, and $q^9$, respectively, we find that

$$f\left(q^{13}, q^{47}\right) - q^2 f\left(q^7, q^{53}\right) = f(-q^{20}) \frac{f\left(-q^2, -q^{18}\right)}{f\left(q^9, q^{11}\right)}, \tag{2.98}$$

$$f\left(q^{19}, q^{41}\right) - q^6 f\left(q, q^{59}\right) = f(-q^{20}) \frac{f\left(-q^6, -q^{14}\right)}{f\left(q^7, q^{13}\right)}, \tag{2.99}$$

$$f\left(q^{29}, q^{31}\right) - q^3 f\left(q^{11}, q^{49}\right) = f(-q^{20}) \frac{f\left(-q^6, -q^{14}\right)}{f\left(q^3, q^{17}\right)}, \tag{2.100}$$

$$f\left(q^{23}, q^{37}\right) - q f\left(q^{17}, q^{43}\right) = f(-q^{20}) \frac{f\left(-q^2, -q^{18}\right)}{f\left(q, q^{19}\right)}. \tag{2.101}$$

Proceeding as in the proof of Lemma 2.17, it can be shown that

$$f^3\left(q^{13}, q^{47}\right) - q^6 f^3\left(q^7, q^{53}\right) = f^3(-q^{20}) \frac{f\left(-q^6, -q^{54}\right)}{f\left(q^{27}, q^{33}\right)}, \tag{2.102}$$

$$f^3\left(q^{19}, q^{41}\right) - q^{18} f\left(q, q^{59}\right) = f^3(-q^{20}) \frac{f\left(-q^{18}, -q^{42}\right)}{f\left(q^{21}, q^{39}\right)}, \tag{2.103}$$

$$f^3\left(q^{29}, q^{31}\right) - q^9 f\left(q^{11}, q^{49}\right) = f^3(-q^{20}) \frac{f\left(-q^{18}, -q^{42}\right)}{f\left(q^9, q^{51}\right)}, \tag{2.104}$$

$$f^3\left(q^{23}, q^{37}\right) - q^3 f^3\left(q^{17}, q^{43}\right) = f^3(-q^{20}) \frac{f\left(-q^6, -q^{54}\right)}{f\left(q^3, q^{57}\right)}. \tag{2.105}$$

However, from (2.22) and (2.23), we have

$$f\left(q^9,q^{11}\right)=\frac{1}{2}\left(f\left(q^2,q^3\right)+f\left(-q^2,-q^3\right)\right),$$
$$f\left(q^7,q^{13}\right)=\frac{1}{2}\left(f\left(q,q^4\right)+f\left(-q,-q^4\right)\right),$$
$$f\left(q,q^{19}\right)=\frac{1}{2q^2}\left(f\left(q^2,q^3\right)-f\left(-q^2,-q^3\right)\right),$$
$$f\left(q^3,q^{17}\right)=\frac{1}{2q}\left(f\left(q,q^4\right)-f\left(-q,-q^4\right)\right).$$

Therefore, (2.98)–(2.105) may be recast as

$$f\left(q^{13},q^{47}\right)-q^2f\left(q^7,q^{53}\right)=f(-q^{20})\frac{2f\left(-q^2,-q^{18}\right)}{f\left(q^2,q^3\right)+f\left(-q^2,-q^3\right)}, \tag{2.106}$$
$$f\left(q^{19},q^{41}\right)-q^6f\left(q,q^{59}\right)=f(-q^{20})\frac{2f\left(-q^6,-q^{14}\right)}{f\left(q,q^4\right)+f\left(-q,-q^4\right)}, \tag{2.107}$$
$$f\left(q^{29},q^{31}\right)-q^3f\left(q^{11},q^{49}\right)=f(-q^{20})\frac{2qf\left(-q^6,-q^{14}\right)}{f\left(q,q^4\right)-f\left(-q,-q^4\right)}, \tag{2.108}$$
$$f\left(q^{23},q^{37}\right)-qf\left(q^{17},q^{43}\right)=f(-q^{20})\frac{2q^2f\left(-q^2,-q^{18}\right)}{f\left(q^2,q^3\right)-f\left(-q^2,-q^3\right)}, \tag{2.109}$$
$$f^3\left(q^{13},q^{47}\right)-q^6f^3\left(q^7,q^{53}\right)=f^3(-q^{20})\frac{2f\left(-q^6,-q^{54}\right)}{f\left(q^6,q^9\right)+f\left(-q^6,-q^9\right)}, \tag{2.110}$$
$$f^3\left(q^{19},q^{41}\right)-q^{18}f\left(q,q^{59}\right)=f^3(-q^{20})\frac{2f\left(-q^{18},-q^{42}\right)}{f\left(q^3,q^{12}\right)+f\left(-q^3,-q^{12}\right)}, \tag{2.111}$$
$$f^3\left(q^{29},q^{31}\right)-q^9f\left(q^{11},q^{49}\right)=f^3(-q^{20})\frac{2q^3f\left(-q^{18},-q^{42}\right)}{f\left(q^3,q^{12}\right)-f\left(-q^3,-q^{12}\right)}, \tag{2.112}$$
$$f^3\left(q^{23},q^{37}\right)-q^3f^3\left(q^{17},q^{43}\right)=f^3(-q^{20})\frac{2q^6f\left(-q^6,-q^{54}\right)}{f\left(q^6,q^9\right)-f\left(-q^6,-q^9\right)}. \tag{2.113}$$

Then proceeding as in the proofs of (2.16) and (2.19), we have the following identities from (2.106) and (2.110), which are somewhat analogous to (2.75) and (2.79):

$$\begin{aligned}
&f^3(-q)f(-q^{60})G^3(q^2)H(q^6)H(q^{12})\left(G(q)H^2(q)+G(q)H(q^2)\right)^3\\
&\quad-4f(-q^3)f^3(-q^{20})H^3(q^2)H^3(q^4)G(q^6)\left(G(q^3)H^2(q^3)+G(q^3)H(q^6)\right)\\
&=\frac{3q^2f^2(-q)f(-q^3)}{f(-q^{20})}G^2(q^2)H(q^2)H(q^4)G(q^6)\left(G(q)H^2(q)+G(q)H(q^2)\right)^2\\
&\quad\times\left(G(q^3)H^2(q^3)+G(q^3)H(q^6)\right)f(q^{13},q^{47})f(q^7,q^{53})
\end{aligned} \tag{2.114}$$

and

$$\begin{aligned}
&f^3(-q)f(-q^{60})G^3(q^2)H(q^6)H(q^{12})\left(G(q)H^2(q)+G(q)H(q^2)\right)^3\\
&\quad-4f(-q^3)f^3(-q^{20})H^3(q^2)H^3(q^4)G(q^6)\left(G(q^3)H^2(q^3)+G(q^3)H(q^6)\right)
\end{aligned}$$

$$= \frac{3q^2 f^3(-q)f(-q^{10})f(-q^{30})f^3(-q^{60})}{f(-q^2)f(-q^6)f^2(-q^{20})} H(q^4) \left(G(q)H^2(q) + G(q)H(q^2)\right)^2$$
$$\times \left(G^2(q)H(q) + G(q^2)H(q)\right). \tag{2.115}$$

In a similar fashion, using (2.107) and (2.111), it can be shown that

$$f^3(-q)f(-q^{60})H^3(q^2)G(q^6)G(q^{12}) \left(G^2(q)H(q) + H(q)G(q^2)\right)^3$$
$$- 4f(-q^3)f^3(-q^{20})G^3(q^2)G^3(q^4)H(q^6) \left(G^2(q^3)H(q^3) + H(q^3)G(q^6)\right)$$
$$= \frac{3q^6 f^2(-q)f(-q^3)}{f(-q^{20})} G(q^2)H^2(q^2)G(q^4)H(q^6) \left(G^2(q)H(q) + H(q)G(q^2)\right)^2$$
$$\times \left(G^2(q^3)H(q^3) + H(q^3)G(q^6)\right) f(q^{19}, q^{41})f(q, q^{59}) \tag{2.116}$$

and

$$f^3(-q)f(-q^{60})H^3(q^2)G(q^6)G(q^{12}) \left(G^2(q)H(q) + H(q)G(q^2)\right)^3$$
$$- 4f(-q^3)f^3(-q^{20})G^3(q^2)G^3(q^4)H(q^6) \left(G^2(q^3)H(q^3) + H(q^3)G(q^6)\right)$$
$$= \frac{3q^4 f^3(-q)f(-q^{10})f(-q^{30})f^3(-q^{60})}{f(-q^2)f(-q^6)f^2(-q^{20})} G(q^4) \left(G^2(q)H(q) + H(q)G(q^2)\right)^2$$
$$\times \left(G(q)H^2(q) - G(q)H(q^2)\right). \tag{2.117}$$

Similar identities may be derived from (2.108),(2.109), (2.112), and (2.113). However, the appearance of terms, like $f(q^{13}, q^{47})f(q^7, q^{53})$ in (2.114) and $f(q^{19}, q^{41})f(q, q^{59})$ in (2.116), might be a hindrance in deriving identities analogous to (2.10)–(2.12).