Chapter 3

Congruences for k-Elongated Plane Partition Diamonds

3.1 Introduction

As defined in Section 1.8, if $d_k(n)$ counts the partitions obtained by adding the links of the k–elongated plane partition diamonds of length n, then the generating function for $d_k(n)$ is given by

$$\sum_{n=0}^{\infty} d_k(n)q^n = \frac{f_2^k}{f_1^{3k+1}}.$$

Andrews and Paule [9] found some elegant generating functions for $d_1(n)$, $d_2(n)$, and $d_3(n)$, where $d_k(n)$ is defined in (1.24). They also proved many Ramanujan-type congruences modulo 2, 3, 4, 5, 8, 9, 27, and 243, mainly by using the Mathematica package *RaduRK* developed by Smoot [114], which uses Radu's Ramanujan-Kolberg algorithm [90].

Andrews and Paule [9] conjectured some congruences in their paper. For example, they conjectured that

$$d_2(n) \equiv 0 \pmod{3^k}, \tag{3.1}$$

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for all positive integers n and k such that $8n \equiv 1 \pmod{3^k}$.

By manipulation of a certain ring of modular functions, Smoot [115] recently proved a refinement of (3.1) as

$$d_2(n) \equiv 0 \pmod{3^{\lfloor k/2 \rfloor + 1}}$$

for all $n \geq 1$ and $k \geq 1$ such that $8n \equiv 1 \pmod{3^k}$.

Andrews and Paule [9] also posed some conjectural congruences $d_2(n)$ modulo 81, 243, and 729. Using elementary q-series techniques, Yao [129] proved these conjectures for $d_2(n)$.

Da Silva, Hirschhorn, and Sellers [50] gave elementary proofs for some of the results of Andrews and Paule [9] and discovered new individual congruences as well as some infinite families of congruences for $d_k(n)$ modulo certain primes. For example, for prime $p \geq 5$, let r, $1 \leq r \leq p-1$, be a quadratic nonresidue modulo p. Then for all $n \geq 0$ and $N \geq 1$,

$$d_{p^N-1}(pn+r) \equiv 0 \pmod{p^N}.$$

Additionally, da Silva, Hirschhorn, and Sellers [50] proved the following overarching theorem, which generalizes the Ramanujan-type congruences modulo prime p with arithmetic progression p.

Theorem 3.1 (Da Silva, Hirschhorn, and Sellers [50])**.** *Let p be a prime, $k \geq 1$, $j \geq 0$, $N \geq 1$ and r be an integer such that $1 \leq r \leq p-1$. If for all $n \geq 0$,*

$$d_k(pn+r) \equiv 0 \pmod{p^N},$$

then

$$d_{p^N j+k}(pn+r) \equiv 0 \pmod{p^N}.$$

Andrews and Paule [9] considered the partition diamonds as the Schmidt-type partitions [108] and in [82], Li and Yee found generating functions for Schmidt k-partitions and unrestricted Schmidt k-partitions in unified combinatorial ways.

Very recently, Sellers and Smoot [109] obtained an infinite family of congruences for $d_7(n)$ modulo arbitrary powers of 8 whereas Banerjee and Smoot [15, 16] found some divisibility properties of $d_5(n)$ and $d_2(n)$ by arbitrary powers of 5 and

7, respectively.

In this chapter, we extend some of the individual congruences of Andrews and Paule [9] and da Silva, Hirschhorn, and Sellers [50] to certain families of congruences. We state them in the following theorem.

Theorem 3.2. *For all $n \geq 0$ and $j \geq 0$,*

$$d_{4j+3}\,(4n+2) \equiv 0 \pmod{2}, \tag{3.2}$$

$$d_{4j+3}\,(4n+3) \equiv 0 \pmod{4}, \tag{3.3}$$

$$d_{8j+7}\,(4n+2) \equiv 0 \pmod{4}, \tag{3.4}$$

$$d_{8j+7}\,(8n+5) \equiv 0 \pmod{4}, \tag{3.5}$$

$$d_{16j+3}\,(16n+9) \equiv 0 \pmod{4}, \tag{3.6}$$

$$d_{8j+7}\,(4n+3) \equiv 0 \pmod{8}, \tag{3.7}$$

$$d_{32j+7}\,(8n+4) \equiv 0 \pmod{8}, \tag{3.8}$$

$$d_{9j+8}\,(9n+3) \equiv 0 \pmod{9}, \tag{3.9}$$

$$d_{27j+2}\,(9n+8) \equiv 0 \pmod{27}, \tag{3.10}$$

$$d_{81j+2}\,(81n+44) \equiv 0 \pmod{81}, \tag{3.11}$$

$$d_{243j+2}\,(27n+8) \equiv 0 \pmod{243}, \tag{3.12}$$

$$d_{729j+2}\,(243n+71) \equiv 0 \pmod{729}. \tag{3.13}$$

Note that the individual cases when $j = 0$ in (3.2), (3.3), (3.10), (3.12) were proved by Andrews and Paule [9]; (3.11), (3.13) were proved by Yao [129] and the remainders of the above theorem were proved by da Silva, Hirschhorn, and Sellers [50], respectively.

We also find some new families of congruences for $d_k(n)$ modulo 8, 16, 32, 64, and 128 in the following theorem.

Theorem 3.3. *For all $n \geq 0$ and $j \geq 0$,*

$$d_{8j+7}\,(8n+6) \equiv 0 \pmod{8}, \tag{3.14}$$

$$d_{8j+7}\,(8n+7) \equiv 0 \pmod{16}, \tag{3.15}$$

$$d_{16j+7}\,(8n+6) \equiv 0 \pmod{16}, \tag{3.16}$$

$$d_{16j+7}(16n+11) \equiv 0 \pmod{16}, \tag{3.17}$$

$$d_{16j+15}(4n+3) \equiv 0 \pmod{16}, \tag{3.18}$$

$$d_{16j+15}(8n+6) \equiv 0 \pmod{16}, \tag{3.19}$$

$$d_{16j+15}(16n+10) \equiv 0 \pmod{16}, \tag{3.20}$$

$$d_{32j+31}(4n+3) \equiv 0 \pmod{32}, \tag{3.21}$$

$$d_{16j+15}(8n+7) \equiv 0 \pmod{64}, \tag{3.22}$$

$$d_{32j+31}(8n+7) \equiv 0 \pmod{128}. \tag{3.23}$$

Next, we provide the following theorem that refines Theorem 3.1, which was found by da Silva, Hirschhorn, and Sellers [50].

Theorem 3.4. *Let p be a prime, $k \geq 1$, $j \geq 0$, $N \geq 1$, $M \geq 1$, and r be integers such that $1 \leq r \leq p^M - 1$. If for all $n \geq 0$,*

$$d_k\left(p^M n + r\right) \equiv 0 \pmod{p^N},$$

then

$$d_{p^{M+N-1}j+k}\left(p^M n + r\right) \equiv 0 \pmod{p^N}.$$

Remark 3.5. *Theorem 3.4 is a refinement of Theorem 3.1 in the sense that it extends individual congruences with arithmetic progressions $p^M n + r$, $M \geq 1$ to their respective families, whereas Theorem 3.1 extends individual congruences with arithmetic progressions $pn + r$ only to their respective families. For example, for all n, we have*

$$d_7(2n+1) \not\equiv 0 \pmod{16},$$

whereas

$$d_7(8n+7) \equiv 0 \pmod{16}. \tag{3.24}$$

So, Theorem 3.1 does not provide any information regarding its extension to an infinite family, whereas Theorem 3.4 and (3.24) *imply that*

$$d_{64j+7}(8n+7) \equiv 0 \pmod{16}.$$

Finally, we present some new families of congruences modulo 5, 7, 11, 13, 17, 19, 23, 25, and 49 in the following theorem.

Theorem 3.6. *For all $n \geq 0$ and $j \geq 0$, we have*

$$d_{25j+1}(25n+23) \equiv 0 \pmod{5}, \tag{3.25}$$

$$d_{625j+1}(125n+k) \equiv 0 \pmod{25}, \quad \textit{where } k \in \{23, 123\}, \tag{3.26}$$

$$d_{125j+2}(125n+k) \equiv 0 \pmod{5}, \quad \textit{where } k \in \{97, 122\}, \tag{3.27}$$

$$d_{49j+1}(49n+k) \equiv 0 \pmod{7}, \quad \textit{where } k \in \{17, 31, 38, 45\}, \tag{3.28}$$

$$d_{49j+2}(49n+43) \equiv 0 \pmod{7}, \tag{3.29}$$

$$d_{49j+3}(49n+41) \equiv 0 \pmod{7}, \tag{3.30}$$

$$d_{2401j+3}(343n+k) \equiv 0 \pmod{49}, \quad \textit{where } k \in \{90, 188, 237\}, \tag{3.31}$$

$$d_{343j+4}(343n+k) \equiv 0 \pmod{7}, \quad \textit{where } k \in \{39, 235, 284\}, \tag{3.32}$$

$$d_{121j+4}(121n+96) \equiv 0 \pmod{11}, \tag{3.33}$$

$$d_{121j+5}(121n+91) \equiv 0 \pmod{11}, \tag{3.34}$$

$$d_{121j+7}(121n+81) \equiv 0 \pmod{11}, \tag{3.35}$$

$$d_{13j+3}(13n+11) \equiv 0 \pmod{13}, \tag{3.36}$$

$$d_{17j+5}(17n+13) \equiv 0 \pmod{17}, \tag{3.37}$$

$$d_{289j+6}(289n+k) \equiv 0 \pmod{17}, \tag{3.38}$$

where $k \in \{52, 69, 137, 171, 188, 205, 222, 239, 273\}$,

$$d_{19j+3}(19n+16) \equiv 0 \pmod{19}, \tag{3.39}$$

$$d_{19j+6}(19n+9) \equiv 0 \pmod{19}, \tag{3.40}$$

$$d_{19j+7}(19n+13) \equiv 0 \pmod{19}, \tag{3.41}$$

$$d_{23j+8}(23n+9) \equiv 0 \pmod{23}. \tag{3.42}$$

The following sections of this chapter contain the proofs of our results. We establish Theorems 3.2, 3.3, 3.4, and 3.6 in Sections 3.2–3.5, respectively.

3.2 Proof of Theorem 3.2

Proof. To prove (3.2), we first find the following exact generating function

$$\sum_{n=0}^{\infty} d_{4j+3}(4n+2)q^n$$

$$= 2\Bigg\{\frac{f_2^{27}f_4^{39}}{f_1^{55}f_8^{18}}\sum_{k=0}^{\lfloor(3j+2)/2\rfloor}\sum_{m=0}^{\lfloor(13j+10)/2\rfloor}2^{2(2k+m)}\binom{6j+5}{4k}\binom{13j+11}{2m+1}q^{k+m}\frac{F_2F_4}{F_1F_8}$$
$$+\frac{f_2^{13}f_4^{53}}{f_1^{51}f_8^{22}}\sum_{k=0}^{\lfloor(3j+1)/2\rfloor}\sum_{m=0}^{\lfloor(13j+11)/2\rfloor}2^{2(2k+m)+1}\binom{6j+5}{4k+2}\binom{13j+11}{2m}q^{k+m}\frac{F_2F_4}{F_1F_8}\Bigg\}, \tag{3.43}$$

where $F_1 := f_1^{65j-8k}$, $F_2 := f_2^{30j-24k+4m}$, $F_4 := f_4^{53j+16k-12m}$, and $F_8 := f_8^{26j-8m}$. From (3.43), congruence (3.2) is evident.

We need the following 2-dissection from [68, (1.9.4)] to establish (3.43):

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5f_{16}^2} + 2q\frac{f_4^2f_{16}^2}{f_2^5f_8}. \tag{3.44}$$

Using (3.44), we have

$$\sum_{n=0}^{\infty}d_{4j+3}(n)q^n = \frac{f_2^{4j+3}}{f_1^{12j+10}} = f_2^{4j+3}\left(\frac{1}{f_1^2}\right)^{6j+5} = f_2^{4j+3}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)^{6j+5}$$
$$= \sum_{k=0}^{6j+5}2^k\binom{6j+5}{k}q^k\frac{f_4^{2k}f_8^{30j-6k+25}}{f_2^{26j+22}f_{16}^{12j-4k+10}}$$
$$= \frac{1}{f_2^{26j+22}}\Bigg\{\sum_{k=0}^{3j+2}2^{2k}\binom{6j+5}{2k}q^{2k}\frac{f_4^{4k}f_8^{30j-12k+25}}{f_{16}^{12j-8k+10}}$$
$$+\sum_{k=0}^{3j+2}2^{2k+1}\binom{6j+5}{2k+1}q^{2k+1}\frac{f_4^{4k+2}f_8^{30j-12k+19}}{f_{16}^{12j-8k+6}}\Bigg\}.$$

Extracting the terms that involve q^{2n} from the above identity, we obtain

$$\sum_{n=0}^{\infty}d_{4j+3}(2n)q^n = \left(\frac{1}{f_1^2}\right)^{13j+11}\sum_{k=0}^{3j+2}2^{2k}\binom{6j+5}{2k}q^k\frac{f_2^{4k}f_4^{30j-12k+25}}{f_8^{12j-8k+10}},$$

which again by using (3.44) can be written as

$$\sum_{n=0}^{\infty}d_{4j+3}(2n)q^n = \left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)^{13j+11}\sum_{k=0}^{3j+2}2^{2k}\binom{6j+5}{2k}q^k\frac{f_2^{4k}f_4^{30j-12k+25}}{f_8^{12j-8k+10}}$$
$$= \sum_{k=0}^{3j+2}2^{2k}\binom{6j+5}{2k}q^k\frac{f_2^{4k}f_4^{30j-12k+25}}{f_8^{12j-8k+10}}\sum_{m=0}^{13j+11}2^m\binom{13j+11}{m}q^m\frac{f_4^{2m}f_8^{65j-6m+55}}{f_2^{65j+55}f_{16}^{26j-4m+22}}.$$

Now, breaking the right side of the above identity on the parity of k and m, we have

$$\sum_{n=0}^{\infty}d_{4j+3}(2n)q^n$$
$$= \Bigg\{\sum_{k=0}^{\lfloor(3j+2)/2\rfloor}2^{4k}\binom{6j+5}{4k}q^{2k}\frac{f_2^{8k}f_4^{30j-24k+25}}{f_8^{12j-16k+10}} + \sum_{k=0}^{\lfloor(3j+1)/2\rfloor}2^{4k+2}\binom{6j+5}{4k+2}q^{2k+1}$$

$$
\times \frac{f_2^{8k+4} f_4^{30j-24k+13}}{f_8^{12j-16k+2}} \Bigg\} \Bigg\{ \sum_{m=0}^{\lfloor (13j+11)/2 \rfloor} 2^{2m} \binom{13j+11}{2m} q^{2m} \frac{f_4^{4m} f_8^{65j-12m+55}}{f_2^{65j+55} f_{16}^{26j-8m+22}}
$$

$$
+ \sum_{m=0}^{\lfloor (13j+10)/2 \rfloor} 2^{2m+1} \binom{13j+11}{2m+1} q^{2m+1} \frac{f_4^{4m+2} f_8^{65j-12m+49}}{f_2^{65j+55} f_{16}^{26j-8m+18}} \Bigg\}. \tag{3.45}
$$

From (3.45), extracting the terms involving q^{2n+1}, we deduce (3.43).

In a similar way, one can find the following generating functions, from which (3.3), (3.10), and (3.7) are evident, respectively:

$$
\sum_{n=0}^{\infty} d_{4j+3}(4n+3) q^n
$$

$$
= 4 \Bigg\{ \frac{f_2^7 f_4^{57}}{f_1^{49} f_8^{22}} \sum_{k=0}^{\lfloor (3j+1)/2 \rfloor} \sum_{m=0}^{\lfloor (13j+11)/2 \rfloor} 2^{2(2k+m)+1} \binom{6j+5}{4k+3} \binom{13j+11}{2m} q^{k+m} \frac{F_2 F_4}{F_1 F_8}
$$

$$
+ \frac{f_2^{21} f_4^{43}}{f_1^{53} f_8^{18}} \sum_{k=0}^{\lfloor (3j+2)/2 \rfloor} \sum_{m=0}^{\lfloor (13j+10)/2 \rfloor} 2^{2(2k+m)} \binom{6j+5}{4k+1} \binom{13j+11}{2m+1} q^{k+m} \frac{F_2 F_4}{F_1 F_8} \Bigg\}, \tag{3.46}
$$

$$
\sum_{n=0}^{\infty} d_{8j+7}(4n+2) q^n
$$

$$
= 4 \frac{f_2^{211}}{f_1^{164} f_4^{62}} \Bigg\{ \sum_{k=0}^{3j+2} \sum_{m=0}^{\lfloor (13j+12)/2 \rfloor} 2^{4(k+m)} \binom{12j+11}{4k} \binom{13j+12}{2m+1} q^{k+m} \frac{G_2}{G_1 G_4}
$$

$$
+ \sum_{k=0}^{3j+2} \sum_{m=0}^{\lfloor (13j+11)/2 \rfloor} 2^{4(k+m)} \binom{12j+11}{4k+2} \binom{13j+12}{2m} q^{k+m} \frac{G_2}{G_1 G_4} \Bigg\}, \tag{3.47}
$$

where $G_1 := f_1^{182j-8k-8m}$, $G_2 := f_2^{242j-24k-24m}$, and $G_4 := f_4^{76j-16k-16m}$,

$$
\sum_{n=0}^{\infty} d_{8j+7}(4n+3) q^n
$$

$$
= 8 \frac{f_2^{205}}{f_1^{162} f_4^{58}} \Bigg\{ \sum_{k=0}^{3j+2} \sum_{m=0}^{\lfloor (13j+12)/2 \rfloor} 2^{4(k+m)} \Bigg(\binom{12j+11}{4k+1} \binom{13j+12}{2m+1} + \binom{12j+11}{4k+3}
$$

$$
\times \binom{13j+12}{2m} \Bigg) q^{k+m} \frac{G_2}{G_1 G_4} \Bigg\}. \tag{3.48}
$$

Note that like the above generating functions, the exponents of f_1 in the generating functions of $d_{8j+7}(4n+1)$, $d_{16j+3}(4n+1)$, and $d_{32j+7}(4n)$ will also involve k. Therefore, the exact generating functions for (3.5), (3.6), and (3.8) can not be found as elegantly as the above exact generating functions. So in the following, we give simple proofs for them as well as for the remaining congruences.

The proofs of (3.5), (3.6), and (3.8) are similar. So, we prove (3.8) only. We have

$$\sum_{n=0}^{\infty} d_{32j+7}(n)q^n = \frac{f_2^{32j+7}}{f_1^{96j+22}} \equiv \frac{f_1^2}{f_2^{16j+5}} \pmod 8. \tag{3.49}$$

Here, we require the following 2-dissections of f_1^2 and $1/f_1^4$ from [68, (1.9.4) and (1.10.1)]:

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{3.50}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{3.51}$$

Employing (3.50) in (3.49) and then extracting the terms that involve q^{2n}, and using (3.51), we obtain

$$\sum_{n=0}^{\infty} d_{32j+7}(2n)q^n \equiv \frac{f_4^5}{f_2^{8j+2} f_8^2}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right) \pmod 8,$$

which gives

$$\sum_{n=0}^{\infty} d_{32j+7}(4n)q^n \equiv \frac{f_2^{19}}{f_1^{8j+16} f_4^6} \equiv \frac{1}{f_2^{4j-11} f_4^6} \pmod 8.$$

The above identity clearly yields (3.8).

Here, we require the following 3-dissections of f_1^2/f_2, f_2^2/f_1, f_1^3, and $1/f_1^3$ from [68, (14.3.2), (14.3.3), (21.3.7), and (39.2.8)]:

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{3.52}$$

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{3.53}$$

$$f_1^3 = a\left(q^3\right) f_3 - 3q f_9^3, \tag{3.54}$$

$$\frac{1}{f_1^3} = a^2\left(q^3\right)\frac{f_9^3}{f_3^{10}} + 3qa^2\left(q^3\right)\frac{f_9^6}{f_3^{11}} + 9q^2\frac{f_9^9}{f_3^{12}}, \tag{3.55}$$

where $a(q)$ is Borweins' cubic theta function defined by $a(q) := \sum_{j,k=-\infty}^{\infty} q^{m^2+mn+n^2}$.

We now prove (3.9), (3.10), and (3.12). Using (3.52), we have

$$\sum_{n=0}^{\infty} d_{9j+8}(n)q^n = \frac{f_2^{9j+8}}{f_1^{27j+25}} \equiv \frac{f_6^{3j+3}}{f_3^{9j+9}} \cdot \frac{f_1^2}{f_2} \equiv \frac{f_6^{3j+3}}{f_3^{9j+9}}\left(\frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}\right) \pmod 9,$$

which gives

$$\sum_{n=0}^{\infty} d_{9j+8}(3n)q^n \equiv \frac{f_2^{3j+3} f_3^2}{f_1^{9j+9} f_6} \equiv \frac{(f_2^3)^{j+1}}{f_3^{3j+1} f_6} \pmod 9.$$

Applying (3.54) in the above identity and then expanding binomially, we find that

$$\sum_{n=0}^{\infty} d_{9j+8}(3n)q^n \equiv \frac{1}{f_3^{3j+1}f_6}\sum_{k=0}^{j+1}(-3)^k\binom{j+1}{k}q^{2k}\left(a\left(q^6\right)f_6\right)^{j-k+1}f_{18}^{3k} \pmod 9.$$

Since in the right side of the above identity, there is no term that involve q^{3n+1} for $k=0$ and 1, extracting the terms that involve q^{3n+1} from the above identity, we deduce (3.9).

We have

$$\sum_{n=0}^{\infty} d_{27j+2}(n)q^n = \frac{f_2^{27j+2}}{f_1^{81j+7}} \equiv \frac{f_6^{9j}}{f_3^{27j}}\cdot\frac{f_2^2}{f_1}\cdot\frac{1}{f_1^6} \pmod{27}.$$

Using (3.53) and (3.55) in the above identity, and then extracting the terms that involve q^{3n+2}, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} d_{27j+2}(3n+2)q^n &\equiv \frac{f_2^{9j}}{f_1^{27j}}\left(27\frac{f_2f_3^{12}}{f_1^{17}f_6}+6a^3(q)\frac{f_3^8f_6^2}{f_1^{21}}+81q\frac{f_3^{17}f_6^2}{f_1^{24}}\right)\\ &\equiv 6\frac{f_6^{3j+2}}{f_3^{9j-2}}\cdot\frac{1}{f_1^3} \pmod{27}.\end{aligned}$$

Now, invoking (3.55) in the above identity and then extracting the terms involving q^{3n+2}, we prove (3.10).

Finally, we prove (3.12) using induction on j. Andrews and Paule [9, (7.12)] proved (3.12) for $j=0$. We assume that (3.12) is true for some integer $j\geq 0$. Now,

$$\begin{aligned}\sum_{n=0}^{\infty} d_{243(j+1)+2}(n)q^n &= \frac{f_2^{243(j+1)+2}}{f_1^{3(243(j+1)+2)+1}} = \frac{f_2^{243j+2}}{f_1^{3(243j+2)+1}}\cdot\frac{f_2^{243}}{f_1^{729}}\\ &= \sum_{n=0}^{\infty} d_{243j+2}(n)q^n\cdot\frac{f_2^{243}}{f_1^{729}}\\ &\equiv \sum_{n=0}^{\infty} d_{243j+2}(n)q^n\cdot\frac{f_6^{81}}{f_3^{243}} \pmod{243}.\end{aligned}$$

Extracting the terms that involve q^{3n+2} from both sides of the above identity, we have

$$\sum_{n=0}^{\infty} d_{243(j+1)+2}(3n+2)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(3n+2)q^n\cdot\frac{f_2^{81}}{f_1^{243}} \pmod{243}.$$

Da Silva, Hirschhorn, and Sellers [50, Eq. (21)] showed that $d_{3j+2}(3n+2)\equiv 0 \pmod 3$. Therefore, the above identity can be written as

$$\sum_{n=0}^{\infty} d_{243(j+1)+2}(3n+2)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(3n+2)q^n\cdot\frac{f_6^{9}}{f_3^{81}} \pmod{243}.$$

Again, extracting the terms that involve q^{3n+2} from both sides of the above identity, we have

$$\sum_{n=0}^{\infty} d_{243(j+1)+2}(9n+8)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(9n+8)q^n \cdot \frac{f_2^9}{f_1^{81}} \pmod{243}.$$

Due to (3.10), the above identity is equivalent to

$$\sum_{n=0}^{\infty} d_{243(j+1)+2}(9n+8)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(9n+8)q^n \cdot \frac{f_6^3}{f_3^{27}} \pmod{243},$$

which gives

$$\sum_{n=0}^{\infty} d_{243(j+1)+2}(27n+8)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(27n+8)q^n \cdot \frac{f_2^3}{f_1^{27}} \pmod{243}.$$

Therefore, by the assumption for induction, we see that (3.12) is true for $j+1$ as well. Thus, (3.12) is true for all $j \geq 0$.

Similar to the proof of (3.12), we can obtain (3.11) and (3.13) by using induction on j together with the help of some intermediate congruences which are already available due to Andrews and Paule [9] . □

3.3 Proof of Theorem 3.3

Proof. First, we prove (3.18). Similar to (3.46)–(3.48), using (3.44) and (3.51), one can find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} d_{16j+15}(4n+3)q^n \\
&= 8\frac{f_2^{447}}{f_1^{344}f_4^{134}}\left\{\sum_{k=0}^{6j+5}\sum_{m=0}^{13j+12} 2^{4(k+m)}\left(\binom{24j+23}{4k+3}\binom{26j+25}{2m}\right.\right. \\
&\quad \left.\left.+\binom{24j+23}{4k+1}\binom{26j+25}{2m+1}\right)q^{k+m}\frac{f_2^{484j-24k-24m}}{f_1^{364j-8k-8m}f_4^{152j-16k-16m}}\right\}.
\end{aligned}$$

Now, we separate the right side of the above identity with the cases $(k,m)=(0,0)$ and $(k,m)\neq(0,0)$ as follows.

$$\begin{aligned}
&\sum_{n=0}^{\infty} d_{16j+15}(4n+3)q^n \\
&= 8\frac{f_2^{447}}{f_1^{344}f_4^{134}}\left\{\left(\binom{24j+23}{3}+\binom{24j+23}{1}\binom{26j+25}{1}\right)\frac{f_2^{484j}}{f_1^{364j}f_4^{152j}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=0}^{13j+11} 2^{4(m+1)}\left(\binom{24j+23}{3}\binom{26j+25}{2m+2}+\binom{24j+23}{1}\binom{26j+25}{2m+3}\right)q^{m+1}\\
&\times\frac{f_2^{484j-24m-24}}{f_1^{364j-8m-8}f_4^{152j-16m-16}}+\sum_{k=0}^{6j+4}2^{4(k+1)}\left(\binom{24j+23}{4k+7}\right.\\
&\left.+\binom{24j+23}{4k+5}\binom{26j+25}{1}\right)q^{k+1}\frac{f_2^{484j-24k-24}}{f_1^{364j-8k-8}f_4^{152j-16k-16}}+\sum_{k=0}^{6j+4}\sum_{m=0}^{13j+11}\\
&\times 2^{4(k+m+2)}\left(\binom{24j+23}{4k+7}\binom{26j+25}{2m+2}+\binom{24j+23}{4k+5}\binom{26j+25}{2m+3}\right)q^{k+m+2}\\
&\left.\times\frac{f_2^{484j-24k-24m-48}}{f_1^{364j-8k-8m-16}f_4^{152j-16k-16m-32}}\right\}.
\end{aligned}
$$

On simplifying the above identity, we find that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}d_{16j+15}(4n+3)q^n\\
&=16\left\{3(24j+23)(16j+17)(j+1)\frac{f_2^{484j+447}}{f_1^{364j+344}f_4^{152j+134}}+8\sum_{m=0}^{13j+11}2^{4m}\right.\\
&\quad\times\left(\binom{24j+23}{3}\binom{26j+25}{2m+2}+(24j+23)\binom{26j+25}{2m+3}\right)q^{m+1}\\
&\quad\times\frac{f_2^{484j-24m+423}}{f_1^{364j-8m+336}f_4^{152j-16m+118}}+8\sum_{k=0}^{6j+4}2^{4k}\left(\binom{24j+23}{4k+7}\right.\\
&\quad\left.+\binom{24j+23}{4k+5}(26j+25)\right)q^{k+1}\frac{f_2^{484j-24k+423}}{f_1^{364j-8k+336}f_4^{152j-16k+118}}+128\sum_{k=0}^{6j+4}\sum_{m=0}^{13j+11}\\
&\quad\times 2^{4(k+m)}\left(\binom{24j+23}{4k+7}\binom{26j+25}{2m+2}+\binom{24j+23}{4k+5}\binom{26j+25}{2m+3}\right)q^{k+m+2}\\
&\quad\left.\times\frac{f_2^{484j-24k-24m+399}}{f_1^{364j-8k-8m+328}f_4^{152j-16k-16m+102}}\right\}. \qquad (3.56)
\end{aligned}
$$

Note that (3.18) is evident from (3.56).

Now, we prove (3.23). Replacing j by $2j+1$ and taking modulo 128 in (3.56), we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty}d_{32j+31}(4n+3)q^n &\equiv 96(48j+47)(32j+33)(j+1)\frac{f_2^{968j+931}}{f_1^{728j+708}f_4^{304j+286}}\\
&\equiv 96(48j+47)(32j+33)(j+1)\frac{f_2^{604j+577}}{f_4^{304j+286}} \pmod{128}.
\end{aligned}
$$

From the above identity, we clearly have (3.23).

Similar to the proof of (3.23), we can obtain (3.14), (3.16), (3.19), and (3.20) from (3.47), (3.15), and (3.17) from (3.48), and (3.21) and (3.22) from (3.56). □

3.4 Proof of Theorem 3.4

Proof. Without loss of generality, we may assume that $r = \sum_{j=0}^{M-1} p^j r_j$ for $0 \le r_j \le p-1$, because $\sum_{j=0}^{M-1} p^j r_j$ can take any value between 1 and $p^M - 1$. For sufficiently large integers $M \ge 1$ and $N \ge 1$, we have

$$\sum_{n=0}^{\infty} d_{p^{M+N-1}j+k}(n)q^n = \frac{f_2^{p^{M+N-1}j+k}}{f_1^{3p^{M+N-1}j+3k+1}} \equiv \frac{f_{2p}^{p^{M+N-2}j}}{f_p^{3p^{M+N-2}j}} \sum_{n=0}^{\infty} d_k(n)q^n \pmod{p^N}.$$

Extracting the terms that involve q^{pn+r_0} from the above identity, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} d_{p^{M+N-1}j+k}(pn+r_0)q^n &\equiv \frac{f_2^{p^{M+N-2}j}}{f_1^{3p^{M+N-2}j}} \sum_{n=0}^{\infty} d_k(pn+r_0)q^n \\ &\equiv \frac{f_{2p}^{p^{M+N-3}j}}{f_p^{3p^{M+N-3}j}} \sum_{n=0}^{\infty} d_k(pn+r_0)q^n \pmod{p^N}.\end{aligned}$$

Now, extracting the terms that involve q^{pn+r_1} from the above identity, we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} d_{p^{M+N-1}j+k}(p^2n+r_0+pr_1)q^n \\ &\equiv \frac{f_2^{p^{M+N-3}j}}{f_1^{3p^{M+N-3}j}} \sum_{n=0}^{\infty} d_k(p^2n+r_0+pr_1)q^n \\ &\equiv \frac{f_{2p}^{p^{M+N-4}j}}{f_p^{3p^{M+N-4}j}} \sum_{n=0}^{\infty} d_k(p^2n+r_0+pr_1)q^n \pmod{p^N}.\end{aligned}$$

From the above identity, we extract the terms that contain q^{pn+r_2}, and from the resulting identity, we again extract the terms that contain q^{pn+r_3}. It can be seen that after the M-th extraction using this iterative scheme, we arrive at

$$\begin{aligned}&\sum_{n=0}^{\infty} d_{p^{M+N-1}j+k}(p^Mn+r_0+pr_1+\cdots+p^{M-1}r_{M-1})q^n \\ &\equiv \frac{f_2^{p^{N-1}j}}{f_1^{3p^{N-1}j}} \sum_{n=0}^{\infty} d_k(p^Mn+r_0+pr_1+\cdots+p^{M-1}r_{M-1})q^n \pmod{p^N}.\end{aligned}$$

Therefore, if we assume that $d_k(p^Mn+r_0+pr_1+\cdots+p^{M-1}r_{M-1}) = d_k(p^Mn+r) \equiv 0 \pmod{p^N}$, from the above identity, we evidently have

$$d_{p^{M+N-1}j+k}(p^M n + r) \equiv 0 \pmod{p^N}.$$

Thus, we complete the proof of Theorem 3.4. $\square$

3.5 Proof of the remaining congruences

In this section, we use modular identities of $T(q)$, which is defined as

$$T(q) := \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots,$$

a 7-dissection of f_1, series representations of certain q-products, and an algorithm developed by Radu [91] to prove the congruences in Theorem 3.6. Note that we only prove the prove the $k = 0$ cases of the congruences in Theorem 3.6 which together with Theorem 3.4 complete the proofs of the congruences.

3.5.1 Required lemmas

Here, we present some background material on the method of Radu [91]. For integers x, let $[x]_m$ denote the residue class of x in $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$, $\mathbb{S}_m$ denote the set of all squares in $\mathbb{Z}_m^*$, and for integers $N \geq 1$, we assume that

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and } ad - bc = 1 \right\},$$

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\},$$

$$[\Gamma : \Gamma_0(N)] := N \prod_{\ell | N} \left(1 + \frac{1}{\ell}\right),$$

where ℓ is a prime.

For integers $M \geq 1$, suppose that $R(M)$ is the set of all the integer sequences

$(r_\delta) := (r_{\delta_1}, r_{\delta_2}, r_{\delta_3}, \ldots, r_{\delta_k})$ indexed by all the positive divisors δ of M, where $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$. For integers $m \geq 1$, $(r_\delta) \in R(M)$, and $t \in \{0, 1, 2, \ldots, m-1\}$, we define the set $P(t)$ as

$$P(t) := \Bigg\{ t' \in \{0, 1, 2, \ldots, m-1\} : t' \equiv ts + \frac{s-1}{24} \sum_{\delta | M} \delta r_\delta \pmod{m} \\ \text{for some } [s]_{24m} \in \mathbb{S}_{24m} \Bigg\}. \tag{3.57}$$

For integers $N \geq 1$, $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $(r_\delta) \in R(M)$, and $(r'_\delta) \in R(N)$, we also define

$$p(\gamma) := \min_{\lambda \in \{0, 1, \ldots, m-1\}} \frac{1}{24} \sum_{\delta | M} r_\delta \frac{(\delta(a + k\lambda c), mc)^2}{\delta m},$$

$$p'(\gamma) := \frac{1}{24} \sum_{\delta | N} r'_\delta \frac{(\delta, c)^2}{\delta}.$$

For integers $m \geq 1$; $2 \nmid m$, $M \geq 1$, $N \geq 1$, $t \in \{0, 1, 2, \ldots, m-1\}$, $k := (m^2 - 1, 24)$, and $(r_\delta) \in R(M)$, define Δ^* to be the set of all tuples $(m, M, N, t, (r_\delta))$ such that all of the following conditions are satisfied

1. Prime divisors of m are also prime divisors of N;
2. If $\delta \mid M$, then $\delta \mid mN$ for all $\delta \geq 1$ with $r_\delta \neq 0$;
3. $24 \mid kN \sum_{\delta | M} \frac{r_\delta m N}{\delta}$;
4. $8 \mid kN \sum_{\delta | M} r_\delta$;
5. $\dfrac{24m}{\left(-24kt - k\sum_{\delta | M} \delta r_\delta, 24m\right)} \mid N$.

The following lemma supports Lemma 3.8 in the proof of Theorem 3.6.

Lemma 3.7. [123, Lemma 4.3] *Let N or $\frac{1}{2}N$ be a square-free integer, then we have*

$$\bigcup_{\delta | N} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \Gamma_\infty = \Gamma.$$

We end this section by stating a result of Radu [91], which is especially useful in completing the proof of Theorem 3.6 in the final section.

Lemma 3.8. [91, Lemma 4.5] *Suppose that* $(m, M, N, t, (r_\delta)) \in \Delta^*$, $(r'_\delta) := (r'_\delta)_{\delta|N} \in R(N)$, $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ *is a complete set of representatives of the double cosets of* $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$, $t_{\min} := \min_{t' \in P(t)} t'$,

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta r'_\delta - \frac{1}{m}\sum_{\delta|M} \delta r_\delta\right) - \frac{t_{min}}{m}, \quad (3.58)$$

$p(\gamma_j) + p'(\gamma_j) \geq 0$ *for all* $1 \leq j \leq n$, *and* $\sum_{n=0}^{\infty} A(n)q^n := \prod_{\delta|M} f_\delta^{r_\delta}$. *If for some integers* $u \geq 1$, *all* $t' \in P(t)$, *and* $0 \leq n \leq \lfloor \nu \rfloor$, $A(mn + t') \equiv 0 \pmod{u}$ *is true, then for integers* $n \geq 0$ *and all* $t' \in P(t)$, *we have* $A(mn + t') \equiv 0 \pmod{u}$.

3.5.2 Proof of Theorem 3.6

Proof of (3.25). First, for integers $\alpha \geq 0$ and β, we let

$$P_{\alpha,\beta} := \frac{1}{T(q)^{\alpha+2\beta}T\left(q^2\right)^{2\alpha-\beta}} + (-1)^{\alpha+\beta}q^{2\alpha}T(q)^{\alpha+2\beta}T\left(q^2\right)^{2\alpha-\beta}. \quad (3.59)$$

We will use the 5-dissections of f_1 and $\frac{1}{f_1}$ from Section 1.3 (see (1.16) and (1.17)) in the proof.

Now,

$$\sum_{n=0}^{\infty} d_1(n)q^n = \frac{f_2}{f_1^4}.$$

Employing the 5-dissections of f_2 and $1/f_1$ from (1.16) and (1.17) in the above identity, then extracting the terms that involve q^{5n+3}, and finally with the help of (3.59), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} d_1(5n+3)q^n = \frac{f_5^{20}f_{10}}{f_1^{24}}\Big(& -4P_{3,6} + 40P_{3,5} - 105qP_{2,5} - 418qP_{2,4} \\ & + 1100qP_{2,3} - 1400q^2P_{1,3} - 1840q^2P_{1,2} \\ & + 1200q^2P_{1,1} - 1500q^3P_{0,1} - 1015q^3\Big). \quad (3.60)\end{aligned}$$

From [19, Lemma 1.3] and [29, (7.4.9)], we have

$$P_{0,1} = 4q\frac{f_1 f_{10}^5}{f_2 f_5^5}, \tag{3.61}$$

$$P_{1,1} = \frac{f_2 f_5^5}{f_1 f_{10}^5} + 2q + 4q^2\frac{f_1 f_{10}^5}{f_2 f_5^5}, \tag{3.62}$$

$$P_{1,2} = \frac{f_1^6}{f_5^6} + 11q, \tag{3.63}$$

and the following relations:

$$P_{1,3} = P_{0,1}P_{1,2} + P_{1,1}, \qquad P_{2,3} = P_{1,1}P_{1,2} - q^2 P_{0,1},$$

$$P_{2,4} = P_{1,2}^2 + 2q^2, \qquad P_{2,5} = P_{0,1}P_{2,4} - P_{2,3},$$

$$P_{3,5} = P_{1,1}P_{2,4} - q^2 P_{1,3}, \qquad P_{3,6} = P_{1,2}P_{2,4} + q^2 P_{1,2}.$$

Employing (3.61)–(3.63) and the above relations in (3.60), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} d_1(5n+3)q^n =& 40\frac{f_2 f_5^{13}}{f_1^{13} f_{10}^4} - 4\frac{f_{10} f_5^2}{f_1^6} - 470q\frac{f_{10} f_5^8}{f_1^{12}} + 1875q\frac{f_2 f_5^{19}}{f_1^{19} f_{10}^4}\\ &+ 15625q^2\frac{f_2 f_5^{25}}{f_1^{25} f_{10}^4} - 8750q^2\frac{f_{10} f_5^{14}}{f_1^{18}} - 260q^2\frac{f_{10}^6 f_5^3}{f_1^{11} f_2}\\ &- 7500q^3\frac{f_{10}^6 f_5^9}{f_1^{17} f_2} - 46875q^3\frac{f_{10} f_5^{20}}{f_1^{24}} - 62500q^4\frac{f_{10}^6 f_5^{15}}{f_1^{23} f_2},\end{aligned}$$

which gives

$$\begin{aligned}\sum_{n=0}^{\infty} d_1(5n+3)q^n \equiv& -4\frac{f_{10} f_5^2}{f_1^6}\\ \equiv& \frac{f_{10} f_5}{f_1} \pmod 5.\end{aligned}$$

Invoking the 5-dissection of $1/f_1$ given by (1.17) in the above identity and then extracting the terms involving q^{5n+4}, we obtain (3.25). □

Proof of (3.29). We have

$$\sum_{n=0}^{\infty} d_2(n)q^n = \frac{f_2^2}{f_1^7} \equiv \frac{f_2^2}{f_7} \pmod 7. \tag{3.64}$$

From [68, (10.5.1)], we recall the following 7-dissection of f_1:

$$\begin{aligned}f_1 =& f_{49}\left(\frac{(q^{14};q^{49})_\infty (q^{35};q^{49})_\infty}{(q^7;q^{49})_\infty (q^{42};q^{49})_\infty} - q\frac{(q^{21};q^{49})_\infty (q^{28};q^{49})_\infty}{(q^{14};q^{49})_\infty (q^{35};q^{49})_\infty}\right.\\ &\left. - q^2 + q^5\frac{(q^7;q^{49})_\infty (q^{42};q^{49})_\infty}{(q^{21};q^{49})_\infty (q^{28};q^{49})_\infty}\right).\end{aligned}$$

With the help of the above identity, we use the 7-dissection of f_2^2 in (3.64) and then extract the terms involving q^{7n+1}. This gives

$$\sum_{n=0}^{\infty} d_2(7n+1)q^n \equiv \frac{f_{14}^2}{f_1} \pmod 7. \tag{3.65}$$

Now, if $p(n)$ counts the unrestricted partitions of an integer $n \geq 0$, we have

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1}$$

and one of Ramanujan's famous three partition congruences

$$p(7n+5) \equiv 0 \pmod 7$$

for all $n \geq 0$.

Therefore, it becomes evident from (3.65) that (3.29) is true. $\square$

Before proceeding to the next proofs, we state some useful product-to-sum identities in the following lemma.

Lemma 3.9. *We have*

$$f_1^3 = \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2}, \tag{3.66}$$

$$\frac{f_2^2}{f_1} = \sum_{j=0}^{\infty} q^{j(j+1)/2}, \tag{3.67}$$

$$\frac{f_2^5}{f_1^2} = \sum_{j=-\infty}^{\infty} (-1)^j (3j+1) q^{3j^2+2j}. \tag{3.68}$$

Proof of Lemma 3.9. The identities (3.66), (3.67), and (3.68) appear as (1.7.1), (1.5.3), and (10.7.7) in Hirschhorn [68]. $\square$

Proofs of (3.36), (3.37), *and* (3.40)–(3.42). We have

$$\sum_{n=0}^{\infty} d_3(n)q^n = \frac{f_2^3}{f_1^{10}} \equiv \frac{f_1^3 f_2^3}{f_{13}} \pmod{13}.$$

Using (3.66) in the above identity, we have

$$\begin{aligned}\sum_{n=0}^{\infty} d_3(n)q^n &\equiv \frac{1}{f_{13}} \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2} \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)} \\ &\equiv \frac{1}{f_{13}} \sum_{j,k=0}^{\infty} (-1)^{j+k} (2j+1)(2k+1) q^{j(j+1)/2+k(k+1)} \pmod{13}. \end{aligned} \tag{3.69}$$

Now,

$$8\left(\frac{j(j+1)}{2}+k(k+1)\right)+3=(2j+1)^2+2(2k+1)^2.$$

If $j(j+1)/2+k(k+1)=13n+11$ for some integer $n\geq 0$, the above equality gives

$$(2j+1)^2+2(2k+1)^2\equiv 0 \pmod{13}.$$

Therefore, $2j+1\equiv 0 \pmod{13}$ and $2k+1\equiv 0 \pmod{13}$. Otherwise, we have $(2j+1)^2\equiv 1,3,4,9,10,12 \pmod{13}$, which gives $(2j+1)^2+2(2k+1)^2\not\equiv 0 \pmod{13}$. This is a contradiction.

Finally, extracting the terms that involve q^{13n+11} from (3.69), we find that for all $n\geq 0$,

$$d_3(13n+11)\equiv 0 \pmod{13}, \tag{3.70}$$

which completes the proof of (3.36).

Congruences (3.37) and (3.40)–(3.42) can be proved similarly as above. So, we provide only the following table containing the product-to-sum identities required for the proofs.

Congruence	Used product-to-sum identities
(3.37)	(3.66), (3.68)
(3.40)	(3.66)
(3.41)	(3.67), (3.68)
(3.42)	(3.66), (3.68)

□

Proofs of the remaining congruences of Theorem 3.6. Proofs of (3.26)–(3.28), (3.30)–(3.35), (3.38), and (3.39) are similar. We elaborate the proof of (3.26) only. We have

$$\sum_{n=0}^{\infty} d_1(n)q^n=\frac{f_2}{f_1^4}\equiv\frac{f_1^{21}f_2}{f_1^{25}}\equiv\frac{f_1^{21}f_2}{f_5^5} \pmod{25}. \tag{3.71}$$

By Conditions 1–5, we have $(m,M,N,t,(r_\delta))=(125,10,10,23,(21,1,-5,0))\in\Delta^*$. So, from (3.57), we obtain $P(t)=\{23,123\}$. Lemma 3.7 gives that $\left\{\begin{pmatrix}1&0\\ \delta&1\end{pmatrix}:\delta\mid N\right\}$

is a complete set of representatives of the double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Using $(r'_\delta) = (18, 0, 0, 0)$, (3.58), and *Mathematica*, we find that

$$p\left(\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\right) + p'\left(\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\right) \geq 0 \quad \text{for all } \delta \mid N,$$

$$\lfloor \nu \rfloor = 25,$$

$$d_1(125n + j) \equiv 0 \pmod{25} \quad \text{for } j \in \{23, 123\},$$

are true for all $0 \leq n \leq \lfloor \nu \rfloor$. Therefore, by Lemma 3.8 and (3.71), (3.26) is true. The proofs of (3.27), (3.28), (3.30)–(3.35), (3.38), and (3.39) follow analogously from Lemma 3.8 and the chart below.

Congruence	$(m, M, N, t, (r_\delta))$ and (r'_δ)	$P(t)$	$\lfloor \nu \rfloor$
(3.27)	$(125, 10, 10, 97, (3, 2, -2, 0))$ and (30,0,0,0)	{97,122}	22
(3.28)	$(49, 14, 14, 45, (3, 1, -1, 0))$ and (4,0,0,0)	{45}	5
	$(49, 14, 14, 17, (3, 1, -1, 0))$ and (4,0,0,0)	{17,31,38}	6
(3.30)	$(49, 14, 14, 41, (4, 3, -2, 0))$ and (9,0,0,0)	{41}	12
(3.31)	$(343, 14, 14, 90, (39, 3, -7, 0))$ and (60,0,0,0)	{90,188,237}	92
(3.32)	$(343, 14, 14, 39, (1, 4, -2, 0))$ and (77,0,0,0)	{39,235,284}	76
(3.33)	$(121, 22, 22, 96, (9, 4, -2, 0))$ and (11,0,0,0)	{96}	31
(3.34)	$(121, 22, 22, 91, (6, 5, -2, 0))$ and (14,0,0,0)	{91}	33
(3.35)	$(121, 22, 22, 81, (0, 7, -2, 0))$ and (19,0,0,0)	{81}	34
(3.38)	$(289, 34, 34, 205, (15, 6, -2, 0))$ and (16,0,0,0)	{205}	77

	$(289, 34, 34, 52, (15, 6, -2, 0))$ and (16,0,0,0)	{52,69,137,171}	77
	$(289, 34, 34, 52, (15, 6, -2, 0))$ and (16,0,0,0)	{188,222,239,273}	77
(3.39)	$(19, 38, 38, 16, (9, 3, -1, 0))$ and (1,0,0,0)	{16}	29

$\square$