# Chapter 4

# Arithmetic Properties of Two Analogues of $t$-Core Partitions

## 4.1 Introduction

We recall the following generating functions for the two analogues $\overline{a}_t(n)$ and $\overline{b}_t(n)$ from (1.25) and (1.26) of Section 1.8:

$$\sum_{n=0}^{\infty} \overline{a}_t(n)q^n = \frac{\varphi^t(-q^t)}{\varphi(-q)} = \frac{f_2 f_t^{2t}}{f_1^2 f_{2t}^t}$$

and

$$\sum_{n=0}^{\infty} \overline{b}_t(n)q^n = \frac{\psi^t(-q^t)}{\psi(-q)} = \frac{f_2 f_t^t f_{4t}^t}{f_1 f_4 f_{2t}^t}.$$

Recently, Gireesh, Ray, and Shivashankar [57] proved several multiplicative formulae and arithmetic identities for $\overline{a}_t(n)$ for $t = 2$, 3, 4, and 8 using Ramanujan's theta functions and $q$-series techniques. Using the theory of modular forms, they studied the divisibility of $\overline{a}_t(n)$ modulo arbitrary powers of primes greater than 5. More precisely, they proved the following theorem.

**Theorem 4.1** (Gireesh, Ray, and Shivashankar [57]). *Let $t = p_1^{a_1} \cdots p_m^{a_m}$ where $p_i$'s*

Theorems 4.3, 4.4, and 4.9 of this chapter have been published in *Bulletin of the Australian Mathematical Society* [117]. The rest of the results of this chapter have been published in *The Ramanujan Journal* [118].

*are primes greater than or equal to 5. Then for every positive integer $j$, we have*

$$\lim_{X\to\infty} \frac{\#\left\{0 \le n \le X : \overline{a}_t(n) \equiv 0 \pmod{p_i^j}\right\}}{X} = 1.$$

Gireesh, Ray, and Shivashankar [57] also deduced a Ramanujan-type congruence for $\overline{a}_5(n)$ modulo 5 by using an algorithm developed by Radu and Sellers [92].

Bandyopadhyay and Baruah [14] proved some new identities connecting $\overline{a}_5(n)$ and $c_5(n)$. Also, for integers $k \ge 2$, they found the following recurrence relation for $\overline{a}_5(n)$:

$$\overline{a}_5\left(5^k n\right) = \left(\frac{5^k - 1}{4}\right)\overline{a}_5(5n) - \left(\frac{5^k - 5}{4}\right)\overline{a}_5(n).$$

Again, Bandyopadhyay and Baruah [14] studied the function $\overline{b}_5(n)$ and deduced some new identities connecting $c_5(n)$, $\overline{a}_5(n)$, and $\overline{b}_5(n)$. They also proved some recurrence relations and vanishing coefficients results for $\overline{b}_5(n)$. For instance, for any nonnegative integer $n$ and $k \ge 2$, they proved the following results:

$$\begin{aligned}\overline{b}_5(10n+6) &= \frac{1}{4}\overline{a}_5(2n+1) + \frac{1}{2}c_5(n),\\ \overline{b}_5(5^k(n+3)-3) &= \left(\frac{5^k - 1}{4}\right)\overline{b}_5(5n+12) - \left(\frac{5^k - 5}{4}\right)\overline{b}_5(n).\end{aligned}$$

Recently, Cotron et al. [48] proved the following theorem on the lacunarity of certain eta-quotients modulo arbitrary powers of primes.

**Theorem 4.2.** *[48, Theorem 1.1] Let* $G(z) = \dfrac{\prod_{i=1}^{u} f_{\alpha_i}^{r_i}}{\prod_{i=1}^{t} f_{\beta_i}^{s_i}}$, *and $p$ is a prime such that $p^a$ divides* $\gcd(\alpha_1, \alpha_2, \cdots, \alpha_u)$ *and*

$$p^a \ge \sqrt{\frac{\sum_{i=1}^{t} \beta_i s_i}{\sum_{i=1}^{u} \frac{r_i}{\alpha_i}}},$$

*then $G(z)$ is lacunary modulo $p^j$ for any positive integer $j$.*

We observe that the eta-quotients associated with $\overline{a}_t(n)$ and $\overline{b}_t(n)$ do not satisfy the conditions of Theorem 4.2, which makes the problem of studying lacunarity of $\overline{a}_t(n)$ and $\overline{b}_t(n)$ more interesting. In this chapter, we obtain the arithmetic densities of $\overline{a}_t(n)$ and $\overline{b}_t(n)$ modulo arbitrary powers of 2 and 3 where $t = 3^\alpha m$. To be specific, we prove the following theorems.

**Theorem 4.3.** *Let $k \geq 1$, $\alpha \geq 0$, and $m \geq 1$ be integers with $\gcd(m, 6) = 1$. Then the set*

$$\left\{n \in \mathbb{N} : \overline{a}_{3^\alpha m}(n) \equiv 0 \pmod{2^k}\right\}$$

*has arithmetic density 1.*

**Theorem 4.4.** *Let $k \geq 1$, $\alpha \geq 0$, and $m \geq 1$ be integers with $\gcd(m, 6) = 1$. Then the set*

$$\left\{n \in \mathbb{N} : \overline{a}_{3^\alpha m}(n) \equiv 0 \pmod{3^k}\right\}$$

*has arithmetic density 1.*

**Theorem 4.5.** *Let $k \geq 1$, $\alpha \geq 0$, and $m \geq 1$ be integers with $\gcd(m, 6) = 1$. Then the set*

$$\left\{n \in \mathbb{N} : \overline{b}_{3^\alpha m}(n) \equiv 0 \pmod{2^k}\right\}$$

*has arithmetic density 1.*

**Theorem 4.6.** *Let $k \geq 1$, $\alpha \geq 0$, and $m \geq 1$ be integers with $\gcd(m, 6) = 1$. Then the set*

$$\left\{n \in \mathbb{N} : \overline{b}_{3^\alpha m}(n) \equiv 0 \pmod{3^k}\right\}$$

*has arithmetic density 1.*

We also study the density of $\overline{b}_t(n)$ modulo arbitrary powers of primes greater than or equal to 5 for certain general values of $t$. In fact, we prove the following result.

**Theorem 4.7.** *Let $k \geq 1$ be a fixed positive integer and $a_1, a_2, \cdots, a_m$ be nonnegative integers for some positive integer $m$, then for $t = p_1^{a_1} \cdots p_m^{a_m}$ where $p_i$'s are primes greater than or equal to 5. The the set*

$$\left\{n \in \mathbb{N} : \overline{b}_t(n) \equiv 0 \pmod{p_i^k}\right\}$$

*has arithmetic density 1.*

As a consequence of the above theorem, we obtain the following divisibility result for $\overline{b}_p(n)$.

**Corollary 4.8.** *Let $k \geq 1$ be a fixed positive integer and $p \geq 5$ be a prime. Then the set*

$$\left\{n \in \mathbb{N} : \overline{b}_p(n) \equiv 0 \pmod{p^k}\right\}$$

*has arithmetic density 1.*

The fact that the action of Hecke algebras on spaces of modular forms of level 1 modulo 2 is locally nilpotent was first observed by Serre and proved by Tate (see [110], [111], [120]). Later, this result was generalized to higher levels by Ono and Taguchi [88]. In this chapter, we observe that the eta-quotients associated to $\overline{a}_3(n)$ and $\overline{b}_3(n)$ are modular forms whose levels are in the list of Ono and Taguchi. Thus, we use a result of Ono and Taguchi to find the following congruences for $\overline{a}_3(n)$ and $\overline{b}_3(n)$.

**Theorem 4.9.** *Let $n$ be a nonnegative integer. Then there exists an integer $c \geq 0$ such that for every $d \geq 1$ and distinct primes $p_1, \ldots, p_{c+d}$ coprime to 6, we have*

$$\overline{a}_3\left(\frac{p_1 \cdots p_{c+d} \cdot n}{24}\right) \equiv 0 \pmod{2^d}$$

*whenever $n$ is coprime to $p_1, \ldots, p_{c+d}$.*

**Theorem 4.10.** *Let $n$ be a nonnegative integer. Then there exists an integer $u \geq 0$ such that for every $v \geq 1$ and distinct primes $q_1, \ldots, q_{u+v}$ coprime to 6, we have*

$$\overline{b}_3\left(\frac{q_1 \cdots q_{u+v} \cdot n - 24}{24}\right) \equiv 0 \pmod{2^v}$$

*whenever $n$ is coprime to $q_1, \ldots, q_{u+v}$.*

Next, we study some modulo 2 behaviours of $\overline{b}_t(n)$. Certain relations between $\overline{a}_t(n)$ and $c_t(n)$ has been deduced in [14] and [57]. In the following result, we show that $\overline{b}_t(n)$ can also be expressed in terms of $c_t(n)$ modulo 2. Recently, Keith and Zanello [78] studied the parity of pure eta-powers $f_1^t$ for different values of $t$. We prove that if the only prime factor of $t$ is 2, then the parity of $\overline{b}_t(n)$ is same as such an eta-power.

**Theorem 4.11.** *Let $n$ be a nonnegative integer.*

*(i) For all $t \geq 2$, we have* $\overline{b}_t(n) \equiv \sum_{k=0}^{\infty} c_t(k)c_t(n-2k) \pmod{2}$.

*(ii) If $t = 2^k$ with $k \geq 1$, then* $\overline{b}_t(n) \equiv f_1^{3(2^{2k}-1)} \pmod{2}$.

In Theorems 4.5–4.7, we have discussed the divisibility of $\overline{b}_t(n)$ for odd values of $t$. Corresponding density properties for even $t$ cannot be studied using similar techniques. However, we adopt a different approach to prove the following theorem.

**Theorem 4.12.** *The series* $\sum_{n=0}^{\infty} \overline{b}_2(n)q^n$ *is lacunary modulo 2.*

Consider $f_1^t = \sum_{n=0}^{\infty} \alpha_t(n)q^n$. Keith and Zanello [78] also deduced several infinite families of congruences for $\alpha_t(n)$ modulo 2. They defined $f_1^t$ to be $p^2$-even at a prime $p$ with base $r \in \{0, \dots, p^2-1\}$ if $\alpha_t(p^2n + kp + r) \equiv 0 \pmod{2}$ for all $k \in \{1, \dots, p-1\}$. Then they showed that $f_1^t$ is $p^2$-even for some specific choices of $p$ and $t$. We use one such result from [78] to prove the following infinite family of congruences for $\overline{b}_2(n)$.

**Theorem 4.13.** *Let $p$ be a prime such that $p \equiv 7 \pmod{8}$ and $r \in \{0, \dots, p^2-1\}$ with $r \equiv -3(1+2^{-3}) \pmod{p^2}$. Then for all $k \in \{1, 2, \dots, p-1\}$, we have*

$$\overline{b}_2\left(p^2n + kp + r\right) \equiv 0 \pmod{2}.$$

We recall that for a prime $p \geq 3$, the Legendre symbol $\left(\frac{a}{p}\right)_L$ is defined by

$$\left(\frac{a}{p}\right)_L := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ 0, & \text{if } p \mid a, \\ -1, & \text{if } a \text{ is a quadratic nonresidue modulo } p\ . \end{cases}$$

Next, we prove an infinite family of congruences for $\overline{b}_2(n)$ for more general choices of primes as stated in the following theorem. To state the theorem, first we define

$$\pi(p) := p_9\left(\frac{3(p^2-1)}{8}\right) + (-1)^{\frac{(p-1)(p-19)}{8}} \left(\frac{\frac{3(p^2-1)}{8}}{p}\right)_L, \tag{4.1}$$

where $p_9(n)$ is defined by

$$\sum_{n=0}^{\infty} p_9(n)q^n = f_1^9. \tag{4.2}$$

**Theorem 4.14.** *Let $p \geq 3$ be a prime. We define*

$$u(p) := \begin{cases} 1, & \text{if } \pi(p) \equiv 0 \pmod 2, \\ 2, & \text{if } \pi(p) \not\equiv 0 \pmod 2. \end{cases} \tag{4.3}$$

*Then, for $n \geq 0$, $k \geq 0$, and $p \nmid n$, we have*

$$\overline{b}_2\left(p^{2(u(p)+1)(k+1)-1}n + \frac{3\left(p^{2(u(p)+1)(k+1)} - 1\right)}{8}\right) \equiv 0 \pmod 2. \tag{4.4}$$

We organize the rest of this chapter as follows. In the next section, we state some preliminary results of the theory modular forms. We then prove Theorems 4.3–4.7 in Sections 4.3–4.7, respectively. We deduce Theorems 4.9 and 4.10 in Section 4.8. Theorems 4.11–4.13 are then established in Section 4.9. In Section 4.10, we prove Theorem 4.14 and we conclude the chapter by mentioning some directions for further work in Section 4.11.

## 4.2 Preliminary results

First, we recall two important theorems regarding eta-quotients and the theory modular forms from [87, p. 18].

**Theorem 4.15.** *[87, Theorem 1.64] If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \qquad \textit{and} \qquad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

*then $f(z)$ satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

*for every* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$*. Here, the character* $\chi$ *is defined by* $\chi(d) := \left(\frac{(-1)^\ell s}{d}\right)$*, where* $s := \prod_{\delta | N} \delta^{r_\delta}$*.*

Consider $f$ to be an eta-quotient which satisfies the conditions of Theorem 4.15 and that the associated weight $\ell$ is a positive integer. If $f(z)$ is holomorphic at all the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell\left(\Gamma_0(N), \chi\right)$. The necessary criterion for determining orders of an eta-quotient at cusps is given by the following theorem.

**Theorem 4.16.** *[87, Theorem 1.64] Let* $c$*,* $d$ *and* $N$ *be positive integers with* $d|N$ *and* $\gcd(c,d)$*=1. If* $f$ *is an eta-quotient satisfying the conditions of Theorem 4.15 for* $N$*, then the order of vanishing of* $f(z)$ *at the cusp* $(c/d)$ *is*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,N/d)d\delta}.$$

Now, we recall a deep theorem of Serre [87, p. 43] which will be used in proving Theorems 4.3–4.7.

**Theorem 4.17.** *[87, p. 43] Let* $g(z) \in M_k(\Gamma_0(N), \chi)$ *has Fourier expansion*

$$g(z) = \sum_{n=0}^{\infty} b(n)q^n \in \mathbb{Z}[[q]].$$

*Then for a positive integer* $r$*, there is a constant* $\alpha > 0$ *such that*

$$\#\{0 < n \le X : b(n) \not\equiv 0 \pmod{r}\} = \mathcal{O}\left(\frac{X}{(logX)^\alpha}\right).$$

*Equivalently,*

$$\lim_{X\to\infty}\frac{\#\{0 < n \le X : b(n) \not\equiv 0 \pmod{r}\}}{X} = 0.$$

## 4.3 Proof of Theorem 4.3

Putting $t = 3^\alpha m$ in (1.25), we have

$$\sum_{n=0}^{\infty} \overline{a}_{3^\alpha m}(n)q^n = \frac{f_2 f_{3^\alpha m}^{2\cdot 3^\alpha m}}{f_1^2 f_{2\cdot 3^\alpha m}^{3^\alpha m}}. \tag{4.5}$$

We define

$$A_{\alpha,m}(z) := \frac{\eta^2\left(2^3 3^{\alpha+1} mz\right)}{\eta\left(2^4 3^{\alpha+1} mz\right)}.$$

For any prime $p$ and positive integer $j$, we have

$$(q;q)_\infty^{p^j} \equiv (q^p;q^p)_\infty^{p^{j-1}} \pmod{p^j}.$$

Using the above relation, for any integer $k \geq 1$, we get

$$A_{\alpha,m}^{2^k}(z) = \frac{\eta^{2^{k+1}}\left(2^3 3^{\alpha+1} mz\right)}{\eta^{2^k}\left(2^4 3^{\alpha+1} mz\right)} \equiv 1 \pmod{2^{k+1}}. \tag{4.6}$$

Next, we define

$$\begin{aligned} B_{\alpha,m,k}(z) &:= \frac{\eta(48z)\eta^{2\cdot 3^\alpha m}\left(2^3 3^{\alpha+1} mz\right)}{\eta^2(24z)\eta^{3^\alpha m}\left(2^4 3^{\alpha+1} mz\right)} A_{\alpha,m}^{2^k}(z) \\ &= \frac{\eta(48z)\eta^{2\cdot 3^\alpha m+2^{k+1}}\left(2^3 3^{\alpha+1} mz\right)}{\eta^2(24z)\eta^{3^\alpha m+2^k}\left(2^4 3^{\alpha+1} mz\right)}. \end{aligned}$$

In view of (4.5) and (4.6), we have

$$\begin{aligned} B_{\alpha,m,k}(z) &\equiv \frac{\eta(48z)\eta^{2\cdot 3^\alpha m}\left(2^3 3^{\alpha+1} mz\right)}{\eta^2(24z)\eta^{3^\alpha m}\left(2^4 3^{\alpha+1} mz\right)} \\ &\equiv \frac{f_{48} f_{2^3 3^{\alpha+1} m}^{2\cdot 3^\alpha m}}{f_{24}^2 f_{2^4\cdot 3^{\alpha+1} m}^{3^\alpha m}} \equiv \sum_{n=0}^{\infty} \overline{a}_{3^\alpha m}(n) q^{24n} \pmod{2^{k+1}}. \end{aligned} \tag{4.7}$$

Next, we will show that $B_{\alpha,m,k}(z)$ is a modular form. Applying Theorem 4.15, we find that the level of $B_{\alpha,m,k}(z)$ is $N = 2^4 3^{\alpha+1} mM$, where $M$ is the smallest positive integer such that

$$2^4 3^{\alpha+1} mM \left(\frac{-2}{24} + \frac{1}{48} + \frac{2\cdot 3^\alpha m + 2^{k+1}}{2^3 3^{\alpha+1} m} + \frac{-3^\alpha m - 2^k}{2^4 3^{\alpha+1} m}\right) \equiv 0 \pmod{24},$$

which implies

$$3 \cdot 2^k M \equiv 0 \pmod{24}.$$

Therefore, $M = 4$ and the level of $B_{\alpha,m,k}(z)$ is $N = 2^6 3^{\alpha+1} m$.

The representatives for the cusps of $\Gamma_0\left(2^6 3^{\alpha+1} m\right)$ are given by fractions $c/d$ where $d | 2^6 3^{\alpha+1} m$ and $\gcd(c, 2^6 3^{\alpha+1} m) = 1$ (see [48, Proposition 2.1]). By Theorem 4.16, $B_{\alpha,m,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$-2\frac{\gcd(d,24)^2}{24} + \frac{\gcd(d,48)^2}{48} + \left(3^\alpha m + 2^k\right)\left(2\frac{\gcd\left(d, 2^3 3^{\alpha+1} m\right)^2}{2^3 3^{\alpha+1} m} - \frac{\gcd\left(d, 2^4 3^{\alpha+1} m\right)^2}{2^4 3^{\alpha+1} m}\right) \geq 0.$$

Equivalently, $B_{\alpha,m,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\mathcal{L} := 3^\alpha m(-4G_1 + G_2 + 4G_3 - 1) + 2^k(4G_3 - 1) \geq 0,$$

where $G_1 = \dfrac{\gcd(d,24)^2}{\gcd\left(d, 2^4 3^{\alpha+1} m\right)^2}$, $G_2 = \dfrac{\gcd(d,48)^2}{\gcd\left(d, 2^4 3^{\alpha+1} m\right)^2}$, and $G_3 = \dfrac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{\gcd\left(d, 2^4 3^{\alpha+1} m\right)^2}$.

Let $d$ be a divisor of $2^6 3^{\alpha+1} m$. We can write $d = 2^{r_1} 3^{r_2} t$ where $0 \leq r_1 \leq 6$, $0 \leq r_2 \leq \alpha+1$, and $t|m$. We now consider the following two cases depending on $r_1$.

Case 1: Let $0 \leq r_1 \leq 3$, $0 \leq r_2 \leq \alpha+1$. Then $G_1 = G_2$, $\frac{1}{3^{2\alpha} t^2} \leq G_1 \leq 1$, and $G_3 = 1$. Therefore $\mathcal{L} = 3^{\alpha+1} m(1 - G_1) + 3 \cdot 2^k \geq 3 \cdot 2^k$.

Case 2: Let $4 \leq r_1 \leq 6$, $0 \leq r_2 \leq \alpha+1$. Then $G_2 = 4G_1$, $\frac{1}{4 \cdot 3^{2\alpha} t^2} \leq G_1 \leq \frac{1}{4}$, and $G_3 = \frac{1}{4}$ which implies $\mathcal{L} = 0$.

Hence, $B_{\alpha,m,k}(z)$ is holomorphic at every cusp $c/d$. The weight of $B_{\alpha,m,k}(z)$ is $\ell = \frac{1}{2}\left(3^\alpha m + 2^k - 1\right)$ which is a positive integer and the associated character is given by

$$\chi_1(\bullet) = \left(\frac{(-1)^\ell 3^{(\alpha+1)\left(3^\alpha m + 2^k\right) - 1} m^{3^\alpha m + 2^k}}{\bullet}\right).$$

Thus, $B_{\alpha,m,k}(z) \in M_\ell\left(\Gamma_0(N), \chi_1\right)$ where $\ell$, $N$, and $\chi_1$ are as above. Therefore, by Theorem 4.17, the Fourier coefficients of $B_{\alpha,m,k}(z)$ are almost divisible by $r = 2^k$. Due to (4.7), this holds for $\overline{a}_{3^\alpha m}(n)$ also. This completes the proof of Theorem 4.3.

## 4.4 Proof of Theorem 4.4

We proceed along the same lines as in the proof of Theorem 4.3. Here, we define

$$C_{\alpha,m}(z) := \frac{\eta^3\left(2^4 3^{\alpha+1} m z\right)}{\eta\left(2^4 3^{\alpha+2} m z\right)}.$$

Using the binomial theorem, for any integer $k \geq 1$, we have

$$C_{\alpha,m}^{3^k}(z) = \frac{\eta^{3^{k+1}}\left(2^4 3^{\alpha+1} m z\right)}{\eta^{3^k}\left(2^4 3^{\alpha+2} m z\right)} \equiv 1 \pmod{3^{k+1}}. \tag{4.8}$$

Next, we define

$$\begin{aligned} D_{\alpha,m,k}(z) &:= \frac{\eta(48z)\eta^{2\cdot 3^\alpha m}\left(2^3 3^{\alpha+1} m z\right)}{\eta^2(24z)\eta^{3^\alpha m}\left(2^4 3^{\alpha+1} m z\right)} C_{\alpha,m}^{3^k}(z) \\ &= \frac{\eta(48z)\eta^{2\cdot 3^\alpha m}\left(2^3 3^{\alpha+1} m z\right)\eta^{3^{k+1} - 3^\alpha m}\left(2^4 3^{\alpha+1} m z\right)}{\eta^2(24z)\eta^{3^k}\left(2^4 3^{\alpha+2} m z\right)}. \end{aligned}$$

From (4.5) and (4.8), we have

$$D_{\alpha,m,k}(z) \equiv \frac{\eta(48z)\eta^{2\cdot 3^\alpha m}\left(2^3 3^{\alpha+1} m z\right)}{\eta^2(24z)\eta^{3^\alpha m}\left(2^4 3^{\alpha+1} m z\right)}$$

$$\equiv \frac{f_{48} f_{2^3 3^{\alpha+1} m}^{2\cdot 3^\alpha m}}{f_{24}^2 f_{2^4\cdot 3^{\alpha+1} m}^{3^\alpha m}} \equiv \sum_{n=0}^{\infty} \overline{a}_{3^\alpha m}(n) q^{24n} \pmod{3^{k+1}}. \tag{4.9}$$

We now prove that $D_{\alpha,m,k}(z)$ is a modular form. Applying Theorem 4.15, we find that the level of $D_{\alpha,m,k}(z)$ is $N = 2^4 3^{\alpha+2} m M$, where $M$ is the smallest positive integer such that

$$2^4 3^{\alpha+2} m M \left( \frac{-2}{24} + \frac{1}{48} + \frac{2\cdot 3^\alpha m}{2^3 3^{\alpha+1} m} + \frac{3^{k+1} - 3^\alpha m}{2^4 3^{\alpha+1} m} + \frac{-3^k}{2^4 3^{\alpha+2} m} \right) \equiv 0 \pmod{24},$$

which gives

$$8\cdot 3^k M \equiv 0 \pmod{24}.$$

Therefore, $M = 1$ and the level of $D_{\alpha,m,k}(z)$ is $N = 2^4 3^{\alpha+2} m$.

The representatives for the cusps of $\Gamma_0(2^4 3^{\alpha+2} m)$ are given by fractions $c/d$ where $d | 2^4 3^{\alpha+2} m$ and $\gcd(c, 2^4 3^{\alpha+2} m) = 1$. By using Theorem 4.16, $D_{\alpha,m,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\begin{aligned} &-2\frac{\gcd(d,24)^2}{24} + \frac{\gcd(d,48)^2}{48} + 2\cdot 3^\alpha m \frac{\gcd\left(d, 2^3 3^{\alpha+1} m\right)^2}{2^3 3^{\alpha+1} m} \\ &+ \left(3^{k+1} - 3^\alpha m\right) \frac{\gcd\left(d, 2^4 3^{\alpha+1} m\right)^2}{2^4 3^{\alpha+1} m} - 3^k \frac{\gcd\left(d, 2^4 3^{\alpha+2} m\right)^2}{2^4 3^{\alpha+2} m} \geq 0. \end{aligned}$$

Equivalently, $D_{\alpha,m,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\mathcal{L} := 3^{\alpha+1} m \left(-4G_1 + G_2 + 4G_3 - G_4\right) + 3^k (9G_4 - 1) \geq 0,$$

where $G_1 = \dfrac{\gcd(d,24)^2}{\gcd\left(d, 2^4 3^{\alpha+2} m\right)^2}$, $G_2 = \dfrac{\gcd(d,48)^2}{\gcd\left(d, 2^4 3^{\alpha+2} m\right)^2}$, $G_3 = \dfrac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{\gcd\left(d, 2^4 3^{\alpha+2} m\right)^2}$, and $G_4 = \dfrac{\gcd(d, 2^4 3^{\alpha+1} m)^2}{\gcd\left(d, 2^4 3^{\alpha+2} m\right)^2}$.

Let $d$ be a divisor of $2^4 3^{\alpha+2} m$. We write $d = 2^{r_1} 3^{r_2} t$ where $0 \leq r_1 \leq 4$, $0 \leq r_2 \leq \alpha + 2$, and $t | m$. We now consider the following four cases depending on the values of $r_1$ and $r_2$.

Case 1: Let $0 \leq r_1 \leq 3$, $0 \leq r_2 \leq \alpha + 1$. Then $G_1 = G_2$, $\dfrac{1}{3^{2\alpha} t^2} \leq G_1 \leq 1$, and $G_3 = G_4 = 1$. Hence, we have $\mathcal{L} = 3^{\alpha+2} m (1 - G_1) + 8\cdot 3^k \geq 8\cdot 3^k$.

Case 2: Let $0 \leq r_1 \leq 3$, $r_2 = \alpha + 2$. Then $G_1 = G_2$, $\dfrac{1}{3^{2(\alpha+1)} t^2} \leq G_1 \leq \dfrac{1}{3^{2(\alpha+1)}}$, and $G_3 = G_4 = \dfrac{1}{9}$. Therefore, $\mathcal{L} = 3^{\alpha+2} m \left(\dfrac{1}{9} - G_1\right) \geq 0$.

Case 3: Let $r_1 = 4$, $0 \leq r_2 \leq \alpha + 1$. Then $G_2 = 4G_1$, $\dfrac{1}{4\cdot 3^{(\alpha+1)} t^2} \leq G_1 \leq \dfrac{1}{4}$,

$G_4 = 4G_3$. and $G_3 = \dfrac{1}{4}$. Hence, we have $\mathcal{L} = 8 \cdot 3^k$.

Case 4: Let $r_1 = 4$, $r_2 = \alpha + 2$. Then $G_2 = 4G_1$, $\dfrac{1}{4 \cdot 3^{(\alpha+1)}t^2} \leq G_1 \leq \dfrac{1}{4 \cdot 3^{2(\alpha+1)}}$, $G_4 = 4G_3$, and $G_3 = \dfrac{1}{36}$. Therefore, $\mathcal{L} = 0$.

Therefore, $D_{\alpha,m,k}(z)$ is holomorphic at every cusp $c/d$. The weight of $D_{\alpha,m,k}(z)$ is $\ell = \dfrac{3^\alpha m - 1}{2} + 3^k$ which is a positive integer and the associated character is given by

$$\chi_2(\bullet) = \left( \frac{(-1)^\ell 3^{2\alpha 3^k + 3^\alpha \alpha m + 3^\alpha m + 3^k - 1} m^{3^\alpha m + 2 \cdot 3^k}}{\bullet} \right).$$

Thus, $D_{\alpha,m,k}(z) \in M_\ell\left(\Gamma_0(N), \chi_2\right)$ where $\ell$, $N$, and $\chi_2$ are as above. Therefore, by Theorem 4.17, the Fourier coefficients of $D_{\alpha,m,k}(z)$ are almost divisible by $r = 3^k$. Due to (4.9), this holds for $\overline{a}_{3^\alpha m}(n)$ also. This completes the proof of Theorem 4.4.

## 4.5 Proof of Theorem 4.5

Putting $t = 3^\alpha m$ in (1.26), we have

$$\sum_{n=0}^{\infty} \overline{b}_{3^\alpha m}(n) q^n = \frac{f_2 f_{3^\alpha m}^{3^\alpha m} f_{4 \cdot 3^\alpha m}^{3^\alpha m}}{f_1 f_4 f_{2 \cdot 3^\alpha m}^{3^\alpha m}}. \tag{4.10}$$

We define

$$E_{\alpha,m}(z) := \frac{\eta^2\left(2^4 3^{\alpha+1} m z\right)}{\eta\left(2^5 3^{\alpha+1} m z\right)}.$$

Applying the binomial theorem, for any integer $k \geq 1$, we have

$$E_{\alpha,m}^{2^k}(z) = \frac{\eta^{2^{k+1}}\left(2^4 3^{\alpha+1} m z\right)}{\eta^{2^k}\left(2^5 3^{\alpha+1} m z\right)} \equiv 1 \pmod{2^{k+1}}. \tag{4.11}$$

Next, we define

$$\begin{aligned} F_{\alpha,m,k}(z) &:= \frac{\eta(48z) \eta^{3^\alpha m}\left(2^3 3^{\alpha+1} m z\right) \eta^{3^\alpha m}\left(2^5 3^{\alpha+1} m z\right)}{\eta(24z) \eta(96z) \eta^{3^\alpha m}\left(2^4 3^{\alpha+1} m z\right)} E_{\alpha,m}^{2^k}(z) \\ &= \frac{\eta(48z) \eta^{3^\alpha m}\left(2^3 3^{\alpha+1} m z\right) \eta^{2^{k+1} - 3^\alpha m}\left(2^4 3^{\alpha+1} m z\right)}{\eta(24z) \eta(96z) \eta^{2^k - 3^\alpha m}\left(2^5 3^{\alpha+1} m z\right)}. \end{aligned}$$

Using (4.10) and (4.11), we obtain

$$F_{\alpha,m,k}(z) \equiv \frac{\eta(48z) \eta^{3^\alpha m}\left(2^3 3^{\alpha+1} m z\right) \eta^{3^\alpha m}\left(2^5 3^{\alpha+1} m z\right)}{\eta(24z) \eta(96z) \eta^{3^\alpha m}\left(2^4 3^{\alpha+1} m z\right)}$$

$$\equiv q^{3\left(3^{2\alpha}m^2-1\right)}\frac{f_{48}f_{2^33^{\alpha+1}m}^{3^\alpha m}f_{2^53^{\alpha+1}m}^{3^\alpha m}}{f_{24}f_{96}f_{2^43^{\alpha+1}m}^{3^\alpha m}}$$
$$\equiv \sum_{n=0}^{\infty}\overline{b}_{3^\alpha m}(n)q^{24n+3\left(3^{2\alpha}m^2-1\right)} \pmod{2^{k+1}}. \tag{4.12}$$

Now, we will show that $F_{\alpha,m,k}(z)$ is a modular form. Applying Theorem 4.15, we find that the level of $F_{\alpha,m,k}(z)$ is $N = 2^53^{\alpha+1}mM$, where $M$ is the smallest positive integer such that

$$2^53^{\alpha+1}mM\left(\frac{-1}{24}+\frac{1}{48}+\frac{-1}{96}+\frac{3^\alpha m}{2^33^{\alpha+1}m}+\frac{2^{k+1}-3^\alpha m}{2^43^{\alpha+1}m}+\frac{-2^k+3^\alpha m}{2^53^{\alpha+1}m}\right)\equiv 0 \pmod{24},$$

which implies

$$3\cdot 2^kM\equiv 0 \pmod{24}.$$

Therefore, $M = 4$ and the level of $F_{\alpha,m,k}(z)$ is $N = 2^73^{\alpha+1}m$.

The cusps of $\Gamma_0\left(2^73^{\alpha+1}m\right)$ are given by fractions $c/d$ where $d|2^73^{\alpha+1}m$ and $\gcd(c,d) = 1$. By Theorem 4.16, $F_{\alpha,m,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$-\frac{\gcd(d,24)^2}{24}+\frac{\gcd(d,48)^2}{48}-\frac{\gcd(d,96)^2}{96}+3^\alpha m\frac{\gcd\left(d,2^33^{\alpha+1}m\right)^2}{2^33^{\alpha+1}m}$$
$$+\left(2^{k+1}-3^\alpha m\right)\frac{\gcd\left(d,2^43^{\alpha+1}m\right)^2}{2^43^{\alpha+1}m}-\left(2^k-3^\alpha m\right)\frac{\gcd\left(d,2^53^{\alpha+1}m\right)^2}{2^53^{\alpha+1}m}\geq 0.$$

Equivalently, $F_{\alpha,m,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\mathcal{L} := 3^\alpha m(-4G_1+2G_2-G_3+4G_4-2G_5+1)+2^k(4G_5-1)\geq 0,$$

where $G_1 = \dfrac{\gcd(d,24)^2}{\gcd\left(d,2^53^{\alpha+1}m\right)^2}$, $G_2 = \dfrac{\gcd(d,48)^2}{\gcd\left(d,2^53^{\alpha+1}m\right)^2}$, $G_3 = \dfrac{\gcd(d,96)^2}{\gcd\left(d,2^53^{\alpha+1}m\right)^2}$,

$G_4 = \dfrac{\gcd(d,2^33^{\alpha+1}m)^2}{\gcd\left(d,2^53^{\alpha+1}m\right)^2}$, and $G_5 = \dfrac{\gcd(d,2^43^{\alpha+1}m)^2}{\gcd\left(d,2^53^{\alpha+1}m\right)^2}$.

Let $d$ be a divisor of $2^73^{\alpha+1}m$. We can write $d = 2^{r_1}3^{r_2}t$ where $0\leq r_1\leq 7$, $0\leq r_2\leq \alpha+1$, and $t|m$. We now consider the following three cases depending on $r_1$.

Case 1: Let $0\leq r_1\leq 3$, $0\leq r_2\leq\alpha+1$. Then $G_1 = G_2 = G_3$, $\dfrac{1}{3^{2\alpha}t^2}\leq G_1\leq 1$, and $G_4 = G_5 = 1$. Hence, $\mathcal{L} = 3^{\alpha+1}m\left(1-G_1\right)+3\cdot 2^k\geq 3\cdot 2^k$.

Case 2: Let $r_1 = 4$, $0\leq r_2\leq\alpha+1$. Then $G_3 = G_2 = 4G_1$, $\dfrac{1}{4\cdot 3^{2\alpha}t^2}\leq G_1\leq\dfrac{1}{4}$, $G_5 = 4G_4$, and $G_4 = \dfrac{1}{4}$. Hence, $\mathcal{L} = 3\cdot 2^k$.

Case 3: Let $5 \leq r_1 \leq 7$, $0 \leq r_2 \leq \alpha+1$. Then $G_3 = 4G_2 = 16G_1$, $\frac{1}{16 \cdot 3^{2\alpha} t^2} \leq G_1 \leq \frac{1}{16}$, $G_5 = 4G_4$, and $G_4 = \frac{1}{16}$. Hence, $\mathcal{L} = 12 \cdot 3^{\alpha} m \left(\frac{1}{16} - G_1\right) \geq 0$.

This proves that $F_{\alpha,m,k}(z)$ is holomorphic at every cusp $c/d$. The weight of $F_{\alpha,m,k}(z)$ is $\ell = \frac{1}{2}\left(3^{\alpha}m + 2^k - 1\right)$, which is a positive integer and the associated character is given by

$$\chi_3(\bullet) = \left(\frac{(-1)^{\ell} 2^{4 \cdot 3^{\alpha}m + 3 \cdot 2^k - 4} 3^{(\alpha+1)\left(3^{\alpha}m + 2^k\right) - 1} m^{3^{\alpha}m + 2^k}}{\bullet}\right).$$

Thus, $F_{\alpha,m,k}(z) \in M_{\ell}\left(\Gamma_0(N), \chi_3\right)$ where $\ell$, $N$, and $\chi_3$ are as above. Therefore, by Theorem 4.17, the Fourier coefficients of $F_{\alpha,m,k}(z)$ are almost divisible by $r = 2^k$. Due to (4.12), this holds for $\overline{b}_{3^{\alpha}m}(n)$ also. This completes the proof of Theorem 4.5.

## 4.6 Proof of Theorem 4.6

We proceed along the same lines as in the proof of Theorem 4.5. Here, we define

$$G_{\alpha,m}(z) := \frac{\eta^3\left(2^5 3^{\alpha+1} mz\right)}{\eta\left(2^5 3^{\alpha+2} mz\right)}.$$

Using the binomial theorem, for any integer $k \geq 1$, we have

$$G_{\alpha,m}^{3^k}(z) = \frac{\eta^{3^{k+1}}\left(2^5 3^{\alpha+1} mz\right)}{\eta^{3^k}\left(2^5 3^{\alpha+2} mz\right)} \equiv 1 \pmod{3^{k+1}}. \tag{4.13}$$

Next, we define

$$\begin{aligned} H_{\alpha,m,k}(z) &:= \frac{\eta(48z)\eta^{3^{\alpha}m}\left(2^3 3^{\alpha+1} mz\right)\eta^{3^{\alpha}m}\left(2^5 3^{\alpha+1} mz\right)}{\eta(24z)\eta(96z)\eta^{3^{\alpha}m}\left(2^4 3^{\alpha+1} mz\right)} G_{\alpha,m}^{3^k}(z) \\ &= \frac{\eta(48z)\eta^{3^{\alpha}m}\left(2^3 3^{\alpha+1} mz\right)\eta^{3^{k+1}+3^{\alpha}m}\left(2^5 3^{\alpha+1} mz\right)}{\eta(24z)\eta(96z)\eta^{3^{\alpha}m}\left(2^4 3^{\alpha+1} mz\right)\eta^{3^k}\left(2^5 3^{\alpha+2} mz\right)}. \end{aligned}$$

From (4.10) and (4.13), we have

$$\begin{aligned} H_{\alpha,m,k}(z) &\equiv \frac{\eta(48z)\eta^{3^{\alpha}m}\left(2^3 3^{\alpha+1} mz\right)\eta^{3^{\alpha}m}\left(2^5 3^{\alpha+1} mz\right)}{\eta(24z)\eta(96z)\eta^{3^{\alpha}m}\left(2^4 3^{\alpha+1} mz\right)} \\ &\equiv q^{3\left(3^{2\alpha}m^2 - 1\right)} \frac{f_{48} f_{2^3 3^{\alpha+1} m}^{3^{\alpha}m} f_{2^5 3^{\alpha+1} m}^{3^{\alpha}m}}{f_{24} f_{96} f_{2^4 3^{\alpha+1} m}^{3^{\alpha}m}} \\ &\equiv \sum_{n=0}^{\infty} \overline{b}_{3^{\alpha}m}(n) q^{24n + 3\left(3^{2\alpha}m^2 - 1\right)} \pmod{3^{k+1}}. \end{aligned} \tag{4.14}$$

Next, we will prove that $H_{\alpha,m,k}(z)$ is a modular form. Applying Theorem 4.15, we find that the level of $H_{\alpha,m,k}(z)$ is $N = 2^5 3^{\alpha+2} mM$, where $M$ is the smallest positive integer such that

$$2^5 3^{\alpha+2} mM \left( \frac{-1}{24} + \frac{1}{48} + \frac{-1}{96} + \frac{3^\alpha m}{2^3 3^{\alpha+1} m} + \frac{-3^\alpha m}{2^4 3^{\alpha+1} m} + \frac{3^{k+1} + 3^\alpha m}{2^5 3^{\alpha+1} m} + \frac{-3^k}{2^5 3^{\alpha+2} m} \right) \equiv 0 \pmod{24},$$

which gives

$$8 \cdot 3^k M \equiv 0 \pmod{24}.$$

Thus, $M = 1$ and the level of $H_{\alpha,m,k}(z)$ is $N = 2^5 3^{\alpha+2} m$.

The cusps of $\Gamma_0(2^5 3^{\alpha+2} m)$ are given by fractions $c/d$ where $d | 2^5 3^{\alpha+2} m$ and $\gcd(c, d) = 1$. By Theorem 4.16, $H_{\alpha,m,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$-\frac{\gcd(d,24)^2}{24} + \frac{\gcd(d,48)^2}{48} - \frac{\gcd(d,96)^2}{96} + 3^a m \frac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{2^3 3^{\alpha+1} m} - 3^\alpha m \frac{\gcd(d, 2^4 3^{\alpha+1} m)^2}{2^4 3^{\alpha+1} m} + \left(3^\alpha m + 3^{k+1}\right) \frac{\gcd(d, 2^5 3^{\alpha+1} m)^2}{2^5 3^{\alpha+1} m} - 3^k \frac{\gcd(d, 2^5 3^{\alpha+2} m)^2}{2^5 3^{\alpha+2} m} \geq 0.$$

Equivalently, $H_{\alpha,m,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\mathcal{L} := 3^{\alpha+1} m(-4G_1 + 2G_2 - G_3 + 4G_4 - 2G_5 + G_6) + 3^k(9G_6 - 1) \geq 0,$$

where $G_1 = \dfrac{\gcd(d,24)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}$, $G_2 = \dfrac{\gcd(d,48)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}$, $G_3 = \dfrac{\gcd(d,96)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}$, $G_4 = \dfrac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}$, $G_5 = \dfrac{\gcd(d, 2^4 3^{\alpha+1} m)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}$, and $G_6 = \dfrac{\gcd(d, 2^5 3^{\alpha+1} m)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}$.

Let $d$ be a divisor of $2^5 3^{\alpha+2} m$. We can write $d = 2^{r_1} 3^{r_2} t$ where $0 \leq r_1 \leq 5$, $0 \leq r_2 \leq \alpha + 2$ and $t | m$. We now consider the following six cases depending on $r_1$ and $r_2$.

Case 1: Let $0 \leq r_1 \leq 3$, $0 \leq r_2 \leq \alpha + 1$. Then $G_1 = G_2 = G_3$, $\dfrac{1}{3^{2\alpha} t^2} \leq G_1 \leq 1$, and $G_4 = G_5 = G_6 = 1$. Therefore, $\mathcal{L} = 3^{\alpha+2} m(1 - G_1) + 8 \cdot 3^k \geq 8 \cdot 3^k$.

Case 2: Let $0 \leq r_1 \leq 3$, $r_2 = \alpha + 2$. Then $G_1 = G_2 = G_3$, $\dfrac{1}{3^{2(\alpha+1)} t^2} \leq G_1 \leq \dfrac{1}{3^{2(\alpha+1)}}$, and $G_4 = G_5 = G_6 = \dfrac{1}{9}$. Therefore, $\mathcal{L} = 3^{\alpha+2} m \left( \dfrac{1}{9} - G_1 \right) \geq 0$.

Case 3: Let $r_1 = 4$, $0 \leq r_2 \leq \alpha + 1$. Then $G_3 = G_2 = 4G_1$, $\dfrac{1}{4 \cdot 3^{2\alpha} t^2} \leq G_1 \leq \dfrac{1}{4}$, $G_5 = G_6 = 4G_4$, and $G_4 = \dfrac{1}{4}$. Hence, $\mathcal{L} = 8 \cdot 3^k$.

Case 4: Let $r_1 = 4$, $r_2 = \alpha + 2$. Then $G_3 = G_2 = 4G_1$, $\dfrac{1}{4 \cdot 3^{2(\alpha+1)}t^2} \leq G_1 \leq \dfrac{1}{4 \cdot 3^{2(\alpha+1)}}$, $G_5 = G_6 = 4G_4$, and $G_4 = \dfrac{1}{36}$. Hence, $\mathcal{L} = 0$.

Case 5: Let $r_1 = 5$, $0 \leq r_2 \leq \alpha + 1$. Then $G_3 = 4G_2 = 16G_1$, $\dfrac{1}{16 \cdot 3^{2\alpha}t^2} \leq G_1 \leq \dfrac{1}{16}$, $G_6 = 4G_5 = 16G_4$, and $G_4 = \dfrac{1}{16}$. Therefore, $\mathcal{L} = 12 \cdot 3^{\alpha+1}m\left(\dfrac{1}{16} - G_1\right) + 8 \cdot 3^k \geq 8 \cdot 3^k$.

Case 6: Let $r_1 = 5$, $r_2 = \alpha + 2$. Then $G_3 = 4G_2 = 16G_1$, $\dfrac{1}{16 \cdot 3^{2(\alpha+1)}t^2} \leq G_1 \leq \dfrac{1}{16 \cdot 3^{2(\alpha+1)}}$, $G_6 = 4G_5 = 16G_4$, and $G_4 = \dfrac{1}{144}$. Therefore, $\mathcal{L} = 12 \cdot 3^{\alpha+1}m\left(\dfrac{1}{144} - G_1\right) \geq 0$.

This proves that $H_{\alpha,m,k}(z)$ is holomorphic at every cusp $c/d$. The weight of $H_{\alpha,m,k}(z)$ is $\ell = \dfrac{3^\alpha m - 1}{2} + 3^k$, which is a positive integer and the associated character is given by

$$\chi_4(\bullet) = \left(\frac{(-1)^\ell 2^{4\cdot 3^\alpha m + 10\cdot 3^k - 4} 3^{2\alpha 3^k + 3^\alpha \alpha m + 3^\alpha m + 3^k - 1} m^{3^\alpha m + 2\cdot 3^k}}{\bullet}\right).$$

Thus, $H_{\alpha,m,k}(z) \in M_\ell\left(\Gamma_0(N), \chi_4\right)$ where $\ell$, $N$, and $\chi_4$ are as above. Therefore, by Theorem 4.17, the Fourier coefficients of $H_{\alpha,m,k}(z)$ are almost divisible by $r = 3^k$. Due to (4.14), the same holds for $\overline{b}_{3^\alpha m}(n)$ also. This completes the proof of Theorem 4.6.

## 4.7 Proof of Theorem 4.7

Consider $t = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where $p_i$'s are primes. Then we have

$$\sum_{n=0}^{\infty} \overline{b}_t(n) q^n = \frac{f_2 f_t^t f_{4t}^t}{f_1 f_4 f_{2t}^t}. \tag{4.15}$$

For a positive integer $i$, we define

$$K_i(z) := \frac{\eta^{p_i^{a_i}}(24z)}{\eta\left(24 p_i^{a_i} z\right)}$$

In view of the binomial theorem, for any integer $k \geq 1$, we have

$$K_i^{p_i^k}(z) = \frac{\eta^{p_i^{a_i+k}}(24z)}{\eta^{p_i^k}\left(24 p_i^{a_i} z\right)} \equiv 1 \pmod{p_i^{k+1}}. \tag{4.16}$$

Define

$$L_{i,k,t}(z) := \frac{\eta(48z)\,\eta^t(24tz)\,\eta^t(96tz)}{\eta(24z)\,\eta(96z)\,\eta^t(48tz)} A_i^{p_i^k}(z)$$
$$= \frac{\eta^{p_i^{a_i+k}-1}(24z)\,\eta(48z)\,\eta^t(24tz)\,\eta^t(96tz)}{\eta(96z)\,\eta^t(48tz)\,\eta^{p_i^k}(24p_i^{a_i})}.$$

From (4.15) and (4.16), we arrive at

$$\begin{aligned} L_{i,k,t}(z) &\equiv \frac{\eta(48z)\,\eta^t(24tz)\,\eta^t(96tz)}{\eta(24z)\,\eta(96z)\,\eta^t(48tz)} \\ &\equiv q^{3(t^2-1)}\frac{f_{48}f_{24t}^t f_{96t}^t}{f_{24}f_{96}f_{48t}^t} \\ &\equiv \sum_{n=0}^{\infty} \overline{b}_t(n) q^{24n+3(t^2-1)} \pmod{p_i^{k+1}}. \end{aligned} \tag{4.17}$$

Next, we show that $L_{i,k,t}(z)$ is a modular form. Applying Theorem 4.15, we first estimate the level of eta quotient $B_{i,k,t}(z)$ . The level of $L_{i,k,t}(z)$ is $N = 96tM$, where $M$ is the smallest positive integer which satisfies

$$96tM\left(\frac{p_i^{a_i+k}-1}{24}+\frac{1}{48}+\frac{-1}{96}+\frac{t}{24t}+\frac{-t}{48t}+\frac{t}{96t}+\frac{-p_i^k}{24p_i^{a_i}}\right) \equiv 0 \pmod{24},$$

which gives

$$4tM\ p_i^k\left(p_i^{a_i}-\frac{1}{p_i^{a_i}}\right) \equiv 0 \pmod{24}.$$

Hence, $M = 6$ and $N = 2^6 3^2 t$.

The cusps of $\Gamma_0(2^6 3^2 t)$ are given by fractions $c/d$ where $d | 2^6 3^2 t$ and $\gcd(c, d) = 1$. By Theorem 4.16, $L_{i,k,t}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\left(p_i^{a_i+k}-1\right)\frac{\gcd(d,24)^2}{24}+\frac{\gcd(d,48)^2}{48}-\frac{\gcd(d,96)^2}{96}-p_i^k\frac{\gcd\left(d,24p_i^{a_i}\right)^2}{24p_i^{a_i}}$$
$$+t\,\frac{\gcd(d,24t)^2}{24t}-t\,\frac{\gcd(d,48t)^2}{48t}+\frac{\gcd(d,96t)^2}{96t} \geq 0.$$

Equivalently, $L_{i,k,t}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\mathcal{L} := -4G_1 + 2G_2 - G_3 + 4G_4 - 2G_5 + 1 + 4\left(p_i^{k+a_i}G_1 - p_i^{k-a_i}G_6\right) \geq 0,$$

where $G_1 = \frac{\gcd(d,24)^2}{\gcd(d,96t)^2}$, $G_2 = \frac{\gcd(d,48)^2}{\gcd(d,96t)^2}$, $G_3 = \frac{\gcd(d,96)^2}{\gcd(d,96t)^2}$, $G_4 = \frac{\gcd(d,24t)^2}{\gcd(d,96t)^2}$, $G_5 = \frac{\gcd(d,48t)^2}{\gcd(d,96t)^2}$, and $G_6 = \frac{\gcd\left(d,24p_i^{a_i}\right)^2}{\gcd(d,96t)^2}$.

Let $d$ be a divisor of $2^6 3^2 t$. We can write $d = 2^{r_1} 3^{r_2} p_i^s u$ where $0 \leq r_1 \leq 6$, $0 \leq r_2 \leq 2$, $0 \leq s \leq a_i$, and $u | t$ but $p_i \nmid u$. We now consider the following three cases depending on $r_1$.

Case 1: Let $0 \le r_1 \le 3$, $0 \le r_2 \le 2$. Then $G_1 = G_2 = G_3 = \frac{1}{u^2 p_i^{2s}}$, $G_4 = G_5 = 1$, and $G_6 = \frac{1}{u^2}$. Therefore,

$$\mathcal{L} = 3\left(1 - \frac{1}{u^2 p_i^{2s}}\right) + 4\left(\frac{p_i^{k+a_i}}{u^2 p_i^{2s}} - \frac{p_i^{k-a_i}}{u^2}\right) = 3\left(1 - \frac{1}{u^2 p_i^{2s}}\right) + 4\frac{p_i^k}{u^2}\left(\frac{p_i^{2a_i} - p_i^{2s}}{p_i^{2s+a_i}}\right).$$

Since $s \le a_i$, we have $\mathcal{L} \ge 0$.

Case 2: Let $r_1 = 4$, $0 \le r_2 \le 2$. Then $G_3 = G_2 = 4G_1$, $G_1 = \frac{1}{4u^2 p_i^{2s}}$, $G_5 = 4G_4$, $G_4 = \frac{1}{4}$, and $G_6 = \frac{1}{4u^2}$. Therefore,

$$\mathcal{L} = 4\left(\frac{p_i^{k+a_i}}{4u^2 p_i^{2s}} - \frac{p_i^{k-a_i}}{4u^2}\right) = \frac{p_i^k}{u^2}\left(\frac{p_i^{2a_i} - p_i^{2s}}{p_i^{2s+a_i}}\right) \ge 0.$$

Case 3: Let $5 \le r_1 \le 6$, $0 \le r_2 \le 2$. Then $G_3 = 4G_2 = 16G_1$, $G_1 = \frac{1}{16u^2 p_i^{2s}}$, $G_5 = 4G_4$, $G_4 = \frac{1}{16}$, and $G_6 = \frac{1}{16u^2}$. Hence,

$$\mathcal{L} = \frac{3}{4}\left(1 - \frac{1}{u^2 p_i^{2s}}\right) + 4\left(\frac{p_i^{k+a_i}}{16u^2 p_i^{2s}} - \frac{p_i^{k-a_i}}{16u^2}\right) = \frac{3}{4}\left(1 - \frac{1}{u^2 p_i^{2s}}\right) + \frac{p_i^k}{4u^2}\left(\frac{p_i^{2a_i} - p_i^{2s}}{p_i^{2s+a_i}}\right) \ge 0.$$

Therefore, $L_{i,k,t}(z)$ is holomorphic at every cusp $c/d$. The weight of $L_{i,k,t}(z)$ is $\ell = \frac{1}{2}\left(p_i^k\left(p_i^{a_i} - 1\right) + t - 1\right)$, which is a positive integer and the associated character is given by

$$\chi_5(\bullet) = \left(\frac{(-1)^\ell 2^{3p_i^{a_i+k} - 3p_i^k + 4t - 4} 3^{p_i^{a_i+k} - p^k + t - 1} t^t \left(p_i^{a_i}\right)^{-p_i^k}}{\bullet}\right).$$

Hence, $L_{i,k,t}(z) \in M_\ell\left(\Gamma_0(N), \chi_5\right)$ where $\ell$, $N$, and $\chi_5$ are as above. Therefore, by Theorem 4.17, the Fourier coefficients of $B_{i,k,t}(z)$ are almost divisible by $r = p_i^k$. Due to (4.17), this holds for $\overline{b}_t(n)$ also. Thus, we complete the proof of Theorem 4.7.

## 4.8 Proofs of Theorems 4.9 and 4.10

First we recall the following result of Ono and Taguchi [88] on the nilpotency of Hecke operators.

**Theorem 4.18.** *[88, Theorem 1.3 (3)] Let $n$ be a nonnegative integer and $k$ be a positive integer. Let $\chi$ be a quadratic Dirichlet character of conductor $9 \cdot 2^a$. Then there is an integer $c \geq 0$ such that for every $f(z) \in M_P(\Gamma_0(9 \cdot 2^a), \chi) \cap \mathbb{Z}[[q]]$ and every $t \geq 1$,*

$$f(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_{c+t}} \equiv 0 \pmod{2^t}$$

*whenever the primes $p_1, \ldots, p_{c+t}$ are coprime to 6.*

Now, we apply the above theorem to the modular forms $B_{1,1,k}(z)$ and $F_{1,1,k}(z)$ to prove Theorem 4.9 and 4.10, respectively.

*Proof of Theorem 4.9.* Putting $\alpha = 1$ and $m = 1$ in (4.7), we find that

$$B_{1,1,k}(z) \equiv \sum_{n=0}^{\infty} \overline{a}_3(n)q^{24n} \pmod{2^{k+1}},$$

which yields

$$B_{1,1,k}(z) := \sum_{n=0}^{\infty} \mathcal{B}_k(n)\, q^n \equiv \sum_{n=0}^{\infty} \overline{a}_3\left(\frac{n}{24}\right) q^n \pmod{2^{k+1}}. \tag{4.18}$$

Now, $B_{1,1,k}(z) \in M_{2^{k-1}+1}(\Gamma_0(9 \cdot 2^6), \chi_6)$ for $k \geq 1$ where $\chi_6$ is the associated character (which is $\chi_1$ evaluated at $\alpha = 1$ and $m = 1$). In view of Theorem 4.18, we find that there is an integer $c \geq 0$ such that for any $d \geq 1$,

$$B_{1,1,k}(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_{c+d}} \equiv 0 \pmod{2^d}$$

whenever $p_1, \ldots, p_{c+d}$ are coprime to 6. It follows from the definition of Hecke operators that if $p_1, \ldots, p_{c+d}$ are distinct primes and if $n$ is coprime to $p_1 \cdots p_{c+d}$, then

$$\mathcal{B}_k(p_1 \cdots p_{c+d} \cdot n) \equiv 0 \pmod{2^d}. \tag{4.19}$$

Combining (4.18) and (4.19), we complete the proof of the theorem.

□

*Proof of Theorem 4.10.* Taking $\alpha = 1$ and $m = 1$ in (4.12), we have

$$F_{1,1,k}(z) \equiv \sum_{n=0}^{\infty} \overline{b}_3(n)q^{24n+24} \pmod{2^{k+1}},$$

which yields

$$F_{1,1,k}(z) := \sum_{n=0}^{\infty} \mathcal{F}_k\,(n)\,q^n \equiv \sum_{n=0}^{\infty} \bar{b}_3\left(\frac{n-24}{24}\right) q^n \pmod{2^{k+1}}. \tag{4.20}$$

Now, $F_{1,1,k}(z) \in M_{2^{k-1}+1}\left(\Gamma_0(9\cdot 2^7), \chi_7\right)$ for $k \geq 1$ where $\chi_7$ is the associated character (which is $\chi_3$ evaluated at $\alpha = 1$ and $m = 1$). In view of Theorem 4.18, we find that there is an integer $u \geq 0$ such that for any $v \geq 1$,

$$F_{1,1,k}(z) \mid T_{q_1} \mid T_{q_2} \mid \cdots \mid T_{q_{u+v}} \equiv 0 \pmod{2^v}$$

whenever $q_1, \ldots, q_{c+d}$ are coprime to 6. From the definition of Hecke operators, we have that if $q_1, \ldots, q_{u+v}$ are distinct primes and if $n$ is coprime to $q_1 \cdots q_{u+v}$, then

$$\mathcal{F}_k\,(q_1 \cdots q_{u+v} \cdot n) \equiv 0 \pmod{2^v}. \tag{4.21}$$

Combining (4.20) and (4.21), we complete the proof of the theorem. □

## 4.9 Proofs of Theorems 4.11–4.13

First, we prove Theorem 4.11.

*Proof of Theorem 4.11.* From (1.26), we have

$$\sum_{n=0}^{\infty} \bar{b}_t(n) q^n = \frac{f_2 f_t^t f_{4t}^t}{f_1 f_4 f_{2t}^t},$$

which under modulo 2 reduces to

$$\sum_{n=0}^{\infty} \bar{b}_t(n) q^n \equiv \frac{f_t^{3t}}{f_1^3} \equiv \frac{f_{2t}^t}{f_2} \cdot \frac{f_t^t}{f_1} \pmod{2}. \tag{4.22}$$

Using (1.19), the above equation can be rewritten as

$$\sum_{n=0}^{\infty} \bar{b}_t(n) q^n \equiv \left(\sum_{n=0}^{\infty} c_t(n) q^{2n}\right)\left(\sum_{n=0}^{\infty} c_t(n) q^n\right) \pmod{2}.$$

Equating the coefficients of $q^n$ from both sides of the above equation, we find that

$$\bar{b}_t(n) \equiv \sum_{k=0}^{\infty} c_t(k) c_t(n-2k) \pmod{2}.$$

Again, putting $t = 2^k$ in (4.22), we have

$$\sum_{n=0}^{\infty} \bar{b}_{2^k}(n) q^n \equiv \frac{f_{2^k}^{3\cdot 2^k}}{f_1^3} \equiv \frac{f_1^{3\cdot 2^{2k}}}{f_1^3} \equiv f_1^{3(2^{2k}-1)} \pmod{2}.$$

Thus, we complete the proof of Theorem 4.11. $\square$

We prove Theorem 4.12 with the aid of the following classical result due to Landau [81].

**Lemma 4.19.** *Let r(n) and s(n) be quadratic polynomials. Then*

$$\left(\sum_{n\in\mathbb{Z}} q^{r(n)}\right)\left(\sum_{n\in\mathbb{Z}} q^{s(n)}\right)$$

*is lacunary modulo 2.*

*Proof of Theorem 4.12.* From Theorem 4.11, we have

$$\overline{b}_{2^k}(n) \equiv f_1^{3(2^{2k}-1)} \pmod{2}.$$

Putting $t = 1$ in the above, we get

$$\overline{b}_2(n) \equiv f_1^9 \equiv f_1 f_8 \pmod{2}. \tag{4.23}$$

Again, from Euler's pentagonal number theorem [29, Corollary 1.3.5], we have

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} \pmod{2}, \quad \text{where } \mid q \mid < 1. \tag{4.24}$$

Magnifying (4.24) by $q \to q^8$, we find that

$$f_8 \equiv \sum_{n=-\infty}^{\infty} q^{4n(3n+1)} \pmod{2}. \tag{4.25}$$

Combining (4.23), (4.24) and (4.25) and then applying Lemma 4.19, we complete the proof. $\square$

Lastly, we prove Theorem 4.13 using a result from [78].

*Proof of Theorem 4.13.* From (4.23), we find that

$$\overline{b}_2(n) \equiv f_1^9 \equiv f_1^{1+2^3} \pmod{2}.$$

Again, from [78, Theorem 10], we have that if $t = 2^d + 1$ and $p$ is a prime such that $p \equiv 7 \pmod{8}$, then $f_1^t$ is $p^2$-even with base $r \equiv -3(2^{d-3} + 2^{-3}) \pmod{p^2}$. Employing this result with $d = 3$, we arrive at Theorem 4.13. $\square$

## 4.10 Proof of Theorem 4.14

First of all, we prove the following two lemmas.

**Lemma 4.20.** *Let $p \geq 3$ be a prime and $p_9(n)$ be defined by (4.2). We have*

$$p_9\left(p^{2k}n + \frac{3(p^{2k}-1)}{8}\right) \equiv P(p,k)p_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + Q(p,k)p_9(n) \pmod 2, \tag{4.26}$$

*where $P(p,k)$ and $Q(p,k)$ are defined by*

$$P(p,k+2) = \pi(p)P(p,k+1) - P(p,k) \tag{4.27}$$

*and*

$$Q(p,k+2) = \pi(p)Q(p,k+1) - Q(p,k) \tag{4.28}$$

*with $P(p,0) = 0$, $P(p,1) = 1$, $Q(p,0) = 1$ and $Q(p,1) = 0$.*

*Proof.* We will prove the lemma by induction on $k$ using the method of Xia [126] based on an identity of Newman [85] and Lucas sequences. We observe that (4.26) is true for $k = 0$ and $k = 1$ since $P(p,0) = 0$, $P(p,1) = 1$, $Q(p,0) = 1$, and $Q(p,1) = 0$. We now assume that (4.26) is true for $k = m$ and $k = m+1$ for some $m \geq 0$, which gives

$$p_9\left(p^{2m}n + \frac{3(p^{2m}-1)}{8}\right) = P(p,m)p_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + Q(p,m)p_9(n), \tag{4.29}$$

and

$$p_9\left(p^{2m+2}n + \frac{3(p^{2m+2}-1)}{8}\right) = P(p,m+1)p_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + Q(p,m+1)p_9(n). \tag{4.30}$$

Newman [85] proved that if $p \geq 3$ is a prime, then

$$p_9\left(p^2n + \frac{3(p^2-1)}{8}\right) = \chi(n)p_9(n) - p^7 p_9\left(\frac{n - \frac{3(p^2-1)}{8}}{p^2}\right), \tag{4.31}$$

where $\chi(n)$ is given by

$$\chi(n) = p_9\left(\frac{3(p^2-1)}{8}\right) + (-1)^{\frac{(p-1)(p-19)}{8}} p^3 \left(\left(\frac{\frac{3(p^2-1)}{8}}{p}\right)_L - \left(\frac{\frac{3(p^2-1)}{8} - n}{p}\right)_L\right).$$

For any prime $p \geq 3$, we have

$$p^3 \equiv p^7 \equiv 1 \pmod{2}. \tag{4.32}$$

We can easily observe that

$$\chi\left(pn + \frac{3(p^2-1)}{8}\right) = \chi\left(p^2 n + \frac{3(p^2-1)}{8}\right) \equiv \pi(p) \pmod{2}, \tag{4.33}$$

where $\pi(p)$ is given by (4.1).

Replacing $n$ by $p^2 n + \dfrac{3(p^2-1)}{8}$ in (4.31), we have

$$p_9\left(p^4 n + \frac{3(p^4-1)}{8}\right) = \chi\left(p^2 n + \frac{3(p^2-1)}{8}\right) p_9\left(p^2 n + \frac{3(p^2-1)}{8}\right) - p^7 p_9(n).$$

Taking modulo 2 on both sides of above and then employing (4.32) and (4.33), we arrive at

$$p_9\left(p^4 n + \frac{3(p^4-1)}{8}\right) \equiv \pi(p) p_9\left(p^2 n + \frac{3(p^2-1)}{8}\right) - p_9(n) \pmod{2}. \tag{4.34}$$

Again, replacing $n$ by $p^{2m} n + \dfrac{3(p^{2m}-1)}{8}$ in (4.34) and then utilizing (4.29) and (4.30), we find that

$$\begin{aligned}
&p_9\left(p^{2m+4} n + \frac{3(p^{2m+4}-1)}{8}\right)\\
&\equiv \pi(p) p_9\left(p^{2m+2} n + \frac{3(p^{2m+2}-1)}{8}\right) - p_9\left(p^{2m} n + \frac{3(p^{2m}-1)}{8}\right)\\
&\equiv \pi(p)\left(P(p,m+1) p_9\left(p^2 n + \frac{3(p^2-1)}{8}\right) + Q(p,m+1) p_9(n)\right)\\
&\quad - \left(P(p,m) p_9\left(p^2 n + \frac{3(p^2-1)}{8}\right) + Q(p,m) p_9(n)\right)\\
&\equiv \left(\pi(p) P(p,m+1) - P(p,m)\right) p_9\left(p^2 n + \frac{3(p^2-1)}{8}\right)\\
&\quad + \left(\pi(p) Q(p,m+1) - Q(p,m)\right) p_9(n)\\
&\equiv P(p,m+2) p_9\left(p^2 n + \frac{3(p^2-1)}{8}\right) + Q(p,m+2) p_9(n) \pmod{2},
\end{aligned}$$

which implies that (4.26) holds for $k = m+2$ also. Hence, by the principle of mathematical induction, we complete the proof of the lemma. □

**Lemma 4.21.** *For $p \geq 3$ prime, we have*

$$\pi(p) P(p, u(p)) + Q(p, u(p)) \equiv 0 \pmod{2}, \tag{4.35}$$

*where $\pi(p)$, $P(p,k)$ and $Q(p,k)$ are given by* (4.2), (4.27) *and* (4.28) *respectively.*

*Proof.* From (4.27) and (4.28), we obtain the first three terms of $P(p,k)$ and $Q(p,k)$ as follows:

$$P(p,0)=0, P(p,1)=1, P(p,2)=\pi(p) \tag{4.36}$$

and

$$Q(p,0)=1, Q(p,1)=0, Q(p,2)=-p. \tag{4.37}$$

Now, the proof is evident from (4.36), (4.37) and (4.3). □

*Proof of Theorem 4.14.* First we substitute (4.31) in (4.26) to arrive at

$$\begin{aligned}
&p_9\left(p^{2k}n+\frac{3(p^{2k}-1)}{8}\right)\\
&\equiv P(p,k)\left(\chi(n)p_9(n)-p_9\left(\frac{n-\frac{3(p^2-1)}{8}}{p^2}\right)\right)+Q(p,k)p_9(n)\\
&\equiv \left(P(p,k)\chi(n)+Q(p,k)\right)p_9(n)-P(p,k)p_9\left(\frac{n-\frac{3(p^2-1)}{8}}{p^2}\right) \pmod 2.
\end{aligned} \tag{4.38}$$

Replacing $n$ by $pn+\frac{3(p^2-1)}{8}$ in (4.38), we find that

$$\begin{aligned}
&p_9\left(p^{2k+1}n+\frac{3(p^{2k+2}-1)}{8}\right)\\
&\equiv \left(P(p,k)\pi(p)\right)+Q(p,k)\right)p_9\left(pn+\frac{3(p^2-1)}{8}\right)-P(p,k)p_9\left(\frac{n}{p}\right) \pmod 2.
\end{aligned} \tag{4.39}$$

Substituting $k$ by $u(p)$ in (4.39) and then employing (4.35) yields

$$p_9\left(p^{2u(p)+1}n+\frac{3(p^{2u(p)+2}-1)}{8}\right)\equiv P(p,u(p))p_9\left(\frac{n}{p}\right) \pmod 2. \tag{4.40}$$

Replacing $n$ by $pn$ in (4.40), we have

$$p_9\left(p^{2u(p)+2}n+\frac{3(p^{2u(p)+2}-1)}{8}\right)\equiv P(p,u(p))p_9\left(n\right) \pmod 2. \tag{4.41}$$

Iterating (4.41) for $k\geq 1$ times, we arrive at

$$p_9\left(p^{2(u(p)+1)k}n+\frac{3(p^{2(u(p)+1)k}-1)}{8}\right)\equiv \left(P(p,u(p))p_9\left(n\right)\right)^k \pmod 2. \tag{4.42}$$

Also, if $p \nmid n$, then (4.40) implies that

$$p_9\left(p^{2u(p)+1}n + \frac{3(p^{2u(p)+2}-1)}{8}\right) \equiv 0 \pmod 2. \tag{4.43}$$

Replacing $n$ by $p^{2u(p)+1}n + \dfrac{3(p^{2u(p)+2}-1)}{8}$ in (4.42) and then employing (4.43), we obtain

$$p_9\left(p^{2(u(p)+1)(k+1)-1}n + \frac{3(p^{2(u(p)+1)(k+1)}-1)}{8}\right) \equiv 0 \pmod 2, \tag{4.44}$$

where $p \nmid n$.

Again, from (4.2) and (4.23), we have

$$\overline{b}_2(n) \equiv p_9(n) \pmod 2. \tag{4.45}$$

Combining (4.44) and (4.45), we complete the proof of the theorem. □

## 4.11 Concluding remarks

(1) Theorems 4.3–4.7 of this chapter and and Theorem 1.8 of [57] discuss the arithmetic densities of $\overline{a}_t(n)$ and $\overline{b}_t(n)$ for odd $t$. But it is not possible to study the arithmetic densities of $\overline{a}_t(n)$ and $\overline{b}_t(n)$ for even $t$ using the similar techniques. We have studied the density of $\overline{b}_2(n)$ using another approach which cannot be used for other even values of $t$. It would be interesting to study the arithmetic densities of $\overline{a}_t(n)$ and $\overline{b}_t(n)$ for the even values of $t$.

(2) Computational evidence suggests that there are Ramanujan-type congruences for $\overline{a}_t(n)$ and $\overline{b}_t(n)$ modulo powers of 2, 3, and other primes $\geq 5$ for various $t$ which are not covered by the results of [14] and [57]. It will be desirable to find new congruences for $\overline{a}_t(n)$ and $\overline{b}_t(n)$.

(3) Asymptotic formulae for partition functions and other related functions have been widely studied in the literature. For instance, the asymptotic formulae for $p(n)$ and $c_t(n)$ were obtained by Hardy and Ramanujan [66] and Anderson [3], respectively. It would be of interest to find an asymptotic formula for $\overline{a}_t(n)$.

(4) Bandyopadhyay and Baruah [14] deduced several arithmetic identities involving $\overline{a}_5(n)$, $\overline{b}_5(n)$, and $c_5(n)$. A combinatorial treatment to $\overline{a}_t(n)$ and $\overline{b}_t(n)$ might reveal more interesting partition theoretic connections of these two functions.