# Chapter 5

# Families of Congruences for 6-Regular Partitions and Partitions $k$-Tuples with $t$-Cores

## 5.1 Introduction

In this chapter, we obtain new infinite families of congruences as well as individual congruences for two restricted partition functions, namely, 6-regular partition function and partition $k$-tuples with 5-cores. The motivation of studying these two functions together lies in the similarity of the proof techniques for their respective congruences which mainly include dissections of certain $q$-products and the theory of Lucas sequences.

Recall that if $b_6(n)$ counts the 6-regular partitions of $n$, then its generating function is given by

$$\sum_{n=0}^{\infty} b_6(n)q^n = \frac{f_6}{f_1}.$$

In 2015, Hou, Sun, and Zhang [72] found infinite families of congruences modulo

Theorems 5.12–5.17 and their proofs of this chapter have been published in *Indian Journal of Pure and Applied Mathematics* [106]. The author thanks Dr. Manjil P. Saikia and Mr. Abhishek Sarma for the collaboration. The other results of this chapter have been submitted for publication [22].

3 for $b_6(n)$ in the arithmetic progression of squares of certain primes. Their results are stated in the following theorem.

**Theorem 5.1** (Hou, Sun, and Zhang [72])**.** *For $\alpha$, $n$ nonnegative integers, $p_i$ primes congruent to 13, 17, 19, 23 (mod 24) and $j \not\equiv 0 \pmod{p_{\alpha+1}}$, we have*

$$b_6\left(p_1^2 \cdots p_{\alpha+1}^2 n + \frac{p_1^2 \cdots p_\alpha^2 p_{\alpha+1}(24j + 5p_{\alpha+1}) - 5}{24}\right) \equiv 0 \pmod 3. \tag{5.1}$$

For $\alpha = 0$, it follows that for all primes $p \equiv 13, 17, 19, 23 \pmod{24}$, $j \not\equiv 0 \pmod p$ and all $n \geq 0$,

$$b_6\left(p^2 n + pj + 5\,\frac{p^2 - 1}{24}\right) \equiv 0 \pmod 3. \tag{5.2}$$

Ahmed and Baruah [2] proved the following infinite family of congruences for $b_6(n)$.

**Theorem 5.2.** *[Ahmed and Baruah [2]] If $p$ is a prime such that $\left(\frac{-6}{p}\right)_L = -1$ and $1 \leq j \leq p-1$, then for all $\alpha$, $n \geq 0$, we have*

$$b_6\left(p^{2\alpha+1}(pn + j) + 5\,\frac{p^{2\alpha+2} - 1}{24}\right) \equiv 0 \pmod 3. \tag{5.3}$$

**Remark 5.3.** *Here, we have corrected a misprint in [2].*

Recently, Ballantine and Merca [13] extended Theorem 5.1 to more choices of primes. They proved the following theorem.

**Theorem 5.4.** *[Ballantine and Merca [13]] Let $\alpha$ be a nonnegative integer and let $p_i \geq 5$, $1 \leq i \leq \alpha + 1$ be primes. If $p_{\alpha+1} \equiv 3 \pmod 4$ and $j \not\equiv 0 \pmod{p_{\alpha+1}}$, then for all integers $n \geq 0$, we have*

$$b_6\left(p_1^2 \cdots p_{\alpha+1}^2 n + \frac{p_1^2 \cdots p_\alpha^2 p_{\alpha+1}(24j + 5p_{\alpha+1}) - 5}{24}\right) \equiv 0 \pmod 3. \tag{5.4}$$

Ballantine and Merca [13] also conjectured some truncated theta series results for $b_6(n)$ and some related partition functions. Using a formula of Cayley [39] on the number of partitions of $n$ into parts not exceeding 3, Yao [128] established two of those conjectures for $b_6(n)$.

Very recently, Zheng [136] proved the existence of infinitely many Ramanujan-type congruences for $b_6(n)$ modulo $m$ for every prime $m \geq 5$. They also deduced new congruences for $b_6(n)$ modulo 5.

In this chapter, we find new infinite families of congruences modulo 3 for $b_6(n)$. We state our results in the following theorems.

First, we note some new individual congruences for $b_6(n)$ modulo 3.

**Theorem 5.5.** *For $n \geq 0$, we have*

$$b_6(125n+r) \equiv 0 \pmod{3} \text{ where } r \in \{30, 55, 80, 105\}, \tag{5.5}$$

$$b_6(15625n+s) \equiv 0 \pmod{3} \text{ where } s \in \{130, 6380, 9505, 12630\}. \tag{5.6}$$

In the following theorem, we state two recurrence relations modulo 3 for $b_6(n)$.

**Theorem 5.6.** *For $n \geq 0$, the following relations hold:*

$$b_6(n) \equiv -b_6\left(5^6 n + 5^5 + 5^3 + 5\right) \pmod{3}, \tag{5.7}$$

$$b_6(n) \equiv (-1)^k b_6\left(5^{6k} n + \frac{5\left(5^{6k}-1\right)}{24}\right) \pmod{3}. \tag{5.8}$$

We now present some new infinite families of congruences modulo 3 for $b_6(n)$.

**Theorem 5.7.** *For $n \geq 0$ and $k \geq 0$, we have*

$$b_6\left(5^{6k+3} n + \frac{5\left(29 \cdot 5^{6k+1}-1\right)}{24}\right) \equiv 0 \pmod{3}, \tag{5.9}$$

$$b_6\left(5^{6k+3} n + \frac{5\left(53 \cdot 5^{6k+1}-1\right)}{24}\right) \equiv 0 \pmod{3}, \tag{5.10}$$

$$b_6\left(5^{6k+3} n + \frac{5\left(77 \cdot 5^{6k+1}-1\right)}{24}\right) \equiv 0 \pmod{3}, \tag{5.11}$$

$$b_6\left(5^{6k+3} n + \frac{5\left(101 \cdot 5^{6k+1}-1\right)}{24}\right) \equiv 0 \pmod{3}, \tag{5.12}$$

$$b_6\left(5^{6(k+1)} n + \frac{5\left(5^{6k+4}-1\right)}{24}\right) \equiv 0 \pmod{3}, \tag{5.13}$$

$$b_6\left(5^{6(k+1)} n + \frac{5\left(49 \cdot 5^{6k+4}-1\right)}{24}\right) \equiv 0 \pmod{3}, \tag{5.14}$$

$$b_6\left(5^{6(k+1)} n + \frac{5\left(73 \cdot 5^{6k+4}-1\right)}{24}\right) \equiv 0 \pmod{3}, \tag{5.15}$$

$$b_6\left(5^{6(k+1)} n + \frac{5\left(97 \cdot 5^{6k+4}-1\right)}{24}\right) \equiv 0 \pmod{3}. \tag{5.16}$$

Note that the cases $k = 0$ of (5.9)–(5.16) are equivalent to the congruences (5.5) and (5.6). We observe that these congruences also hold for modulo 9. That is, for $n \geq 0$, the following congruences are true:

$$b_6(125n + r) \equiv 0 \pmod 9, \text{ where } r \in \{30, 55, 80, 105\}, \tag{5.17}$$

$$b_6(15625n + s) \equiv 0 \pmod 9, \text{ where } s \in \{130, 6380, 9505, 12630\}. \tag{5.18}$$

Using an approach different from the proof of Theorem 5.5, we prove (5.17) and (5.18).

Next, we state two new infinite families of congruences modulo 3 for $b_6(n)$ in the arithmetic progression of certain primes.

**Theorem 5.8.** *Let $p$ be a prime with $p \equiv 1 \pmod{24}$. We define*

$$g(p) = \begin{cases} 1, & \text{if } c\left(\frac{p-1}{24}\right) \equiv 0 \pmod 3, \\ 2, & \text{if } c\left(\frac{p-1}{24}\right) \not\equiv 0 \pmod 3 \text{ and } \left(\frac{2}{p}\right)_L = 1, \\ 3, & \text{if } c\left(\frac{p-1}{24}\right) \not\equiv 0 \pmod 3 \text{ and } \left(\frac{2}{p}\right)_L = -1, \end{cases} \tag{5.19}$$

*where*

$$\sum_{n=0}^{\infty} c(n)q^n := \frac{f_1^3}{f_2}. \tag{5.20}$$

*Then, for $n \geq 0$, $k \geq 0$ with $p \nmid (24n + 1)$, we have*

$$b_6\left(125p^{(g(p)+1)k+g(p)}n + \frac{125p^{(g(p)+1)k+g(p)} - 5}{24}\right) \equiv 0 \pmod 3. \tag{5.21}$$

**Theorem 5.9.** *Let $p$ be a prime with $p \equiv 1 \pmod{24}$. We define*

$$h(p) = \begin{cases} 1, & \text{if } d\left(\frac{5(p-1)}{24}\right) \equiv 0 \pmod 3, \\ 2, & \text{if } d\left(\frac{5(p-1)}{24}\right) \not\equiv 0 \pmod 3 \text{ and } \left(\frac{2}{p}\right)_L = 1, \\ 3, & \text{if } d\left(\frac{5(p-1)}{24}\right) \not\equiv 0 \pmod 3 \text{ and } \left(\frac{2}{p}\right)_L = -1, \end{cases} \tag{5.22}$$

*where*

$$\sum_{n=0}^{\infty} d(n)q^n := \frac{f_2^3}{f_1}. \tag{5.23}$$

*Then, for $n \geq 0$, $k \geq 0$ with $p \nmid (24n + 5)$, we have*

$$b_6\left(p^{(h(p)+1)k+h(p)}n + \frac{5\left(p^{(h(p)+1)k+h(p)} - 1\right)}{24}\right) \equiv 0 \pmod 3. \tag{5.24}$$

**Remark 5.10.** *It is easy to observe that congruences in Theorem 5.8 are different from those in Theorems 5.1, 5.2, and 5.4. We also note that Theorem 5.9 is not same as Theorem 5.2 as for* $p = 73$, *we have* $73 \equiv 1 \pmod{24}$, *but* $\left(\frac{-6}{73}\right) \neq -1$. *Also, the choices of primes for Theorems 5.4 and 5.9 are not the same as the first one is given for* $p_{\alpha+1} \equiv 3 \pmod 4$ *and for the later one, all primes are congruent to 1 modulo 24.*

We also prove some Kolberg-type congruences modulo 3 for $b_6(n)$ as given in the following theorem.

**Theorem 5.11.** *Let*

$$\sum_{n=0}^{\infty} A(n)q^n = \left(\sum_{m=0}^{\infty} b_6(5m+1)q^m\right)\left(\sum_{m=0}^{\infty} b_6(5m+4)q^m\right),$$
$$\sum_{n=0}^{\infty} B(n)q^n = \left(\sum_{m=0}^{\infty} b_6(25m)q^m\right)\left(\sum_{m=0}^{\infty} b_6(25m+10)q^m\right),$$
$$\sum_{n=0}^{\infty} C(n)q^n = \left(\sum_{m=0}^{\infty} b_6(625m+380)q^m\right)\left(\sum_{m=0}^{\infty} b_6(625m+605)q^m\right),$$
$$\sum_{n=0}^{\infty} D(n)q^n = \left(\sum_{m=0}^{\infty} b_6(3125m+1380)q^m\right)\left(\sum_{m=0}^{\infty} b_6(3125m+2005)q^m\right),$$
$$\sum_{n=0}^{\infty} E(n)q^n = \left(\sum_{m=0}^{\infty} b_6(m)q^m\right)\left(\sum_{m=0}^{\infty} b_6(125m+5)q^m\right).$$

*Then the following hold:*

$$\sum_{n=0}^{\infty} A(5n+1)q^n \equiv -\left(\sum_{m=0}^{\infty} b_6(m)q^m\right)\left(\sum_{m=0}^{\infty} b_6(25m+5)q^m\right) \pmod 3, \tag{5.25}$$
$$\sum_{n=0}^{\infty} B(n)q^n \equiv \left(\sum_{m=0}^{\infty} b_6(25m+5)q^m\right)^2 \pmod 3, \tag{5.26}$$
$$\sum_{n=0}^{\infty} D(5n+1)q^n \equiv \left(\sum_{m=0}^{\infty} b_6(m)q^m\right)^2 \pmod 3. \tag{5.27}$$

*Also, we have*

$$B(5n+r) \equiv 0 \pmod 3, \quad \textit{where} \quad r \in \{1,2,3,4\}, \tag{5.28}$$
$$C(n) \equiv C(5n+4) \pmod 3, \tag{5.29}$$
$$D(5n+r) \equiv 0 \pmod 3, \quad \textit{where} \quad r \in \{0,2,3,4\}, \tag{5.30}$$
$$E(3n+2) \equiv 0 \pmod 3. \tag{5.31}$$

Again, we recall from Section 1.4 that the generating function of partitions $k$-

tuples with $t$-cores is given by

$$\sum_{n=0}^{\infty} \mathcal{A}_{t,k}(n)q^n = \frac{(q^t;q^t)_\infty^{kt}}{(q;q)_\infty^k} = \frac{f_t^{kt}}{f_1^k}. \tag{5.32}$$

There have been several studies involving the congruence properties of $\mathcal{A}_{t,k}(n)$ for different values of $t$ and $k$. For more details of the works done on this function, we refer the readers to [51] and the references cited therein.

Recently, Dasappa [100] proved the following infinite family of congruences for $\mathcal{A}_{5,2}(n)$:

$$\mathcal{A}_{5,2}(5^\alpha n + 5^\alpha - 2) \equiv 0 \pmod{5^\alpha}, \quad \alpha \geq 1. \tag{5.33}$$

In a similar vein, Majid and Fathima [83] proved the following result:

$$\mathcal{A}_{5,3}(5^\alpha n + 5^\alpha - 3) \equiv 0 \pmod{5^\alpha}, \quad \alpha \geq 1. \tag{5.34}$$

Both of these results were proved using elementary techniques which involved using dissection formulae and induction. We extend these results in the following theorem.

**Theorem 5.12.** *For all $n \geq 0$ and $\alpha \geq 1$, we have*

$$\mathcal{A}_{5,4}(5^{\alpha+1} n + 5^{\alpha+1} - 4) \equiv 0 \pmod{5^{\alpha+4}}. \tag{5.35}$$

We further find some new infinite family of congruences for $\mathcal{A}_{t,k}(n)$ for some general values of $k$ and $t$, as stated in the following results.

**Theorem 5.13.** *Let $p \geq 5$ be a prime and let $r \in \mathbb{N}$ with $1 \leq r \leq p-1$, be such that $24r+1$ is a quadratic nonresidue modulo $p$. Then, for all $n \geq 0$, $i \geq 1$, and $N \geq 1$, we have*

$$\mathcal{A}_{p,p^N i-1}(pn+r) \equiv 0 \pmod{p^N}.$$

**Theorem 5.14.** *Let $p \geq 5$ be a prime and let $r \in \mathbb{N}$ with $1 \leq r \leq p-1$, be such that $8r+1$ is a quadratic nonresidue modulo $p$. Then, for all $n \geq 0$, $i \geq 1$, and $N \geq 1$, we have*

$$\mathcal{A}_{p,p^N i-3}(pn+r) \equiv 0 \pmod{p^N}.$$

**Theorem 5.15.** *Let $p \geq 5$ be a prime and let $r \in \mathbb{N}$ with $1 \leq r \leq p-1$, be the* *unique value such that* $8r+1 \equiv 0 \pmod{p}$*. Then, for all* $n \geq 0$ *and* $i \geq 1$*, we have*

$$\mathcal{A}_{p,pi-3}(pn+r) \equiv 0 \pmod{p}.$$

We also prove some new individual congruences for $\mathcal{A}_{5,t}(n)$ for some specific values of $t$.

**Theorem 5.16.** *For all* $n \geq 0$*, the following congruences are true:*

$$\mathcal{A}_{5,2}(25n+23) \equiv 0 \pmod{5^2}, \tag{5.36}$$

$$\mathcal{A}_{5,2}(125n+123) \equiv 0 \pmod{5^3}, \tag{5.37}$$

$$\mathcal{A}_{5,3}(25n+22) \equiv 0 \pmod{5}, \tag{5.38}$$

$$\mathcal{A}_{5,3}(125n+122) \equiv 0 \pmod{5^2}, \tag{5.39}$$

$$\mathcal{A}_{5,4}(25n+21) \equiv 0 \pmod{5^5}, \tag{5.40}$$

$$\mathcal{A}_{5,4}(125n+121) \equiv 0 \pmod{5^6}. \tag{5.41}$$

Next, we establish a congruence result for $\mathcal{A}_{t,k}(n)$ modulo powers of primes, which can be also viewed as an existence result for infinite family of congruences.

**Theorem 5.17.** *Let* $p$ *be a prime,* $k \geq 1$*,* $j \geq 0$*,* $N \geq 1$*,* $M \geq 1$*, and* $r$ *be integers such that* $1 \leq r \leq p^M - 1$*. If for all* $n \geq 0$*,*

$$\mathcal{A}_{p,k}(p^M n + r) \equiv 0 \pmod{p^N},$$

*then for all* $n \geq 0$*, we have*

$$\mathcal{A}_{p,p^{M+N-1}i+k}(p^M n + r) \equiv 0 \pmod{p^N}.$$

The following is an easy corollary.

**Corollary 5.18.** *For all* $i \geq 0$ *and* $n \geq 0$*, we have*

$$\mathcal{A}_{5,5^2i+2}(25n+23) \equiv 0 \pmod{5}, \tag{5.42}$$

$$\mathcal{A}_{5,5^3i+2}(25n+23) \equiv 0 \pmod{5^2}, \tag{5.43}$$

$$\mathcal{A}_{5,5^5i+2}(125n+123) \equiv 0 \pmod{5^3}, \tag{5.44}$$

$$\mathcal{A}_{5,5^2i+3}(25n+22) \equiv 0 \pmod{5}, \tag{5.45}$$

$$\mathcal{A}_{5,5^4i+3}(125n+122) \equiv 0 \pmod{5^2}, \tag{5.46}$$

$$\mathcal{A}_{5,5i+4}(5n+3,4) \equiv 0 \pmod{5}, \tag{5.47}$$

$$\mathcal{A}_{5,5^3i+4}(25n+21) \equiv 0 \pmod{5^2}, \tag{5.48}$$

$$\mathcal{A}_{5,5^2i+4}(25n+21) \equiv 0 \pmod{5}, \tag{5.49}$$

$$\mathcal{A}_{5,5^3i+4}(25n+21) \equiv 0 \pmod{5^2}, \tag{5.50}$$

$$\mathcal{A}_{5,5^4i+4}(25n+21) \equiv 0 \pmod{5^3}, \tag{5.51}$$

$$\mathcal{A}_{5,5^5i+4}(25n+21) \equiv 0 \pmod{5^4}, \tag{5.52}$$

$$\mathcal{A}_{5,5^6i+4}(25n+21) \equiv 0 \pmod{5^5}, \tag{5.53}$$

$$\mathcal{A}_{5,5^8i+4}(125n+121) \equiv 0 \pmod{5^6}. \tag{5.54}$$

The rest of the chapter is organized as follows. In Section 5.2, we state some preliminary results that we require for our proofs. In Section 5.3, we use some 5-dissections of certain $q$-products and two identities from the list of forty identities for the Rogers-Ramanujan functions of Ramanujan to obtain Theorems 5.5–5.7. We prove Theorems 5.8 and 5.9 in Section 5.4 using two identities of Newman [86]. In Section 5.5, we establish the Kolberg-type congruences of Theorem 5.11. We prove (5.17) and (5.18) in Section 5.6 using an approach due to Radu [91]. Theorem 5.12 is then proved in Section 5.7, Theorems 5.13–5.15 are proved in Section 5.8, Theorem 5.16 is proved in Section 5.9. We deduce Theorem 5.17 and Corollary 5.18 in Section 5.10. Finally we close the chapter with some concluding remarks and conjectures in Section 5.11.

## 5.2 Preliminaries

In the following lemma, we recall the 5-dissections of $\chi(q)$ and $1/\chi(-q)$ given by (2.68) and (2.70).

**Lemma 5.19** (Baruah and Talukdar [21]). *We have*

$$\chi(q) = \frac{1}{f(-q^5)f(-q^{20})}\Big(\varphi(q^{25})f(q^{15},q^{35}) + qf^2(q^{15},q^{35}) + q^3\varphi(q^{25})f(q^5,q^{45}) + q^4 f(q^5,q^{45})f(q^{15},q^{35}) + q^7 f^2(q^5,q^{45})\Big), \tag{5.55}$$

$$\frac{1}{\chi(-q)} = \frac{f(-q^{10})}{f^3(-q^5)}\Big(f^2(q^{10},q^{15}) + qf(q^5,q^{20})f(q^{10},q^{15}) + q^2 f^2(q^5,q^{20})$$

$$+ 2q^3\psi(q^{25})f(q^{10},q^{15}) + 2q^4\psi(q^{25})f(q^5,q^{20})\Big). \tag{5.56}$$

Also, by the binomial theorem, for positive integers $j,k \geq 1$, we have

$$G^{3k}(q^j) \equiv G^k\left(q^{3j}\right) \pmod 3 \text{ and } H^{3k}(q^j) \equiv H^k\left(q^{3j}\right) \pmod 3.$$

We will use these congruences without referring to them.

In the following lemma, we state two identities from this list of forty identities of Ramanujan for $G(q)$ and $H(q)$.

**Lemma 5.20** (Berndt, Choi, Choi, Hahn, Yeap, Yee, Yesilyurt, and Yi [31])**.** *The following hold:*

$$H(q)G(q^6) - qG(q)H(q^6) = \frac{\chi(-q)}{\chi(-q^3)}, \tag{5.57}$$

$$G(q^2)G(q^3) + qH(q^2)H(q^3) = \frac{\chi(-q^3)}{\chi(-q)}. \tag{5.58}$$

The next four lemmas contain some generating functions needed for the proof of Theorem 5.12.

**Lemma 5.21.** *Let* $\sum_{n=0}^{\infty} P_4(n)q^n = \frac{1}{f_1^4}$*. Then we have*

$$\sum_{n=0}^{\infty} P_4(5n+1)q^n = 4\frac{f_5^2}{f_1^6} + 550q\frac{f_5^8}{f_1^{12}} + 12500q^2\frac{f_5^{14}}{f_1^{18}} + 78125q^3\frac{f_5^{20}}{f_1^{24}}.$$

*Proof.* Using equation (1.17), extracting the terms involving $q^{5n+1}$, dividing by $q$ and then replacing $q^5$ by $q$, we arrive at

$$\sum_{n=0}^{\infty} P_4(5n+1)q^n = \frac{f_5^{20}}{f_1^{24}}\Bigg(\frac{4}{T^{15}(q)} + \frac{418q}{T^{10}(q)} + \frac{1840q^2}{T^5(q)}$$
$$+ 1015q^3 - 1840q^4T^5(q) + 418q^5T^{10}(q) - 4q^6T^{15}(q)\Bigg). \tag{5.59}$$

We use the following formula [29, Theorem 7.4.4]:

$$\frac{1}{T(q)^5} - 11q - q^2T(q)^5 = \frac{f_1^6}{f_5^6}, \tag{5.60}$$

to obtain from equation (5.59)

$$\sum_{n=0}^{\infty} P_4(5n+1)q^n = 4\frac{f_5^2}{f_1^6} + 550q\frac{f_5^8}{f_1^{12}} + 12500q^2\frac{f_5^{14}}{f_1^{18}} + 78125q^3\frac{f_5^{20}}{f_1^{24}}. \tag{5.61}$$

□

**Lemma 5.22.** *If* $\sum_{n=0}^{\infty} Q_4(n)q^n = f_5^2 f_1^{14}$*, then we have* $\sum_{n=0}^{\infty} Q_4(5n+4)q^n = -15625q^2 f_5^{14} f_1^2$.

*Proof.* Using equation (1.16), we obtain

$$\sum_{n=0}^{\infty} Q_4(n)q^n = f_5^2 f_{25}^{14}\Bigg(\frac{1}{T^{14}(q)} - 14\frac{q}{T^{13}(q)} + 77\frac{q^2}{T^{12}(q)} - 182\frac{q^3}{T^{11}(q)} + 910\frac{q^5}{T^9(q)} - 1365\frac{q^6}{T^8(q)} - 1430\frac{q^7}{T^7(q)} + 5005\frac{q^8}{T^6(q)} - 10010\frac{q^{10}}{T^4(q)} + 3640\frac{q^{11}}{T^3(q)} + 14105\frac{q^{12}}{T^2(q)} - 6930\frac{q^{13}}{T(q)} - 15625q^{14} + 6930q^{15}T(q) + 14105q^{16}T^2(q) - 3640q^{17}T^3(q) - 10010q^{18}T^4(q) + 5005q^{20}T^6(q) + 1430q^{21}T^7(q) - 1365q^{22}T^8(q) - 910q^{23}T^9(q) + 182q^{25}T^{11}(q) + 77q^{26}T^{12}(q) + 14q^{27}T^{13}(q) + q^{28}T^{14}(q)\Bigg).$$

Extracting the terms involving $q^{5n+4}$ and then dividing by $q^4$ and replacing $q^5$ by $q$, we have

$$\sum_{n=0}^{\infty} Q_4(5n+4)q^n = -15625q^2 f_5^{14} f_1^2.$$

□

**Lemma 5.23.** *If* $\sum_{n=0}^{\infty} Q_5(n)q^n = qf_5^8 f_1^8$*, then we have* $\sum_{n=0}^{\infty} Q_5(5n+4)q^n = -125qf_5^8 f_1^8$.

**Lemma 5.24.** *If* $\sum_{n=0}^{\infty} Q_6(n)q^n = q^2 f_5^{14} f_1^2$*, then we have* $\sum_{n=0}^{\infty} Q_6(5n+4)q^n = -f_5^2 f_1^{14}$.

The proofs of Lemmas 5.23 and 5.24 are exactly similar to the proof of Lemma 5.22. So, we skip them.

## 5.3 Proofs of the Theorems 5.5–5.7

*Proof of Theorem 5.5.* We have

$$\sum_{n=0}^{\infty} b_6(n)q^n = \frac{f_6}{f_1} \equiv \frac{f_2^3}{f_1} \equiv f_2\psi(q) \pmod 3. \tag{5.62}$$

Using the 5-dissections of $f_1$ and $\psi(q)$ given by (1.16) and (2.37) in (5.62), we have

$$\sum_{n=0}^{\infty} b_6(n)q^n \equiv f_{50}\left(T(q^{10}) - q^2 - \frac{q^4}{T(q^{10})}\right)$$

$$\times\left(f(q^{10},q^{15})+qf(q^5,q^{20})+q^3\psi(q^{25})\right) \pmod 3, \tag{5.63}$$

from which we extract,

$$\begin{aligned}
\sum_{n=0}^{\infty} b_6(5n)q^n &\equiv -qf_{10}\psi(q^5)+f_{10}\left(f(q^2,q^3)\frac{f(-q^4,-q^6)}{f(-q^2,-q^8)}-qf(q,q^4)\frac{f(-q^2,-q^8)}{f(-q^4,-q^6)}\right)\\
&\equiv -qf_{10}\psi(q^5)+\frac{f_{10}}{f(-q^2,-q^8)f(-q^4,-q^6)}\\
&\quad\times\left(f(q^2,q^3)f^2(-q^4,-q^6)-qf(q,q^4)f^2(-q^2,-q^8)\right)\\
&\equiv -qf_{10}\psi(q^5)+\frac{f_2^2f_5}{f_{10}}\left(H(q)G^3(q^2)-qG(q)H^3(q^2)\right)\\
&\equiv -qf_{10}\psi(q^5)+\frac{f_2^2f_5}{f_{10}}\left(H(q)G(q^6)-qG(q)H(q^6)\right)\\
&\equiv -qf_{10}\psi(q^5)+\frac{f_2^2f_5}{f_{10}}\frac{\chi(-q)}{\chi(-q^3)} \quad \text{(from (5.57))}\\
&\equiv -q\frac{f_5}{f_{10}}\psi^2(q^5)+\frac{f_5}{f_{10}}\frac{f_2^4}{f_1^2}\\
&\equiv \frac{f_5}{f_{10}}\left(-q\psi^2(q^5)+\psi^2(q)\right)\\
&\equiv \frac{f_5}{f_{10}}f(q,q^4)f(q^2,q^3) \quad \text{( by (2.44))}\\
&\equiv \frac{f_5^2}{f_{10}}\frac{\varphi(-q^5)}{\chi(-q)}\\
&\equiv \frac{f_5^4}{f_{10}^2}\frac{1}{\chi(-q)} \pmod 3.
\end{aligned} \tag{5.64}$$

Employing (5.56) in (5.64), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} b_6(5n)q^n \equiv \frac{f_5^4}{f_{10}^2}\frac{f_{10}}{f_5^3}\Big(&f^2(q^{10},q^{15})+qf(q^5,q^{20})f(q^{10},q^{15})+q^2f^2(q^5,q^{20})\\
&+2q^3\psi(q^{25})f(q^{10},q^{15})+2q^4\psi(q^{25})f(q^5,q^{20})\Big) \pmod 3,
\end{aligned} \tag{5.65}$$

which by extraction gives

$$\sum_{n=0}^{\infty} b_6(25n+5)q^n \equiv \frac{f_1}{f_2}f(q,q^4)f(q^2,q^3) \pmod 3. \tag{5.66}$$

$$\begin{aligned}
&\equiv \frac{f_1}{f_2}\frac{f_1}{f_2}f_5^2\frac{G(q)H(q)}{G(q^2)H(q^2)}\\
&\equiv \frac{f_5^3}{f_{10}} \pmod 3.
\end{aligned} \tag{5.67}$$

Thus, extracting the terms involving $q^{5n+r}$, $1 \le r \le 4$ from both sides of the above, we have

$$b_6(125n + r) \equiv 0 \pmod 3, \quad \text{where } r \in \{30, 55, 80, 105\}$$

which proves (5.5).

Again, extracting the terms involving $q^{5n}$ from both sides of the (5.67), and then replacing $q^5$ by $q$, we find that

$$\sum_{n=0}^{\infty} b_6(125n + 5)q^n \equiv \frac{f_1^3}{f_2} \equiv f_1\varphi(-q) \pmod 3. \tag{5.68}$$

Employing (2.36) and (1.16) in (5.68), we find that

$$\sum_{n=0}^{\infty} b_6(125n + 5)q^n \equiv f_{25}\left(T(q^5) - q - \frac{q^2}{T(q^5)}\right)\Big(\varphi(-q^{25}) \\ - 2qf(-q^{15}, -q^{35}) + 2q^4 f(-q^5, -q^{45})\Big) \pmod 3, \tag{5.69}$$

which implies that

$$\begin{aligned}
\sum_{n=0}^{\infty} b_6(625n + 130)q^n &\equiv f_5\left( - 2T(q)f(-q^3, -q^7) - \varphi(-q^5) - \frac{2q}{T(q)}f(-q, -q^9)\right) \\
&\equiv 2f_5\varphi(-q^5) + f_5\left(\frac{f(-q^2, -q^3)}{f(-q, -q^4)}f(-q^3, -q^7)\right. \\
&\qquad \left. + q\frac{f(-q, -q^4)}{f(-q^2, -q^3)}f(-q, -q^9)\right) \\
&\equiv 2f_5\varphi(-q^5) + \frac{f_5}{f(-q, -q^4)f(-q^2, -q^3)} \\
&\qquad \times \left(f^2(-q^2, -q^3)f(-q^3, -q^7) + qf^2(-q, -q^4)f(-q, -q^9)\right) 
\end{aligned} \tag{5.70}$$

$$\begin{aligned}
&\equiv 2f_5\varphi(-q^5) + \frac{f_1^2 f_{10}}{f_5}\left(G^3(q)G(q^2) + qH^3(q)H(q^2)\right) \\
&\equiv 2f_5\varphi(-q^5) + \frac{f_1^2 f_{10}}{f_5}\left(G(q^3)G(q^2) + qH(q^3)H(q^2)\right) \\
&\equiv 2f_5\varphi(-q^5) + \frac{f_1^2 f_{10}}{f_5}\frac{\chi(-q^3)}{\chi(-q)} \quad (\text{ by } (5.58)) \\
&\equiv 2\frac{f_{10}}{f_5}\varphi^2(-q^5) + \frac{f_{10}}{f_5}\varphi^2(-q) \qquad (5.71) \\
&\equiv 2\frac{f_{10}}{f_5}\varphi^2(-q^5) \\
&\qquad + \frac{f_{10}}{f_5}\left(\varphi^2(-q^5) - 4qf(-q, -q^9)f(-q^3, -q^7)\right)
\end{aligned}$$

$$\equiv 2q\frac{f_{10}}{f_5}f(-q,-q^9)f(-q^3,-q^7) \tag{5.72}$$

$$\equiv 2q\frac{f_{10}}{f_5}\chi(-q)f(q^5)f_{20} \pmod 3. \tag{5.73}$$

Again, using (5.55) in (5.73), we arrive at

$$\sum_{n=0}^{\infty} b_6(625n+130)q^n \equiv 2q\frac{f_{10}}{f_5}\frac{f(q^5)f_{20}}{f(q^5)f_{20}}\Big(\varphi(-q^{25})f(-q^{15},-q^{35}) - qf^2(-q^{15},-q^{35}) - q^3\varphi(-q^{25})f(-q^5,-q^{45}) + q^4f(-q^5,-q^{45})f(-q^{15},-q^{35}) - q^7f^2(-q^5,-q^{45})\Big) \pmod 3, \tag{5.74}$$

from which we extract,

$$\sum_{n=0}^{\infty} b_6(3125n+130)q^n \equiv 2q\frac{f_2}{f_1}f(-q,-q^9)f(-q^3,-q^7) \pmod 3. \tag{5.75}$$

Using (2.31) and (2.32) in (5.75) and then employing (1.8), we find that

$$\sum_{n=0}^{\infty} b_6(3125n+130)q^n \equiv 2q\frac{1}{\chi(-q)}\chi(-q)f(q^5)f_{20} \equiv 2q\frac{f_{10}^3}{f_5f_{20}}f_{20} \equiv -q\frac{f_{10}^3}{f_5} \pmod 3. \tag{5.76}$$

Now, extracting the terms involving $q^{5n}$, $q^{5n+2}$, $q^{5n+3}$, and $q^{5n+4}$ from the right hand side of the above, we have

$$b_6(15625n+s) \equiv 0 \pmod 3, \quad \text{where } s \in \{130, 6380, 9505, 12630\},$$

which is (5.6). Thus, we complete the proof of Theorem 5.5. □

*Proof of Theorem 5.6.* Extracting the terms involving $q^{5n+1}$ from both sides of the (5.74), dividing by $q$ and then replacing $q^5$ by $q$, we find that

$$\sum_{n=0}^{\infty} b_6(15625n+3255)q^n \equiv -\frac{f_2^3}{f_1} \pmod 3,$$

which with the aid of (5.62) reduces to

$$b_6(15625n+3255) \equiv -b_6(n) \pmod 3,$$

which proves (5.7).

Now, by successive iterations of (5.7), we have

$$\begin{aligned} b_6(n) &\equiv -b_6\left(5^6n+5^5+5^3+5\right) \\ &\equiv b_6\left(5^6\left(5^6n+5^5+5^3+5\right)+5^5+5^3+5\right) \end{aligned}$$

$$
\begin{aligned}
&\equiv b_6\left(5^{12}n+5^{11}+5^9+5^7+5^5+5^3+5\right)\\
&\vdots\\
&\equiv (-1)^k b_6\left(5^{6k}n+5^{6k-1}+5^{6k-3}+\cdots+5^3+5\right)\\
&\equiv (-1)^k b_6\left(5^{6k}n+\frac{5\left(5^{6k}-1\right)}{24}\right) \pmod 3,
\end{aligned}
$$

which gives (5.8). Thus, we complete the proof of Theorem 5.6. □

*Proof of Theorem 5.7.* From (5.8), we have

$$
b_6(n) \equiv (-1)^k b_6\left(5^{6k}n+\frac{5\left(5^{6k}-1\right)}{24}\right) \pmod 3. \tag{5.77}
$$

Replacing $n$ by $125n+30$ in (5.77), we obtain

$$
\begin{aligned}
b_6(125n+30) &\equiv (-1)^k b_6\left(5^{6k}\left(125n+30\right)+\frac{5\left(5^{6k}-1\right)}{24}\right)\\
&\equiv (-1)^k b_6\left(5^{6k+3}n+\frac{5\left(29\cdot 5^{6k+1}-1\right)}{24}\right) \pmod 3.
\end{aligned} \tag{5.78}
$$

Similarly, replacing $n$ by $125n+55$, $125n+80$, and $125n+105$ in (5.77) respectively, we arrive at the following:

$$
b_6(125n+55) \equiv (-1)^k b_6\left(5^{6k+3}n+\frac{5\left(53\cdot 5^{6k+1}-1\right)}{24}\right) \pmod 3, \tag{5.79}
$$

$$
b_6(125n+80) \equiv (-1)^k b_6\left(5^{6k+3}n+\frac{5\left(77\cdot 5^{6k+1}-1\right)}{24}\right) \pmod 3, \tag{5.80}
$$

$$
b_6(125n+105) \equiv (-1)^k b_6\left(5^{6k+3}n+\frac{5\left(101\cdot 5^{6k+1}-1\right)}{24}\right) \pmod 3. \tag{5.81}
$$

Therefore, (5.9)–(5.12) are evident from (5.78)–(5.81) and (5.5).

Again, replacing $n$ by $15625n+130$ in (5.77), we find that

$$
\begin{aligned}
b_6(15625n+130) &\equiv (-1)^k b_6\left(5^{6k}\left(15625n+130\right)+\frac{5\left(5^{6k}-1\right)}{24}\right)\\
&\equiv (-1)^k b_6\left(5^{6(k+1)}n+\frac{5\left(5^{6k+4}-1\right)}{24}\right) \pmod 3.
\end{aligned} \tag{5.82}
$$

In a similar way, replacing $n$ by $15625n+6380$, $15625n+9505$, and $15625n+12630$

in (5.77) respectively, we obtain the following:

$$b_6(15625n+6380) \equiv (-1)^k b_6\left(5^{6(k+1)}n + \frac{5\left(49\cdot 5^{6k+4}-1\right)}{24}\right) \pmod 3, \quad (5.83)$$

$$b_6(15625n+9505) \equiv (-1)^k b_6\left(5^{6(k+1)}n + \frac{5\left(73\cdot 5^{6k+4}-1\right)}{24}\right) \pmod 3, \quad (5.84)$$

$$b_6(15625n+12630) \equiv (-1)^k b_6\left(5^{6(k+1)}n + \frac{5\left(97\cdot 5^{6k+4}-1\right)}{24}\right) \pmod 3. \quad (5.85)$$

Employing (5.6) in (5.82)–(5.85), we arrive at (5.13)–(5.16) which complete the proof of Theorem 5.7. □

## 5.4 Proofs of Theorems 5.8 and 5.9

First of all, we prove the following two lemmas.

**Lemma 5.25.** *Let $p$ be a prime with $p \equiv 1 \pmod{24}$ and $c(n)$ be defined by (5.20). We have*

$$c\left(p^k n + \frac{p^k-1}{24}\right) = U(p,k)c\left(pn + \frac{p-1}{24}\right) + V(p,k)c(n), \quad (5.86)$$

*where $U(p,k)$ and $V(p,k)$ are defined by*

$$U(p,k+2) = c\left(\frac{p-1}{24}\right)U(p,k+1) - \left(\frac{2}{p}\right)_L U(p,k) \quad (5.87)$$

*and*

$$V(p,k+2) = c\left(\frac{p-1}{24}\right)V(p,k+1) - \left(\frac{2}{p}\right)_L V(p,k) \quad (5.88)$$

*with $U(p,0)=0$, $U(p,1)=1$, $V(p,0)=1$, and $V(p,1)=0$.*

*Proof.* We will prove the lemma by induction on $k$ using the method of Xia [126] based on Newman's identities [86] and Lucas sequences. We observe that (5.86) is true for $k=0$ and $k=1$ since $U(p,0)=0$, $U(p,1)=1$, $V(p,0)=1$, and $V(p,1)=0$. We now assume that (5.86) is true for $k=m$ and $k=m+1$ for some $m \geq 0$, which gives

$$c\left(p^m n + \frac{p^m-1}{24}\right) = U(p,m)c\left(pn + \frac{p-1}{24}\right) + V(p,m)c(n), \quad (5.89)$$

and

$$c\left(p^{m+1}n+\frac{p^{m+1}-1}{24}\right)=U(p,m+1)c\left(pn+\frac{p-1}{24}\right)+V(p,m+1)c(n). \quad (5.90)$$

Newman [86, Theorem 3] proved the following identity for $c(n)$:

$$c\left(pn+\frac{p-1}{24}\right)=c\left(\frac{p-1}{24}\right)c(n)-\left(\frac{2}{p}\right)_L c\left(\frac{n-\frac{p-1}{24}}{p}\right) \quad (5.91)$$

where $p$ is a prime with $p\equiv 1 \pmod{24}$.

Replacing $n$ by $pn+\frac{p-1}{24}$ in (5.91), we have

$$c\left(p^2n+\frac{p^2-1}{24}\right)=c\left(\frac{p-1}{24}\right)c\left(pn+\frac{p-1}{24}\right)-\left(\frac{2}{p}\right)_L c(n). \quad (5.92)$$

Again, replacing $n$ by $p^mn+\frac{p^m-1}{24}$ in (5.92) and then employing (5.89) and (5.90), we find that

$$\begin{aligned}
&c\left(p^{m+2}n+\frac{p^{m+2}-1}{24}\right)\\
&=c\left(\frac{p-1}{24}\right)c\left(p^{m+1}n+\frac{p^{m+1}-1}{24}\right)-\left(\frac{2}{p}\right)_L c\left(p^mn+\frac{p^m-1}{24}\right)\\
&=c\left(\frac{p-1}{24}\right)\left(U(p,m+1)c\left(pn+\frac{p-1}{24}\right)+V(p,m+1)c(n)\right)\\
&\quad-\left(\frac{2}{p}\right)_L\left(U(p,m)c\left(pn+\frac{p-1}{24}\right)+V(p,m)c(n)\right)\\
&=\left(c\left(\frac{p-1}{24}\right)U(p,m+1)-\left(\frac{2}{p}\right)_L U(p,m)\right)c\left(pn+\frac{p-1}{24}\right)\\
&\quad+\left(c\left(\frac{p-1}{24}\right)V(p,m+1)-\left(\frac{2}{p}\right)_L V(p,m)\right)c(n)\\
&=U(p,m+2)c\left(pn+\frac{p-1}{24}\right)+V(p,m+2)c(n),
\end{aligned}$$

which implies that (5.86) holds for $k=m+2$ also. Hence, by the principle of mathematical induction we complete the proof of the lemma. □

**Lemma 5.26.** *Let $p$ be a prime with $p\equiv 1 \pmod{24}$. We have*

$$c\left(\frac{p-1}{24}\right)U(p,g(p))+V(p,g(p))\equiv 0 \pmod 3, \quad (5.93)$$

*where $c(n)$, $U(p,k)$, and $V(p,k)$ are given by* (5.20), (5.87) *and* (5.88) *respectively.*

*Proof.* From (5.87) and (5.88), we obtain the first four terms of $U(p,k)$ and $V(p,k)$

as follows:

$$U(p,0)=0, U(p,1)=1, U(p,2)=c\left(\frac{p-1}{24}\right), U(p,3)=c\left(\frac{p-1}{24}\right)^2-\left(\frac{2}{p}\right)_L \tag{5.94}$$

and

$$V(p,0)=1, V(p,1)=0, V(p,2)=-\left(\frac{2}{p}\right)_L, V(p,3)=-c\left(\frac{p-1}{24}\right)\left(\frac{2}{p}\right)_L. \tag{5.95}$$

Now, the proof is evident from (5.94), (5.95) and (5.19). □

*Proof of Theorem 5.8.* First we substitute (5.91) in (5.86) to arrive at

$$\begin{aligned} &c\left(p^k n+\frac{p^k-1}{24}\right) \\ &=U(p,k)\left(c\left(\frac{p-1}{24}\right)c(n)-\left(\frac{2}{p}\right)_L c\left(\frac{n-\frac{p-1}{24}}{p}\right)\right)+V(p,k)c(n) \\ &=\left(c\left(\frac{p-1}{24}\right)U(p,k)+V(p,k)\right)c(n)-\left(\frac{2}{p}\right)_L U(p,k)c\left(\frac{n-\frac{p-1}{24}}{p}\right). \end{aligned} \tag{5.96}$$

Replacing $k$ by $g(p)$ in (5.96) and then using (5.93), we find that

$$c\left(p^{g(p)}n+\frac{p^{g(p)}-1}{24}\right)\equiv-\left(\frac{2}{p}\right)_L U(p,g(p))c\left(\frac{n-\frac{p-1}{24}}{p}\right) \pmod 3. \tag{5.97}$$

Again, replacing $n$ by $pn+\dfrac{p-1}{24}$ in (5.97), we have

$$c\left(p^{g(p)+1}n+\frac{p^{g(p)+1}-1}{24}\right)\equiv-\left(\frac{2}{p}\right)_L U(p,g(p))c(n) \pmod 3. \tag{5.98}$$

Iterating (5.98) for $k\geq 1$ times, we arrive at

$$c\left(p^{(g(p)+1)k}n+\frac{p^{(g(p)+1)k}-1}{24}\right)\equiv\left(-\left(\frac{2}{p}\right)_L U(p,g(p))\right)^k c(n) \pmod 3. \tag{5.99}$$

Now, if $p\nmid(24n+1)$, then $c\left(\dfrac{n-\frac{p-1}{24}}{p}\right)=0$ and from (5.97), we have

$$c\left(p^{g(p)}n+\frac{p^{g(p)}-1}{24}\right)\equiv 0 \pmod 3. \tag{5.100}$$

Replacing $n$ by $p^{g(p)}n + \frac{p^{g(p)}-1}{24}$ in (5.99) and employing (5.100), we obtain

$$c\left(p^{(g(p)+1)k+g(p)}n + \frac{p^{(g(p)+1)k+g((p)}-1}{24}\right) \equiv 0 \pmod 3, \tag{5.101}$$

where $p \nmid (24n+1)$.

Again, from (5.20) and (5.68), we have

$$b_6(125n+5) \equiv c(n) \pmod 3. \tag{5.102}$$

Combining (5.101) and (5.102), we complete the proof of the theorem. □

*Proof of Theorem 5.9.* The proof of Theorem 5.9 is similar to the proof of Theorem 5.8. Thus, we omit the details and mention only the required lemmas and an identity due to Newman [86].

**Lemma 5.27.** *Let $p$ be a prime with $p \equiv 1 \pmod{24}$ and $d(n)$ be defined by (5.23). We have*

$$d\left(p^k n + \frac{5(p^k-1)}{24}\right) = U_1(p,k)d\left(pn + \frac{5(p-1)}{24}\right) + V_1(p,k)d(n), \tag{5.103}$$

*where $U_1(p,k)$ and $V_1(p,k)$ are defined by*

$$U_1(p,k+2) = d\left(\frac{5(p-1)}{24}\right)U_1(p,k+1) - \left(\frac{2}{p}\right)_L U_1(p,k) \tag{5.104}$$

*and*

$$V_1(p,k+2) = d\left(\frac{5(p-1)}{24}\right)V_1(p,k+1) - \left(\frac{2}{p}\right)_L V_1(p,k) \tag{5.105}$$

*with $U_1(p,0)=0$, $U_1(p,1)=1$, $V_1(p,0)=1$, and $V_1(p,1)=0$.*

**Lemma 5.28.** *Let $p$ be a prime with $p \equiv 1 \pmod{24}$. We have*

$$d\left(\frac{5(p-1)}{24}\right)U_1(p,h(p)) + V_1(p,h(p)) \equiv 0 \pmod 3 \tag{5.106}$$

*where $d(n)$, $U_1(p,k)$, and $V_1(p,k)$ are given by (5.23), (5.104) and (5.105) respectively.*

Newman [86] also proved the following identity for $d(n)$:

$$d\left(pn + \frac{5(p-1)}{24}\right) = d\left(\frac{5(p-1)}{24}\right)d(n) - \left(\frac{2}{p}\right)_L d\left(\frac{n - \frac{5(p-1)}{24}}{p}\right) \tag{5.107}$$

where $p$ is a prime with $p \equiv 1 \pmod{24}$. □

## 5.5 Proof of Theorem 5.11

*Proof of* (5.25). From (5.63), we have

$$\sum_{n=0}^{\infty} b_6(n)q^n \equiv f_{50}\left(T(q^{10}) - q^2 - \frac{q^4}{T(q^{10})}\right) \times \left(f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25})\right) \pmod 3,$$

from which we extract

$$\sum_{n=0}^{\infty} b_6(5n+1)q^n \equiv f_{10}f(q,q^4)\frac{f(-q^4,-q^6)}{f(-q^2,-q^8)} \pmod 3$$

and

$$\sum_{n=0}^{\infty} b_6(5n+4)q^n \equiv -f_{10}f(q^2,q^3)\frac{f(-q^2,-q^8)}{f(-q^4,-q^6)} \pmod 3.$$

Therefore, from the above and Lemma 2.8, we have

$$\begin{aligned}\sum_{n=0}^{\infty} A(n)q^n &= \left(\sum_{n=0}^{\infty} b_6(5n+1)q^n\right)\left(\sum_{n=0}^{\infty} b_6(5n+4)q^n\right)\\ &\equiv -f_{10}^2 f(q,q^4)f(q^2,q^3)\\ &\equiv -f_5^2 f_{10}^2\frac{G(q)H(q)}{G(q^2)H(q^2)}\\ &\equiv -f_5^2 f_{10}^2\frac{f_5 f_2}{f_1 f_{10}}\\ &\equiv -f_5^3 f_{10}\frac{1}{\chi(-q)} \pmod 3.\end{aligned}$$

Employing (5.56) in the above and then extracting, we find that

$$\sum_{n=0}^{\infty} A(5n+1)q^n \equiv -f_2^2 f(q,q^4)f(q^2,q^3) \equiv -\frac{f_2^3}{f_1}\frac{f_5^3}{f_{10}} \pmod 3. \tag{5.108}$$

Now, (5.25) follows from (5.62), (5.67) and (5.108). □

*Proofs of* (5.26) *and* (5.28). From (5.65), we have

$$\sum_{n=0}^{\infty} b_6(5n)q^n \equiv \frac{f_5^4}{f_{10}^2}\frac{f_{10}}{f_5^3}\Big(f^2(q^{10},q^{15}) + qf(q^5,q^{20})f(q^{10},q^{15}) + q^2f^2(q^5,q^{20}) + 2q^3\psi(q^{25})f(q^{10},q^{15}) + 2q^4\psi(q^{25})f(q^5,q^{20})\Big) \pmod 3.$$

Thus, in view of Lemma 2.8, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B(n)q^n &= \left(\sum_{n=0}^{\infty} b_6(25n)q^n\right)\left(\sum_{n=0}^{\infty} b_6(25n+10)q^n\right)\\ &\equiv \left(\frac{f_1}{f_2}f^2(q^2,q^3)\right)\left(\frac{f_1}{f_2}f^2(q,q^4)\right)\\ &\equiv \frac{f_1^2}{f_2^2}f_5^4\frac{G^2(q)H^2(q)}{G^2(q^2)H^2(q^2)} \equiv \left(\frac{f_5^3}{f_{10}}\right)^2 \pmod 3. \end{aligned} \tag{5.109}$$

The proof of (5.26) is evident from (5.67) and (5.109).

Again, extracting the terms involving $q^{5n+r}$, $1 \le r \le 4$ from (5.109), we obtain (5.28). □

*Proofs of* (5.27) *and* (5.30). From (5.74), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_6(625n+130)q^n \equiv 2q\frac{f_{10}}{f_5}\frac{f(q^5)f_{20}}{f(q^5)f_{20}}\Big(&\varphi(-q^{25})f(-q^{15},-q^{35}) - qf^2(-q^{15},-q^{35})\\ &- q^3\varphi(-q^{25})f(-q^5,-q^{45}) + q^4f(-q^5,-q^{45})f(-q^{15},-q^{35})\\ &- q^7f^2(-q^5,-q^{45})\Big) \pmod 3.\end{aligned}$$

From the above and Lemma 2.8, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} D(n)q^n &= \left(\sum_{n=0}^{\infty} b_6(3125n+1380)q^n\right)\left(\sum_{n=0}^{\infty} b_6(3125n+2005)q^n\right)\\ &\equiv \left(\frac{f_2}{f_1}f^2(-q^3,-q^7)\right)\left(q\frac{f_2}{f_1}f^2(-q,-q^9)\right)\\ &\equiv q\frac{f_2^2}{f_1^2}f_{10}^4\frac{G^2(q^2)H^2(q^2)}{G^2(q)H^2(q)} \equiv q\frac{f_{10}^6}{f_5^2} \pmod 3,\end{aligned} \tag{5.110}$$

which implies that

$$\sum_{n=0}^{\infty} D(5n+1)q^n \equiv \left(\frac{f_2^3}{f_1}\right)^2 \pmod 3. \tag{5.111}$$

The proof of (5.27) follows from (5.62) and (5.111).

Again, extracting the terms involving $q^{5n+r}$, $r = 0, 2, 3, 4$ from (5.110), we complete the proof of (5.30). □

*Proof of* (5.29). From (5.69), we have the following

$$\sum_{n=0}^{\infty} b_6(125n+5)q^n \equiv f_{25}\left(T(q^5) - q - \frac{q^2}{T(q^5)}\right)\Big(\varphi(-q^{25})$$

$$- 2qf(-q^{15}, -q^{35}) + 2q^4 f(-q^5, -q^{45})\Big) \pmod 3,$$

from which we extract

$$\sum_{n=0}^{\infty} b_6(625n + 380)q^n \equiv 2f_5 f(-q^3, -q^7) \frac{f(-q, -q^4)}{f(-q^2, -q^3)} \pmod 3$$

and

$$\sum_{n=0}^{\infty} b_6(625n + 605)q^n \equiv 2f_5 f(-q, -q^9) \frac{f(-q^2, -q^3)}{f(-q, -q^4)} \pmod 3.$$

Thus, from the above two equations and Lemma 2.8, we have

$$\sum_{n=0}^{\infty} C(n)q^n = \left(\sum_{n=0}^{\infty} b_6(625n + 380)q^n\right)\left(\sum_{n=0}^{\infty} b_6(625n + 605)q^n\right)$$

$$\equiv f_5^2 f(-q^3, -q^7) f(-q, -q^9) \tag{5.112}$$

$$\equiv f_5^2 f_{10}^2 \frac{G(q^2)H(q^2)}{G(q)H(q)} \equiv f_5 f_{10}^3 \chi(-q) \pmod 3.$$

Employing (5.55) in the above and then extracting, we find that

$$\sum_{n=0}^{\infty} C(5n + 4)q^n \equiv f_5^2 f(-q^3, -q^7) f(-q, -q^9) \pmod 3. \tag{5.113}$$

Now, (5.29) follows from (5.112) and (5.113). □

*Proof of* (5.31). From (5.62) and (5.68), we have

$$\sum_{n=0}^{\infty} E(n)q^n = \left(\sum_{m=0}^{\infty} b_6(m)q^m\right)\left(\sum_{m=0}^{\infty} b_6(125m + 5)q^m\right)$$

$$\equiv \frac{f_1^3 f_2^3}{f_1 f_2} \equiv \frac{f_3 f_6}{f_1 f_2} \pmod 3. \tag{5.114}$$

Also, it is known from Chan [41] that if $\sum_{n=0}^{\infty} a(n)q^n = \dfrac{1}{f_1 f_2}$, then

$$\sum_{n=0}^{\infty} a(3n + 2)q^n = 3\frac{f_3^3 f_6^3}{f_1^4 f_2^4}. \tag{5.115}$$

The proof of (5.31) is evident from (5.114) and (5.115). □

## 5.6 Proofs of (5.17) and (5.18)

We prove (5.17) and (5.18) by employing a method of Radu [91]. The background materials required for the method has been presented in Subsection 3.5.1 of Chapter

3.

Proofs of (5.17) and (5.18) are similar. Hence, we elaborate the proof of (5.17) only. We have

$$\sum_{n=0}^{\infty} b_6(n)q^n = \frac{f_6}{f_1} \equiv \frac{f_1^8 f_6}{f_3^3} \pmod 9. \tag{5.116}$$

Using Conditions 1–5 of Subsection 3.5.1, we have $(m, M, N, t, (r_\delta)) = (125, 6, 6, 30, (8, 0, -3, 1)) \in \Delta^*$. So, by (3.57), we obtain $P(t) = \{30, 105\}$. Lemma 3.7 gives that $\left\{ \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} : \delta \mid N \right\}$ is a complete set of representatives of the double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Using $(r'_\delta) = (1, 0, 0, 0)$, (3.58), and *Mathematica*, we find that

$$p\left(\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}\right) + p'\left(\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}\right) \geq 0 \quad \text{for all } \delta \mid N,$$

$$\lfloor \nu \rfloor = 3,$$

$$b_6(125n + j) \equiv 0 \pmod 9 \quad \text{for } j \in \{30, 105\},$$

are true for all $0 \leq n \leq \lfloor \nu \rfloor$. Therefore, by Lemma 3.8 and (5.116), for all $n \geq 0$, we have

$$b_6(125n + j) \equiv 0 \pmod 9 \quad \text{for } j \in \{30, 105\}.$$

Again, if we choose $(m, M, N, t, (r_\delta)) = (125, 6, 6, 55, (8, 0, -3, 1))$, then by (3.57), we have $P(t) = \{55, 80\}$. Taking $(r'_\delta) = (1, 0, 0, 0)$, $\lfloor \nu \rfloor = 3$ and proceeding as above, for all $n \geq 0$, we have

$$b_6(125n + j) \equiv 0 \pmod 9 \quad \text{for } j \in \{55, 80\}.$$

Thus, we complete the proof of (5.17).

The proof of (5.18) follows analogously from Lemma 3.8 and the following table.

| Congruence | $(m, M, N, t, (r_\delta))$ and $(r'_\delta)$ | $P(t)$ | $\lfloor \nu \rfloor$ |
|---|---|---|---|
| (5.6) | $(15625, 6, 6, 130, (8, 0, -3, 1))$ and $(1, 0, 0, 0)$ | $\{130, 6380\}$ | 3 |
| | $(15625, 6, 6, 9505, (8, 0, -3, 1))$ and $(1, 0, 0, 0)$ | $\{9505, 12630\}$ | 2 |

## 5.7 Proof of Theorem 5.12

We prove Theorem 5.12 using elementary $q$-series techniques, remniscent of the proof of the result of Majid and Fathima [83]. But, before that we need the following result.

**Theorem 5.29.** *For all integers $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(5^{\alpha+1}n + 5^{\alpha+1} - 4)q^n = \mathcal{A}_\alpha f_5^2 f_1^{14} + B_\alpha q f_5^8 f_1^8 + C_\alpha q^2 f_5^{14} f_1^2 + D_\alpha q^3 \sum_{n=0}^{\infty} \mathcal{A}_{5,4}(n)q^n, \tag{5.117}$$

*where $A_0 = 4$, $B_0 = 550$, $C_0 = 12500$, $D_0 = 78125$, and for any integer $n \geq 1$, $A_n$, $B_n$, $C_n$, and $D_n$ are defined as*

$$A_n = -C_{n-1} + 4D_{n-1}, \tag{5.118}$$

$$B_n = -125B_{n-1} + 550D_{n-1}, \tag{5.119}$$

$$C_n = -15625A_{n-1} + 12500D_{n-1}, \tag{5.120}$$

$$D_n = D_0^{n+1}. \tag{5.121}$$

*Proof.* From equation (5.32), we have

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(n)q^n = \frac{f_5^{20}}{f_1^4}.$$

From equation (5.61), we have

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(5n+1)q^n = 4f_5^2 f_1^{14} + 550q f_5^8 f_1^8 + 12500q^2 f_5^{14} f_1^2 + 78125q^3 \frac{f_5^{20}}{f_1^4}$$
$$= 4f_5^2 f_1^{14} + 550q f_5^8 f_1^8 + 12500q^2 f_5^{14} f_1^2 + 78125q^3 \sum_{n=0}^{\infty} \mathcal{A}_{5,4}(n)q^n. \tag{5.122}$$

Equation (5.122), is the case for $\alpha = 0$.

Now assume that the result holds for all values up to $\alpha + 1$ ($\alpha \geq 0$). Replacing $n$ by $5n + 4$, and by using Lemmas 5.22, 5.23, 5.24, and (5.122), we have

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(5^{\alpha+2}n + 5^{\alpha+2} - 4)q^n$$
$$= A_\alpha(-15625q^2 f_1^2 f_5^{14}) + B_\alpha(-125q f_1^8 f_5^8) + C_\alpha(-f_1^{14} f_5^2) + D_\alpha(4f_5^2 f_1^{14}$$

$$+ 550qf_5^8f_1^8 + 12500q^2f_5^{14}f_1^2 + 78125q^3\sum_{n=0}^{\infty}\mathcal{A}_{5,4}(n)q^n)$$

$$= (-C_\alpha + 4D_\alpha)f_1^{14}f_5^2 + (-125B_\alpha + 550D_\alpha)qf_1^8f_5^8$$

$$+ (-15625A_\alpha + 12500D_\alpha)q^2f_1^2f_5^{14} + D_\alpha 78125q^3\sum_{n=0}^{\infty}\mathcal{A}_{5,4}(n)q^n$$

$$= A_{\alpha+1}f_1^{14}f_5^2 + B_{\alpha+1}qf_1^8f_5^8 + C_{\alpha+1}q^2f_1^2f_5^{14} + D_{\alpha+1}q^3\sum_{n=0}^{\infty}\mathcal{A}_{5,4}(n)q^n.$$

Hence, the result is true by induction. □

Now, we are in a position to prove Theorem 5.12.

*Proof of Theorem 5.12.* From equations (5.118), (5.119), (5.120), and (5.121), we see that

$$\begin{array}{llll}
A_1 \equiv 0 \pmod{5^5}, & B_1 \equiv 0 \pmod{5^6}, & C_1 \equiv 0 \pmod{5^6}, & D_1 \equiv 0 \pmod{5^7}, \\
A_2 \equiv 0 \pmod{5^6}, & B_2 \equiv 0 \pmod{5^7}, & C_2 \equiv 0 \pmod{5^7}, & D_2 \equiv 0 \pmod{5^8}, \\
\vdots & \vdots & \vdots & \vdots \\
A_\alpha \equiv 0 \pmod{5^{\alpha+4}}, & B_\alpha \equiv 0 \pmod{5^{\alpha+5}}, & C_\alpha \equiv 0 \pmod{5^{\alpha+5}}, & D_\alpha \equiv 0 \pmod{5^{\alpha+6}}.
\end{array}$$

Now, it is easy to see that (5.117) implies Theorem 5.12. □

## 5.8 Proofs of Theorems 5.13–5.15

*Proof of Theorem 5.13.* From the generating function of $\mathcal{A}_{p,p^Ni-1}(n)$, we have

$$\sum_{n=0}^{\infty}\mathcal{A}_{p,p^Ni-1}(n)q^n = \frac{f_p^{p(p^Ni-1)}}{f_1^{p^Ni-1}} \equiv \frac{f_p^{p(p^Ni-1)}}{f_p^{p^{N-1}i}}f_1 \pmod{p^N}.$$

With the help of equation (4.24), we obtain

$$\sum_{n=0}^{\infty}\mathcal{A}_{p,p^Ni-1}(n)q^n \equiv \frac{f_p^{p(p^Ni-1)}}{f_p^{p^{N-1}i}}\left(\sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m-1)/2}\right) \pmod{p^N}.$$

For some $m$ and $n$, we are interested in finding out whether $m(3m-1)/2 = pn + r$. This is equivalent to asking whether $24pn + 24r + 1 = (6m-1)^2$, which implies $24r + 1 \equiv (6m-1)^2 \pmod{p}$. However $24r + 1$ is a quadratic non residue modulo

$p$. It follows that

$$\mathcal{A}_{p,p^N i-1}(pn+r) \equiv 0 \pmod{p^N}.$$

□

*Proof of Theorem 5.14.* Like before, we have

$$\sum_{n=0}^{\infty} \mathcal{A}_{p,p^N i-3}(n)q^n = \frac{f_p^{p(p^N i-3)}}{f_1^{p^N i-3}} \equiv \frac{f_p^{p(p^N i-3)}}{f_p^{p^{N-1} i}} f_1^3 \pmod{p^N}.$$

With the help of equation (3.66), we obtain

$$\sum_{n=0}^{\infty} \mathcal{A}_{p,p^N i-3}(n)q^n \equiv \frac{f_p^{p(p^N i-3)}}{f_p^{p^{N-1} i}} \left( \sum_{m \geq 0} (-1)^m (2m+1) q^{m(m+1)/2} \right) \pmod{p^N}. \quad (5.123)$$

For some $m$ and $n$, we are interested in finding out whether $m(m+1)/2 = pn+r$. This is equivalent to asking whether $8pn + 8r + 1 = (2m+1)^2$, which implies $8r+1 \equiv (2m+1)^2 \pmod{p}$. However $8r+1$ is a quadratic nonresidue modulo $p$. It follows that

$$\mathcal{A}_{p,p^N i-}(pn+r) \equiv 0 \pmod{p^N}.$$

□

*Proof of Theorem 5.15.* Due to equations (3.66) and (5.123), we must determine whether $pn+r = m(m+1)/2$ for some integers $m$ and $n$. Completing the square and considering the result modulo $p$ gives $(2m+1)^2 \equiv 8r+1 \equiv 0 \pmod{p}$. Therefore, $p$ divides $(2m+1)^2$, implying that $p$ divides $2m+1$. Since the coefficient of $q^{m(m+1)/2}$ in the series representation in equation (3.66) is exactly $2m+1$, it follows that the coefficient we are interested in is congruent to 0 modulo $p$. □

## 5.9 Proof of Theorem 5.16

The proofs of the congruences are similar in nature. So, we only present proofs of (5.40) and (5.41). For others, we just give the corresponding generating functions.

*Proofs of* (5.40) *and* (5.41). First, we prove (5.40). We have,

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(n)q^n = \frac{f_5^{20}}{f_1^4} = f_5^{20} \sum_{n=0}^{\infty} P_4(n)q^n.$$

Then, extracting the terms involving $q^{5n+1}$ and dividing by $q$ and then replacing $q^5$ by $q$, we obtain

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(5n+1)q^n = f_1^{20} \sum_{n=0}^{\infty} P_4(5n+1)q^n.$$

With the help of Lemma 5.21, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(5n+1)q^n &= f_1^{20}\left(4\frac{f_5^2}{f_1^6} + 550q\frac{f_5^8}{f_1^{12}} + 12500q^2\frac{f_5^{14}}{f_1^{18}} + 78125q^3\frac{f_5^{20}}{f_1^{24}}\right)\\ &= 4f_5^2 f_1^{14} + 550q f_5^8 f_1^8 + 12500q^2 f_5^{14} f^2 + 78125q^3\frac{f_5^{20}}{f_1^4}.\end{aligned}$$

Using equations (1.17) and (1.16), extracting the terms involving $q^{5n+4}$, dividing by $q$ and then replacing $q^5$ by $q$, we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(25n+21)q^n = 5^5\Bigg(&-4f_1^{14}f_5^2 + 100\frac{f_5^{20}}{f_1^4 T^{15}(q)} + q\left(10450\frac{f_5^{20}}{f_1^4 T^{10}(q)} - 22f_1^8 f_5^8\right)\\ &+ q^2\left(46000\frac{f_5^{20}}{f_1^4 T^5(q)} - 62500f_1^2 f_5^{14}\right) + 25375q^3\frac{f_5^{20}}{f_1^4}\\ &- 46000q^4\frac{f_5^{20}T^5(q)}{f_1^4} + 10450q^5\frac{f_5^{20}T^{10}(q)}{f_1^4} - 100q^6\frac{f_5^{20}T^{15}(q)}{f_1^4}\Bigg),\end{aligned}$$

which on usage of (5.60) reduces to

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(25n+21)q^n = 5^5\left(96f_1^{14}f_5^2 + 13728qf_1^8f_5^8 + 312480q^2f_1^2f_5^{14} + 1953125q^3\frac{f_5^{20}}{f_1^4}\right),$$

which implies (5.40).

Proceeding in a similar way, we can also deduce

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{A}_{5,4}(125n+121)q^n = 5^6\Bigg(&1500004f_1^{14}f_5^2 + 214500550qf_1^8f_5^8 + 4882512500q^2f_1^2f_5^{14}\\ &+ 30517578125q^3\frac{f_5^{20}}{f_1^4}\Bigg)\end{aligned}$$

which proves (5.41). □

Now, we note the following generating functions which will complete the proofs of the other congruences stated in the result:

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,2}(25n+23)q^n = 5^2\left(48f_1^4f_5^4 + 625q\frac{f_5^{10}}{f_1^2}\right),$$

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,2}(125n+123)q^n = 5^3\left(1202f_1^4f_5^4 + 15625q\frac{f_5^{10}}{f_1^2}\right),$$

$$\sum_{n=0}^{\infty}\mathcal{A}_{5,3}(25n+22)q^n = 5\left(5838f_1^9f_5^3+233250qf_1^3f_5^9+1953125q^2\frac{f_5^{15}}{f_1^3}\right),$$
$$\sum_{n=0}^{\infty}\mathcal{A}_{5,3}(125n+122)q^n = 5^2\left(3643791f_1^9f_5^3+145754625qf_1^3f_5^9+1220703125q^2\frac{f_5^{15}}{f_1^3}\right).$$

This completes the proof of Theorem 5.16.

## 5.10 Proofs of Theorem 5.17 and Corollary 5.18

*Proof of Theorem 5.17.* Without loss of generality, we may assume that $r = \sum_{j=0}^{M-1} p^j r_j$ for $0 \le r_j \le p-1$, as $\sum_{j=0}^{M-1} p^j r_j$ can take any value between 1 and $p^M - 1$.

For integers $M \ge 1$ (sufficiently large) and $N \ge 1$, we have

$$\sum_{n=0}^{\infty}\mathcal{A}_{p,p^{M+N-1}i+k}(n)q^n = \frac{f_p^{p(p^{M+N-1}i+k)}}{f_1^{p^{M+N-1}i+k}} = \frac{f_p^{p^{M+N}i}}{f_1^{p^{M+N-1}i}}\sum_{n=0}^{\infty}\mathcal{A}_{p,k}(n)q^n$$
$$\equiv f_p^{p^{M+N-2}(p^2-1)i}\sum_{n=0}^{\infty}\mathcal{A}_{p,k}(n)q^n \pmod{p^N}.$$

Extracting the terms that involve $q^{pn+r_0}$ from the above identity, we obtain

$$\sum_{n=0}^{\infty}\mathcal{A}_{p,p^{M+N-1}i+k}(pn+r_0)q^n \equiv f_1^{p^{M+N-2}(p^2-1)i}\sum_{n=0}^{\infty}\mathcal{A}_{p,k}(pn+r_0)q^n$$
$$\equiv f_p^{p^{M+N-3}(p^2-1)i}\sum_{n=0}^{\infty}\mathcal{A}_{p,k}(pn+r_0)q^n \pmod{p^N}.$$

Again, extracting the terms that involve $q^{pn+r_1}$ from the above identity, we have

$$\sum_{n=0}^{\infty}\mathcal{A}_{p,p^{M+N-1}i+k}(p^2n+r_0+pr_1)q^n \equiv f_1^{p^{M+N-3}(p^2-1)i}\sum_{n=0}^{\infty}\mathcal{A}_{p,k}(p^2n+r_0+pr_1)q^n$$
$$\equiv f_p^{p^{M+N-4}(p^2-1)i}\sum_{n=0}^{\infty}\mathcal{A}_{p,k}(p^2n+r_0+pr_1)q^n \pmod{p^N}.$$

From the above identity, we extract the terms that contain $q^{pn+r_2}$, and from the resulting identity, we again extract the terms that contain $q^{pn+r_3}$ and so on. It can be seen that after the $M$-th extraction using this iterative scheme, we arrive at

$$\sum_{n=0}^{\infty}\mathcal{A}_{p,p^{M+N-1}i+k}(p^Mn+r_0+pr_1+\cdots+p^{M-1}r_{M-1})q^n$$

$$\equiv f_1^{p^{N-1}(p^2-1)i} \sum_{n=0}^{\infty} \mathcal{A}_{p,k}(p^M n + r_0 + pr_1 + \cdots + p^{M-1} r_{M-1}) q^n \pmod{p^N}. \quad (5.124)$$

Therefore, if we assume that $\mathcal{A}_{p,k}(p^M n + r_0 + pr_1 + \cdots + p^{M-1} r_{M-1}) = \mathcal{A}_{p,k}(p^M n + r) \equiv 0 \pmod{p^N}$, then from the above identity, we have

$$\mathcal{A}_{p,p^{M+N-1}i+k}(p^M n + r) \equiv 0 \pmod{p^N}.$$

This completes the proof of Theorem 5.17. □

**Remark 5.30.** *We have the following easy corollary, which follows from equation* (5.124) *when* $M = 1$.

**Corollary 5.31.** *Let* $p$ *be a prime,* $k \geq 1$, $j \geq 0$, $N \geq 1$, *and* $r$ *be integers such that* $1 \leq r \leq p-1$. *Then for all* $n \geq 0$, *we have*

$$\sum_{n=0}^{\infty} \mathcal{A}_{p,p^N i+k}(pn + r) q^n \equiv f_1^{p^{N-1}(p^2-1)i} \sum_{n=0}^{\infty} \mathcal{A}_{p,k}(pn + r) q^n \pmod{p^N}.$$

*Proof of Corollary 5.18.* The proofs of the above congruences follow from Theorems 5.16 and 5.17 and are similar in nature. Hence, here we only present the proof of (5.48).

The case for $i = 0$ is true by (5.40).

Using Theorem 5.17, and the case for $i = 0$, we deduce that

$$\sum_{n=0}^{\infty} \mathcal{A}_{5,125i+4}(25n + 21) q^n \equiv \sum_{n=0}^{\infty} \mathcal{A}_{5,4}(25n + 21) q^n \equiv 0 \pmod{25},$$

which completes the proof. □

## 5.11 Concluding remarks

1. In Theorem 5.6, we have proved recurrence relations for $b_6(n)$ modulo 3 using elementary $q$-series and theta functions manipulations, whereas we have deduced the individual congruences of $b_6(n)$ modulo 9 in (5.17) and (5.18) employing algorithmic techniques based on the theory of modular forms. It may be interesting to prove (5.17) and (5.18) via elementary approach. Another question for further consideration is to find whether Theorems 5.6 and 5.7 hold for modulo 9 or not.

2. We have found several congruences modulo powers of 5, individual as well as infinite families similar to those stated in Theorem 5.17, for $\mathcal{A}_{5,k}(n)$ for higher values of $k$. The proofs of these are routine execises similar to the proofs of (5.40) and (5.41) hence, they are not proved here.

   For instance, the following congruences are true:

$$\mathcal{A}_{5,6}(25n+14,19,24) \equiv 0 \pmod{5^2}, \tag{5.125}$$

$$\mathcal{A}_{5,6}(125n+119) \equiv 0 \pmod{5^3}, \tag{5.126}$$

$$\mathcal{A}_{5,7}(25n+13,18,23) \equiv 0 \pmod{5^2}, \tag{5.127}$$

$$\mathcal{A}_{5,7}(125n+118) \equiv 0 \pmod{5^3}, \tag{5.128}$$

$$\mathcal{A}_{5,5^2i+6}(25n+14,19,24) \equiv 0 \pmod{5}, \tag{5.129}$$

$$\mathcal{A}_{5,5^3i+6}(25n+14,19,24) \equiv 0 \pmod{5^2}, \tag{5.130}$$

$$\mathcal{A}_{5,5^5i+6}(125n+119) \equiv 0 \pmod{5^3}, \tag{5.131}$$

$$\mathcal{A}_{5,5^2i+7}(25n+13,18,23) \equiv 0 \pmod{5}, \tag{5.132}$$

$$\mathcal{A}_{5,5^3i+7}(25n+13,18,23) \equiv 0 \pmod{5^2}, \tag{5.133}$$

$$\mathcal{A}_{5,5^5i+7}(125n+118) \equiv 0 \pmod{5^3}. \tag{5.134}$$

3. Looking at the sequence of results in (5.33), (5.34) and Theorem 5.12, one interesting problem may be to study the behaviour of
   $\mathcal{A}_{5,5i+k}(5^\alpha n + 5^\alpha - k) \pmod{5^\alpha}$ where $k \in \{2,3,4\}$.

4. Experiments suggest some additional infinite family of congruences modulo powers of 5 which are stronger results than those given in Theorem 5.17. We present them in the following conjecture.

   **Conjecture 5.32.** *For $n \geq 0$,*

$$\mathcal{A}_{5,5i+1}(25n+24) \equiv 0 \pmod{5^2}, \tag{5.135}$$

$$\mathcal{A}_{5,5^2i+2}(25n+23) \equiv 0 \pmod{5^2}, \tag{5.136}$$

$$\mathcal{A}_{5,5^3i+2}(125n+123) \equiv 0 \pmod{5^3}, \tag{5.137}$$

$$\mathcal{A}_{5,5^3i+3}(125n+122) \equiv 0 \pmod{5^2}, \tag{5.138}$$

$$\mathcal{A}_{5,5^4i+4}(125n+121) \equiv 0 \pmod{5^5}, \tag{5.139}$$

$$\mathcal{A}_{5,5^5i+4}(125n+121) \equiv 0 \pmod{5^6}, \tag{5.140}$$

$$\mathcal{A}_{5,5^2i+6}(25n+14,19) \equiv 0 \pmod{5^2}, \tag{5.141}$$

$$\mathcal{A}_{5,5^2i+6}(125n+119) \equiv 0 \pmod{5^3}, \tag{5.142}$$

$$\mathcal{A}_{5,5^2i+7}(25n+13,18,23) \equiv 0 \pmod{5^2}, \tag{5.143}$$

$$\mathcal{A}_{5,5^3i+7}(125n+118) \equiv 0 \pmod{5^3}. \tag{5.144}$$