"Physicists have come to realize that mathematics, when used with sufficient care, is a proven pathway to truth."

— Brian Greene

CHAPTER 2

BK EQUATION AND ITS ANALYTICAL SOLUTION

In this chapter, we study the Balitsky-Kovchegov (BK) evolution equation, which is a very important nonlinear parton evolution equation at small-x. We present an approximate analytical solution of the BK equation using a method called the homotopy perturbation method (HPM). Based on the obtained solution, we extract the gluon saturation momentum and compare it with the numerical analysis of the BK equation. The solution presented in this chapter follows published work, ***"An analytical solution of Balitsky-Kovchegov equation using homotopy perturbation method", International Journal of Modern Physics A 37 (31n32) (2022): 2250190.***

2.1 Introduction

The QCD evolution equations are a valuable tool in high-energy physics phenomenology, detailing the progression of parton distribution functions (PDFs) within a nuclear medium. The evolution of PDFs with respect to x and Q^2 can be analyzed under various limits of QCD. At sufficiently large scales where $Q^2 >> \Lambda^2_{QCD}$, the Q^2 evolutions can be obtained from the DGLAP evolution equation utilizing perturbative

QCD. Conversely, at small-x, indicative of a high energy limit, the evolution in x is characterized by the BFKL evolution equation [1, 2].When the center of mass energy in a collision significantly exceeds the fixed hard scale (for instance, in deep inelastic scattering (DIS), where the photon virtuality represents the hard scale and $s >> Q^2$), parton densities augment with increasing energy. As collision energy escalates, parton densities correspondingly increase with rising energy levels. HERA data confirms a significant increase in gluon density at small-x. As energy escalates, the gluon density increases more rapidly at small-x, resulting in an exponential development of the total cross-section with energy. The rapid growth of gluons must be moderated to maintain the unitarity constraint on the total cross-section, as established by Froissart and Martin [3]. Consequently, at exceedingly high energies, the BFKL equation contravenes the Froissart-Martin bound; therefore, its applicability is constrained. This equation is not applicable at too high energies.

To address the above problems faced by the BFKL equation, we need to look for solutions to the related problems. The solution is that at high energy and small-x, partons themselves start to recombine and get saturated. The first idea of parton-parton recombination was addressed in Ref. [4–8]. Some analytical solutions of nonlinear QCD evolution equations incorporating parton-parton recombination can be found in Ref. [9–12]. The parton-parton recombination will tame down the gluon density in the high gluon density region of the scattering process. The BFKL equation, being linear, could not address this nonlinear effect of parton recombination and saturation and hence was unable to explain underlying physics at high-density QCD. It is imperative to understand the implicit physics in saturation regions of partons at small-x. In this region, nonlinear QCD evolution equations come into play, which helps in understanding the physics in that region. Therefore, to describe the parton-parton recombination and saturation effect, the linear evolution equations have to be replaced by the nonlinear QCD evolution equations. The nonlinear evolution equations have important features dealing with the saturation effect. They contain damping terms that reflect the saturation effect arising out of parton-parton recombination. So, studying nonlinear QCD evolution equations and their solutions is crucial for phenomenological studies.

The JIMWLK (Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner) equation [13–16] permits gluon saturation in a high gluon density region. The JIMWLK equation addresses the nonlinear correction using the Wilson renormalization group approach. Nevertheless, it is complicated to solve the JIMWLK equation because of its complex nature. Instead, its mean-field approximation BK equation [17–20] has been widely used in the context of the saturation effect. Because of its simple nature, the BK equation can be solved numerically. However, it is tough to solve the BK equation using general methods. It is an integrodifferential equation in coordinate space that can be transformed into momentum space, resulting in a partial differential equation. Resolving the BK equation in momentum space is advantageous for phenomenological analyses related to several high-energy hadron scattering investigations. Notwithstanding many numerical investigations, a precise analytical solution to this problem continues to be unattainable owing to its intricate character.

This chapter presents an analytical solution to the BK equation using the homotopy perturbation method (HPM) in relation to the FKPP (Fisher-Kolmogorov-Petrovsky-Piscounov) equation. The FKPP equation is a partial differential equation classified as a reaction-diffusion equation in statistical physics. The small-x geometric scaling phenomenon observed at HERA can be associated with the traveling wave solution of the FKPP equation [25]. The seminal research in Ref. [26–28] demonstrates that the BK equation in momentum space may be converted into the FKPP equation through a variable transformation [26–28]. The transition of the scattering amplitude into the saturation zone resembles the creation of the front of the traveling wave in the FKPP equation [26]. The BK evolution equation and its solution may prove beneficial for subsequent phenomenological investigations in the context of current and forthcoming accelerator facilities.

The chapter is organized as follows: Section 2.2 provides a concise overview of the BK equation. An overview of the HPM is provided in Section 2.3. An analytical solution to the BK equation utilizing the HPM is shown in Section 2.4. We graph the scattering amplitude derived from the analytical solution of the BK equation at different rapidities and compare it with the scattering amplitude obtained numerically. Additionally, we derive the saturation momentum from the acquired solution and compare it with the numerical analysis of the BK equation. Section 2.5 is dedicated to the discussion and summary of the chapter.

2.2 BK Equation

The BK equation is formulated by I. Balitsky within the context of the effective theory of high-energy interactions and, independently, by Y. V. Kovchegov utilizing the color dipole model formalism. The BK equation pertains to the energy dependence of scattering amplitude; it is frequently advantageous to conduct analyses inside the pQCD dipole framework of DIS. [29–32]. For clarity, we examine deep inelastic scattering of a virtual photon on a hadron or nucleus. In the dipole model, the system is analyzed from the reference frame of the target, whereby the hadron or nucleus remains stationary. Consequently, all QCD evolution is encompassed inside the wave function of the incident photon. The primary benefit of the dipole model of DIS is the decomposition of the scattering process into multiple stages. In the dipole model, an incoming virtual photon, upon fluctuation, transforms into a quark-antiquark dipole. The quark-antiquark pair subsequently interacts with the target proton and recombines to produce final state particles. The color dipole representation of electron-proton deep inelastic scattering is schematically illustrated in Figure 2.1. The transverse dimension of the quark-antiquark pair is denoted as $\mathbf{r}$, whereas $\mathbf{b}$ represents the impact parameter for the dipole-proton interaction. The quark possesses a fraction of momentum (z) of the photon's light-cone momentum, while the antiquark possesses $(1 - z)$.

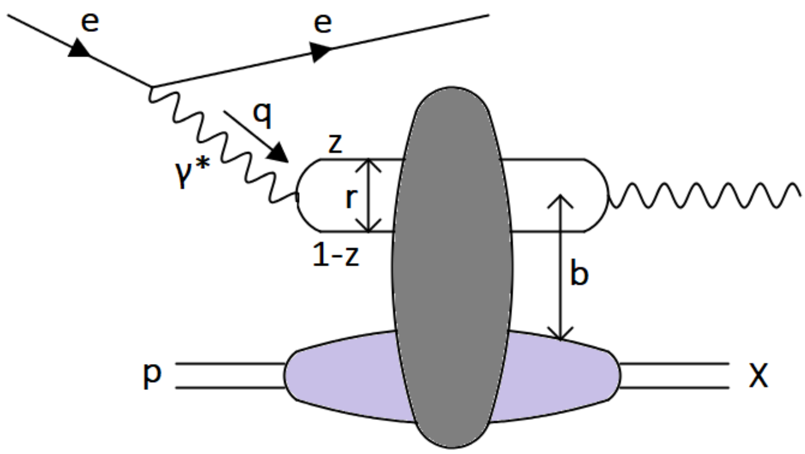

Figure 2.1: Color dipole picture of e-p DIS.

When the dipole is accelerated to a higher rapidity or supplied with additional energy, an increased phase space becomes accessible, facilitating the emission of a gluon from the dipole. If the gluon is real, the parent dipole is partitioned into two dipoles of sizes r' and $r - r'$. These dipoles persist in evolving and interacting autonomously from the target proton. Gluon emission constitutes a higher-order modification to the virtual photon wave function, and we anticipate gaining insight into the energy dependence of the dipole-proton scattering amplitude. We may derive an evolution equation for the dipole-proton scattering amplitude by accounting for virtual corrections and interactions with the target proton, leading to the evolution of the parent dipole as

$$\frac{\partial}{\partial Y}(r, b, Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 r' \frac{r^2}{r'^2 (r - r')^2} \Big[N(r', b + \frac{r - r'}{2}, Y) + N(r - r', \frac{r'}{2}, Y) - N(r, b, Y) - N(r', b + \frac{r - r'}{2}, Y) N(r - r', \frac{r'}{2}, Y) \Big]. \tag{2.1}$$

This is the BK evolution equation that was first derived by Balitsky in Ref. [13] and by Kovchegov in Ref. [18]. The equation (2.1) is an integro-differential equation that gives the scattering amplitude $N(r, Y)$ at any given rapidity $Y > 0$ if the initial condition $N(r, Y = 0)$ is known.

The BK equation (2.1) possesses a clear physical significance: A colorless dipole of size r decays at a constant rapidity into two dipoles of sizes r' and $r - r'$. The initial two linear terms after the equal sign indicate that one dipole evolves and interacts with the target while the other remains passive, or the nonlinear term suggests that both dipoles evolve and interact with the target. The inclusion of two separate interactions inflates the results, as evidenced by the negative sign preceding the nonlinear term. At elevated rapidities, this nonlinear term becomes essential, resulting in saturation and a reduction in energy growth.

In the computation of the BK equation (2.1), it is justifiable to presume that when b fluctuates over distance scales comparable to the dipole size $|r|$, the change in the amplitude $N(r, b, Y)$ with respect to the impact parameter b is minimal. This is indeed accurate for scattering on a significantly large nucleus that is far from its periphery. The variations in the impact parameter on the right side of equation (2.1) can be disregarded due to this assumption. Additionally, we can disregard the angular dependence of r on the assumption that the nucleus is isotropic. Therefore, we may substitute $N(r,b,Y)$ with roughly $N(r,Y)$ in equation (2.1), yielding

$$\frac{\partial}{\partial Y}N(r,Y) = \frac{\alpha_s N_c}{2\pi^2}\int d^2r' \frac{r^2}{r'^2(r-r')^2} \times [N(r') + N(r-r') - N(r) - N(r')N(r-r')]. \tag{2.2}$$

Let us transform this coordinate space BK equation into momentum space performing the following Fourier transformation

$$N(r,Y) = r^2 \int \frac{d^2k}{2\pi} e^{ik.r} N(k,Y), \tag{2.3}$$

we write [19]

$$\frac{\partial N(k,Y)}{\partial Y} = \bar{\alpha}_s \chi(-\partial_L)N(k,Y) - \bar{\alpha}_s N^2(k,Y), \tag{2.4}$$

where $\bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$ and $\chi(\xi) = 2\psi(1) - \psi(\xi) - \psi(1-\xi)$ is the BFKL kernel with $\xi = -\partial_L$, where $L = \ln(k^2/k_0^2)$ with k_0 being some fixed low momentum scale. The equation (2.4) is useful for obtaining approximate solutions for the BK equation that we will present in this chapter.

The expansion of the BFKL kernel around $\xi = \frac{1}{2}$ has been suggested in Ref. [26], and with this expansion equation (2.4) reduces to the nonlinear partial differential equation given by

$$\partial_Y N = \bar{\alpha}\bar{\chi}(-\partial_L)N - \bar{\alpha}N^2, \tag{2.5}$$

where

$$\bar{\chi}(-\partial_L) = \chi(\frac{1}{2}) + \frac{\chi''(\frac{1}{2})}{2}\left(\partial_L + \frac{1}{2}\right)^2. \tag{2.6}$$

In reference to the above expansion and defining $\bar{\xi} = 1 - \frac{1}{2}\sqrt{1 + 8\frac{\chi(\frac{1}{2})}{\chi''(\frac{1}{2})}}$, with the following change of variables [26]

$$t = \frac{\bar{\alpha}\chi''(\frac{1}{2})}{2}(1-\bar{\xi})^2 Y, \quad x = (1-\bar{\xi})\left(L + \frac{\bar{\alpha}\chi''(\frac{1}{2})}{2}Y\right),$$

$$u(t,x) = \frac{2}{\chi''(\frac{1}{2})(1-\bar{\xi})^2} \times N\left(\frac{2t}{\bar{\alpha}\chi''(\frac{1}{2})(1-\bar{\xi})^2}, \frac{x}{1-\bar{\xi}} - \frac{t}{(1-\bar{\xi})^2}\right),$$

the equation (2.5) turns into the FKPP equation [23, 24] for $u(x,t)$, can be expressed as [26]

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + u(t,x) - u^2(t,x). \tag{2.7}$$

Thus, with some variable transformation, it is seen that the BK equation (2.5) can be transformed to the above equation (2.7), which is the famous FKPP equation.

2.3 Homotopy Perturbation Method

The homotopy perturbation method (HPM) is a widely used method to get analytical solutions to numerous linear to nonlinear ordinary and partial differential equations, first proposed by Ji-Huan He. This method is a combined sort of both homotopy in topology and also the traditional perturbation technique. Recently, this method has been widely used to solve different problems in engineering, physics, and other fields [33–39]. One can see the original papers for a better understanding of the HPM [21, 22]. For completeness, let us move to the fundamental idea of the HPM and consider the following differential equation nonlinear in nature as

$$A(q) - f(r) = 0, \quad r \in \Lambda, \tag{2.8}$$

constrained to the boundary precondition

$$B(q, \frac{\partial q}{\partial n}) = 0, \quad r \in \lambda, \tag{2.9}$$

where A is a general differential operator, B is a boundary operator, and $f(r)$ is a known analytic function, and λ is the boundary of the domain Λ.

The general differential operator A within the equation (2.8) can be divided into two parts: the linear part L, which is simple to handle, and the remaining nonlinear part N. Thus, equation (2.8) can be expressed in the following form:

$$L(q) + N(q) - f(r) = 0. \tag{2.10}$$

Now, applying the homotopy technique, one can construct a homotopy $h(r, p)$, which satisfies

$$H(h, p) = (1 - p)[L(h) - L(q_0)] + p[A(h) - f(r)] = 0, \tag{2.11}$$

where $p \in [0, 1]$ is an embedding parameter, and q_0 is an initial guess for the equation (2.8) satisfying the initial boundary condition(s). We have

$$\begin{aligned} p &= 0, \quad H(h, 0) = L(h) - L(q_0) = 0, \\ p &= 1, \quad H(h, 1) = A(h) - f(r) = 0. \end{aligned} \tag{2.12}$$

According to the homotopy technique, we can use the embedding parameter p as an expanding parameter, provided p is a small parameter. Then we can assume the solution of equation (2.11) as a power series in p as

$$h = h_0 + ph_1 + p^2 h_2 + \dots \tag{2.13}$$

Setting $p = 1$ within the above equation, the approximate solution of equation (2.11) is

$$q = \lim_{p \to 1} h = h_0 + h_1 + h_2 + \dots \tag{2.14}$$

In general, the equation (2.14) is a convergent series and hence results in the precise solution of the equation (2.8).

2.4 Analytical Solution to BK Equation

In this section, we provide an analytical solution to the BK equation (2.5) for the scattering amplitude $N(k,Y)$. Since BK equation (2.5) can be transformed to FKPP equation (2.7), we write the following equation for the scattering amplitude $N(k,Y)$ in connection with BK equation (2.5)

$$\partial_Y N(k,Y) = \partial_k^2 N(k,Y) + N(k,Y) - N^2(k,Y), \tag{2.15}$$

constrained to a initial condition (say)

$$N(k,0) = A. \tag{2.16}$$

Using the HPM discussed in the section 2.3, we can construct the homotopy:

$$\frac{\partial N}{\partial Y} - \frac{\partial N_0}{\partial Y} = p\left(\frac{\partial^2 N}{\partial^2 k} + N(1-N) - \frac{\partial N_0}{\partial Y}\right), \tag{2.17}$$

and using equation (2.13), we get the approximate solution of equation (2.17) in series as

$$N = N_0 + pN_1 + p^2 N_2 + p^3 N_3 + \dots \quad . \tag{2.18}$$

Equating the coefficients of powers of p, one can obtain the following set of differential equations on substitution of equations (2.16) and (2.18) into equation (2.17)

:

$$
\begin{aligned}
&p^0 : \frac{\partial N_0}{\partial Y} - \frac{\partial N_0}{\partial Y} = 0, \;\; N_0(k,0) = A, \\
&p^1 : \frac{\partial N_1}{\partial Y} = \frac{\partial^2 N_0}{\partial^2 k} + N_0 - N_0^2, \;\; N_1(k,0) = 0, \\
&p^2 : \frac{\partial N_2}{\partial Y} = \frac{\partial^2 N_1}{\partial^2 k} - N_0 N_1 + N_1(1 - N_0), \;\; N_2(k,0) = 0, \\
&p^3 : \frac{\partial N_3}{\partial Y} = \frac{\partial^2 N_2}{\partial^2 k} - N_0 N_2 + N_1^2 + N_2(1 - N_0), \;\; N_3(k,0) = 0, \\
&\vdots \\
&\text{etc.}
\end{aligned}
\tag{2.19}
$$

Proceeding in this way to solve the system, we obtain

$$
\begin{aligned}
&N_0(k,Y) = A, \\
&N_1(k,Y) = A(1-A)Y, \\
&N_2(k,Y) = A(1-A)(1-2A)\frac{Y^2}{2!}, \\
&N_3(k,Y) = A(1-A)(1-6A+A^2)\frac{Y^3}{3!}, \\
&\vdots \\
&\text{etc.}
\end{aligned}
\tag{2.20}
$$

Setting $p = 1$ in equation (2.18), we can approximate the solution of equation (2.18) as $N = N_0 + N_1 + N_2 + N_3 + \ldots$. Thus, in view of the equation (2.20), we can express the solution of equation (2.18) in a series form as

$$
\begin{aligned}
N(k,Y) = A + A(1-A)Y + A(1-A)(1-2A)\frac{Y^2}{2!} + \\
A(1-A)(1-6A+A^2)\frac{Y^3}{3!} + \ldots \quad .
\end{aligned}
\tag{2.21}
$$

The above solution can be put into the following simplified form using some algebra with the help of symbolic computation tool as

$$N(k, Y) = \frac{Ae^Y}{1 - A + Ae^Y}. \tag{2.22}$$

This is the solution of the BK equation (2.5) in connection with the equation (2.7) for the scattering amplitude $N(k, Y)$. Once the initial condition is known to us, the solution of the BK equation gives the scattering amplitude $N(k, Y)$ at any given rapidity $Y > 0$. In this work, we will use the following initial condition given by K. Golec-Biernat and M. Wüsthoff (GBW), introduced first in Ref. [40] as

$$N^{GBW}(r, Y = 0) = 1 - exp\left[-\left(\frac{r^2 Q_{s0}^2}{4}\right)\right]. \tag{2.23}$$

Q_{s0}^2 is the fit parameter, called the initial saturation scale squared. This initial condition has been chosen since we are working with the BK equation in momentum space, and it is simple to conduct an analytical Fourier transformation of this initial condition into momentum space. The momentum space outcome of the GBW initial condition can be written as

$$\begin{aligned} N^{GBW}(k, Y = 0) &= \int \frac{d^2 r}{2\pi r^2} e^{ik.r} N^{GBW}(r, Y = 0) \\ &= \frac{1}{2}\Gamma\left(0, \frac{k^2}{Q_{s0}^2}\right). \end{aligned} \tag{2.24}$$

$\Gamma(0, k^2/Q_{s0}^2)$ is the incomplete gamma function. At large values of k^2/Q_{s0}^2, this behaves as

$$\Gamma\left(0, \frac{k^2}{Q_{s0}^2}\right) = exp\left(-\frac{k^2}{Q_{s0}^2}\right).$$

Therefore,

$$N^{GBW}(k, Y = 0) = \frac{1}{2} exp\left(-\frac{k^2}{Q_{s0}^2}\right). \tag{2.25}$$

Substitution of the above equation in (2.22) for the initial condition, we obtain the scattering amplitude $N(k, Y)$ with GBW as the initial condition

$$N(k, Y) = \frac{e^{Y - k^2/Q_{s0}^2}}{1 - e^{-k^2/Q_{s0}^2} + e^{Y - k^2/Q_{s0}^2}}. \tag{2.26}$$

The aforementioned equation serves as an approximate analytical solution to the BK equation (2.5). The scattering amplitude $N(k, Y)$ characterizes the traversal of a quark-antiquark dipole across the target color field inside the dipole model. The evolution of the scattering amplitude at different rapidities can be seen in Figure 2.2.

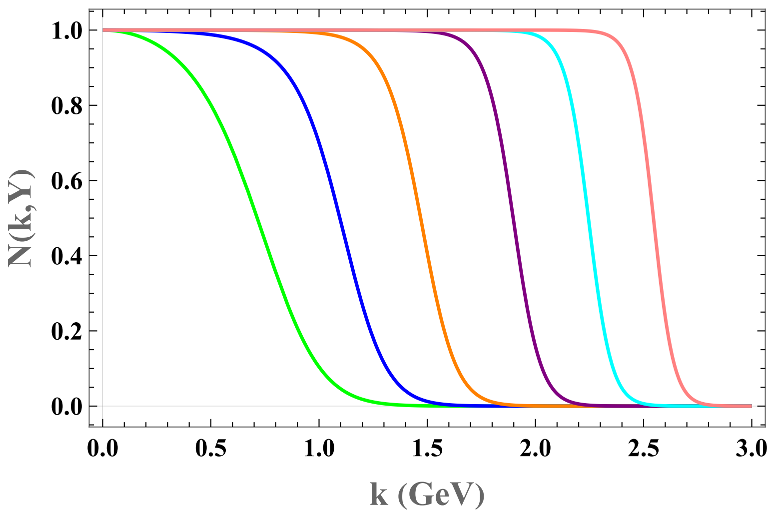

Figure 2.2: The solution of the BK equation in momentum space, $N(k)$, at various rapidities $Y = 2$, $Y = 5$, $Y = 9$, $Y = 15$, $Y = 21$ and $Y = 27$.

Let us now extract the saturation momentum $Q_s^2(Y)$, which is a function of rapidity Y, using the solution obtained in the equation (2.26) by requiring the following condition given in Ref. [41] as

$$N(k, Y) = \frac{1}{2} \quad \text{for} \quad k = Q_s. \tag{2.27}$$

We extracted the saturation momentum $Q_s^2(Y)$ for different rapidities Y and expressed in Table 2.1. The extracted saturation momentum Q_s^2 is plotted as a function of Y in the Figure 2.4. The saturation momentum as a function of rapidity is estimated in the numerical analysis of the BK equation given by [42]

$$Q_s^2(Y) = \Lambda^2 exp(\Delta'\sqrt{Y+X}), \tag{2.28}$$

where $X = (\Delta')^{-2}\ln(Q_0^2/\Lambda^2)$ and $\Delta' = 3.2$. We plot this estimate of saturation momentum in Figure 2.4.

Y	0	0.5	1	1.5	2
Q_s^2 (GeV^2)	0.1663	0.3659	0.4678	0.6643	0.7958

Table 2.1: The extracted saturation momentum Q_s^2 from (2.26) at different rapidities Y.

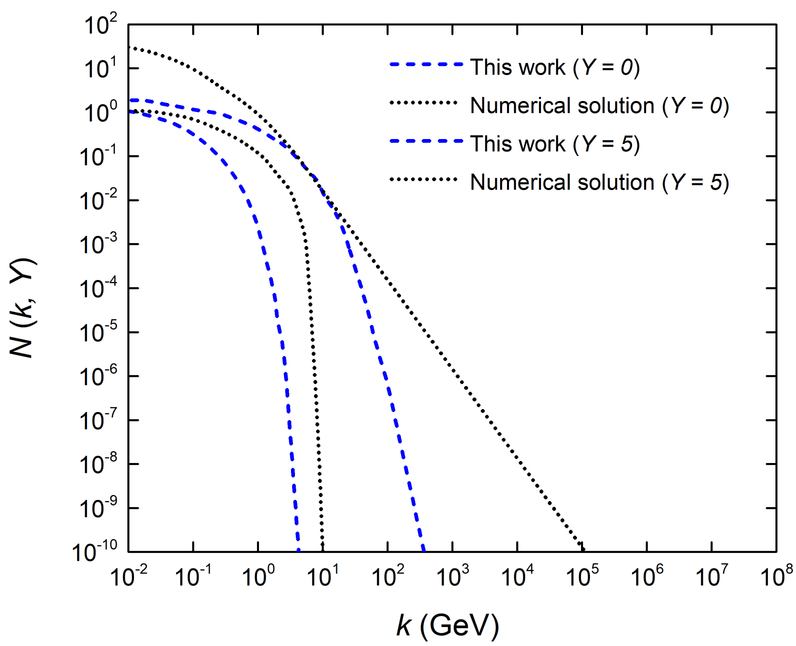

Figure 2.3: Scattering amplitude $N(k, Y)$ obtained in this work compared to the scattering amplitude $N(k, Y)$ obtained from numerical solution [43] of the BK equation.

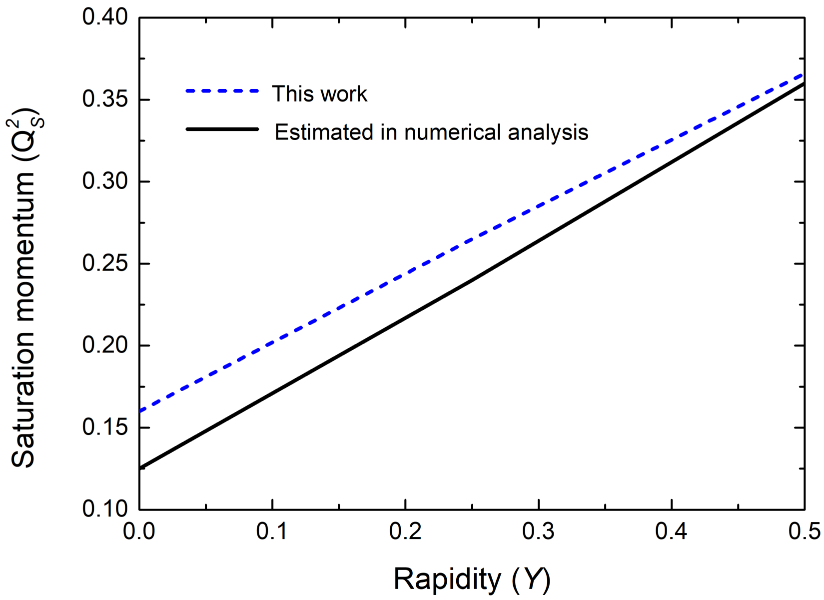

Figure 2.4: Extracted saturation momentum Q_s^2 from the solution of the BK equation in this work (dashed line) at different rapidities Y, compared to the estimated saturation momentum by numerical analysis of the BK equation [42] (solid line).

2.5 Summary

This chapter provided an approximate analytical solution for the BK equation using the HPM. The connection between the geometric scaling phenomena of solution of the BK equation and the travelling wave solution of the FKPP equation, proposed by S. Munier and R. Peschanski in their seminal work, has influenced the scientific community working in the area of saturation physics. In this chapter, we began with a quick overview of the BK equation and its relationship to the FKPP equations. We carried out studies on the pQCD dipole model of DIS, where the measured scattering amplitude $N(k, Y)$ obeys the BK equation in momentum space. The momentum space frame is often regarded as the most natural space frame for working with at least a travelling wave solution and geometric scaling. Afterward, with some change of variables and a slight approximation in the BK equation, we obtained the approximated analytical solution of the BK equation in momentum space. We displayed the resulting solution, equation (2.26), at different rapidities in Figure 2.2 to check the solution's travelling wave nature. Indeed, the solution has the appearance of a travelling wave. It illustrates that at high energies, the scattering amplitude acts like a wave, moving from the region $N = 1$ to $N = 0$ as k grows without changing the profile. This is indeed a vital physical result of this travelling wave approach.

To contrast the analytical analysis of this study with the numerical analysis of the BK equation, Figure 2.3 illustrates a comparison of the scattering amplitude's characteristics with the numerical findings of the BK equation. We observed a comparable characteristic of the scattering amplitude (traveling wave) with the scattering amplitude calculated by the numerical solution of the BK equation [43]. Additionally, we derived the saturation momentum from the acquired solution to examine the characteristics of saturation momentum as rapidity varies. Figure 2.4 presents a comparison between the saturation momentum derived from this study and that estimated from the numerical analysis of the BK equation [42]. We observed a strong correlation between our findings and the estimates derived from the numerical study of the BK equation. Figure 2.4 illustrates that the saturation momentum escalates with increasing rapidity. Consequently, the analytical solution (2.26) helps elucidate parton saturation events in high-density QCD.

The solution derived in this study may assist in subsequent phenomenological investigations of high-density QCD and saturation domains. It will be intriguing to ascertain the existence of this form of traveling wave solution and geometric scaling at extremely high energy during the operational phase of the Electron-Ion Collider (EIC) and other forthcoming projects. Nonetheless, the BK equation, when truncated with the BFKL kernel, effectively elucidates the observed geometric scaling and the traveling wave characteristics of its solution at contemporary accelerator facilities.

We must depend on forthcoming accelerator facilities for accurate measurements of observed phenomena and their validation.

Bibliography

[1] EA Kuraev, LN Lipatov, and VS Fadin. The pomeranchuk singularity in nonabelian gauge theories. *Sov. Phys. JETP*, 45(199):3, 1977.

[2] Yu Yu Balitsky and LN Lipatov. The pomeranchuk singularity in quantum chromodynamics. *Yad. Fiz.*, 28:1597–1611, 1978.

[3] Marcel Froissart. Asymptotic behavior and subtractions in the mandelstam representation. *Physical Review*, 123(3):1053, 1961.

[4] LV Gribov, EM Levin, and MG Ryskin. Singlet structure function at small x: Unitarization of gluon ladders. *Nuclear Physics B*, 188(3):555–576, 1981.

[5] Leonid Vladimirovič Gribov, Eugene M Levin, and Michail G Ryskin. Semi-hard processes in qcd. *Physics Reports*, 100(1-2):1–150, 1983.

[6] Alfred H Mueller and Jianwei Qiu. Gluon recombination and shadowing at small values of x. *Nuclear Physics B*, 268(2):427–452, 1986.

[7] Alfred H Mueller. Small-x behavior and parton saturation: A qcd model. *Nuclear Physics B*, 335(1):115–137, 1990.

[8] Wei Zhu and Jianhong Ruan. A new modified altarelli-parisi evolution equation with parton recombination in proton. *Nuclear Physics B*, 559(1-2):378–392, 1999.

[9] M Lalung, P Phukan, and JK Sarma. Nonlinear effects in gluon distribution predicted by glr-mq evolution equation at next-to-leading order in lhc data. *International Journal of Theoretical Physics*, 56(11):3625–3637, 2017.

[10] P Phukan, M Lalung, and JK Sarma. Nnlo solution of nonlinear glr–mq evolution equation to determine gluon distribution function using regge like ansatz. *Nuclear Physics A*, 968:275–286, 2017.

[11] Pragyan Phukan, Madhurjya Lalung, and Jayanta Kumar Sarma. Studies on gluon evolution and geometrical scaling in kinematic constrained unitarized bfkl equation: application to high-precision hera dis data. *The European Physical Journal C*, 79(6):1–25, 2019.

[12] M Lalung, P Phukan, and JK Sarma. Small-x analysis on the effect of gluon recombinations inside hadrons in light of the glr-mq-zrs equation. *Nuclear Physics A*, 992:121615, 2019.

[13] Ian Balitsky. Operator expansion for high-energy scattering. *Nuclear Physics B*, 463(1):99–157, 1996.

[14] Jamal Jalilian-Marian, Alex Kovner, Andrei Leonidov, and Heribert Weigert. The bfkl equation from the wilson renormalization group. *Nuclear Physics B*, 504(1-2):415–431, 1997.

[15] Edmond Iancu, Andrei Leonidov, and Larry McLerran. Nonlinear gluon evolution in the color glass condensate: I. *Nuclear Physics A*, 692(3-4):583–645, 2001.

[16] Heribert Weigert. Unitarity at small bjorken x. *Nuclear Physics A*, 703(3-4):823–860, 2002.

[17] Ian Balitsky. Operator expansion for diffractive high-energy scattering. In *AIP Conference Proceedings*, volume 407, pages 953–957. American Institute of Physics, 1997.

[18] Yuri V Kovchegov. Small-x f 2 structure function of a nucleus including multiple pomeron exchanges. *Physical Review D*, 60(3):034008, 1999.

[19] Yuri V Kovchegov. Unitarization of the bfkl pomeron on a nucleus. *Physical Review D*, 61(7):074018, 2000.

[20] Ian Balitsky. Effective field theory for the small-x evolution. *Physics Letters B*, 518(3-4):235–242, 2001.

[21] Ji-Huan He. Homotopy perturbation technique. *Computer methods in applied mechanics and engineering*, 178(3-4):257–262, 1999.

[22] Ji-Huan He. A coupling method of a homotopy technique and a perturbation technique for non-linear problems. *International journal of non-linear mechanics*, 35(1):37–43, 2000.

[23] Ronald Aylmer Fisher. The wave of advance of advantageous genes. *Annals of eugenics*, 7(4):355–369, 1937.

[24] Petrovskii Kolmogorov and Piskunov. A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem. *Bull. Moscow Univ. Math. Mech.*, 1.

[25] AM Staśto, K Golec-Biernat, and J Kwieciński. Geometric scaling for the total γ* p cross section in the low x region. *Physical Review Letters*, 86(4):596, 2001.

[26] Stéphane Munier and Robi Peschanski. Geometric scaling as traveling waves. *Physical Review Letters*, 91(23):232001, 2003.

[27] Stéphane Munier and Robi Peschanski. Traveling wave fronts and the transition to saturation. *Physical Review D*, 69(3):034008, 2004.

[28] Stéphane Munier and Robi Peschanski. Universality and tree structure of high-energy qcd. *Physical Review D*, 70(7):077503, 2004.

[29] Nikolai N Nikolaev and BG Zakharov. Colour transparency and scaling properties of nuclear shadowing in deep inelastic scattering. In *30 Years Of The Landau Institute—Selected Papers*, pages 733–744. World Scientific, 1996.

[30] Alfred H Mueller. Soft gluons in the infinite-momentum wave function and the bfkl pomeron. *Nuclear Physics B*, 415(2):373–385, 1994.

[31] Alfred H Mueller and Bimal Patel. Single and double bfkl pomeron exchange and a dipole picture of high energy hard processes. *Nuclear Physics B*, 425(3):471–488, 1994.

[32] Alfred H Mueller. Unitarity and the bfkl pomeron. *Nuclear Physics B*, 437(1):107–126, 1995.

[33] Ji-Huan He. Application of homotopy perturbation method to nonlinear wave equations. *Chaos, Solitons & Fractals*, 26(3):695–700, 2005.

[34] Ji-Huan He. Homotopy perturbation method for solving boundary value problems. *Physics letters A*, 350(1-2):87–88, 2006.

[35] Ahmet Yıldırım. Solution of bvps for fourth-order integro-differential equations by using homotopy perturbation method. *Computers & Mathematics with Applications*, 56(12):3175–3180, 2008.

[36] A Golbabai and B Keramati. Modified homotopy perturbation method for solving fredholm integral equations. *Chaos, Solitons & Fractals*, 37(5):1528–1537, 2008.

[37] Zaid Odibat and Shaher Momani. A reliable treatment of homotopy perturbation method for klein–gordon equations. *Physics Letters A*, 365(5-6):351–357, 2007.

[38] MSH Chowdhury and I Hashim. Application of homotopy-perturbation method to klein–gordon and sine-gordon equations. *Chaos, Solitons & Fractals*, 39(4):1928–1935, 2009.

[39] J a nd Biazar and H Ghazvini. Exact solutions for non-linear schrödinger equations by he's homotopy perturbation method. *Physics Letters A*, 366(1-2):79–84, 2007.

[40] K Golec-Biernat and M Wüsthoff. Saturation effects in deep inelastic scattering at low q 2 and its implications on diffraction. *Physical Review D*, 59(1):014017, 1998.

[41] Edmond Iancu and Raju Venugopalan. The color glass condensate and high energy scattering in qcd. In *Quark–Gluon Plasma 3*, pages 249–363. World Scientific, 2004.

[42] JL Albacete, N Armesto, JG Milhano, CA Salgado, and UA Wiedemann. Numerical analysis of the balitsky-kovchegov equation with running coupling: Dependence of the saturation scale on nuclear size and rapidity. *Physical Review D*, 71(1):014003, 2005.

[43] JT Amaral, MA Betemps, Gay Ducati, and Gregory Soyez. Bk equation and traveling wave solutions. *Brazilian journal of physics*, 37:648–651, 2007.

[44] A Accardi, JL Albacete, M Anselmino, N Armesto, EC Aschenauer, A Bacchetta, D Boer, WK Brooks, T Burton, N-B Chang, et al. Electron-ion collider: The next qcd frontier. *The European Physical Journal A*, 52(9):1–100, 2016.