Chapter-5

NONLINEAR ANALYSIS OF REGULAR PLASMA FIREBALL SHEATH INSTABILITY

Abstract: *The steady plasma fireball sheath (PFBS) instability is studied herein using bifluidic plasma model with spherical geometry. The host plasma is assumed to undergo higher-order nonlinear local perturbation in terms of the various plasma parameters about their hydrostatic homogeneous equilibrium values. The fluid governing equations are perturbed through the substitution of these perturbed plasma parameters of required order. The individual governing equations are finally substituted in the closure Poisson equation, yielding a non-homogeneous differential equation of variable coefficients. An order-by-order numerical analysis of these perturbed governing equations generate the spatially variant dependent plasma parameters of the corresponding order. This chapter analyses the perturbed plasma parameters up to their fourth-order. The dependent plasma parameters comprise of potential, Mach number, number density, and electric field. Increasing the order of perturbation is noticed to steepen spatial variation of the perturbed plasma parameters, i.e., nonlinearity acts as a fluctuation steepening (anti-dispersing) agent*[‡]*. The obtained peakonic multi-parametric behaviours are observed to be well-corroborating with experimental reporting. The various applications of PFBS research in both pure and applied fields are summarily highlighted.*

5.1 INTRODUCTION

The fundamental research on plasma-electrode interaction mechanisms primarily involves the basic physics of plasma sheath formation dynamics and active cross-border effects. The research has, since, initiated originally by Dr. I. Langmuir in 1923, continued to widen with different laboratory arrangements [1]. The fundamental philosophy of active plasma sheath has been extremely useful in the overall scientific progress in this direction achieved extensively in diverse astrolabcosmic plasma explorations [2]. One of the most interesting outcomes of plasma-electrode interactions in such circumstances is realizable through the formation of a plasma sheath region enveloping a plasma-embedded electrode (anode) and its encompassing regular fireball (RFB) evolutionary structures [3, 4].

It is noteworthy that both the sheath and DL are hosts of numerous nonlinear instabilities in the plasma fireball (FB) system, such as secondary ionisation instability,

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electron transit time instability [5], Rayleigh-Taylor instability (RTI) [6], etc. The free energy for the eruption of the RTI in a PFBS system originates from the stiff equilibrium charge density gradient across it. The density gradient results from the un-even intrinsic electric field strength across the sheath [6]. In other words, one may speculate that the PFBS instability source is seeded in the existence of the zeroth-order non-zero plasma currents in the system. Consequently, there exists such free energies and hence, associated instabilities, and an extensive plethora of collective wave phenomena with diversified utilities in multiscale domains in different orders [3]. However, it is observed that the PFBS instability dynamics, particularly on the nonlinear multi-order parametric regime, has still been lying as an open challenge yet to be dealt with for years [2]. This serves as the main motivation of the current work highlighting the nonlinear multi-order PFBS fluctuation dynamics of practical value and subsequent eigen-pattern analyses.

We, hence, further intend to explore the nonlinearities involved with the various plasma parameters in the PFBS operation zones. The FB events are spatiotemporally unstable in both laboratory [7] and astrophysical [8] environs. An appropriate model formalism to see the multi-order nonlinear PFBS instability dynamics against this backdrop is hence needed. A nonlinear bifluidic plasma model approach is adopted herein for the nonlinear stability analysis. It self-consistently reduces the entire PFBS system into a unique set of inhomogeneous differential equations on the multi-order perturbations with multiparametric variable coefficients. This indeed is the main novelty of this study founded on the nonlinear multi-order instability patterns in the PFBS context.

After an imposition of relevant physical boundary conditions [9], the new plasma parametric fluctuation patterns are obtained with the help of numerical software programming. It yields mainly atypical nonlinear PFBS eigenmode structures as a unique peakon family, tangibly corroborating with the experimental findings on direct FB sheath system, already reported elsewhere [3]. The analytical facts and faults of our theoretic investigation are finally discussed together with a refined indication to its future applicability in diverse astrolabcosmic PFBS circumstances.

5.2 PHYSICAL MODEL AND ANALYTIC FORMALISM

A plasma FB sheath formed around a spherical electrode is considered herein in the framework of a basic bifluidic plasma model on the relevant laboratory scales of space and time. The plasma medium is assumed to be exclusively composed of electrons and ions with meagre neutral species in a hydrostatic homogeneous equilibrium configuration maintaining a global quasi-neutrality. The FB is assumed to be spherically symmetric in geometrical structure so as to simplify the problem as a 1-D one with a one radial degree of freedom only. It hereby enables us to get rid of the complications likely to be originating from the polar and azimuthal degrees of freedom. The closure of the bifluidic model is obtained with the electrostatic Poisson equation giving the net electrostatic potential distribution arising from the individual charge density fields, and so forth.

We are, herein, interested in analysing the asymptotic steady-state evolution of the PFBS system on the various sensible orders of nonlinearity in a spherically symmetric geometry. It means the temporal fluctuations of the PFBS system are ignored ($\partial/\partial\tau = 0$), but all the spatial counter parts are retained ($\partial/\partial\xi \neq 0$). Here, the effective radial distance (ξ) is normalized with respect to the Debye length as $\xi = r/\lambda_D$; where, the plasma Debye length is given in generic notations by $\lambda_D = \sqrt{(T_e/4\pi n_e^2)}$. The normalized continuity and momentum equations (Eqs. (3.6)-(3.9)) depicting the electronic dynamics here are respectively given as

$$M_e\partial_\xi N_e + N_e\partial_\xi M_e + \left(\frac{2}{\xi}\right) M_e N_e = 0, \tag{5.1}$$

$$N_e\partial_\xi \Phi = N_e M_e \left(\frac{m_e}{m_i}\right) \partial_\xi M_e + \partial_\xi N_e. \tag{5.2}$$

Similarly, the normalized set of equations governing the ion dynamics are cast respectively as

$$M_i\partial_\xi N_i + N_i\partial_\xi M_i + \left(\frac{2}{\xi}\right) M_i N_i = 0, \tag{5.3}$$

$$N_i\partial_\xi \Phi = N_i M_i \partial_\xi M_i + \left(\frac{T_i}{T_e}\right) \partial_\xi N_i. \tag{5.4}$$

Lastly, the electrostatic Poisson equation coupling the electron and ion dynamics is written as

$$\partial_\xi^2 \Phi + \left(\frac{2}{\xi}\right) \partial_\xi \Phi = N_e - N_i. \tag{5.5}$$

Here, $\partial_\xi \equiv \partial/\partial\xi$, $\partial_\xi^2 \equiv \partial^2/\partial\xi^2$, and so forth. The terms $M_{e(i)}(= v_{e(i)}/c_s)$, and $N_{e(i)}(= n_{e(i)}/n_{e(o)})$ denote the normalized electron (ion) Mach number and normalized electron (ion) population density, respectively. Here, $c_s = \sqrt{(T_e/m_i)}$ is the ion-sound phase speed. The electrostatic plasma potential in the normalized form is given as $\Phi = e\phi/T_e$; where, the normalizing parameter, T_e/e, is called the electron thermal potential. The various normalized and normalizing parameters along with their respective typical values are tabulated in Table 5.1. The multiparametric numerical values of the different physical variables of current relevance are estimated with the inputs available in different sources in the literature [10]. Besides, the unnormalized forms of Eqs. (5.1)-(5.5) are available in Chapter-3 (Eqs. (3.1)-(3.5)).

Table 5.1: Adopted normalization scheme and parametric values

S. No.	Normalized parameter	Normalizing parameter	Typical value
1.	Radial distance ($\xi = r/\lambda_D$)	Debye length (λ_D)	3.32×10^{-4} m
2.	Population density ($N_{e(i)} = n_{e(i)}/n_o$)	Equilibrium density (n_o)	5×10^{14} m^{-3}
3.	Mach number ($M_{e(i)} = v_{e(i)}/c_s$)	Ion-sound phase speed (c_s)	10^4 m s^{-1}
4.	Electrostatic potential ($\Phi = e\phi/T_e$)	Electron thermal potential (T_e/e)	1 J C^{-1}
5.	Electric field ($E = e\phi/T_e$)	Electron thermal field ($T_e/e\lambda_D$)	3.01×10^3 N C^{-1}

The relevant plasma parameters ($F_\alpha(\xi)$) are assumed to undergo nonlinear local perturbation in the ϵ-order against their respective normalized equilibrium values (F_o) as

$$F(\xi) = F_o + \sum_{\alpha=1}^{\infty} \epsilon^\alpha F_\alpha(\xi), \tag{5.6}$$

$$F(\xi) = [N_e\ M_e\ N_i\ M_i\ \Phi]^T, \tag{5.7}$$

$$F_o(\xi) = [1\ 0\ 1\ 0\ 0]^T, \tag{5.8}$$

$$F_\alpha(\xi) = [N_\alpha\ M_\alpha\ N_\alpha\ M_\alpha\ \Phi_\alpha]^T. \tag{5.9}$$

Here, ϵ denotes an order (smallness) parameter signifying the strength of nonlinearity [11]. The order-wise analyses of the governing equations (Eqs. (5.1)-(5.5)) yield the following order specific perturbed set of equations.

5.2.1 FIRST-ORDER PERTURBED PARAMETERS

The first-order perturbed plasma parameters of current relevancy are obtained systematically with the help of the order-by-order analysis of Eqs. (5.1)-(5.5) generating the corresponding differential equations given as

$$\partial_\xi M_{e1} + \left(\frac{2}{\xi}\right) M_{e1} = 0, \tag{5.10}$$

$$\partial_\xi \Phi_1 = \partial_\xi N_{e1}, \tag{5.11}$$

$$\partial_\xi M_{i1} + \left(\frac{2}{\xi}\right) M_{i1} = 0, \tag{5.12}$$

$$\partial_\xi \Phi_1 = \left(\frac{T_i}{T_e}\right) \partial_\xi N_{i1}, \tag{5.13}$$

$$\partial_\xi^2 \Phi_1 + \left(\frac{2}{\xi}\right) \partial_\xi \Phi_1 = N_{e1} - N_{i1}. \tag{5.14}$$

The spatial integration of Eqs. (5.10)-(5.13) with respect to ξ yields the following first-order quasilinear plasma parameters presented respectively as

$$M_{e1} = \frac{c_{eM}}{\xi^2}, \tag{5.15}$$

$$N_{e1} = \Phi_1 + c_{eN}, \tag{5.16}$$

$$M_{i1} = \frac{c_{iM}}{\xi^2}, \tag{5.17}$$

$$N_{i1} = \theta_{ei}\Phi_1 + c_{iN}. \tag{5.18}$$

Here, $\theta_{ei}(= T_e/T_i)$ in Eq. (5.18), denotes the electron-to-ion temperature ratio. Further, c_{eM}, c_{iM}, and c_3 are the integration constants arising due to the indefinite spatial integration of Eqs. (5.10)-(5.13). The values of these constants can be determined using the boundary conditions at the local extrema.

The first-order perturbed Poisson equation (Eq. (5.14)) encompasses both the perturbed electron and ion densities, viz., N_{e1} and N_{i1}, respectively. Substituting the expressions for both N_{e1} from Eq. (5.16) and N_{i1} from Eq. (5.18) in Eq. (5.14) within the purview of standard asymptotic local conditions [2], e.g., $\Phi_1(\xi \to \pm\infty) = 0$, etc., the first-order perturbed electrostatic potential is worked out as

$$\Phi_1(\xi) = \frac{c_3}{\xi}. \tag{5.19}$$

It may be noted here that the $1/\xi$-term involved in Eq. (5.19) originates on account of the inevitable geometric curvature effects in the spherical symmetry of the PFBS [12, 13]. Replacing the expression for $\Phi_1(\xi) = c_3/\xi$ from Eq. (5.19) in Eq. (5.16) and Eq. (5.18), we get the spatially varying expressions for N_{e1} and N_{i1} respectively as

$$N_{e1} = \frac{c_3}{\xi} + c_{eN}, \tag{5.20}$$

$$N_{i1} = \theta_{ei}\frac{c_3}{\xi} + c_{iN}. \tag{5.21}$$

The corresponding first-order perturbed electric field is evaluated from Eq. (5.19) as per the fundamental definition of a conservative force field as

$$E_1 = -\partial_\xi \Phi_1 = \frac{c_3}{\xi^2}. \tag{5.22}$$

5.2.2 SECOND-ORDER PERTURBED PARAMETERS

The governing equations (Eqs. (5.1)-(5.5)) are now analysed systematically up to the second-order nonlinear perturbation. The second-order perturbation of Eq. (5.1) yields

$$M_{e1}\partial_\xi N_{e1} + \partial_\xi M_{e2} + \frac{2}{\xi}(M_{e2} + M_{e1}N_{e1}) = 0. \tag{5.23}$$

We, now, substitute the first-order plasma parameters, M_{e1} and N_{e1}, from Eq. (5.15) and Eq. (5.20), respectively in Eq. (5.23). A spatial integration of Eq. (5.23) yields

$$M_{e2} = \frac{c_5}{\xi^2} - \frac{c_3 c_{eM}}{\xi^3}. \tag{5.24}$$

Here, c_5 is an integration constant appearing as a result of the indefinite integration. Redefining the constant $-c_3 c_{eM}$ as C_{eM} in Eq. (5.24), we find a modified form of Eq. (5.24) given as

$$M_{e2} = \frac{c_5}{\xi^2} + \frac{C_{eM}}{\xi^3}. \tag{5.25}$$

Treating Eq. (5.3) alike with Eq. (5.1) for ion-dynamics in its second-order, the corresponding plasma parameter, the second-order perturbed ionic Mach number, M_{i2}, is deduced with $C_{iM} = -\theta_{ei} c_3 c_{iM}$ as

$$M_{i2} = \frac{c_6}{\xi^2} + \frac{C_{iM}}{\xi^3}. \tag{5.26}$$

Moreover, Eq. (5.2) and Eq. (5.4) are perturbed nonlinearly in their respective second-order form. The existent first-order plasma parameters $M_{e1}, N_{e1}, M_{i1}, N_{i1}$, and Φ_1 are now substituted from Eq. (5.15), Eq. (5.20), Eq. (5.17), Eq. (5.21), and Eq. (5.19) in Eq. (5.2) and Eq. (5.4), respectively. The resulting perturbed density differential equations for the electrons and ions are respectively obtained as

$$\partial_\xi N_{e2} = \partial_\xi \Phi_2 - \frac{c_3 c_{eM}}{\xi^2} - \frac{c_3^2}{\xi^3} + 2\left(\frac{m_e}{m_i}\right)\frac{c_{eM}^2}{\xi^5}, \tag{5.27}$$

$$\partial_\xi N_{i2} = \theta_{ei}\left(\partial_\xi \Phi_2 - \frac{c_3 c_{iM}}{\xi^2} - \frac{c_3^2}{\xi^3} + 2\frac{c_{eM}^2}{\xi^5}\right). \tag{5.28}$$

A spatial differentiation (with respect to ξ) of the second-order perturbed form of Eq. (5.5) yields a third-order ODE (Eq. (5.29)) involving both $\partial_\xi N_{e2}$ and $\partial_\xi N_{i2}$ given as

$$\partial_\xi^3 \Phi_2 + \left(\frac{2}{\xi}\right)\partial_\xi^2 \Phi_2 - \left(\frac{2}{\xi^2}\right)\partial_\xi \Phi_2 = \partial_\xi N_{e2} - \partial_\xi N_{i2}. \tag{5.29}$$

Substitution for $\partial_\xi N_{e2}$ from Eq. (5.27) and $\partial_\xi N_{i2}$ from Eq. (5.28) in Eq. (5.29) results in

$$\partial_\xi^3 \Phi_2 + \left(\frac{2}{\xi}\right)\partial_\xi^2 \Phi_2 - \left(\frac{2}{\xi^2}\right)\partial_\xi^2 \Phi_2 + A\partial_\xi \Phi_2 = \frac{B}{\xi^2} + \frac{C}{\xi^3} + \frac{D}{\xi^5}. \tag{5.30}$$

The coefficients involved in Eq. (5.30) are given as

$$A = (\theta_{ei} - 1), \tag{5.31}$$

$$B = c_3(\theta_{ei} c_{iN} + c_{eN}), \tag{5.32}$$

$$C = c_3^2(\theta_{ei} - 1), \tag{5.33}$$

$$D = 2\left[\left(\frac{m_e}{m_i}\right)c_{eM}^2 - \theta_{ei} c_{iM}^2\right]. \tag{5.34}$$

We integrate Eq. (5.30) spatially with respect to ξ to reduce its order of differentiation to second-order. The simplified second-order perturbed electrostatic Poisson equation (Eq. (5.30)) is cast as

$$\partial_\xi^2 \Phi_2 + \left(\frac{2}{\xi}\right)\partial_\xi \Phi_2 + A\Phi_2 + \frac{B}{\xi} + \frac{C}{2\xi} + \frac{D}{4\xi^4} + E = 0; \tag{5.35}$$

Where, E is an integration constant appearing due to the indefinite spatial integration of Eq. (5.30).

The differential equation (Eq. (5.35)) is solved analytically. The solution of Eq. (5.35) is given as follows

$$\Phi_2(\xi) = c_7 \frac{exp(\sqrt{A}\xi)}{\xi} - \frac{c_8}{2\sqrt{-A}} \frac{exp(-\sqrt{-A}\xi)}{\xi} - \frac{1}{\xi}\left[\left(\frac{1}{8\sqrt{-A}\xi^2}\left(\frac{D}{2} + 4\left(\frac{E}{A} - \frac{C}{\sqrt{-A}}\right)\xi^2 - \frac{4E}{\sqrt{-A}}\xi^3 - \frac{\sqrt{-A}D}{2}\xi\right) - \frac{(4B-AD)}{16\sqrt{-A}}\exp\left(\sqrt{-A}\xi\right) Ei\left(-\sqrt{-A}\xi\right)\right)\right] + \frac{1}{2\sqrt{-A}\xi}\left[\frac{1}{4\xi^2}\left(\frac{D}{2} + 4\left(\frac{E}{A} + \frac{C}{\sqrt{-A}}\right)\xi^2 + \frac{4E}{\sqrt{-A}}\xi^3 + \frac{\sqrt{-A}D}{2}\xi\right) - \left(\frac{B}{2} - \frac{AD}{8}\right) Ei\left(\sqrt{-A}\xi\right)\right]. \tag{5.36}$$

Here, $Ei\left(\sqrt{-A}\xi\right)=\int_{-\infty}^{\sqrt{-A}\xi}(\exp(t)/t)\,dt$ and $Ei\left(-\sqrt{-A}\xi\right)=\int_{-\infty}^{-\sqrt{-A}\xi}(\exp(t)/t)dt$ denote the one argument exponential integral function (appears in a while in the analytical solutions generated with specialized numerical software) [14]. As these two integrals fail to converge to any finite value with increasing distance (ξ), hence, the two terms comprising these integrals are ignored. Due to our interest in the localised potential solution around the electrode centre, i.e., $\xi \approx 0$, we further expand the exponentials in Eq. (5.36). The terms which lead to the violation of the boundary conditions ($\Phi_2(\xi \to \pm\infty) = 0$) and source to unnecessary complications (imaginary terms) are neglected. Considering only the physically admissible terms which abide by all necessary conditions, the second-order perturbed electrostatic potential equation (Eq. (5.36)) reduces to

$$\Phi_2(\xi) = \frac{F}{\xi} + \frac{G}{\xi^2}. \tag{5.37}$$

Here, the different involved constants are given as

$$F = c_7 - 1 - \theta_{ei}, \tag{5.38}$$

$$G = \frac{D}{8}. \tag{5.39}$$

There is no integration constant in Eq. (5.37) because of the global quasi-neutrality condition in the considered bulk plasma system. The corresponding second-order perturbed electric field is evaluated from Eq. (5.37) as per the fundamental definition of a conservative force field given as

$$E_2(\xi) = -\partial_\xi \Phi_2 = \frac{F}{\xi^2} + \frac{2G}{\xi^3}. \tag{5.40}$$

A spatial integration of Eqs. (5.27)-(5.28) in the light of Eq. (5.37) yields the algebraic expressions for N_{e2} and N_{i2} respectively presented as

$$N_{e2} = \frac{A_{e1}}{\xi} + \frac{B_{e1}}{\xi^2} + \frac{C_{e1}}{\xi^4} + c_9; \tag{5.41}$$

$$N_{i2} = \frac{A_{i1}}{\xi} + \frac{B_{i1}}{\xi^2} + \frac{C_{i1}}{\xi^4} + c_{10}; \tag{5.42}$$

$$A_{e1} = c_3 c_{eN} + F, \tag{5.43}$$

$$B_{e1} = 2c_3^2 + G, \tag{5.44}$$

$$C_{e1} = -\frac{1}{2}\left(\frac{m_e}{m_i}\right) c_{eM}^2, \tag{5.45}$$

$$A_{i1} = \theta_{ei}(c_3 c_{iN} + F), \tag{5.46}$$

$$B_{i1} = \theta_{ei}(c_3^2 \theta_{ei} + G), \tag{5.47}$$

$$C_{i1} = -\frac{1}{2}\theta_{ei} c_{im}^2. \tag{5.48}$$

In addition to the above, the two distinct integration constants, c_9 and c_{10}, asymptotically should vanish in the local approximation of the considered multi-order parametric fluctuations idealistically.

5.2.3 THIRD-ORDER PERTURBED PARAMETERS

The governing equations (Eqs. (5.1)-(5.5)) are now analysed systematically up to the third-order nonlinear perturbation. Now, the third-order perturbation of Eq. (5.1) yields

$$M_{e1}\partial_\xi N_{e2} + M_{e2}\partial_\xi N_{e1} + \partial_\xi M_{e3} + N_{e1}\partial_\xi M_{e2} + N_{e2}\partial_\xi M_{e1}$$

$$+\frac{2}{\xi}\left(M_{e3}+M_{e2}N_{e1}+M_{e1}N_{e2}\right)=0. \quad (5.49)$$

We now substitute for M_{e1} from Eq. (5.15), N_{e2} from Eq. (5.41), M_{e2} from Eq. (5.25), and N_{e1} from Eq. (5.20) in Eq. (5.49), the second-order perturbed continuity equation takes the new form as

$$\left(\frac{A_{e1}}{\xi^2}+\frac{2B_{e1}}{\xi^2}-\frac{4C_{e1}}{\xi^5}\right)\left(\frac{c_{eM}}{\xi^2}\right)-\left(\frac{c_{eM}c_3}{\xi^3}-\frac{c_5}{\xi^2}\right)\left(\frac{c_3}{\xi^2}\right)-\partial_\xi M_{e3}-\left(c_{eN}+\frac{c_3}{\xi}\right)\left(\frac{3c_{eM}c_3}{\xi^4}-\frac{2c_5}{\xi^3}\right)$$
$$+\left(\frac{A_{e1}}{\xi}+\frac{B_{e1}}{\xi^2}-\frac{C_{e1}}{\xi^4}+c_9\right)\left(\frac{2c_{eM}}{\xi^3}\right)-\left(\frac{2}{\xi}\right)\left(M_{e3}-\left(\frac{c_{eM}c_3}{\xi^3}-\frac{c_5}{\xi^2}\right)\left(c_{eN}+\frac{c_3}{\xi}\right)+\left(\frac{A_{e1}}{\xi}+\frac{B_{e1}}{\xi^2}-\right.\right.$$
$$\left.\left.\frac{C_{e1}}{\xi^4}+c_9\right)\left(\frac{c_{eM}}{\xi^2}\right)\right)=0. \quad (5.50)$$

The various terms in Eq. (5.50) are rearranged so as to yield a first-order differential equation with involved variable coefficients given as

$$\partial_\xi M_{e3}+2\frac{M_{e3}}{\xi}-\frac{A_1}{\xi^4}-\frac{B_1}{\xi^5}+\frac{C_1}{\xi^7}=0; \quad (5.51)$$

$$A_1=c_3c_{eM}c_{eN}-c_3c_5-A_{e1}c_{eM}, \quad (5.52)$$

$$B_1=2(c_3^2-B_{e1})c_{eM}, \quad (5.53)$$

$$C_1=-4c_{eM}C_{e1}. \quad (5.54)$$

A spatial indefinite integration of Eq. (5.51) with respect to ξ yields the following third-order electronic Mach number perturbation as

$$M_{e3}=\frac{c_{11}}{\xi^2}+\frac{A_1}{\xi^3}+\frac{B_1}{2\xi^4}+\frac{C_1}{4\xi^6}; \quad (5.55)$$

Where, c_{11} is an integration constant evaluable with fair boundary conditions.

Similarly, for the ionic dynamics, we substitute for M_{i1} from Eq. (5.17), N_{i2} from Eq. (5.42), M_{i2} from Eq. (5.26), and N_{i1} from Eq. (5.21) in the third-order perturbed form of Eq. (5.3) and integrate it spatially with respect to ξ to yield the third-order perturbed ionic Mach number given as

$$M_{i3} = \frac{c_{12}}{\xi^2} + \frac{A_2}{\xi^3} + \frac{B_2}{2\xi^4} + \frac{C_2}{4\xi^6}; \tag{5.56}$$

Where, c_{12} is a new integration constant and other involved constants read as

$$A_2 = \theta_{ei} c_3 c_{iM} c_{iN} - 3A_{i1} c_{iM} - \theta_{ei} c_3 c_6, \tag{5.57}$$

$$B_2 = 2B_{i1} c_{iM} + 2\theta_{ei}^2 c_3^2 c_{iM}, \tag{5.58}$$

$$C_2 = 4C_{i1} c_{iM}. \tag{5.59}$$

We now perturb Eq. (5.2) up to the third-order of nonlinearity so as to yield what is given below

$$\partial_\xi \Phi_3 + N_{e1} \partial_\xi \Phi_2 + N_{e2} \partial \Phi_1 = \left(\frac{m_e}{m_i}\right) \left(M_{e1} \partial_\xi M_{e2} + M_{e2} \partial_\xi M_{e1} + N_{e1} M_{e1} \partial_\xi M_{e1}\right) + \partial_\xi N_{e3}. \tag{5.60}$$

We substitute for N_{e1} from Eq. (5.20), Φ_2 from Eq. (5.37), N_{e2} from Eq. (5.41), Φ_1 from Eq. (5.19), M_{e1} from Eq. (5.15), and M_{e2} from Eq. (5.25) in Eq. (5.60) so as to obtain the ordinary evolution equation of the third-order perturbed electronic density as

$$\partial_\xi N_{e3} = \partial_\xi \Phi_3 - \left(c_{eN} + \frac{c_2}{\xi}\right)\left(\frac{F}{\xi^2} + \frac{D}{4\xi^3}\right) - \left(\frac{A_{e1}}{\xi} + \frac{B_{e1}}{\xi^2} - \frac{C_{e1}}{\xi^4} + c_9\right)\left(\frac{c_3}{\xi^2}\right) - \left(\frac{m_e}{m_i}\right)\left(\frac{c_{eM}}{\xi^2}\left(\frac{3c_3 c_{eM}}{\xi^4} - \frac{2c_5}{\xi^3}\right) - \left(\frac{c_3 c_{eM}}{\xi^3} + \frac{c_5}{\xi^2}\right)\left(\frac{2c_{eM}}{\xi^3}\right) + \left(c_{eN} + \frac{c_2}{\xi}\right)\left(\frac{c_{eM}}{\xi^2}\right)\left(\frac{2c_{eM}}{\xi^3}\right)\right). \tag{5.61}$$

A systematic rearrangement of Eq. (5.61) yields

$$\partial_\xi N_{e3} = \partial_\xi \Phi_3 - \frac{A_3}{\xi^2} - \frac{B_3}{\xi^3} - \frac{C_3}{\xi^4} + \frac{D_3}{\xi^5} + \frac{F_3}{\xi^6}; \tag{5.62}$$

$$A_3 = c_3 c_9 + c_{eN} F, \tag{5.63}$$

$$B_3 = \frac{1}{4} c_3 c_{eN} D F, \tag{5.64}$$

$$C_3 = \frac{1}{4} C_2 D + c_3 B_{e1}, \tag{5.65}$$

$$D_3 = \left(\frac{m_e}{m_i}\right)(2c_{em}^2 c_{eN} + 4c_5 c_{eM}), \tag{5.66}$$

$$F_3 = c_3\left[2\left(\frac{m_e}{m_i}\right) c_{eM}^2 - c_{e1}\right]. \tag{5.67}$$

Repeating the same perturbation and rearrangement with Eq. (5.4) for the ions, as done in Eq. (5.61), we get

$$\partial_\xi N_{i3} = \theta_{ei}\,\partial_\xi \Phi_3 - \frac{A_4}{\xi^2} - \frac{B_4}{\xi^3} - \frac{C_4}{\xi^4} + \frac{D_4}{\xi^5} + \frac{F_4}{\xi^6}; \tag{5.68}$$

$$A_4 = \theta_{ei}(c_3 c_{10} + c_{iN} F), \tag{5.69}$$

$$B_4 = \theta_{ei}\left(c_3 A_{i1} + \theta_{ei} c_3 F + \frac{1}{4} C_{iN} D\right), \tag{5.70}$$

$$C_4 = -\theta_{ei}\left(B_{i1} c_3 + \frac{1}{4}\theta_{ei} c_3 D\right), \tag{5.71}$$

$$D_4 = -\theta_{ei}(4 c_6 C_{iM} + 2 C_{iM}^2 C_{iN}), \tag{5.72}$$

$$F_4 = \theta_{ei}(2\theta_{ei} c_3 C_{iM}^2 + c_6 C_{i1}). \tag{5.73}$$

The third-order perturbed Poisson equation (Eq. (5.5)) differentiated once further with respect to ξ yields

$$\partial_\xi^3 \Phi_3 + \left(\frac{2}{\xi}\right)\partial_\xi^2 \Phi_3 - \left(\frac{2}{\xi^2}\right)\partial_\xi \Phi_3 = \partial_\xi N_{e3} - \partial_\xi N_{i3}. \tag{5.74}$$

We substitute for $\partial_\xi N_{e3}$ from Eq. (5.62) and $\partial_\xi N_{i3}$ from Eq. (5.68) in Eq. (5.74) to obtain

$$\partial_\xi^3 \Phi_3 + \left(\frac{2}{\xi}\right)\partial_\xi^2 \Phi_3 - \left(\frac{2}{\xi^2}\right)\partial_\xi \Phi_3 + B\partial_\xi \Phi_3 + \frac{A_5}{\xi^2} + \frac{B_5}{\xi^3} + \frac{C_5}{\xi^4} - \frac{D_5}{\xi^5} - \frac{F_5}{\xi^6} = 0; \tag{5.75}$$

$$B = \theta_{ei} - 1, \tag{5.76}$$

$$A_5 = A_4 - A_3, \tag{5.77}$$

$$B_5 = B_4 - B_3, \tag{5.78}$$

$$C_5 = -(C_4 + C_3), \tag{5.79}$$

$$D_5 = D_3 - D_4, \tag{5.80}$$

$$F_5 = F_3 - F_4. \tag{5.81}$$

An indefinite integration of Eq. (5.75), with respect to ξ gives the third-order electrostatic potential fluctuations as

$$\Phi_3(\xi) = c_{13}\frac{\exp(i\sqrt{B}\xi)}{\xi} - c_{14}\frac{\exp(-i\sqrt{B}\xi)}{2i\sqrt{B}\xi} - \frac{ei(i\sqrt{B}\xi)}{2i\sqrt{B}\xi}\left(\frac{B_6}{2} + \frac{iC_6}{3} + \frac{D_6 B}{8} + \frac{iF_6 B^{\frac{3}{2}}}{30}\right)$$

$$-\frac{1}{120i\sqrt{B}\xi^4}\left[\left(4F_6+\left(2iE_6\sqrt{B}+\frac{15}{2}D_6\right)\xi-\left(2BF_6+20C_6-15iD_6\sqrt{B}\right)\xi^2-\frac{i60}{\sqrt{B}}A_6\xi^3\right)\right]-\frac{1}{\xi}\left[ei\left(-i\sqrt{B}\xi\right)\left(\frac{BF_6}{60}-\frac{B_6}{4i\sqrt{B}}+\frac{C_6}{6}+\frac{iD_6\sqrt{B}}{16}\right)\exp\left(i\sqrt{B}\xi\right)-\frac{1}{120i\sqrt{B}\xi^3}\left(4F_6+\left(\frac{15}{2}D_6-2F\sqrt{B}\right)\xi-\left(2F_6B+20C_6+i\frac{15}{6}D_6\sqrt{B}\right)\xi^2+i\frac{60A_6}{\sqrt{B}}\xi^3\right)\right]. \tag{5.82}$$

Repeating the same procedure, as done in the case of $\Phi_2(\xi)$ in Eq. (5.36), we finally deduce the third-order perturbed electrostatic potential as

$$\Phi_3(\xi)=\frac{A_6}{\xi}+\frac{B_6}{\xi^2}+\frac{C_6}{\xi^3}; \tag{5.83}$$

$$A_6=\frac{A_5}{2B}-\frac{A_5}{2\sqrt{B}}+c_{13}, \tag{5.84}$$

$$B_6=\frac{3}{16}D_5+\frac{1}{60\sqrt{B}}(BF_5+10C_5), \tag{5.85}$$

$$C_6=\frac{1}{60}F_5. \tag{5.86}$$

The corresponding third-order perturbed electric field ($E_3(\xi)$) is evaluated from Eq. (5.83) as per the fundamental rule of a conservative force field as

$$E_3(\xi)=-\partial_\xi\Phi_3=\frac{A_6}{\xi^2}+\frac{2B_6}{\xi^3}+\frac{3C_6}{\xi^4}. \tag{5.87}$$

We substitute for $\Phi_3(\xi)$ from Eq. (5.83) in Eq. (5.62) and integrate it spatially with respect to ξ so as to extract third-order perturbed electronic population density

$$N_{e3}=A_6\ln(\xi)+(A_3-B_6)\frac{1}{\xi}+\frac{1}{2}(B_3+C_6)\left(\frac{1}{\xi^2}\right)+\frac{C_3}{3\xi^3}+\frac{D_3}{4\xi^4}+\frac{F_3}{5\xi^5}+c_{15}. \tag{5.88}$$

We take $A_6=0$, because the logarithmic term asymptotically yields an unacceptable divergence ($N_{e3}(\xi\to\pm\infty)=\infty$). Consequently, an admissible form of N_{e3} is given as

$$N_{e3}=\frac{A_{e7}}{\xi}+\frac{B_{e7}}{2\xi^2}+\frac{C_3}{3\xi^3}+\frac{D_3}{4\xi^4}+\frac{F_3}{5\xi^5}+c_{15}; \tag{5.89}$$

$$A_{e7}=A_3-B_6, \tag{5.90}$$

$$B_{e7}=B_3+C_6. \tag{5.91}$$

In addition to the above, the integration constant, c_{15}, asymptotically should vanish in the local approximation of the considered multi-order parametric fluctuations idealistically. Repeating the same for the ionic dynamics with Eq. (5.68), as in the case of N_{e3} in Eq. (5.88), one gets

$$N_{i3} = \frac{A_{i7}}{\xi} + \frac{B_{i7}}{2\xi^2} + \frac{C_4}{3\xi^3} + \frac{D_4}{4\xi^4} + \frac{F_4}{5\xi^5} + c_{16}; \quad (5.92)$$

$$A_{i7} = A_4 - \theta_{ei}B_6, \quad (5.93)$$

$$B_{i7} = B_4 - C_6. \quad (5.94)$$

The new integration constant, c_{16} in Eq. (5.92), should also vanish asymptotically without any loss of generality during the multi-order FB sheath instability dynamics.

5.2.4 FOURTH-ORDER PERTURBED PARAMETERS

The PFBS governing equations (Eqs. (5.1)-(5.5)) are now perturbed nonlinearly up to the fourth-order so as to perceive the effects of associated harmonic generations. The lower-order plasma parameters already deduced previously are now substituted in the perturbed governing equations to derive the fourth-order parameters. The fourth-order perturbation of the electron momentum equation (Eq. (5.2)) now reads as

$$\partial_\xi \Phi_4 + N_{e1}\partial_\xi \Phi_3 + N_{e2}\partial_\xi \Phi_2 + N_{e3}\partial \Phi_1 = \partial_\xi N_{e4} + \left(\frac{m_e}{m_i}\right)\left(M_{e1}\partial_\xi M_{e3} + M_{e2}\partial M_{e2} + M_{e3}\partial_\xi M_{e1} + N_{e1}M_{e1}\,\partial_\xi M_{e2} + N_{e1}M_{e2}\partial_\xi M_{e1} + N_{e2}M_{e1}\partial_\xi M_{e1}\right). \quad (5.95)$$

It is now intended to derive a steady-state evolution equation for the fourth-order perturbed electronic population density, N_{e4}, with the help of a standard method of decoupling among the above various perturbed equations on the plasma parameters systematically. Accordingly, we substitute for N_{e1} from Eq. (5.20), Φ_3 from Eq. (5.83), N_{e2} from Eq. (5.41), Φ_2 from Eq. (5.37), N_{e3} from Eq. (5.89), Φ_1 from Eq. (5.19), M_{e1} from Eq. (5.15), M_{e3} from Eq. (5.55), and M_{e2} from Eq. (5.25) in Eq. (5.95) to obtain the N_{e4}-evolution equation given as

$$\partial_\xi N_{e4} = \partial_\xi \Phi_4 - \frac{A_{e7}}{\xi^2} - \frac{B_{e7}}{\xi^3} - \frac{C_{e7}}{\xi^4} - \frac{D_{e7}}{\xi^5} + \frac{E_{e7}}{\xi^6} + \frac{F_{e7}}{\xi^7} - \frac{G_{e7}}{\xi^9}; \quad (5.96)$$

$$A_{e7} = A_6 c_{eN} + c_3 c_{15}, \tag{5.97}$$

$$B_{e7} = A_6 c_3 + A_{e1} F_2 B_6 c_{eN} - A_{e7} c_3 + \frac{1}{4} c_9 D, \tag{5.98}$$

$$C_{e7} = \frac{1}{4} A_{e1} D + 2B_6 c_3 - 3c_{eN} C_6 + B_{e1} F + \frac{1}{2} B_{e7} c_3 + 3\left(\frac{m_e}{m_i}\right) c_2 c_{eM}, \tag{5.99}$$

$$D_{e7} = \frac{1}{4} B_{e1} D_3 - 3c_3 C_6 + \frac{1}{3} c_3 C_3 + \left(\frac{m_e}{m_i}\right)(4c_5 c_{eM} c_{eN} - 2c_5^2 + 2c_9 c_{eM}^2 + 4c_{11} c_{eM}), \tag{5.100}$$

$$E_{e7} = \frac{1}{4} c_3 D_3 + C_{e1} F - \left(\frac{m_e}{m_i}\right)(5c_3 c_5 c_{eM} + 2c_{eM}(A_1 - c_3 c_5) - 2c_3 c_5 + 5c_3 c_{eM}^2 c_{eN} - 2A_{e1} c_{eM}^2), \tag{5.101}$$

$$F_{e7} = \frac{1}{5} c_3 E_3 + \frac{1}{4} c_{e1} D - \left(\frac{m_e}{m_i}\right)(2B_2 c_{eM} - c_3^2 c_{eM}^2 + B_1 c_{eM} - 3c_3^2 c_{eM} - 2B_{e1} c_{eM}^2), \tag{5.102}$$

$$G_{e7} = \left(\frac{m_e}{m_i}\right)\left(\frac{7}{2} C_1 c_{eM} + 2c_{e1} c_{eM}^2\right). \tag{5.103}$$

A similar mathematical treatment, as already applied in the case of finding a valid expression for N_{e4} in Eq. (5.96), now in Eq. (5.4) followed by a rigorous exercise of elimination and simplification results in the N_{i4}-evolution equation presented in the generic notations given as

$$\partial_\xi N_{i4} = \partial_\xi \Phi_4 - \frac{A_{i7}}{\xi^2} - \frac{B_{i7}}{\xi^3} - \frac{C_{i7}}{\xi^4} - \frac{D_{i7}}{\xi^5} + \frac{E_{i7}}{\xi^6} + \frac{F_{i7}}{\xi^7} - \frac{G_{i7}}{\xi^9}; \tag{5.104}$$

$$A_{i7} = \theta_{ei}(A_6 c_{iN} + c_{10} F), \tag{5.105}$$

$$B_{i7} = \theta_{ei}(A_6 c_3 \theta_{ei} + 2B_6 c_{iN} + A_{i1} F + \frac{1}{4} c_{10} D + A_{i7} c_3, \tag{5.106}$$

$$C_{i7} = \theta_{ei}\left(2B_6 c_3 \theta_{ei} - 3C_6 c_{iN} + \frac{1}{4} A_{i1} D + B_{i1} F + \frac{1}{2} B_{i7} c_3\right), \tag{5.107}$$

$$D_{i7} = \theta_{ei}\left(3c_3 C_6 \theta_{ei} - \frac{1}{4} B_{i1} D_3 - \frac{c_3 C_5}{3} + 2c_{12} c_{iM} + 2c_6^2 + c_3 c_{12} + 4c_6 c_{iM} c_{iN} + 2c_{10} c_{iM}^2\right), \tag{5.108}$$

$$E_{i7} = \theta_{ei}\left(2A_{i1} c_{iM}^2 + C_{i1} F + \frac{1}{4} c_3 D_4 - 3A_2 c_{iM} - 3\theta_{ei} c_3 c_6 c_{iN} - 2c_3 c_6 \theta_{ei} c_{iM} - A_2 c_3 + 2c_3 c_6 c_{iM} \theta_{ei} - 3\theta_{ei} c_3 c_{iM}^2 c_{iN} + 2\theta_{ei} c_3 c_6 c_{iM} - 2c_3 \theta_{ei} c_{iM}^2 c_{iN}\right), \tag{5.109}$$

$$F_{i7} = \theta_{ei}\left(2B_{i1} c_{iM}^2 - \frac{1}{2} B_2 c_3 + \frac{1}{4} C_{i1} D + \frac{1}{5} c_3 E_4 - 2B_2 c_{iM} + 3\theta_{ei}^2 c_3^2 c_{iM}^2 - 5\theta_{ei} c_3^2 c_{iM}^2\right), \tag{5.110}$$

$$G_{i7} = \theta_{ei}\left(\frac{1}{2} c_3 c_{iM} + \frac{1}{4} c_3 C_2 + 2c_{i1} c_{iM}^2\right). \tag{5.111}$$

In order to derive the fourth-order electrostatic potential fluctuation, Φ_4, we perturb the electrostatic Poisson equation (Eq. (5.5)) up to the required order to yield

$$\partial_\xi^3\Phi_4+\left(\frac{2}{\xi}\right)\partial_\xi^2\Phi_4-\left(\frac{2}{\xi^2}\right)\partial_\xi\Phi_4=\partial_\xi N_{e4}-\partial_\xi N_{i4}. \qquad (5.112)$$

Substituting for $\partial_\xi N_{e4}$ from Eq. (5.96) and $\partial_\xi N_{i4}$ from Eq. (5.104), we find the following third-order differential equation for Φ_4 given as

$$\partial_\xi^3\Phi_4+\left(\frac{2}{\xi}\right)\partial_\xi^2\Phi_4-\left(\frac{2}{\xi^2}\right)\partial_\xi\Phi_4+B\partial_\xi\Phi_4+\frac{A_7}{\xi^2}+\frac{B_7}{\xi^3}+\frac{C_7}{\xi^4}-\frac{D_7}{\xi^5}+\frac{E_7}{\xi^6}+\frac{F_7}{\xi^7}+\frac{G_7}{\xi^9}=0. \qquad (5.113)$$

A spatial indefinite integration of Eq. (5.113) with respect to ξ results in the following Φ_4-evolution equation with multi-order contributory terms given as

$$\partial_\xi^2\Phi_4+\left(\frac{2}{\xi}\right)\partial_\xi\Phi_4+B\partial_\xi\Phi_4-\frac{A_7}{\xi}-\frac{B_7}{2\xi^2}-\frac{C_7}{3\xi^3}+\frac{D_7}{4\xi^4}+\frac{F_7}{5\xi^5}-\frac{G_7}{6\xi^6}-\frac{H_7}{8\xi^8}+J_7=0; \qquad (5.114)$$

$$A_7=A_{e7}+A_{i7}, \qquad (5.115)$$

$$B_7=B_{e7}+B_{i7}, \qquad (5.116)$$

$$C_7=C_{e7}+C_{i7}, \qquad (5.117)$$

$$D_7=D_{i7}-D_{e7}, \qquad (5.118)$$

$$F_7=E_{e7}-E_{i7}, \qquad (5.119)$$

$$G_7=F_{e7}-F_{i7}, \qquad (5.120)$$

$$H_7=G_{e7}-G_{i7}. \qquad (5.121)$$

Where, J_7 is an integration constant appearing here as a direct consequence of the indefinite spatial integration of Eq. (5.113) with respect to ξ. The presence of geometrical curvature effects makes it tedious to find an exact solution manually. Therefore, an analytic solution of Eq. (5.114) with the help of software programs is obtained as

$$\Phi_4(\xi)=c_{17}\frac{\exp\left(i\sqrt{B}\xi\right)}{\xi}-\frac{c_{18}}{2i\sqrt{B}}\frac{\exp\left(-i\sqrt{B}\xi\right)}{\xi}-\frac{1}{\xi}\left[\left(\frac{C_7}{6}+\frac{F_7B}{60}-\frac{B_7}{4i\sqrt{B}}+\frac{iD_7\sqrt{B}}{16}-\frac{iG_8B^{\frac{3}{2}}}{288}-\right.\right.$$

$$\left.\frac{iG_7B^{\frac{5}{2}}}{11520}\right)Fi\left(-i\sqrt{B}\xi\right)\exp\left(i\sqrt{B}\xi\right)+\frac{1}{240i\sqrt{B}\xi}\left(\frac{5H_7}{2}+\left(40C_7+4F_7B+15iD_7\sqrt{G}+\right.\right.$$

$$i\frac{5}{6}G_7B^{\frac{3}{2}} - i\frac{H_7B^{\frac{5}{2}}}{48}\Big)\xi^5 + \xi^6\left(\frac{120J_7}{B} + \frac{120A_7}{i\sqrt{B}}\right)\Bigg) - \left(\frac{5iF_7\sqrt{B}}{3} + 8F_7 - \frac{iH_7B^{\frac{3}{2}}}{24}\right)\xi^3 +$$

$$\left(5G_7 - \frac{H_7B}{8}\right)\xi^2 - \left(15D_7 + \frac{5G_7g}{6} - \frac{H_7B^2}{48} - 4iF_7\sqrt{B}\right)\xi^4 - \frac{120J_7}{i\sqrt{B}}\xi^7 - i\frac{H_7}{2}\sqrt{B}\xi\Bigg]$$

$$-\frac{1}{2i\sqrt{B}\xi}\Bigg[\left(\frac{B_7}{2} + \frac{D_7B}{8} - \frac{G_7B^2}{144} - \frac{H_7B^3}{5760} + \frac{iC_7\sqrt{B}}{3} + \frac{iF_7B^{\frac{3}{2}}}{30}\right)Ei\left(i\sqrt{B}\xi\right)\exp\left(i\sqrt{B}\xi\right) -$$

$$\frac{1}{120\xi^6}\Bigg(\frac{5H_7}{2} + \Bigg(40C_7 + 4F_7B - 15iD_7\sqrt{B} - i\frac{5}{6}G_7B^{\frac{3}{2}} - \frac{iH_7B^{\frac{5}{2}}}{48}\Bigg)\xi^5 + \xi^6\left(\frac{120J_7}{B} -\right.$$

$$\left.\frac{120A_7}{i\sqrt{B}}\right)\Bigg) + \left(\frac{5iG_7\sqrt{B}}{3} + 8F_7 - \frac{iH_7B^{\frac{3}{2}}}{24}\right)\xi^3 + \left(5G_7 - \frac{H_7B}{8}\right)\xi^2 - \left(15D_7 + \frac{5G_7g}{6} -\right.$$

$$\left.\frac{H_7B^2}{48} - 4iF_7\sqrt{B}\right)\xi^4 + \frac{120J_7}{i\sqrt{B}}\xi^7 + i\frac{H_7}{2}\sqrt{B}\xi\Bigg]. \qquad (5.122)$$

It may be noted here that the various numerical figures in the above equation originate naturally from the programming employed for the analytic evaluation and simplification. We expand the exponential terms in Eq. (5.122). The component terms involving $Ei\left(\pm\sqrt{(-A)}\xi\right) = \int_{-\infty}^{\pm\sqrt{(-A)}\xi}[\exp(t)/t]dt$ may be summarily ignored on the local grounds of obtaining physically acceptable converging solutions of Eq. (5.122). Accordingly, we apply the admissible boundary conditions on the relevant perturbed physical variables so as to obtain a validated asymptotic expression for $\Phi_4(\xi)$ as

$$\Phi_4(\xi) = \frac{A_8}{\xi} + \frac{B_8}{\xi^2} + \frac{C_8}{\xi^3} + \frac{D_8}{\xi^4}; \qquad (5.123)$$

$$A_8 = c_{17} + \frac{A_7}{2B}, \qquad (5.124)$$

$$B_8 = \frac{1}{5760}(720D_7 + 40B - B^2H_7), \qquad (5.125)$$

$$C_8 = \frac{1}{30}F_3, \qquad (5.126)$$

$$D_8 = \frac{1}{2880}(40G_7 - BH_7). \qquad (5.127)$$

We now substitute for $\Phi_4(\xi)$ from Eq. (5.123) in Eq. (5.96) and finally deduce the fourth-order perturbed population density for the electrons after a spatial integration of the resultant equation with respect to ξ as

$$N_{e4} = \frac{A_{e9}}{\xi} + \frac{B_{e9}}{\xi^2} + \frac{C_{e9}}{\xi^3} + \frac{D_{e9}}{\xi^4} + \frac{E_{e7}}{5\xi^5} + \frac{F_{e7}}{6\xi^6} + \frac{G_{e7}}{8\xi^8}; \tag{5.128}$$

$$A_{e9} = A_8 + A_{e7}, \tag{5.129}$$

$$B_{e9} = B_8 - \frac{1}{2}B_{e7}, \tag{5.130}$$

$$C_{e9} = C_8 - \frac{1}{3}C_{e7}, \tag{5.131}$$

$$D_{e9} = \frac{1}{4}D_8(4D_8 + D_{e7}). \tag{5.132}$$

The involved constants E_{e7}, F_{e7}, and G_{e7} are already explicitly given with the help of Eqs. (5.101)-(5.103). Repeating similarly for N_{i4} as done with N_{e4} in Eq. (5.128) yields

$$N_{i4} = \frac{A_{i9}}{\xi} + \frac{B_{i9}}{\xi^2} + \frac{C_{i9}}{\xi^3} + \frac{D_{i9}}{\xi^4} + \frac{E_{i7}}{5\xi^5} + \frac{F_{i7}}{6\xi^6} + \frac{G_{i7}}{8\xi^8}; \tag{5.133}$$

$$A_{i9} = \theta_{ei}A_8 + A_{i7}, \tag{5.134}$$

$$B_{i9} = 2\theta_{ei}B_8 + B_{i7}, \tag{5.135}$$

$$C_{i9} = 3\theta_{ei}C_8 + C_{e7}, \tag{5.136}$$

$$D_{i9} = 4\theta_{ei}(D_8 - D_{i7}), \tag{5.137}$$

The involved constants E_{i7}, F_{i7}, G_{i7} are already given in Eqs. (5.109)-(5.111). We again perturb Eq. (5.1) nonlinearity up to its fourth-order to yield

$$M_{e1}\partial_\xi N_{e3} + M_{e2}\partial_\xi N_{e2} + M_{e3}\partial_\xi N_{e1} + \partial_\xi M_{e4} + N_{e1}\partial_\xi M_{e3}$$
$$+N_{e2}\partial_\xi M_{e2} + N_{e3}\partial_\xi M_{e1} + \frac{2}{\xi}(M_{e4} + M_{e3}N_{e1} + M_{e2}N_{e2} + M_{e1}N_{e3}) = 0. \tag{5.138}$$

We substitute the evaluated lower order plasma parameters with required order of differentiations in Eq. (5.138), same as Eq. (5.50) for M_{e3}. We rearrange the terms with ascending order of power in the denominators to yield

$$\partial_\xi M_{e4} + \frac{2M_{e4}}{\xi} + \frac{A_{e8}}{\xi^4} + \frac{B_{e8}}{\xi^5} + \frac{C_{e8}}{\xi^6} + \frac{D_{e8}}{\xi^7} + \frac{E_{e8}}{\xi^8} = 0. \tag{5.139}$$

The analytic solution of Eq. (5.139) is given as

$$M_{e4} = \frac{c_{19}}{\xi^2} + \frac{A_{e8}}{\xi^3} + \frac{B_{e8}}{2\xi^4} + \frac{C_{e8}}{3\xi^5} + \frac{D_{e8}}{\xi^6} + \frac{E_{e8}}{5\xi^7}; \tag{5.140}$$

$$A_{e8} = 2A_{e1}c_5 + 2A_{e7}c_{eM} - 3A_1c_{eN} - A_{e7}c_5 + 2B_{e1}c_5 + c_5F + c_3c_{11}, \tag{5.141}$$

$$B_{e8} = 2B_{e7}c_{eM} + \frac{1}{4}c_5D + 4c_3^2c_5 - c_2c_{eM}F - c_3^2c_{eM}c_{eN} - 4A_1c_3 - 2B_2c_{eN}, \tag{5.142}$$

$$C_{e8} = 2B_{e7}c_3c_{eM} + \frac{1}{2}B_1c_3 + c_4c_{eM} - \frac{1}{4}c_3c_{eM}D - 4c_3^3c_{eM}, \tag{5.143}$$

$$D_{e8} = c_2c_{eN} + c_{eM}D_3 + 2\left(\frac{m_e}{m_i}\right)c_5c_{eM}^2 + 2c_5(c_{eN} - c_{e1}), \tag{5.144}$$

$$E_{e8} = \frac{3}{5}c_{eM}E_3 - 2\left(\frac{m_e}{m_i}\right)c_3c_{eM}^2 + \frac{7}{4}c_2c_3 - c_3c_{eM}c_{e1}. \tag{5.145}$$

In addition, c_{19} in Eq. (5.140), is an integration constant resulting due to indefinite integration of Eq. (5.139) with respect to ξ. We repeat the same procedure to Eq. (5.3) as done in Eq. (5.55) and obtain the fourth-order perturbed ion Mach number given as

$$M_{i4} = \frac{c_{20}}{\xi^2} + \frac{A_{i8}}{\xi^3} + \frac{B_{i8}}{2\xi^4} + \frac{C_{i8}}{3\xi^5} + \frac{D_{i8}}{\xi^6} + \frac{E_{i8}}{5\xi^7}; \tag{5.146}$$

$$A_{i8} = 2A_1c_{iN} - 3A_2c_{iN} + 2A_{e1}c_5 + A_{i7}c_{iM} - 2A_{i1}c_3 + \theta_{ei}c_3c_{12} - 3C_{m4}c_{10} + B_2c_{iN} + 2c_{M3}c_9, \tag{5.147}$$

$$B_{i8} = 2A_2\theta_{ei}c_3 + 2A_{e1}c_{M3} + 2B_{e1}c_5 + 3\theta_{ei}c_3A_1 - 3\theta_{ei}c_3A_2 - 2B_2c_{iN} - 3A_{i1}c_{M4} - 2B_{i7}c_{iM}, \tag{5.148}$$

$$C_{i8} = 2B_{e1}c_{M3} + \frac{2}{3}C_4c_{iM} - \frac{4}{3}c_5c_{iM} + \frac{3}{2}\theta_{ei}B_1c_3 - 2\theta_{ei}B_2c_3 - 3B_{i1}c_{M4}, \tag{5.149}$$

$$D_{i8} = \frac{1}{2}Cc_{iN} + 2c_5c_{i1} - \frac{1}{2}c_{iM}D_4 - \frac{3}{2}C_2c_{iN} - 2c_{i1}c_3, \tag{5.150}$$

$$E_{i8} = \theta_{ei}c_3c_2 - \frac{3}{2}C_2c_3\theta_{ei} - 3\,c_{i1}c_{M4} - \frac{2}{5}E_4c_{iM} + \frac{1}{2}\theta_{ei}c_{12}c_3. \tag{5.151}$$

The corresponding fourth-order perturbed electric field ($E_4(\xi)$) is evaluated from Eq. (5.123) as per the fundamental definition of a conservative force field as already applied in the following form

$$E_4(\xi) = \frac{A_8}{\xi^2} + \frac{2B_8}{\xi^3} + \frac{3C_8}{\xi^4} + \frac{4D_8}{\xi^5}. \tag{5.152}$$

The deduced fourth-order perturbed electric field expression, $E_4(\xi)$ as in Eq. (5.152), determines the electrostatic force experienced by a unit charge of primary or secondary origin in the sheath region as a fifth-degree polynomial form with the $1/\xi$-dependency. It may be important to note here that the perturbed field $E_4(\xi)$ developed due to the electron accumulation around the embedded central anode forces the other adjacent electrons and ions in the sheath region to diverge outward and converge inward as per the universal long-range Coulombic force law, respectively.

It is admitted that the analysis is constrained up to the fourth-order plasma parametric PFBS perturbation dynamics only. As already mentioned, the free energy source for the excitation of the PFBS instability originates from the equilibrium plasma currents across the sheath structures. The fourth-order truncation is basically on account of the ascending power of the $1/\xi$-dependency of the plasma field variables ($E_n(\xi) \sim n/\xi^{n+1}$, $n \epsilon Z^+$). Besides, our formalism is stemmed in a weakly nonlinear perturbative analysis. Hence, the higher-order perturbed parameters asymptotically get redundant due to their algebraically weaker strength with enhanced perturbation order (Fig. 5.2).

5.3 RESULTS AND DISCUSSIONS

A theoretical bifluidic PFBS model is constructed herein to analyse its instability dynamics in the multi-order nonlinear perturbative analysis framework on the relevant laboratory spatiotemporal scale. As a typical scenario, we choose the ion-acoustic wave evolutionary scales as our referral standardization unit of the spatiotemporal coordinates as a first step. The order-wise nonlinearly perturbed governing equations of the considered fluctuating PFBS system decouple systematically into a set of third-order perturbed Poisson equations in a unique nonhomogeneous differential form with variable coefficients. The multi-order perturbed plasma parameters in the presence of geometrical curvature effects are independently derived in the framework of judicious boundary conditions relevant for a realistic PFBS instability to evolve. The derived lower-order plasma parameters are further substituted in the higher-order perturbed differential equations, rearranged and integrated further to yield the higher-order perturbed plasma parameters. The power of $1/\xi$-factors get polynomially enhanced in the algebraic expression of the perturbed quantities with the order of parametric nonlinearity. It is further noteworthy here that the $1/\xi$-terms appear in the parametric equations due to the spherical geometry of the considered system. The $1/\xi$-factor would not appear in the PFBS formalism if it were carried out in a plane parallel geometry instead [12, 13].

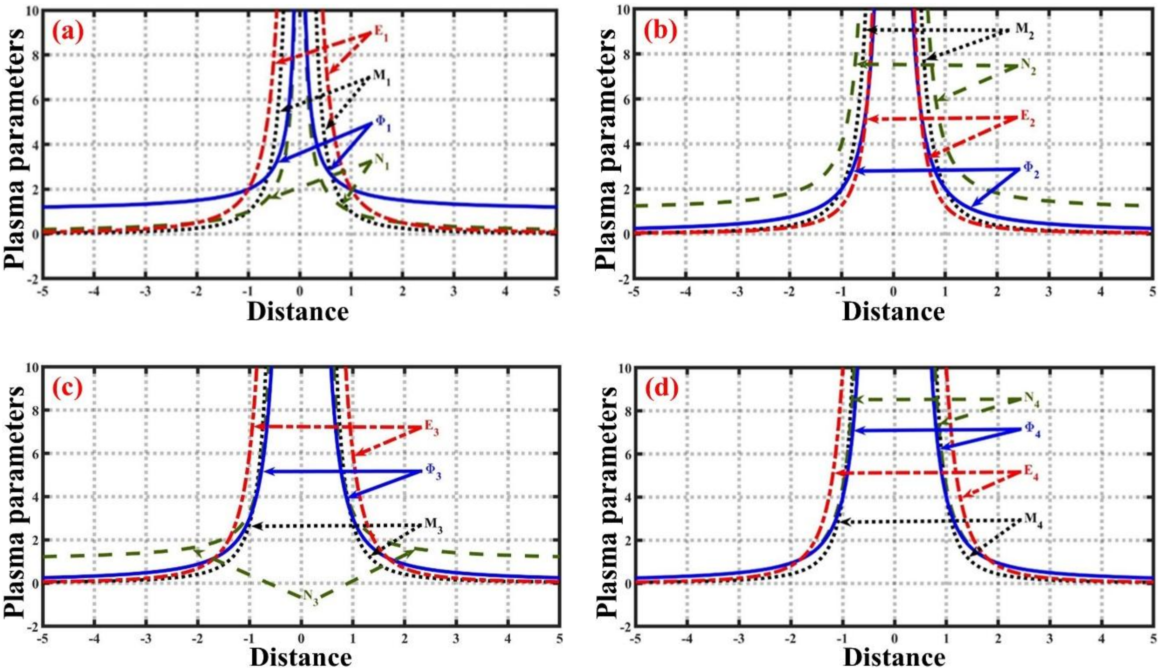

Figure 5.1: *Spatial profile of the (a) first-order, (b) second-order, (c) third-order, and (d) fourth-order perturbed plasma parameters given as: (i) Electrostatic potential (Φ, blue solid line), (ii) Electric field (E, red dashed-dotted line), (iii) Density (N, green dashed line), and (iv) Mach number (M, black dotted line). The various fine details for our numerical analysis are given in the text.*

The PFBS instability governed by the multi-order perturbed Poisson equations are graphically illustrated after judicious numerical analysis in Figs. (5.1(a)-5.1(d)). The analytically developed spatial plots and colormaps manifest a negative rate of change of the plasma parameters with respect to the radial distance (ξ). As it is evident from Figs. (5.1)-(5.2), the highest rate of change of the plasma parameters is observed across one Debye length (λ_D) only. The spatial variation of the plasma parameters is negligibly small at scale larger than one Debye length, $(dP/d\xi)_{\xi>|\lambda_D|} \approx 0$; here, $P =(\Phi, \text{E}, N, M)$. The narrow spike in the electrostatic potential (Fig. 5.2(a)) beside the anode region (within one Debye length, λ_D), is a consequence of the consistent higher electrostatic potential sustained owing to the anode voltage in the dissipative plasma media.

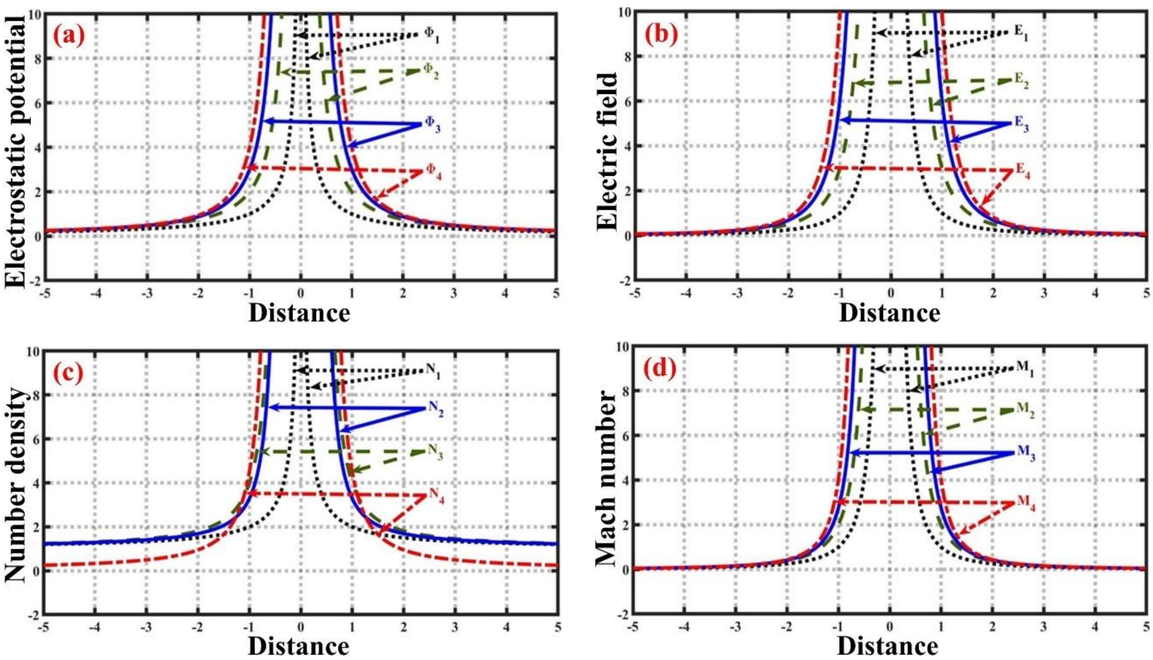

Figure 5.2: *Spatial comparative profile of the (a) electrostatic potential, (b) electric field, (c) number density, and (d) Mach number across their order of nonlinearity, denoted by their suffixes. The detailed analyses are given in the text.*

The simultaneous existence of the Mach number spikes (Fig. 5.2(d)) manifests higher electron (ion) velocities at the periphery of the anode, originating from denser electric field lines, thereby pulling (pushing) the electron (ions) inwards (outwards). Here, the field line density determines the electrostatic force strength experienced by the charged particles. It is found that the analytically developed potential plot (Fig. 5.2(a)) smoothly corroborates with the experimental [2] and theoretical [3] findings available in the literature. It justifies and validates the reliability of our bifluidic plasma model conjectures for the steady-state description of the PFBS dynamics [2, 3].

The resembling surface colormaps (Figs. 5.3(a)-5.3(d)) with 2-D plots (Figs. 5.2(a)-5.2(b)) further embolden the above inferences drawn from the behavioural analyses of the PFBS dynamics. The surface colormaps manifest the magnitude of the collective spatial variation of three adjoint plasma parameters (Φ, $-\partial_\xi \Phi$, $-\partial_\xi^2 \Phi$) with four order-wise surface plots. The inconsistent colour gradient in the surface plots of the electrostatic potential (Φ), electric field ($E = -\partial_\xi \Phi$), and potential curvature ($-\partial_\xi^2 \Phi$), with respect to the radial distance (ξ) denote the collective parametric variations with respect to each other. It is highlighted that the rapid colour variations of the surface plots beside the anode ($\xi \rightarrow 0$) signifies a greater potential curvature effect in comparison to the wider distance ($\xi \geq \lambda_D$). The spatially attenuating plasma parameters (diminished colour gradient) outwards is a consequence of the intrinsic dissipative mechanism of the plasma medium hindering in the propagation of disturbance outwards from the central anode.

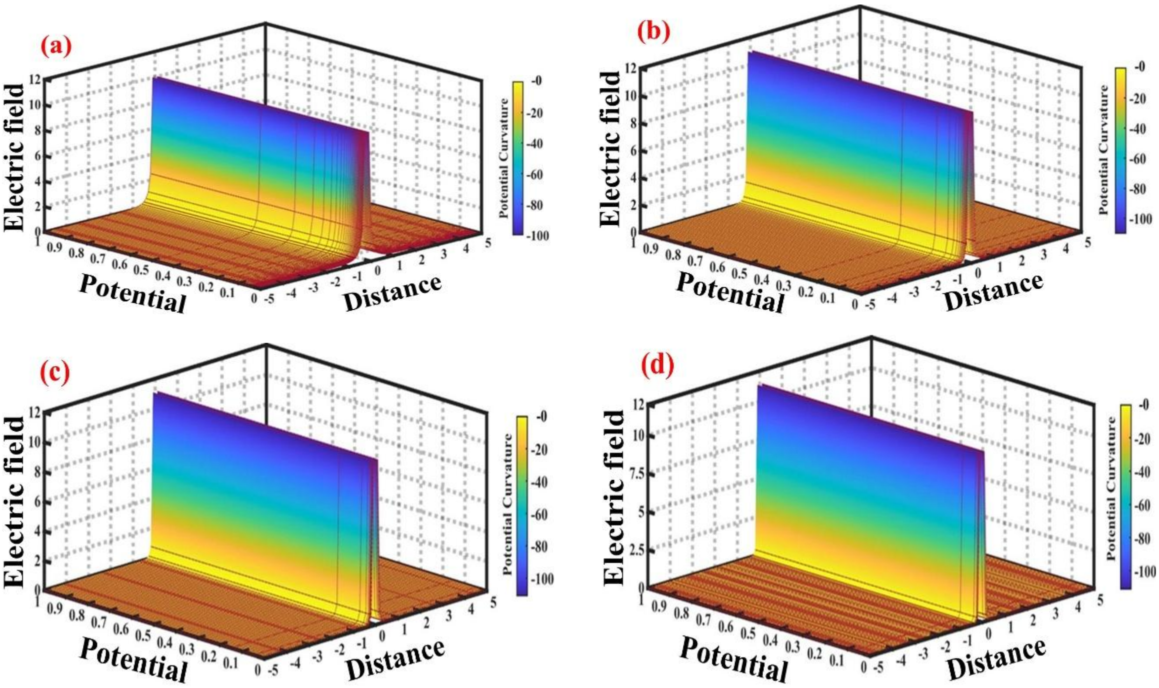

Figure 5.3: *Profile of the (a) first-order, (b) second-order, (c) third-order, and (d) fourth-order perturbed plasma parameters in a 4-D phase space defined by (i) Distance (ξ), (ii) Electrostatic potential (Φ_1), (iii) Electric field (E_1), (iv) Potential curvature ($\partial^2\Phi_1/\partial\xi^2$). The initial and input parameters used here are the same as Fig. 5.1.*

To obtain the 2-D profiles (Figs. 5.1-5.2) smoothly, we assign a numerical value of unity to the natural constants (F, G, A_{e1}, B_{e1}, C_{e1}, A_{i1}, B_{i1}, C_{i1}, A_1, B_1, C_1, A_2, B_2, C_2, A_6, B_6, C_6, A_{e7}, B_{e7}, C_3, D_3, F_3, A_{i7}, B_{i7}, A_8, B_8, C_8, A_{e8}, B_{e8}, C_{e8}, D_{e8}, E_{e8}, A_{i8}, B_{i8}, C_{i8}, D_{i8}, E_{i8}, A_{e9}, B_{e9}, C_{e9}, D_{e9}, A_{i9}, B_{i9}, C_{i9}, D_{i9}) and integration constants (C_{eM}, C_{iM}, C_{eN}, c_3, c_{eN}, c_5, c_6, c_{11}, c_{12}, c_{19}, c_{20}). In order to simplify the surface plotting, the order-wise common constants taken in the first-order (Fig. 5.3(a)), second-order (Fig. 5.3(b)), third-order (Fig. 5.3(c)), and fourth-order (Fig. 5.3(d)) surface plots are 1 (for c_3), 0.1 (for c_5, C_{eM}, C_{iM}, C_{eN}, c_6, G, H, A_{e1}, B_{e1}, C_{e1}, A_{i1}, B_{i1}, and C_{i1}), 0.01 (for c_{11}, c_{12}, A_1, B_1, C_1, A_2, B_2, C_2, A_6, B_6, C_6, A_{e7}, B_{e7}, A_{i7}, and B_{i7}), and 0.001 (for c_{19}, c_{20}, A_8, B_8, C_8, D_8, A_{e8}, B_{e8}, C_{e8}, D_{e8}, E_{e8}, A_{i8}, B_{i8}, C_{i8}, D_{i8}, E_{i8}, A_{e9}, B_{e9}, C_{e9}, D_{e9}, A_{i9}, B_{i9}, C_{i9}, and D_{i9}), respectively. The same magnitudes of the unknown constants are considered to shrink the plasma parametric variations within a physically admissible range in accordance with the relevant normalized laboratory spatiotemporal scales.

5.4 CONCLUSIONS

This chapter analyses the nonlinear multi-order fluctuation dynamics of a spherical PFBS developed around a plasma embedded anode. The analysis is done with the help of a steady-state bifluidic plasma model approach on the relevant laboratory spatial scales. The bifluidic model-based multi-order nonlinear PFBS perturbative analysis proves to be efficaciously successful in recreating the experimental results in terms of the plasma parametric variations in the diversified peakonic pattern forms without any loss of generality [3, 17, 18]. The electrostatic potential and other parametric variations agree with each other throughout their spatial evolution as well as the colour-phase mapping profiles. The model enables us to find out a unique class of peakon-type plasma potential profiles with the interrelation of the electron and ion dynamics through a new set of the third-order linear differential equations of the perturbed electrostatic Poisson formalism.

The governing ODEs for the PFBS system are analysed herein with multi-order nonlinear perturbation up to their fourth-order of nonlinearity. The escalating (amplifying) plasma parameters with their order of nonlinearity indicates the unstable equilibrium state of the PFBS system as a whole [9]. A stable equilibrium state of the PFBS system would otherwise reduce the magnitude of the plasma parameters with sensible dispersive effects. The slight parametric amplification (with the order of nonlinearity) may also indicate the abrupt fluid (plasma) density variation across the associated double layer and the plasma sheath region leading to the eruption of the Rayleigh-Taylor instability (RTI) or other fluid instabilities caused by the equilibrium density gradients acting as free energy source therein [6].

In other words, we can alternatively say that the free energy source for the PFBS instability excitation originates from the equilibrium plasma currents across the sheath structures under investigation. The nonlinear PFBS fluctuation dynamics has been known to evolve in the form of diverse peakonic structures only up to the first-order of perturbation as of now [2]. The proposed semi-analytic investigation strengthens the same density gradient-driven instability to evolve up to the fourth-order nonlinearity with a higher level of fluctuation amplitude exhibiting quantitatively modified characteristics.

It is further manifested that an increase in the order of plasma nonlinearity in our analysis increases the amplitude peakiness of the peakonic structures explored here, and vice-versa. The investigated peakonic pattern structures are fairly bolstered by both the asymptotic boundary conditions predicted theoretically [9] and PFBS instability observations reported experimentally [3]. The harmonic amplitude enhancement with the order of plasma fluid nonlinearity in this perturbation analysis is quite in accordance with the previous predictions available in the literature [15-18]. It, therefore, enables us to guarantee the reliability of our multi-order nonlinear perturbative analysis of the steady-state PFBS instability dynamics. Although it is semi-analytically restricted only up to the fourth-order parametric perturbations, but evolving self-consistently as a unique class of multi-order peakonic eigen-structures with varied amplitudes, collectively.

It is relevant to reiterate here that the PFBS instability dynamics analysed herein, tentatively supports a plethora of utilities in diversified astrolabcosmic environments. To enumerate a few, the pressure exerted by the plasma FB on the constitutive ions and neutral atoms around it builds a considerable macroscopic force. A streamlined flow of these moving ions and neutrals can develop into a power jet, proving the significance of PFBS research for aviation technology and various astrophysical explorations as well [19]. A systematic analysis of the PFBS evolution would reveal the excitation of various microscopic instabilities and their saturations summarily left here now for our future course of studies with a refined incorporation of wave-turbulence effects.

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