CHAPTER 6

Existence and Exact Controllability of a Hybrid Evolution Inclusion

6.1 Introduction

In 2000, Hilfer [65] introduced a new definition of fractional derivative, known as the Hilfer fractional derivative, which generalizes the Riemann-Liouville fractional derivative and acts as an interpolation between the Caputo and Riemann-Liouville derivatives. This derivative has significant applications in various fields, including polymer science, viscoelasticity, rheological modelling, generalized anomalous diffusion, and financial mathematics [37,51,57,90,133]. Gu and Trujillo [59] investigated an evolution equation involving the Hilfer fractional derivative, deriving the mild solution using the Laplace transform and density function, and established the existence of solutions using the noncompact measures approach. Varun and Udhayakumar [127] studied the existence of solutions for a Hilfer fractional differential inclusion by applying fixed point theorems in the context of almost sectorial operators.

Recent advancements in controllability theory [52, 71, 81, 119, 120] have addressed various forms, including approximate, null, and exact controllability etc. in both finite and infinite-dimensional spaces. Wang and Zhou [131] provided a comprehensive analysis of the exact controllability of a Caputo fractional differential inclusion via the fixed point theorem. Similarly, Kumar et al. [80] investigated the controllability of Sobolev-type Hilfer fractional integro-differential equations. Recently Priyadharshini and Vijayakumar [101] investigated the approximate controllability of a fractional stochastic differential equation with Hilfer fractional derivatives and non-dense domain in Hilbert spaces. The results were derived using fractional calculus, semigroup theory, Wiener process, and fixed point techniques.

In this work we consider the following semi-linear hybrid fractional differential inclusion with nonlocal condition

$$\begin{gathered} {}_0\mathcal{D}_t^{\epsilon,\rho}\left(\frac{\omega(t)}{\mathfrak{F}(t,\omega(t))}\right) \in \mathcal{A}\left(\frac{\omega(t)}{\mathfrak{F}(t,\omega(t))}\right) + \mathcal{G}(t,\omega(t)), \qquad t \in (0,\mathscr{T}] = \mathscr{J}', \\ \mathcal{I}^{(1-\epsilon)(1-\rho)}\left(\frac{\omega(0)}{\mathfrak{F}(0,\omega(0))}\right) - \mathfrak{H}(\omega) = \omega_0. \end{gathered} \tag{6.1.1}$$

In this case, the Hilfer fractional derivative of type $\rho \in [0,1]$ and order $\epsilon \in (0,1)$ is denoted by ${}_0\mathcal{D}_t^{\epsilon,\rho}$. For a bounded linear operator $\{\mathcal{T}(t)\}_{t\geq 0}$, let $\mathcal{A}$ be the infinitesimal generator of a strongly continuous semigroup in a Banach space $\mathcal{X}$. We denote the interval $[0,\mathscr{T}]$ as $\mathscr{J}$. Consider $\mathfrak{F} \in \mathcal{C}(\mathscr{J} \times \mathcal{X}, \mathcal{X} \setminus \{0\})$, $\mathfrak{H} : \Omega \to \mathcal{X}$ be a continuous and compact map. Additionally, $\mathcal{G} : \mathscr{J} \times \mathcal{X} \to 2^{\mathcal{X}} \setminus \{\phi\}$ is a nonempty, closed, convex and bounded multivalued map.

This chapter has been set up as: In Section 6.2, we introduce certain fundamental concepts and Lemmas based on our requirements, as well as find out the corresponding integral equation of (6.1.1). Section 6.3 is dedicated to proving the existence results for the proposed system. In Section 6.4, we examine the controllability of the hybrid class of fractional differential inclusions. Finally, in Section 6.5, we provide an illustrative example to demonstrate and clarify the key findings of our study.

6.2 Preliminaries

In this context, let $\bar{\mathscr{C}} = \mathcal{C}(\mathscr{J}', \mathcal{X})$ represents the spaces of continuous functions mapping from $\mathscr{J}'$ to $\mathcal{X}$ and $\mathscr{C} = \mathcal{C}(\mathscr{J}, \mathcal{X})$ represents from $\mathscr{J}$ to $\mathcal{X}$.

Now define the space Ω as

$$\Omega = \left\{\omega \in \bar{\mathscr{C}} : \lim_{t\to 0} t^{(1-\epsilon)(1-\rho)}\omega(t) \quad \text{exists and finite}\right\},$$

equipped with the norm $\|\cdot\|_\Omega$, where $\|\omega\|_\Omega = \sup_{t\in\mathscr{J}'}\left\{t^{(1-\epsilon)(1-\rho)}\|\omega(t)\|\right\}$. Thus Ω forms a Banach space. Additionally, the following statements are accurate

1. For $\rho = 1$ we get that $\Omega = \mathscr{C}$ and $\|\omega\|_\Omega = \sup_{t\in\mathscr{J}'}\|\omega(t)\|$.
2. Let for $t \in \mathscr{J}'$, $\omega(t) = t^{(\rho-1)(1-\epsilon)}\varpi(t)$. Then $\omega \in \Omega$ if and only if $\varpi \in \mathscr{C}$, also $\|\omega\|_\Omega = \|\varpi\|$.

Let us assume, $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J}) = \{\upsilon \in \mathscr{C} : \|\upsilon\| \leq \mathfrak{r}\}$ and $\mathfrak{B}_{\mathfrak{r}}^{\Omega}(\mathscr{J}') = \{\omega \in \Omega : \|\omega\|_\Omega \leq \mathfrak{r}\}$. Thus both of them are closed, convex and bounded subsets of $\mathscr{C}$ and Ω respectively.

Lemma 6.2.1. *The equivalent integral inclusion of the considered hybrid fractional differential inclusion* (6.1.1) *is the following*

$$\omega(t) \in \mathfrak{F}(t,\omega(t))\Bigg[\frac{\omega_0 + \mathfrak{H}(\omega)}{\Gamma(\rho(1-\epsilon)+\epsilon)} t^{(\epsilon-1)(1-\rho)} + \frac{1}{\Gamma(\epsilon)}\int_0^t (t-s)^{\epsilon-1}\Bigg\{\mathcal{A}\Bigg(\frac{\omega(s)}{\mathfrak{F}(s,\omega(s))}\Bigg) + \mathcal{G}(s,\omega(s))\Bigg\}ds\Bigg]. \tag{6.2.1}$$

Proof. The proof of the result can be found in [57]. □

To construct a mild solution of the system (6.1.1) we present the Wright function $\mathcal{M}_\epsilon(\phi)$ and defined as

$$\mathcal{M}_\epsilon(\phi) = \sum_{\eta=1}^{\infty} \frac{(-\phi)^{\eta-1}}{(\eta-1)!\Gamma(1-\epsilon\eta)}, \quad 0<\epsilon<1, \quad \phi\in\mathbb{C},$$

which satisfies

$$\int_0^\infty \phi^\gamma \mathcal{M}_\epsilon(\phi)d\phi = \frac{\Gamma(1+\gamma)}{\Gamma(1+\epsilon\gamma)}, \quad \text{for} \quad \phi \geq 0.$$

Lemma 6.2.2. *If the integral equation* (6.2.1) *satisfies and there exists a* $\mathfrak{g} \in L^1(\mathscr{J},\mathcal{X})$ *for all* $t\in\mathscr{J}$*, such that* $\mathfrak{g}(t)\in\mathcal{G}(t,\omega(t))$*, then we have*

$$\omega(t) = \mathfrak{F}(t,\omega(t))\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0+\mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds\big], \quad t\in\mathscr{J}, \tag{6.2.2}$$

where

$$\mathcal{Q}_\epsilon(t) = t^{\epsilon-1}\mathcal{P}_\epsilon(t), \quad \mathcal{P}_\epsilon(t) = \int_0^\infty \epsilon\phi\mathcal{M}_\epsilon(\phi)\mathcal{T}(t^\epsilon\phi)d\phi, \quad \textit{and} \quad \mathcal{S}_{\epsilon,\rho}(t) = \mathcal{I}_{0+}^{\rho(1-\epsilon)}\mathcal{Q}_\epsilon(t).$$

Proof. The proof of the result can be found in [59]. □

In order to establish a few key remarks, we need the following assumption:

(A_0) In the uniform operator topology, the family $\{\mathcal{T}(t)\}_{t\geq 0}$ is continuous for $t\geq 0$ and uniformly bounded. This implies the existence of a constant $\mathcal{M}>1$ such that $\sup_{t\in[0,\infty)}|\mathcal{T}(t)| < \mathcal{M}$.

Remark 6.2.3. *[59] Based on the presumption* (A_0)*,* $\mathcal{P}_\epsilon(t)$ *is continuous for* $t>0$ *in the uniform operator topology.*

Remark 6.2.4. *[59] Based on the presumption* (A_0)*, for any fixed* $t>0$*,* $\{\mathcal{Q}_\epsilon(t)\}_{t>0}$ *and* $\{\mathcal{S}_{\epsilon,\rho}(t)\}_{t>0}$ *are linear operators and for any given* $\omega\in\mathcal{X}$*,*

$$\|\mathcal{Q}_\epsilon(t)\omega\| \leq \frac{\mathcal{M}t^{\epsilon-1}}{\Gamma(\epsilon)}\|\omega\| \quad \textit{and} \quad \|\mathcal{S}_{\epsilon,\rho}(t)\omega\| \leq \frac{\mathcal{M}t^{(\rho-1)(1-\epsilon)}}{\Gamma(\rho(1-\epsilon)+\epsilon)}\|\omega\|.$$

Remark 6.2.5. *[59] Based on the presumption* (A_0)*,* $\{\mathcal{Q}_\epsilon(t)\}_{t>0}$ *and* $\{\mathcal{S}_{\epsilon,\rho}(t)\}_{t>0}$ *are strongly continuous, i.e., for any* $\omega\in\mathcal{X}$ *and* $0<\zeta_1<\zeta_2\leq\mathscr{T}$ *we have*

$$\|\mathcal{Q}_\epsilon(\zeta_1)\omega - \mathcal{Q}_\epsilon(\zeta_2)\omega\| \to 0 \quad \textit{and} \quad \|\mathcal{S}_{\epsilon,\rho}(\zeta_1)\omega - \mathcal{S}_{\epsilon,\rho}(\zeta_2)\omega\| \to 0, \quad \textit{as} \quad \zeta_2\to\zeta_1.$$

Lemma 6.2.6. *[131] Assuming $\mathscr{J}$ to be a real interval that is compact, $\mathcal{X}_{BCC}$ represents the nonempty, bounded, closed and convex subset of $\mathcal{X}$. For each fixed $\omega \in \mathcal{X}$, consider the multivalued map $\mathcal{G} : \mathscr{J} \times \mathcal{X} \to \mathcal{X}_{BCC}$ is measurable to t, for each $t \in \mathscr{J}$ and upper semi continuous with respect to ω. Additionally, the set $\mathfrak{S}_{\mathcal{G},\omega} = \big\{\mathfrak{g} \in L^1(\mathscr{J}, \mathcal{X}) : \mathfrak{g}(t) \in \mathcal{G}(t, \omega(t)), \quad t \in \mathscr{J}\big\}$ for every $\omega \in \mathscr{C}$ is nonempty. If $\Delta : L^1(\mathscr{J}, \mathcal{X}) \to \mathscr{C}$ is a continuous and linear operator, then the operator*

$$\Delta \circ \mathfrak{S}_{\mathcal{G}} : \mathscr{C} \to \mathscr{C}_{BCC}, \quad \omega \mapsto (\Delta \circ \mathfrak{S}_{\mathcal{G}})(\omega) = \Delta(\mathfrak{S}_{\mathcal{G},\omega})$$

in $\mathscr{C} \times \mathscr{C}$ is a closed graph operator.

6.3 Existence Result

Definition 6.3.1. *If, for all $t \in \mathscr{J}$ there exists $\mathfrak{g} \in L^1(\mathscr{J}, \mathcal{X})$ such that $\mathfrak{g}(t) \in \mathcal{G}(t, \omega(t))$, then a function $\omega \in \bar{\mathscr{C}}$ is a mild solution of the considered problem* (6.1.1) *that satisfies*

$$\omega(t) = \mathfrak{F}(t, \omega(t))\Big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds\Big], \quad t \in \mathscr{J}'. \tag{6.3.1}$$

6.3.1 Hypotheses

We provide the following hypotheses before discussing and proving our main findings.

(A_1) $\{\mathcal{T}(t)\}_{t>0}$ is the compact operator.

(A_2) $\mathcal{G} : \mathscr{J} \times \mathcal{X} \to \mathcal{X}_{BCC}$, the multivalued map is such that it is measurable to t, for each $t \in \mathscr{J}$ and exhibits upper semicontinuity with respect to ω. Define the selection set corresponding to each $\omega \in \mathscr{C}$ as

$$\mathfrak{S}_{\mathcal{G},\omega} = \big\{\mathfrak{g} \in L^1(\mathscr{J}, \mathcal{X}) : \mathfrak{g}(t) \in \mathcal{G}(t, \omega(t)), \quad t \in \mathscr{J}\big\},$$

which is nonempty.

(A_3) There exists a constant $\epsilon_1 \in (0, \epsilon)$ and $\mathfrak{L}_{\mathfrak{g}}(\cdot)$ belonging to the space $L^{\frac{1}{\epsilon_1}}(\mathscr{J}, \mathbb{R}^+)$ such that

$$\sup\{\|\mathfrak{g}\| : \mathfrak{g}(t) \in \mathcal{G}(t, \omega(t))\} \le \mathfrak{L}_{\mathfrak{g}}(t)$$

for all $\omega(t) \in \mathfrak{B}_{\mathfrak{r}}^{\Omega}(\mathscr{J}')$ and for almost all $t \in \mathscr{J}$ also,

$$\lim_{t \to 0^+} t^{(1-\epsilon)(1-\rho)} \mathcal{I}_{0^+}^{\epsilon} \mathfrak{L}_{\mathfrak{g}}(t) = 0, \quad \text{for almost all} \ \ t \in \mathscr{J}.$$

(A_4) For bounded functions φ and $\varrho \in \mathscr{C}$ with the bounds $\|\varphi\|$ and $\|\varrho\|$ respectively, the functions $\mathfrak{F} : \mathscr{J} \times \mathcal{X} \to \mathcal{X} \setminus \{0\}$ and $\mathfrak{H} : \Omega \to \mathcal{X}$ satisfy the following

$$\|\mathfrak{F}(t, \omega_1(t)) - \mathfrak{F}(t, \omega_2(t))\| \le \varphi(t) t^{(1-\epsilon)(1-\rho)} \|\omega_1(t) - \omega_2(t))\|$$

for a.e. $t \in \mathscr{J}$, $\omega_1, \omega_2 \in \mathcal{X}$ and

$$\|\mathfrak{H}(\omega)\| \leq \varrho(t)$$

for a.e. $t \in \mathscr{J}$.

6.3.2 Main Result

Theorem 6.3.2. *Suppose that the hypotheses* (A_1)-(A_4) *are valid. Then there exists a mild solution for the hybrid system* (6.1.1) *on* $\mathscr{J}$ *provided that*

$$\mathfrak{R} > \frac{\mathcal{F}_0 \mathfrak{P}}{1 - \mathfrak{P}\|\varphi\|}, \tag{6.3.2}$$

where $\mathfrak{P} = \dfrac{\mathcal{M}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\|+\|\varrho\|)+\dfrac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)}\left(\dfrac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1}\mathscr{T}^{\epsilon-\epsilon_1}\|\mathfrak{L}_{\mathfrak{g}}\|_{L^{\frac{1}{\epsilon_1}}}$, $\mathcal{F}_0 = \|\mathfrak{F}(t,0)\|$, *and* $\mathfrak{P}\|\varphi\| < 1$.

Proof. Let us define an operator $\Xi : \mathscr{C} \to 2^{\mathscr{C}}$ as $\Xi\omega$, which is the set of $\Theta \in \mathscr{C}$ such that

$$\Theta(t) = \mathfrak{F}(t,\omega(t))\left[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds\right]$$

for all $t \in \mathscr{J}'$ and $\mathfrak{g} \in \mathfrak{S}_{\mathcal{G},\omega}$.

Now let us define an operator Λ for any $\varpi \in \mathscr{C}$ and assume that $\omega(t) = t^{(\rho-1)(1-\epsilon)}\varpi(t)$ as follows

$$(\Lambda\varpi)(t) = \begin{cases} t^{(1-\rho)(1-\epsilon)}\Theta(t), & t \in (0,\mathscr{T}], \\ \dfrac{\mathfrak{F}_0(\omega_0 + \mathfrak{H}(\omega))}{\Gamma(\rho(1-\epsilon)+\epsilon)}, & t = 0, \end{cases} \tag{6.3.3}$$

where $\mathfrak{F}_0 = \mathfrak{F}(0,\omega(0))$. This means that ω is a mild solution of (6.1.1) in Ω if and only if there exists a solution $\varpi \in \mathscr{C}$ for the operator equation $\varpi = \Lambda\varpi$.

In order to show the fixed point of Λ, we consider two operators $\Lambda_1, \Lambda_2 : \mathfrak{B}_{\mathfrak{r}}(\mathscr{J}) \to \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ as follows:

$$(\Lambda_1\varpi)(t) = \begin{cases} \mathfrak{F}(t,\omega(t)), & t \in (0,\mathscr{T}], \\ \mathfrak{F}_0, & t = 0, \end{cases} \tag{6.3.4}$$

$$(\Lambda_2\varpi)(t) = \begin{cases} t^{(1-\rho)(1-\epsilon)}\left[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds\right], & t \in (0,\mathscr{T}], \\ \dfrac{\omega_0 + \mathfrak{H}(\omega)}{\Gamma(\rho(1-\epsilon)+\epsilon)}, & t = 0. \end{cases} \tag{6.3.5}$$

Therefore,

$$(\Lambda\varpi)(t) = (\Lambda_1\varpi)(t) \times (\Lambda_2\varpi)(t) \quad \text{for} \quad t \in \mathscr{J}.$$

Now we have to establish that both the operators Λ_1 and Λ_2 meet all requirements of Theorem 1.6.22.

Step I: To prove that on $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, Λ_1 is Lipschitz.

Assume that $\varpi_1, \varpi_2 \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ and $\omega_\iota(t) = t^{(\rho-1)(1-\epsilon)}\varpi_\iota(t), \quad t \in \mathscr{J}'$ for $\iota = 1, 2$. Hence $\omega_\iota \in \mathfrak{B}_{\mathfrak{r}}^{\Omega}(\mathscr{J}')$ for $\iota = 1, 2$. Now

$$\begin{aligned}\|\Lambda_1\varpi_1(t) - \Lambda_1\varpi_2(t)\| &= \|\mathfrak{F}(t, \omega_1(t)) - \mathfrak{F}(t, \omega_2(t))\| \\ &\le \varphi(t)t^{(1-\rho)(1-\epsilon)}\|\omega_1(t) - \omega_2(t)\| \\ &= \varphi(t)\|\varpi_1(t) - \varpi_2(t)\|.\end{aligned}$$

By considering the supremum over t, we obtain

$$\|\Lambda_1\varpi_1 - \Lambda_2\varpi_2\| \le \|\varphi\|\|\varpi_1 - \varpi_2\|.$$

This shows that Λ_1 is Lipschitz on $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ with a Lipschitz constant $\|\varphi\|$.

Step II: We have to show on $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, Λ_2 is compact and upper semi-continuous.

Claim A: Assume that $\varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, for $t \in \mathscr{J}'$, $\omega(t) = t^{(\rho-1)(1-\epsilon)}\varpi(t)$. Therefore, $\omega \in \mathfrak{B}_{\mathfrak{r}}^{\Omega}(\mathscr{J}')$. For all $\varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ to show that $\Lambda_2\varpi$ is convex.

Let $\lambda_1, \lambda_2 \in \{\Lambda_2\varpi(t)\}$ and for $t \in \mathscr{J}$ there exist $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathfrak{S}_{\mathcal{G},\omega}$ such that

$$\lambda_\iota(t) = t^{(1-\rho)(1-\epsilon)}\Big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}_\iota(s)ds\Big], \quad \iota = 1, 2.$$

For any $\delta \in [0, 1]$ and $t \in \mathscr{J}$ we have

$$\begin{aligned}\delta\lambda_1(t) + (1-\delta)\lambda_2(t) =& t^{(1-\rho)(1-\epsilon)}\Big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) \\ &+ \int_0^t \mathcal{Q}_\epsilon(t-s)\big\{\delta\mathfrak{g}_1(s) + (1-\delta)\mathfrak{g}_2(s)\big\}ds\Big].\end{aligned}$$

As $\mathcal{G}$ is convex, thus for each $t \in \mathscr{J}$, $\delta\lambda_1(t) + (1-\delta)\lambda_2(t) \in \mathcal{G}(t, \omega(t))$. Therefore,

$$\delta\lambda_1(t) + (1-\delta)\lambda_2(t) \in \mathfrak{S}_{\mathcal{G},\omega}.$$

Hence $\delta\lambda_1(t) + (1-\delta)\lambda_2(t) \in \{\Lambda_2\varpi(t)\}$ which implies that Λ_2 is convex.

Claim B: To establish that Λ_2 maps bounded sets into bounded sets in $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$.

For $\varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, assume $\omega(t) = t^{(\rho-1)(1-\epsilon)}\varpi(t), \quad t \in \mathscr{J}'$. Hence, $\omega \in \mathfrak{B}_{\mathfrak{r}}^{\Omega}(\mathscr{J}')$. Consequently we have the following: for $t \in \mathscr{J}$

$$\begin{aligned}\|\Lambda_2\varpi(t)\| =& \Big\|t^{(1-\rho)(1-\epsilon)}\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds\big]\Big\| \\ \le& t^{(1-\rho)(1-\epsilon)}\Big[\frac{\mathcal{M}t^{(\rho-1)(1-\epsilon)}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\| + \|\varrho\|) + \frac{\mathcal{M}}{\Gamma(\epsilon)}\int_0^t (t-s)^{\epsilon-1}\mathfrak{g}(s)ds\Big] \\ \le& \frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\| + \|\varrho\|)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)}\left[\left(\int_0^t (t-s)^{\frac{\epsilon-1}{1-\epsilon_1}}ds\right)^{1-\epsilon_1}\left(\int_0^t |\mathfrak{g}(s)|^{\frac{1}{\epsilon_1}}ds\right)^{\epsilon_1}\right]\\
\leq&\frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\|+\|\varrho\|)+\frac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)}\left(\frac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1}\mathscr{T}^{\epsilon-\epsilon_1}\|\mathfrak{L}_{\mathfrak{g}}\|_{L^{\frac{1}{\epsilon_1}}}.
\end{aligned}
$$

Therefore it is bounded.

Claim C: Next, we need to demonstrate that Λ_2 maps bounded sets into equicontinuous sets of $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$. For $\varpi\in\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, define $\omega(t)=t^{(\rho-1)(1-\epsilon)}\varpi(t)$, where $t\in\mathscr{J}'$. For any $\zeta_1,\zeta_2\in\mathscr{J}'$ with $\zeta_1<\zeta_2$ and $\omega\in\mathfrak{B}^{\Omega}_{\mathfrak{r}}(\mathscr{J}')$. For $\mathfrak{g}\in\mathfrak{S}_{\mathcal{G},\omega}$ we have

$$
\begin{aligned}
\|\Lambda_2\varpi(\zeta_2)-\Lambda_2\varpi(\zeta_1)\|=&\left\|\zeta_2^{(1-\rho)(1-\epsilon)}\left[\mathcal{S}_{\epsilon,\rho}(\zeta_2)(\omega_0+\mathfrak{H}(\omega))+\int_0^{\zeta_2}\mathcal{Q}_{\epsilon}(\zeta_2-s)\mathfrak{g}(s)ds\right]\right.\\
&\left.-\zeta_1^{(1-\rho)(1-\epsilon)}\left[\mathcal{S}_{\epsilon,\rho}(\zeta_1)(\omega_0+\mathfrak{H}(\omega))+\int_0^{\zeta_1}\mathcal{Q}_{\epsilon}(\zeta_1-s)\mathfrak{g}(s)ds\right]\right\|\\
\leq&\left\|\zeta_2^{(1-\rho)(1-\epsilon)}\mathcal{S}_{\epsilon,\rho}(\zeta_2)(\omega_0+\mathfrak{H}(\omega))-\zeta_1^{(1-\rho)(1-\epsilon)}\mathcal{S}_{\epsilon,\rho}(\zeta_1)(\omega_0+\mathfrak{H}(\omega))\right\|\\
&+\zeta_2^{(1-\rho)(1-\epsilon)}\left\|\int_{\zeta_1}^{\zeta_2}(\zeta_2-s)^{\epsilon-1}\mathcal{P}_{\epsilon}(\zeta_2-s)\mathfrak{g}(s)ds\right\|\\
&+\left\|\int_0^{\zeta_1}\left[\zeta_2^{(1-\rho)(1-\epsilon)}(\zeta_2-s)^{\epsilon-1}-\zeta_1^{(1-\rho)(1-\epsilon)}(\zeta_1-s)^{\epsilon-1}\right]\right.\\
&\left.\times\mathcal{P}_{\epsilon}(\zeta_2-s)\mathfrak{g}(s)ds\right\|+\zeta_1^{(1-\rho)(1-\epsilon)}\left\|\int_0^{\zeta_1}(\zeta_1-s)^{\epsilon-1}\left[\mathcal{P}_{\epsilon}(\zeta_2-s)\right.\right.\\
&\left.\left.-\mathcal{P}_{\epsilon}(\zeta_1-s)\right]\mathfrak{g}(s)ds\right\|\\
=&\sum_{\iota=1}^{4}\mathscr{I}_{\iota}.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\mathscr{I}_1=&\left\|\zeta_2^{(1-\rho)(1-\epsilon)}\mathcal{S}_{\epsilon,\rho}(\zeta_2)(\omega_0+\mathfrak{H}(\omega))-\zeta_1^{(1-\rho)(1-\epsilon)}\mathcal{S}_{\epsilon,\rho}(\zeta_1)(\omega_0+\mathfrak{H}(\omega))\right\|\\
\leq&\frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon))\Gamma(\epsilon)}\left[\zeta_2^{(1-\rho)(1-\epsilon)}\left\|\int_{\zeta_1}^{\zeta_2}(\zeta_2-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}(\omega_0+\mathfrak{H}(\omega))ds\right\|\right.\\
&+\zeta_2^{(1-\rho)(1-\epsilon)}\left\|\int_0^{\zeta_1}\left\{(\zeta_2-s)^{\rho(1-\epsilon)-1}-(\zeta_1-s)^{\rho(1-\epsilon)-1}\right\}s^{\epsilon-1}(\omega_0+\mathfrak{H}(\omega))ds\right\|\\
&\left.+\left(\zeta_2^{(1-\rho)(1-\epsilon)}-\zeta_1^{(1-\rho)(1-\epsilon)}\right)\left\|\int_0^{\zeta_1}(\zeta_1-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}(\omega_0+\mathfrak{H}(\omega))ds\right\|\right]\\
=&\frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon))\Gamma(\epsilon)}\times\sum_{\iota=1}^{3}\mathscr{I}_{1\iota},
\end{aligned}
$$

where

$$
\mathscr{I}_{11}=\zeta_2^{(1-\rho)(1-\epsilon)}\left\|\int_{\zeta_1}^{\zeta_2}(\zeta_2-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}(\omega_0+\mathfrak{H}(\omega))ds\right\|
$$

$$\leq \frac{\mathscr{T}^{(1-\rho)(1-\epsilon)}\zeta_1^{\epsilon-1}}{\rho(1-\epsilon)}(|\omega_0| + \|\varrho\|)(\zeta_2-\zeta_1)^{\rho(1-\epsilon)}$$
$$\to 0 \quad \text{as} \quad \zeta_2 \to \zeta_1.$$

$$\begin{aligned}\mathscr{I}_{12} =& \zeta_2^{(1-\rho)(1-\epsilon)}\Big\| \int_0^{\zeta_1} \Big\{(\zeta_2-s)^{\rho(1-\epsilon)-1} - (\zeta_1-s)^{\rho(1-\epsilon)-1}\Big\} s^{\epsilon-1}(\omega_0 + \mathfrak{H}(\omega))ds\Big\| \\ \leq& \mathscr{T}^{(1-\rho)(1-\epsilon)}(|\omega_0| + \|\varrho\|)\Big\| \int_0^{\zeta_1} \Big\{(\zeta_2-s)^{\rho(1-\epsilon)-1} - (\zeta_1-s)^{\rho(1-\epsilon)-1}\Big\} s^{\epsilon-1}ds\Big\|.\end{aligned}$$

As,

$$\Big\| \int_0^{\zeta_1} \Big\{(\zeta_2-s)^{\rho(1-\epsilon)-1} - (\zeta_1-s)^{\rho(1-\epsilon)-1}\Big\} s^{\epsilon-1}ds\Big\| \leq 2\int_0^{\zeta_1} (\zeta_1-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}ds$$

exists, then by applying Lebesgue's dominated convergence theorem we obtain that

$$\Big\| \int_0^{\zeta_1} \Big\{(\zeta_2-s)^{\rho(1-\epsilon)-1} - (\zeta_1-s)^{\rho(1-\epsilon)-1}\Big\} s^{\epsilon-1}ds\Big\| \to 0 \quad \text{as} \quad \zeta_2 \to \zeta_1.$$

And

$$\mathscr{I}_{13} = \Big(\zeta_2^{(1-\rho)(1-\epsilon)} - \zeta_1^{(1-\rho)(1-\epsilon)}\Big)\Big\| \int_0^{\zeta_1} (\zeta_1-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}(\omega_0 + \mathfrak{H}(\omega))ds\Big\|.$$

As,

$$\zeta_2^{(1-\rho)(1-\epsilon)} - \zeta_1^{(1-\rho)(1-\epsilon)} \leq (\zeta_2-\zeta_1)^{(1-\rho)(1-\epsilon)},$$

thus $\mathscr{I}_{13} \to 0$ whenever $\zeta_2 \to \zeta_1$.

Combining all the results we get that $\mathscr{I}_1 \to 0$ as $\zeta_2 \to \zeta_1$.

Next

$$\begin{aligned}\mathscr{I}_2 =& \zeta_2^{(1-\rho)(1-\epsilon)}\Big\| \int_{\zeta_1}^{\zeta_2} (\zeta_2-s)^{\epsilon-1}\mathcal{P}_\epsilon(\zeta_2-s)\mathfrak{g}(s)ds\Big\| \\ \leq& \frac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)}\left(\frac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)\left[(\zeta_2-\zeta_1)^{\frac{\epsilon-\epsilon_1}{1-\epsilon_1}}\right]^{1-\epsilon_1}\|\mathfrak{L}_\mathfrak{g}\|_{L^{\frac{1}{\epsilon_1}}} \to 0 \quad \text{as} \quad \zeta_2 \to \zeta_1.\end{aligned}$$

$$\begin{aligned}\mathscr{I}_3 =& \Big\| \int_0^{\zeta_1} \Big[\zeta_2^{(1-\rho)(1-\epsilon)}(\zeta_2-s)^{\epsilon-1} - \zeta_1^{(1-\rho)(1-\epsilon)}(\zeta_1-s)^{\epsilon-1}\Big]\mathcal{P}_\epsilon(\zeta_2-s)\mathfrak{g}(s)ds\Big\| \\ \leq& \frac{\mathcal{M}}{\Gamma(\epsilon)}\Bigg[\Big\| \int_0^{\zeta_1} \zeta_2^{(1-\rho)(1-\epsilon)}\Big\{(\zeta_2-s)^{\epsilon-1} - (\zeta_1-s)^{\epsilon-1}\Big\}\mathfrak{g}(s)ds\Big\| \\ &+ \Big(\zeta_2^{(1-\rho)(1-\epsilon)} - \zeta_1^{(1-\rho)(1-\epsilon)}\Big)\Big\| \int_0^{\zeta_1} (\zeta_1-s)^{\epsilon-1}\mathfrak{g}(s)ds\Big\|\Bigg] \\ \leq& \frac{\mathcal{M}}{\Gamma(\epsilon)}\Bigg[\mathscr{T}^{(1-\rho)(1-\epsilon)}\left(\int_0^{\zeta_1} \Big\|(\zeta_2-s)^{\epsilon-1} - (\zeta_1-s)^{\epsilon-1}\Big\|^{\frac{1}{1-\epsilon_1}}ds\right)^{1-\epsilon_1}\left(\int_0^{\zeta_1} \|\mathfrak{g}(s)\|^{\frac{1}{\epsilon_1}}ds\right)^{\epsilon_1} \\ &+ (\zeta_2-\zeta_1)^{(1-\rho)(1-\epsilon)}\Big\| \int_0^{\zeta_1} (\zeta_1-s)^{\epsilon-1}\mathfrak{g}(s)ds\Big\|\Bigg]\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mathcal{M}}{\Gamma(\epsilon)}\Bigg[\mathscr{T}^{(1-\rho)(1-\epsilon)}\left(\frac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1}\left\{\left(\zeta_2-\zeta_1\right)^{\frac{\epsilon-\epsilon_1}{1-\epsilon_1}}-\left(\zeta_2^{\frac{\epsilon-\epsilon_1}{1-\epsilon_1}}-\zeta_1^{\frac{\epsilon-\epsilon_1}{1-\epsilon_1}}\right)\right\}^{1-\epsilon_1}\|\mathfrak{L}_{\mathfrak{g}}\|_{\frac{1}{\epsilon_1}}\\
&+\left(\zeta_2-\zeta_1\right)^{(1-\rho)(1-\epsilon)}\left\|\int_0^{\zeta_1}(\zeta_1-s)^{\epsilon-1}\mathfrak{g}(s)ds\right\|\Bigg]\\
&\to 0 \quad \text{as} \quad \zeta_2\to\zeta_1.
\end{aligned}
$$

And

$$
\mathscr{I}_4=\zeta_1^{(1-\rho)(1-\epsilon)}\left\|\int_0^{\zeta_1}(\zeta_1-s)^{\epsilon-1}\Big[\mathcal{P}_\epsilon(\zeta_2-s)-\mathcal{P}_\epsilon(\zeta_1-s)\Big]\mathfrak{g}(s)ds\right\|.
$$

From the continuity of $\mathcal{P}_\epsilon(t)$ in the uniform operator topology it can be easily shown that $\mathscr{I}_4\to 0$ as $\zeta_2\to\zeta_1$.

Combining all the results we get that $\|\Lambda_2\varpi(\zeta_2)-\Lambda_2\varpi(\zeta_1)\|\to 0$ as $\zeta_2\to\zeta_1$ independent of ϖ. Since $t^{(1-\rho)(1-\epsilon)}\mathcal{S}_{\epsilon,\rho}$ is uniformly continuous on $\mathscr{J}$, thus Λ_2 is equicontinuous on $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$.

Claim D: To proof that Λ maps $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ into itself. Assume that $\varpi\in\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, for $t\in\mathscr{J}'$, $\omega(t)=t^{(\rho-1)(1-\epsilon)}\varpi(t)$. Hence, $\omega\in\mathfrak{B}_{\mathfrak{r}}^{\Omega}(\mathscr{J}')$. For $t\in\mathscr{J}$ we have that

$$
\begin{aligned}
\|\Lambda\varpi(t)\| \leq& t^{(1-\rho)(1-\epsilon)}\mathfrak{F}(t,\omega(t))\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0+\mathfrak{H}(\omega))+\int_0^t\mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds\big]\\
\leq&\big[\|\mathfrak{F}(t,\omega(t))-\mathfrak{F}(t,0)\|+\|\mathfrak{F}(t,0)\|\big]\\
&\times\left[\frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\|+\|\varrho\|)+\frac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)}\left(\frac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1}\mathscr{T}^{\epsilon-\epsilon_1}\|\mathfrak{L}_{\mathfrak{g}}\|_{L^{\frac{1}{\epsilon_1}}}\right]\\
\leq&\big[\|\varphi\|\|\varpi\|+\mathcal{F}_0\big]\mathfrak{P}\\
\leq&\big[\|\varphi\|\mathfrak{r}+\mathcal{F}_0\big]\mathfrak{P}\\
\leq&\mathfrak{r}.
\end{aligned}
$$

Hence $\|\Lambda\varpi\|\leq\mathfrak{r}$ for any $\varpi\in\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$.

Claim E: To establish that Λ_2 is completely continuous. For that we assume $\varpi\in\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, for any $t\in\mathscr{J}'$, let $\omega(t)=t^{(\rho-1)(1-\epsilon)}\varpi(t)$. Therefore, $\omega\in\mathfrak{B}_{\mathfrak{r}}^{\Omega}(\mathscr{J}')$.

Now, define $\mathcal{V}(t)=\{\Lambda_2\varpi(t):\varpi\in\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})\}$. This set is relatively compact in $\mathcal{X}$ for any $t\in\mathscr{J}$. It is clear that $\mathcal{V}(0)$ is relatively compact in $\mathcal{X}$. Let $t\in\mathscr{J}'$ be fixed. As we already have

$$
\begin{aligned}
\Lambda_2\varpi(t)=&t^{(1-\rho)(1-\epsilon)}\Big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0+\mathfrak{H}(\omega))+\int_0^t\mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds\Big]\\
=&t^{(1-\rho)(1-\epsilon)}\Bigg[\frac{1}{\Gamma\big(\rho(1-\epsilon)\big)}\int_0^t(t-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}\int_0^\infty\epsilon\phi\mathcal{M}_\epsilon(\phi)\mathcal{T}(s^\epsilon\phi)\\
&\times(\omega_0+\mathfrak{H}(\omega))d\phi ds+\int_0^t(t-s)^{\epsilon-1}\int_0^\infty\epsilon\phi\mathcal{M}_\epsilon(\phi)\mathcal{T}\big((t-s)^\epsilon\phi\big)\mathfrak{g}(s)d\phi ds\Bigg],
\end{aligned}
$$

then for all $\mu\in(0,t)$ and for all $\eta>0$, define

$$
\Lambda_2^{\mu,\eta}\varpi(t)=t^{(1-\rho)(1-\epsilon)}\Bigg[\frac{1}{\Gamma\big(\rho(1-\epsilon)\big)}\int_0^{t-\mu}(t-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}\int_\eta^\infty\epsilon\phi\mathcal{M}_\epsilon(\phi)\mathcal{T}(s^\epsilon\phi)
$$

$$\times (\omega_0 + \mathfrak{H}(\omega))d\phi ds + \int_0^{t-\mu}(t-s)^{\epsilon-1}\int_\eta^\infty \epsilon\phi\mathcal{M}_\epsilon(\phi)\mathcal{T}\big((t-s)^\epsilon\phi\big)\mathfrak{g}(s)d\phi ds\Bigg]$$
$$= t^{(1-\rho)(1-\epsilon)}\Bigg[\frac{1}{\Gamma\big(\rho(1-\epsilon)\big)}\int_0^{t-\mu}(t-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}\int_\eta^\infty \epsilon\phi\mathcal{M}_\epsilon(\phi)\mathcal{T}(s^\epsilon\phi)$$
$$\times (\omega_0 + \mathfrak{H}(\omega))d\phi ds + \mathcal{T}(\mu^\epsilon\phi)\int_0^{t-\mu}\int_\eta^\infty \epsilon\phi(t-s)^{\epsilon-1}\mathcal{M}_\epsilon(\phi)\mathcal{T}\big((t-s)^\epsilon\phi - \mu^\epsilon\phi\big)$$
$$\times \mathfrak{g}(s)d\phi ds\Bigg].$$

Given that $\mathcal{T}(\mu^\epsilon\phi)$ is compact for $\mu^\epsilon > 0$, it follows that the set $\mathcal{V}^{\mu,\eta}(t) = \{\Lambda_2^{\mu,\eta}\varpi(t) : \varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})\}$ is relatively compact in $\mathcal{X}$ for every $\mu \in (0,t)$ and $\eta > 0$. From this observation, we can deduce

$$\begin{aligned}
&\|\Lambda_2\varpi(t) - \Lambda_2^{\mu,\eta}\varpi(t)\| \\
\leq &\Bigg\|t^{(1-\rho)(1-\epsilon)}\Bigg[\frac{1}{\Gamma\big(\rho(1-\epsilon)\big)}\int_0^t (t-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}\int_0^\eta \epsilon\phi\mathcal{M}_\epsilon(\phi)\mathcal{T}(s^\epsilon\phi) \\
&\times(\omega_0 + \mathfrak{H}(\omega))d\phi ds\Bigg]\Bigg\| + \Bigg\|t^{(1-\rho)(1-\epsilon)}\Bigg[\frac{1}{\Gamma\big(\rho(1-\epsilon)\big)}\int_{t-\mu}^t (t-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1} \\
&\times \int_\eta^\infty \epsilon\phi\mathcal{M}_\epsilon(\phi)\mathcal{T}(s^\epsilon\phi)(\omega_0 + \mathfrak{H}(\omega))d\phi ds\Bigg]\Bigg\| + \Bigg\|t^{(1-\rho)(1-\epsilon)}\Bigg[\int_0^t\int_0^\eta \epsilon\phi(t-s)^{\epsilon-1}\mathcal{M}_\epsilon(\phi) \\
&\times\mathcal{T}\big((t-s)^\epsilon\phi\big)\mathfrak{g}(s)d\phi ds\Bigg]\Bigg\| + \Bigg\|t^{(1-\rho)(1-\epsilon)}\Bigg[\int_{t-\mu}^t\int_\eta^\infty \epsilon\phi(t-s)^{\epsilon-1}\mathcal{M}_\epsilon(\phi)\mathcal{T}\big((t-s)^\epsilon\phi\big) \\
&\quad\mathfrak{g}(s)d\phi ds\Bigg]\Bigg\| \\
\leq &\frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon))}B\big(\rho(1-\epsilon),\epsilon\big)\big(\|\omega_0\| + \|\varrho\|\big)\int_0^\eta \epsilon\phi\mathcal{M}_\epsilon(\phi)d\phi \\
+ &\frac{\mathcal{M}\epsilon}{\Gamma(\rho(1-\epsilon))\Gamma(1+\epsilon)}\big(\|\omega_0\| + \|\varrho\|\big)\mathscr{T}^{(1-\rho)(1-\epsilon)}\int_{t-\mu}^t (t-s)^{\rho(1-\epsilon)-1}s^{\epsilon-1}ds \\
+ &\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}\int_0^t\int_0^\eta \epsilon\phi(t-s)^{\epsilon-1}\mathcal{M}_\epsilon(\phi)\mathfrak{L}_{\mathfrak{g}}(s)d\phi ds \\
+ &\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}\frac{\epsilon}{\Gamma(1+\epsilon)}\int_{t-\mu}^t (t-s)^{\epsilon-1}\mathfrak{L}_{\mathfrak{g}}(s)ds.
\end{aligned}$$

By using the absolute continuity of the Lebesgue integral, we can determine that the RHS of the above inequality tends to 0 as $\mu,\eta \to 0$. Therefore, $\mathcal{V}^{\mu,\eta}(t)$ are arbitrarily close to the set $\mathcal{V}(t)$. From Arzelá-Ascoli theorem we can conclude that $\mathcal{V}(t)$ is relatively compact. Thus, Λ_2 is a completely continuous operator obtained from the continuity of Λ_2 (with the help of [59]) and the relatively compactness of $\mathcal{V}(t)$.

Claim F: To show that Λ_2 is closed graph.

Let $\omega_n \to \omega_*$ as $n \to \infty$, $\Lambda_{2,n} \in \Xi(\omega_n)$ and $\Lambda_{2,n} \to \Lambda_{2,*}$ as $n \to \infty$. We need to show that $\Lambda_{2,*} \in \Xi(\omega_*)$. As $\Lambda_{2,n} \in \Xi(\omega_n)$, then there exists a function $\mathfrak{g}_n \in \mathfrak{S}_{\mathcal{G},\omega_n}$ such that

$$\Lambda_{2,n}\varpi(t) = t^{(1-\rho)(1-\epsilon)}\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_n)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}_n(s)ds\big].$$

Now we have to prove that there exists a $\mathfrak{g}_* \in \mathfrak{S}_{\mathcal{G},\omega_*}$ such that

$$\Lambda_{2,*}\varpi(t) = t^{(1-\rho)(1-\epsilon)}\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_*)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}_*(s)ds\big].$$

Clearly

$$\Big\|\Big\{\Lambda_{2,n}\varpi(t) - t^{(1-\rho)(1-\epsilon)}\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_n))\big]\Big\} - \Big\{\Lambda_{2,*}\varpi(t) - t^{(1-\rho)(1-\epsilon)}\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_*))\big]\Big\}\Big\| \to 0 \quad \text{as} \quad n \to \infty.$$

Next we define a operator $\Delta : L^1(\mathscr{J}, \mathcal{X}) \to \mathscr{C}$ as

$$\Delta\mathfrak{g}(t) = \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds.$$

Thus from Lemma 6.2.6 we get that $\Delta \circ \mathfrak{S}_{\mathcal{G}}$ is a closed graph operator. Hence, by referring to Δ we have

$$\Lambda_{2,n}\varpi(t) - t^{(1-\rho)(1-\epsilon)}\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_n))\big] \in \Delta(\mathfrak{S}_{\mathcal{G},\omega_n}).$$

Since $\omega_n \to \omega_*, \quad \mathfrak{g}_n \to \mathfrak{g}_*$ as $n \to \infty$, follows from Lemma 6.2.6 we get that

$$\Lambda_{2,*}\varpi(t) - t^{(1-\rho)(1-\epsilon)}\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_*))\big] \in \Delta(\mathfrak{S}_{\mathcal{G},\omega_*}).$$

Therefore Λ_2 is closed graph.

Hence the proof of Step II is completed.

Step III: Assume that $\varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, for $t \in \mathscr{J}'$, $\omega(t) = t^{(\rho-1)(1-\epsilon)}\varpi(t)$. Hence, $\omega \in \mathfrak{B}^{\Omega}_{\mathfrak{r}}(\mathscr{J}')$.

We have to show that $\alpha\Upsilon < 1$ i.e., (iii) of Theorem 1.6.22.

Clearly this comes from (6.3.2). As we have

$$\begin{aligned}\Upsilon &= \|\Lambda_2(\mathfrak{B}_{\mathfrak{r}}(\mathscr{J}))\| = \sup\{\|\Lambda_2\varpi\| : \varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})\}\\ &\leq \frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\| + \|\varrho\|) + \frac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)}\left(\frac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1}\mathscr{T}^{\epsilon-\epsilon_1}\|\mathfrak{L}_{\mathfrak{g}}\|_{L^{\frac{1}{\epsilon_1}}}\end{aligned}$$

and $\alpha = \|\varphi\|$.

Thus all the conditions of Theorem 1.6.22 are satisfisfied. Therefore either (a) or (b) is possible.

Step IV: Next we have to show that (b) of 1.6.22 is not true.

Let $\bar{\chi}(t) \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ is arbitrary. Assume that $\chi(t) = t^{(\rho-1)(1-\epsilon)}\bar{\chi}(t), \quad t \in \mathscr{J}'$, hence $\chi \in \mathfrak{B}^{\Omega}_{\mathfrak{r}}(\mathscr{J}')$. Also $\sigma\bar{\chi} \in \Lambda_1\bar{\chi}(t) \times \Lambda_2\bar{\chi}(t)$. There exists $\mathfrak{g} \in \mathfrak{S}_{\mathcal{G},\chi}$ and for $\sigma > 1$ we have

$$\begin{aligned}\bar{\chi}(t) \leq& \sigma^{-1}\left[t^{(1-\rho)(1-\epsilon)}\mathfrak{F}(t,\chi(t))\Big\{\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\chi)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds\Big\}\right]\\ \leq& \big[\|\mathfrak{F}(t,\chi(t)) - \mathfrak{F}(t,0)\| + \|\mathfrak{F}(t,0)\|\big]\end{aligned}$$

$$\times\left[\frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\|+\|\varrho\|)+\frac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)}\left(\frac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1}\mathscr{T}^{\epsilon-\epsilon_1}\|\mathfrak{L}_{\mathfrak{g}}\|_{L^{\frac{1}{\epsilon_1}}}\right]$$
$$\implies \bar{\chi}(t) \leq \left[\|\varphi\|\|\bar{\chi}\|+\mathcal{F}_0\right]\mathfrak{P}$$
$$\implies \|\bar{\chi}\| \leq \frac{\mathcal{F}_0\mathfrak{P}}{1-\mathfrak{P}\|\varphi\|} \leq \mathfrak{R},$$

where we consider $\mathcal{F}_0 = \sup_{t\in\mathscr{J}}\|\mathfrak{F}(t,0)\|$. Thus condition (b) of Theorem 1.6.22 does not hold by 6.3.2. Therefore the operator equation $\Lambda\varpi = \Lambda_1\varpi \times \Lambda_2\varpi$ has a fixed point ϖ in $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$. Thus the considered problem (6.1.1) has a mild solution.

6.4 Controllability Result

In this section we study the exact controllability of the following hybrid fractional differential inclusion

$$\begin{gathered}{}_0\mathcal{D}_t^{\epsilon,\rho}\left(\frac{\omega(t)}{\mathfrak{F}(t,\omega(t))}\right) \in \mathcal{A}\left(\frac{\omega(t)}{\mathfrak{F}(t,\omega(t))}\right)+\mathcal{G}(t,\omega(t))+\mathscr{B}z(t), \qquad t\in(0,\mathscr{T}]=\mathscr{J}', \\ \mathcal{I}^{(1-\epsilon)(1-\rho)}\left(\frac{\omega(0)}{\mathfrak{F}(0,\omega(0))}\right)-\mathfrak{H}(\omega)=\omega_0.\end{gathered} \tag{6.4.1}$$

In this problem, the operators $\mathcal{A}$, $\mathfrak{F}$, $\mathcal{G}$, and $\mathfrak{H}$ are defined as in the previous problem. Let $z(\cdot)$ be the control function, which belongs to $L^2(\mathscr{J},\mathcal{Z})$, where $\mathcal{Z}$ is a Banach space. The operator $\mathscr{B}:\mathcal{Z}\to\mathcal{X}$ is assumed to be bounded and linear. In this section, we do not assume the compactness of the semigroup operator $\{\mathcal{T}(t)\}_{t>0}$.

Definition 6.4.1. *A function $\omega\in\bar{\mathscr{C}}$ is termed as a mild solution of the equation* (6.4.1) *if, for each $t\in\mathscr{J}$, there exists a function $\mathfrak{g}\in L^1(\mathscr{J},\mathcal{X})$ such that $\mathfrak{g}(t)\in\mathcal{G}(t,\omega(t))$. Moreover, $\omega\in\bar{\mathscr{C}}$ must satisfy the following integral equation*

$$\omega(t)=\mathfrak{F}(t,\omega(t))\Big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0+\mathfrak{H}(\omega))+\int_0^t\mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds+\int_0^t\mathcal{Q}_\epsilon(t-s)\mathscr{B}z(s)ds\Big], \quad t\in\mathscr{J}'. \tag{6.4.2}$$

Definition 6.4.2. *The system* (6.4.1) *is said to be controllable on $\mathscr{J}=[0,\mathscr{T}]$ if, for every $\omega_0,\omega_1\in\mathcal{X}$, there exist a control $z\in L^2(\mathscr{J},\mathcal{Z})$ such that the mild solution $\omega(t)$ of the system* (6.4.1) *satisfies $\mathcal{I}^{(1-\epsilon)(1-\rho)}\left(\frac{\omega(0)}{\mathfrak{F}(0,\omega(0))}\right)-\mathfrak{H}(\omega)=\omega_0$ and $\omega(\mathscr{T})=\omega_1$.*

Besides the hypotheses (A_2)-(A_4), also we need the following hypotheses to prove our result:

(A_5) $\mathfrak{g}:\mathscr{J}\to\mathcal{X}$ is compact.

(A_6) The operator $\mathscr{B} : L^2(\mathscr{J}, \mathcal{Z}) \to L^1(\mathscr{J}, \mathcal{X})$ is both linear and bounded. Moreover, the operator $\mathcal{Y} : L^2(\mathscr{J}, \mathcal{Z}) \to \mathcal{X}$ is linear and defined by the expression

$$\mathcal{Y}z = \int_0^{\mathscr{T}} \mathcal{Q}_\epsilon(\mathscr{T} - s)\mathscr{B}z(s)\, ds.$$

This operator $\mathcal{Y}$ has an inverse $\mathcal{Y}^{-1}$, that maps to $L^2(\mathscr{J}, \mathcal{Z})/\ker(\mathcal{Y})$, and there exists a constant $\mathfrak{L}_Y > 0$ such that

$$\|\mathcal{Y}^{-1}\|_{L^2} \leq \mathfrak{L}_Y.$$

For each $\omega_1 \in X$, by using hypothesis (A_6) we define the control as

$$z_\omega(t) = \mathcal{Y}^{-1}\left[\frac{\omega_1}{\mathfrak{F}(\mathscr{T}, \omega_1)} - \mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) - \int_0^{\mathscr{T}} \mathcal{Q}_\epsilon(\mathscr{T} - s)\mathfrak{g}(s)ds\right](t), \tag{6.4.3}$$

where $\mathfrak{g} \in \mathfrak{S}_{\mathcal{G},\omega}, \quad \omega \in \bar{\mathscr{C}}$.

Theorem 6.4.3. *Suppose that the assumptions (A_2)-(A_6) hold. Then the system (6.4.1) is controllable on $\mathscr{J}$ provided that*

$$\tilde{\mathfrak{R}} > \frac{\mathcal{F}_0\tilde{\mathfrak{P}}}{1 - \tilde{\mathfrak{P}}\|\varphi\|}, \tag{6.4.4}$$

where $\tilde{\mathfrak{P}} = \dfrac{\mathcal{M}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\| + \|\varrho\|) + \dfrac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)}\left(\dfrac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1}\mathscr{T}^{\epsilon-\epsilon_1}\|\mathfrak{L}_\mathfrak{g}\|_{L^{\frac{1}{\epsilon_1}}} + \dfrac{\mathcal{M}\mathscr{T}^{\epsilon(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon+1)}\|\mathscr{B}\|\mathfrak{L}_z$, $\mathcal{F}_0 = \|\mathfrak{F}(t, 0)\|$, *and* $\tilde{\mathfrak{P}}\|\varphi\| < 1$.

Proof. Using control let us define an operator $\tilde{\Xi} : \mathscr{C} \to 2^{\mathscr{C}}$ as $\tilde{\Xi}\omega$, which is the set of $\tilde{\Theta} \in \mathscr{C}$ such that

$$\tilde{\Theta}(t) = \mathfrak{F}(t, \omega(t))\left[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}z_\omega(s)ds\right]$$

for all $t \in \mathscr{J}'$ and $\mathfrak{g} \in \mathfrak{S}_{\mathcal{G},\omega}$.

Like the previous problem now let us define an operator $\tilde{\Lambda}$ for any $\varpi \in \mathscr{C}$ and assume that $\omega(t) = t^{(\rho-1)(1-\epsilon)}\varpi(t)$ as follows

$$(\tilde{\Lambda}\varpi)(t) = \begin{cases} t^{(1-\rho)(1-\epsilon)}\tilde{\Theta}(t), & t \in (0, \mathscr{T}], \\ \dfrac{\mathfrak{F}_o(\omega_0 + \mathfrak{H}(\omega))}{\Gamma(\rho(1-\epsilon)+\epsilon)}, & t = 0. \end{cases} \tag{6.4.5}$$

This implies that ω is a mild solution of (6.1.1) in Ω if and only if the operator equation $\varpi = \Lambda\varpi$ has a solution $\varpi \in \mathscr{C}$. Moreover, if ω is a mild solution of (6.1.1) with the control given by (6.4.3), then $\omega(\mathscr{T}) = \omega_1$.

To show that the operator $\tilde{\Lambda}$ has a fixed point, similarly we consider two operators $\tilde{\Lambda}_1, \tilde{\Lambda}_2 : \mathfrak{B}_{\mathfrak{r}}(\mathscr{J}) \to \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ such that $\tilde{\Lambda}_1$ is same as (6.3.4) and $\tilde{\Lambda}_2$ is as follows:

$$(\tilde{\Lambda}_2\varpi)(t) = \begin{cases} t^{(1-\rho)(1-\epsilon)}\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds \\ + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}z_\omega(s)ds\big], & t \in (0, \mathscr{T}], \\ \dfrac{\omega_0 + \mathfrak{H}(\omega)}{\Gamma(\rho(1-\epsilon)+\epsilon)}, & t = 0. \end{cases} \tag{6.4.6}$$

Therefore,

$$(\tilde{\Lambda}\varpi)(t) = (\tilde{\Lambda}_1\varpi)(t) \times (\tilde{\Lambda}_2\varpi)(t) \quad \text{for} \quad t \in \mathscr{J}.$$

Now we have to establish that both the operators Λ_1 and Λ_2 meet all the requirements of Theorem 1.6.22. □

Note that

$$\begin{aligned} \|z_\omega(t)\| &\leq \mathfrak{L}_Y \left[\left\| \frac{\omega_1}{\mathfrak{F}(\mathscr{T}, \omega_1)} \right\| + \frac{\mathcal{M}\mathscr{T}^{(\rho-1)(1-\epsilon)}}{\Gamma(\rho(1-\epsilon)+\epsilon)} \big(\|\omega_0\| + \|\varrho\|\big) \right. \\ &\quad \left. + \frac{\mathcal{M}}{\Gamma(\epsilon)} \left(\frac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1} \mathscr{T}^{\epsilon-\epsilon_1} \|\mathfrak{L}_{\mathfrak{g}}\|_{L^{\frac{1}{\epsilon_1}}} \right] \\ &= \mathfrak{L}_z. \end{aligned}$$

We can proceed Step I and Claim A - Claim C of Step II like the previous way. Now
Claim D: To proof that $\tilde{\Lambda}$ maps $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ into itself. Assume that $\varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$, for $t \in \mathscr{J}'$, $\omega(t) = t^{(\rho-1)(1-\epsilon)}\varpi(t)$. Thus, $\omega \in \mathfrak{B}^{\Omega}_{\mathfrak{r}}(\mathscr{J}')$. For $t \in \mathscr{J}$ we have that

$$\begin{aligned} \|\tilde{\Lambda}\varpi(t)\| &\leq t^{(1-\rho)(1-\epsilon)}\mathfrak{F}(t,\omega(t))\big[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds \\ &\quad + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}z_\omega(s)ds\big] \\ &\leq \big[\|\mathfrak{F}(t,\omega(t)) - \mathfrak{F}(t,0)\| + \|\mathfrak{F}(t,0)\|\big] \\ &\quad \times \left[\frac{\mathcal{M}}{\Gamma(\rho(1-\epsilon)+\epsilon)}(\|\omega_0\| + \|\varrho\|) + \frac{\mathcal{M}\mathscr{T}^{(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon)} \left(\frac{1-\epsilon_1}{\epsilon-\epsilon_1}\right)^{1-\epsilon_1} \mathscr{T}^{\epsilon-\epsilon_1}\|\mathfrak{L}_{\mathfrak{g}}\|_{L^{\frac{1}{\epsilon_1}}} \right. \\ &\quad \left. + \frac{\mathcal{M}\mathscr{T}^{\epsilon(1-\rho)(1-\epsilon)}}{\Gamma(\epsilon+1)}\|\mathscr{B}\|\mathfrak{L}_z \right] \\ &\leq \big[\|\varphi\|\mathfrak{r} + \mathcal{F}_0\big]\tilde{\mathfrak{P}} \\ &\leq \mathfrak{r}. \end{aligned}$$

Hence $\|\tilde{\Lambda}\varpi\| \leq \mathfrak{r}$ for any $\varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$.
Claim E: To establish that the operator $\tilde{\Lambda}_2$ is completely continuous, we start by assuming that $\varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$. For $t \in \mathscr{J}'$, let $\omega(t) = t^{(\rho-1)(1-\epsilon)}\varpi(t)$. Consequently, ω belongs to $\mathfrak{B}^{\Omega}_{\mathfrak{r}}(\mathscr{J}')$.

We then analyse the set $\tilde{\mathcal{V}}(t) = \{\tilde{\Lambda}_2\varpi(t) : \varpi \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})\}$. It is clear that $\tilde{\mathcal{V}}(0)$ is relatively compact in $\mathcal{X}$. For any fixed $t \in \mathscr{J}'$, we need to demonstrate that $\tilde{\mathcal{V}}(t)$ is also relatively compact in $\mathcal{X}$.

By applying assumption (A_5) and Remark 6.2.5, we can infer that the set

$$\mathscr{S} = \left\{t^{(1-\rho)(1-\epsilon)}\mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s) : t \in \mathscr{J}', s \in [0,t]\right\}$$

is relatively compact in $\mathcal{X}$. Consequently,

$$\mathscr{S}' = \left\{t^{(1-\rho)(1-\epsilon)}\int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)\,ds\right\} \subset t\overline{CH}(\mathscr{S})$$

is also relatively compact in $\mathcal{X}$, where $\overline{CH}(\mathscr{S})$ denotes the closure of the convex hull of $\mathscr{S}$ in $\mathcal{X}$.

Furthermore, by assumption (A_6), we have that

$$\mathscr{S}'' = \left\{z_\omega = \mathcal{Y}^{-1}\left[\frac{\omega_1}{\mathfrak{F}(\mathscr{T},\omega_1)} - \mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega)) - \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)\,ds\right] : \omega \in \mathfrak{B}_{\mathfrak{r}}^{\Omega}\right\}$$

is relatively compact in $L^2(\mathscr{J},\mathcal{Z})$. Since $\mathscr{B} : L^2(\mathscr{J},\mathcal{Z}) \to L^1(\mathscr{J},\mathcal{X})$ is a bounded operator, the set $\mathscr{B}\mathscr{S}''$ is relatively compact in $L^1(\mathscr{J},\mathcal{X})$. Thus, the set

$$\mathscr{S}''' = \left\{t^{(1-\rho)(1-\epsilon)}\int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}u(s)\,ds : u \in \mathscr{S}''\right\}$$

is relatively compact in $\mathcal{X}$, as the mapping

$$\mathscr{U} \to t^{(1-\rho)(1-\epsilon)}\int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{U}(s)\,ds$$

is continuous from $L^1(\mathscr{J},\mathcal{X})$ to $\mathcal{X}$.

Combining these results, we obtain that

$$\tilde{\mathcal{V}}(t) \subset t^{(1-\rho)(1-\epsilon)}\left[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega))\right] + \mathscr{S}' + \mathscr{S}'''$$

is relatively compact in $\mathcal{X}$ for $t \in \mathscr{J}'$. Therefore, $\tilde{\Lambda}_2$ is shown to be a completely continuous operator due to its continuity and the relatively compact nature of $\tilde{\mathcal{V}}(t)$.

Claim F: To show that $\tilde{\Lambda}_2$ is closed graph.

Let $\omega_n \to \omega_*$ as $n \to \infty$, $\tilde{\Lambda}_{2,n} \in \tilde{\Xi}(\omega_n)$ and $\tilde{\Lambda}_{2,n} \to \tilde{\Lambda}_{2,*}$ as $n \to \infty$. We need to show that $\tilde{\Lambda}_{2,*} \in \tilde{\Xi}(\omega_*)$. As $\tilde{\Lambda}_{2,n} \in \tilde{\Xi}(\omega_n)$, then there exists a function $\mathfrak{g}_n \in \mathfrak{S}_{\mathcal{G},\omega_n}$ such that

$$\begin{aligned}\Lambda_{2,n}\varpi(t) =& t^{(1-\rho)(1-\epsilon)}\Bigg[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_n)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}_n(s)ds \\ &+ \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}\mathcal{Y}^{-1}\Bigg\{\frac{\omega_1}{\mathfrak{F}(\mathscr{T},\omega_1)} - \mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_n)) \\ &- \int_0^{\mathscr{T}} \mathcal{Q}_\epsilon(\mathscr{T}-\tau)\mathfrak{g}_n(\tau)d\tau\Bigg\}(s)ds\Bigg].\end{aligned}$$

Now we have to prove that there exists a $\mathfrak{g}_* \in \mathfrak{S}_{\mathcal{G},\omega}$ such that

$$\begin{aligned}\Lambda_{2,*}\varpi(t) =& t^{(1-\rho)(1-\epsilon)}\Bigg[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_*)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}_*(s)ds \\ &+ \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}\mathcal{Y}^{-1}\Bigg\{\frac{\omega_1}{\mathfrak{F}(\mathscr{T},\omega_1)} - \mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_*)) \\ &- \int_0^{\mathscr{T}} \mathcal{Q}_\epsilon(\mathscr{T}-\tau)\mathfrak{g}_*(\tau)d\tau\Bigg\}(s)ds\Bigg].\end{aligned}$$

Set

$$\tilde{z}_\omega(t) = \mathcal{Y}^{-1}\left[\frac{\omega_1}{\mathfrak{F}(\mathscr{T},\omega_1)} - \mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega))\right].$$

As $\mathcal{Y}^{-1}$ is continuous, then $\tilde{z}_{\omega_n}(t) \to \tilde{z}_{\omega_*}(t)$ as $n \to \infty$. Hence it is clear that

$$\begin{aligned}&\Bigg\|\Bigg[\Lambda_{2,n}\varpi(t) - t^{(1-\rho)(1-\epsilon)}\Bigg\{\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_n)) - \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}\tilde{z}_{\omega_n}(s)ds\Bigg\}\Bigg] \\ &- \Bigg[\Lambda_{2,*}\varpi(t) - t^{(1-\rho)(1-\epsilon)}\Bigg\{\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_*)) - \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}\tilde{z}_{\omega_*}(s)ds\Bigg\}\Bigg]\Bigg\| \\ &\to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

Next we define a operator $\tilde{\Delta} : L^1(\mathscr{J}, \mathcal{X}) \to \mathscr{C}$ as

$$\tilde{\Delta}\mathfrak{g}(t) = \int_0^t \mathcal{Q}_\epsilon(t-s)\left[\mathfrak{g}(s) - \mathscr{B}\mathcal{Y}^{-1}\left\{\int_0^{\mathscr{T}} \mathcal{Q}_\epsilon(\mathscr{T}-\tau)\mathfrak{g}(\tau)d\tau\right\}(s)\right]ds.$$

Thus from Lemma 6.2.6 we get that $\Delta \circ \mathfrak{S}_{\mathcal{G}}$ is a closed graph operator. Therefore by referring to Δ we have

$$\Lambda_{2,n}\varpi(t) - t^{(1-\rho)(1-\epsilon)}\left[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_n)) - \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}\tilde{z}_{\omega_n}(s)ds\right] \in \Delta(\mathfrak{S}_{\mathcal{G},\omega_n}).$$

Since $\omega_n \to \omega_*$, $\mathfrak{g}_n \to \mathfrak{g}_*$ as $n \to \infty$, follows from Lemma 6.2.6 we get that

$$\Lambda_{2,*}\varpi(t) - t^{(1-\rho)(1-\epsilon)}\left[\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\omega_*)) - \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}\tilde{z}_{\omega_*}(s)ds\right] \in \Delta(\mathfrak{S}_{\mathcal{G},\omega_*}).$$

Therefore Λ_2 is closed graph.

Hence the proof of Step II is completed. Also Step III can be obtained from (6.4.4).

Step IV: Next we have to show that (b) of 1.6.22 is not true.

Let $\bar{\chi}(t) \in \mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$ is arbitrary. Assume that $\chi(t) = t^{(\rho-1)(1-\epsilon)}\bar{\chi}(t)$, $t \in \mathscr{J}'$. Hence, $\chi \in \mathfrak{B}_{\mathfrak{r}}^{\Omega}(\mathscr{J}')$. Also $\sigma\bar{\chi} \in \Lambda_1\bar{\chi}(t) \times \Lambda_2\bar{\chi}(t)$. There exists $\mathfrak{g} \in \mathfrak{S}_{\mathcal{G},\chi}$ and for $\sigma > 1$ we have

$$\begin{aligned}\bar{\chi}(t) \leq& \sigma^{-1}\Bigg[t^{(1-\rho)(1-\epsilon)}\mathfrak{F}(t,\chi(t))\Bigg\{\mathcal{S}_{\epsilon,\rho}(t)(\omega_0 + \mathfrak{H}(\chi)) + \int_0^t \mathcal{Q}_\epsilon(t-s)\mathfrak{g}(s)ds \\ &+ \int_0^t \mathcal{Q}_\epsilon(t-s)\mathscr{B}z_\omega(s)ds\Bigg\}\Bigg] \\ \implies \bar{\chi}(t) \leq& \big[\|\varphi\|\|\bar{\chi}\| + \mathcal{F}_0\big]\tilde{\mathfrak{P}}\end{aligned}$$

$$\implies \|\bar{\chi}\| \leq \frac{\mathcal{F}_0 \tilde{\mathfrak{P}}}{1-\tilde{\mathfrak{P}}\|\varphi\|} \leq \tilde{\mathfrak{R}},$$

where we consider $\mathcal{F}_0 = \sup_{t\in\mathscr{J}} \|\mathfrak{F}(t,0)\|$. Thus condition (b) of Theorem 1.6.22 does not hold by 6.4.4. Therefore the operator equation $\tilde{\Lambda}\varpi = \tilde{\Lambda}_1\varpi \times \tilde{\Lambda}_2\varpi$ has a fixed point ϖ in $\mathfrak{B}_{\mathfrak{r}}(\mathscr{J})$. Thus the system (6.4.1) is controllable.

6.5 Example

As an application we provide the following example

$$\begin{aligned}
{}_0\mathcal{D}_t^{\frac{2}{3},\frac{3}{4}}\left(\frac{\eta(\tau,\mathfrak{z})}{\frac{1}{20\pi}\Big(\tan^{-1}\eta(\tau,\mathfrak{z})+\frac{\pi}{2}\Big)}\right) &\in \Delta\left(\frac{\eta(\tau,\mathfrak{z})}{\frac{1}{20\pi}\Big(\tan^{-1}\eta(\tau,\mathfrak{z}))+\frac{\pi}{2}\Big)}\right)+\mathcal{G}(\tau,\eta(\tau,\mathfrak{z})),\\
&\qquad \tau\in(0,1], \quad \mathfrak{z}\in[0,\pi],\\
\mathcal{I}^{(1-\frac{2}{3})(1-\frac{3}{4})}\left(\frac{\eta(0,\mathfrak{z})}{\mathfrak{F}(0,\eta(0,\mathfrak{z}))}\right) - 4\int_0^1 \sin\big(\eta(s,\mathfrak{z})\big)ds &= \frac{1}{15}, \quad \mathfrak{z}\in[0,\pi],\\
\eta(\tau,0)=\eta(\tau,\pi)&=0, \quad \tau\in(0,1).
\end{aligned} \tag{6.5.1}$$

$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ is defined by $\mathcal{A} = \Delta$ with domain $D(\mathcal{A}) = \big\{\eta \in H^2(0,\pi) : \eta(0) = \eta(\pi) = 0\big\}$ and the semigroup $\mathcal{T}(t)$ generated by $\mathcal{A} = \Delta$ is contractive, i.e $\|\mathcal{T}(t)\| \leq 1, \ \forall \ t \geq 0$. The multivalued map $\mathcal{G}$ is defined as

$$(\tau,\mathfrak{z}) \mapsto \mathcal{G}(\tau,\eta(\tau,\mathfrak{z})) = \left[\frac{1}{30}\big(2\tanh\eta(\tau,\mathfrak{z})+1\big), \frac{|\cos\eta(\tau,\mathfrak{z})|}{20\big(|\sin\eta(\tau,\mathfrak{z})|+1\big)}+\frac{1}{30}\right].$$

Considered system (6.5.1) satisfies all the assumptions (A_1)-(A_4). Putting all the values in (6.3.2) we get that $\mathfrak{P}\|\varphi\| = 0.063183 < 1$.

Thus the system (6.5.1) has at least a mild solution by Theorem 1.6.22.

□