CHAPTER 7

Results on a Hybrid Type of Multipoint BVP using Topological Degree Theory

7.1 Introduction

The ψ-Caputo fractional derivative is a generalized form of the classical fractional derivative. It extends the concept of fractional calculus by incorporating a ψ-function, which allows for greater flexibility and adaptability in modelling complex systems, particularly in mathematical physics, engineering, and applied sciences. As fractional calculus grew in popularity, researchers explored ways to generalize classical derivatives to address diverse problems. The ψ-Caputo [6, 14, 17, 27, 34, 40] derivative came as part of these generalizations, where the ψ-function plays a central role in redefining the differential operator.

To simplify the various definitions of fractional operators, one approach is to use general operators with specific kernels to recover classical fractional derivatives and integrals. For example, choosing $k(x,t) = x - t$ and the differential operator $\frac{d}{dx}$ yields the Riemann-Liouville derivative, while $k(x,t) = \ln(x/t)$ and the differential operator $x\frac{d}{dx}$ gives the Hadamard derivative. However, the arbitrary nature of the kernel restricts the ability to derive fundamental properties. Almeida [13] suggested a more effective approach by considering a special case where $k(x,t) = \psi(x) - \psi(t)$ and the derivative operator is $\frac{1}{\psi'(x)}\frac{d}{dx}$, which generalizes both the Riemann-Liouville and Hadamard derivatives by putting $\psi(x) = x$ and $\psi(x) = \ln(x)$ respectively and present some key properties. Adjimi et al. [6] studied a neutral hybrid nonlinear differential equation with the ψ-Caputo operator, using noncompactness measures and Darbo's criterion to extend existing results. Recently, Chabane et al. [34] studied a generalized impulsive ψ-Caputo differential equation with a $p-$Laplacian operator, proving existence and uniqueness of solutions via fixed point theorems.

Boundary value problems play a pivotal role in advancing the theory and applications of fractional differential equations and represent a key area of research in this field. While extensive studies have focused on two-point boundary value problems for fractional ordinary differential equations [33, 86, 113, 123], multipoint boundary value problems provide a broader framework. These involve boundary conditions specified at multiple points within an interval, rather than only at its endpoints, thereby generalizing traditional boundary value problems. Such problems arise in various physical and engineering applications where system behaviour is influenced by conditions at several locations rather than just two.

Despite numerous significance of multipoint boundary value problem, study on these types of problems for fractional differential equations remains relatively limited [8, 23, 66, 114, 130]. To bridge this gap, further study is important to establish broader results, develop efficient numerical methods, and investigate new applications in emerging scientific and engineering fields.

In this work, we aim to study the following perturbed fractional differential equation involving $\psi-$Caputo fractional derivative with multipoint boundary condition

$$
{}_0\mathcal{D}_m^{\sigma,\psi}\big[\omega(\tau) - \mathfrak{F}(\tau, \omega(\tau))\big] = \mathcal{G}(\tau, \omega(\tau)), \quad \tau \in \mathscr{J} = [m, n], \tag{7.1.1}
$$

$$
\omega(m) = 0, \tag{7.1.2}
$$

$$
\alpha\big[\omega(\tau) - \mathfrak{F}(\tau, \omega(\tau))\big]_{\tau=n} = \beta\big[\omega(\tau) - \mathfrak{F}(\tau, \omega(\tau))\big]_{\tau=\zeta}. \tag{7.1.3}
$$

Here, ${}_0\mathcal{D}_m^{\sigma,\psi}$ stands for σ order $\psi-$Caputo fractional derivative with $\sigma \in (1, 2]$. Let $\mathfrak{F}, \mathcal{G} \in \mathcal{C}(\mathscr{J} \times \mathbb{R}, \mathbb{R})$ also $\alpha, \beta \in \mathbb{R}$ and $\zeta \in (m, n)$.

For many years, fixed point theory has been a fundamental tool in demonstrating the existence of solutions to differential equations, as highlighted by the numerous studies referenced in the literature above. One key development in this field was the application of topological degree theory by Mawhin [92], who was among the first to use it to solve integral equations, marking a notable progress in mathematical methods. Following this, Isaia [67] further advanced the use of topological degree theory, applying it in a theoretical framework to analyse various integral equations. This progression underscores the growing significance of topological degree theory in addressing complex problems across different areas of mathematics [16, 20, 56, 117].

This paper is organized as follows: In Section 7.2, we derive the integral equation corresponding to the considered problem, outline the necessary assumptions, and provide the proof of the main result. In Section 7.3, we include an analytical example and in Section 7.4, an example with numerical results is provided as an application of the considered problem.

7.2 Existence Result

Lemma 7.2.1. *The considered problem* (7.1.1)-(7.1.3) *has a solution* $\omega \in \mathcal{C}(\mathscr{J}, \mathbb{R})$ *of the following integral form*

$$\begin{aligned}\omega(\tau) &= \mathfrak{F}(\tau, \omega(\tau)) + \frac{1}{\Gamma(\sigma)} \int_m^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\sigma-1} \mathcal{G}(s, \omega(s)) ds - \mathfrak{F}(m, 0) \\ &+ \frac{\psi(\tau) - \psi(m)}{\Theta} \Bigg[(\alpha - \beta)\mathfrak{F}(m, 0) - \frac{\alpha}{\Gamma(\sigma)} \int_m^n \psi'(s)\big(\psi(n) - \psi(s)\big)^{\sigma-1} \mathcal{G}(s, \omega(s)) ds \\ &+ \frac{\beta}{\Gamma(\sigma)} \int_m^\zeta \psi'(s)\big(\psi(\zeta) - \psi(s)\big)^{\sigma-1} \mathcal{G}(s, \omega(s)) ds\Bigg],\end{aligned} \tag{7.2.1}$$

where

$$\Theta = \alpha\big(\psi(n) - \psi(m)\big) - \beta\big(\psi(\zeta) - \psi(m)\big) \neq 0. \tag{7.2.2}$$

Proof.

$$\omega(\tau) - \mathfrak{F}(\tau, \omega(\tau)) = \mathcal{I}_{m^+}^{\sigma,\psi} \mathcal{G}(\tau, \omega(\tau)) + \mathfrak{C}_0 + \mathfrak{C}_1\big(\psi(\tau) - \psi(m)\big), \tag{7.2.3}$$

where $\mathfrak{C}_0$ and $\mathfrak{C}_1 \in \mathbb{R}$.
Applying (7.1.2) we obtain that $\mathfrak{C}_0 = -\mathfrak{F}_m$, where $\mathfrak{F}_m = \mathfrak{F}(m, 0)$.
Again from the condition (7.1.3) we get that

$$\alpha\big[\mathcal{I}_{m^+}^{\sigma,\psi} \mathcal{G}(n, \omega(n)) - \mathfrak{F}_m + \mathfrak{C}_1\big(\psi(n) - \psi(m)\big)\big] = \beta\big[\mathcal{I}_{m^+}^{\sigma,\psi} \mathcal{G}(\zeta, \omega(\zeta)) - \mathfrak{F}_m + \mathfrak{C}_1\big(\psi(\zeta) - \psi(m)\big)\big].$$

Then,

$$\mathfrak{C}_1 = \frac{1}{\Theta}\big[(\alpha - \beta)\mathfrak{F}_m - \alpha \mathcal{I}_{m^+}^{\sigma,\psi} \mathcal{G}(n, \omega(n)) + \beta \mathcal{I}_{m^+}^{\sigma,\psi} \mathcal{G}(\zeta, \omega(\zeta))\big],$$

where

$$\Theta = \alpha\big(\psi(n) - \psi(m)\big) - \beta\big(\psi(\zeta) - \psi(m)\big) \neq 0.$$

Substituting the values of $\mathfrak{C}_0$ and $\mathfrak{C}_1$ in (7.2.3) then we get the following integral solution

$$\begin{aligned}\omega(\tau) &= \mathfrak{F}(\tau, \omega(\tau)) + \mathcal{I}_{m^+}^{\sigma,\psi} \mathcal{G}(\tau, \omega(\tau)) - \mathfrak{F}_m + \frac{\psi(\tau) - \psi(m)}{\Theta}\big[(\alpha - \beta)\mathfrak{F}_m \\ &- \alpha \mathcal{I}_{m^+}^{\sigma,\psi} \mathcal{G}(n, \omega(n)) + \beta \mathcal{I}_{m^+}^{\sigma,\psi} \mathcal{G}(\zeta, \omega(\zeta))\big], \quad \tau \in \mathscr{J}.\end{aligned}$$

□

7.2.1 Hypotheses

The following are the assumptions we consider in order to prove our main results.

(H_1) There exist constants $\mathfrak{L}_f, \mathfrak{L}_g > 0$ such that $\tau \in \mathscr{J}$ and for each $\omega, \varpi \in \mathbb{R}$, the continuous functions $\mathfrak{F} : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{G} : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ satisfy

$$|\mathfrak{F}(\tau,\omega) - \mathfrak{F}(\tau,\varpi)| \le \mathfrak{L}_f|\omega - \varpi|$$

and

$$|\mathcal{G}(\tau,\omega) - \mathcal{G}(\tau,\varpi)| \le \mathfrak{L}_g|\omega - \varpi|.$$

(H_2) There exist constants $\mathfrak{M}_f, \mathfrak{M}_g, \mathfrak{N}_f, \mathfrak{N}_g > 0$ and $\mu, \nu \in (0,1)$ then the continuous functions $\mathfrak{F} : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{G} : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ satisfy the following growth conditions

$$|\mathfrak{F}(\tau,\omega)| \le \mathfrak{M}_f|\omega|^{\mu} + \mathfrak{N}_f$$

and

$$|\mathcal{G}(\tau,\omega)| \le \mathfrak{M}_g|\omega|^{\nu} + \mathfrak{N}_g.$$

7.2.2 Main Results

For the further calculations we consider that

$$\phi = \frac{|\psi(n) - \psi(m)|}{|\Theta|\Gamma(\sigma+1)}\Big[\alpha\big\{\psi(n) - \psi(m)\big\}^{\sigma} + \beta\big\{\psi(\zeta) - \psi(m)\big\}^{\sigma}\Big]. \tag{7.2.4}$$

Now, we define the operator Λ on $\mathcal{C}(\mathscr{J}, \mathbb{R})$ as

$$\begin{aligned}\Lambda\omega(\tau) &= \mathfrak{F}(\tau,\omega(\tau)) + \frac{1}{\Gamma(\sigma)}\int_m^{\tau} \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\sigma-1}\mathcal{G}(s,\omega(s))ds - \mathfrak{F}(m,0)\\ &+ \frac{\psi(\tau) - \psi(m)}{\Theta}\Bigg[(\alpha-\beta)\mathfrak{F}(m,0) - \frac{\alpha}{\Gamma(\sigma)}\int_m^{n} \psi'(s)\big(\psi(n) - \psi(s)\big)^{\sigma-1}\mathcal{G}(s,\omega(s))ds\\ &+ \frac{\beta}{\Gamma(\sigma)}\int_m^{\zeta} \psi'(s)\big(\psi(\zeta) - \psi(s)\big)^{\sigma-1}\mathcal{G}(s,\omega(s))ds\Bigg], \quad \tau \in \mathscr{J}.\end{aligned} \tag{7.2.5}$$

We define two operators $\Lambda_1, \Lambda_2 : \mathcal{C}(\mathscr{J}, \mathbb{R}) \to \mathcal{C}(\mathscr{J}, \mathbb{R})$ as:

$$\Lambda_1\omega(\tau) = \mathfrak{F}(\tau,\omega(\tau)) + \frac{1}{\Gamma(\sigma)}\int_m^{\tau} \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\sigma-1}\mathcal{G}(s,\omega(s))ds - \mathfrak{F}(m,0), \quad \tau \in \mathscr{J},$$

and

$$\begin{aligned}\Lambda_2\omega(\tau) &= \frac{\psi(\tau) - \psi(m)}{\Theta}\Bigg[(\alpha-\beta)\mathfrak{F}(m,0) - \frac{\alpha}{\Gamma(\sigma)}\int_m^{n} \psi'(s)\big(\psi(n) - \psi(s)\big)^{\sigma-1}\mathcal{G}(s,\omega(s))ds\\ &+ \frac{\beta}{\Gamma(\sigma)}\int_m^{\zeta} \psi'(s)\big(\psi(\zeta) - \psi(s)\big)^{\sigma-1}\mathcal{G}(s,\omega(s))ds\Bigg], \quad \tau \in \mathscr{J}.\end{aligned}$$

Then the operator equation (7.2.5) can be written as

$$\Lambda\omega(\tau) = \Lambda_1\omega(\tau) + \Lambda_2\omega(\tau), \quad \tau \in \mathscr{J}.$$

Lemma 7.2.2. *The operator Λ_2 is Lipschitz with a Lipschitz constant $\mathfrak{L}_g\phi$ also satisfies the following*

$$\|\Lambda_2\omega\| \leq \big(\mathfrak{M}_g\|\omega\|^\nu + \mathfrak{N}_g\big)\phi + \frac{|\psi(\tau)-\psi(m)|}{|\Theta|}|(\alpha-\beta)\mathfrak{F}(m,0)|.$$

Proof. To show that the operator Λ_2 is Lipschitz. Let us consider $\omega, \varpi \in \mathcal{C}(\mathscr{J},\mathbb{R})$ and for all $\tau \in \mathscr{J}$ we have

$$\begin{aligned}
|\Lambda_2\omega(\tau)-\Lambda_2\varpi(\tau)| \leq\ & \frac{|\psi(\tau)-\psi(m)|}{|\Theta|} \\
& \times\Bigg[\frac{\alpha}{\Gamma(\sigma)}\int_m^n \psi'(s)\big(\psi(n)-\psi(s)\big)^{\sigma-1}\big|\mathcal{G}(s,\omega(s))-\mathcal{G}(s,\varpi(s))\big|ds \\
& + \frac{\beta}{\Gamma(\sigma)}\int_m^\zeta \psi'(s)\big(\psi(\zeta)-\psi(s)\big)^{\sigma-1}\big|\mathcal{G}(s,\omega(s))-\mathcal{G}(s,\varpi(s))\big|ds\Bigg] \\
\leq\ & \frac{\mathfrak{L}_g|\psi(\tau)-\psi(m)|}{|\Theta|}\Bigg[\frac{\alpha}{\Gamma(\sigma)}\int_m^n \psi'(s)\big(\psi(n)-\psi(s)\big)^{\sigma-1}\|\omega-\varpi\|ds \\
& + \frac{\beta}{\Gamma(\sigma)}\int_m^\zeta \psi'(s)\big(\psi(\zeta)-\psi(s)\big)^{\sigma-1}\|\omega-\varpi\|ds\Bigg] \\
\leq\ & \frac{\mathfrak{L}_g\|\omega-\varpi\||\psi(n)-\psi(m)|}{|\Theta|\Gamma(\sigma+1)}\Big[\alpha\big\{\psi(n)-\psi(m)\big\}^\sigma+\beta\big\{\psi(\zeta)-\psi(m)\big\}^\sigma\Big] \\
\leq\ & \mathfrak{L}_g\phi\|\omega-\varpi\|.
\end{aligned}$$

Taking supremum over τ we get that

$$\|\Lambda_2\omega-\Lambda_2\varpi\| \leq \mathfrak{L}_g\phi\|\omega-\varpi\|.$$

Hence, $\Lambda_2 : \mathcal{C}(\mathscr{J},\mathbb{R}) \to \mathcal{C}(\mathscr{J},\mathbb{R})$ is a Lipschitzian with a Lipschitz constant $\mathfrak{L}_g\phi$.
By Proposition 1.6.15, Λ_2 is M-Lipschitz with constant $\mathfrak{L}_g\phi$.
Also

$$\begin{aligned}
|\Lambda_2\omega(\tau)| \leq\ & \frac{|\psi(\tau)-\psi(m)|}{|\Theta|}\Bigg[|(\alpha-\beta)\mathfrak{F}(m,0)| + \frac{\alpha}{\Gamma(\sigma)}\int_m^n \psi'(s)\big(\psi(n)-\psi(s)\big)^{\sigma-1}\big|\mathcal{G}(s,\omega(s))\big|ds \\
& + \frac{\beta}{\Gamma(\sigma)}\int_m^\zeta \psi'(s)\big(\psi(\zeta)-\psi(s)\big)^{\sigma-1}\big|\mathcal{G}(s,\omega(s))\big|ds\Bigg] \\
\leq\ & \frac{\big(\mathfrak{M}_g\|\omega\|^\nu+\mathfrak{N}_g\big)|\psi(\tau)-\psi(m)|}{|\Theta|}\Bigg[\frac{\alpha}{\Gamma(\sigma)}\int_m^n \psi'(s)\big(\psi(n)-\psi(s)\big)^{\sigma-1}ds \\
& + \frac{\beta}{\Gamma(\sigma)}\int_m^\zeta \psi'(s)\big(\psi(\zeta)-\psi(s)\big)^{\sigma-1}ds\Bigg] + \frac{|\psi(\tau)-\psi(m)|}{|\Theta|}|(\alpha-\beta)\mathfrak{F}(m,0)| \\
\leq\ & \frac{\big(\mathfrak{M}_g\|\omega\|^\nu+\mathfrak{N}_g\big)|\psi(n)-\psi(m)|}{|\Theta|\Gamma(\sigma+1)}\Big[\alpha\big\{\psi(n)-\psi(m)\big\}^\sigma+\beta\big\{\psi(\zeta)-\psi(m)\big\}^\sigma\Big] \\
& + \frac{|\psi(\tau)-\psi(m)|}{|\Theta|}|(\alpha-\beta)\mathfrak{F}(m,0)| \\
\leq\ & \big(\mathfrak{M}_g\|\omega\|^\nu+\mathfrak{N}_g\big)\phi + \frac{|\psi(n)-\psi(m)|}{|\Theta|}|(\alpha-\beta)\mathfrak{F}(m,0)|.
\end{aligned}$$

Taking supremum over τ we get that the operator Λ_2 satisfies the following growth condition

$$\|\Lambda_2\omega\| \leq \big(\mathfrak{M}_g\|\omega\|^\nu + \mathfrak{N}_g\big)\phi + \frac{|\psi(n)-\psi(m)|}{|\Theta|}|(\alpha-\beta)\mathfrak{F}(m,0)|. \tag{7.2.6}$$

□

Lemma 7.2.3. Λ_1 *is continuous and satisfies the following growth condition*

$$\|\Lambda_1\omega\| \leq \mathfrak{M}_f\|\omega\|^\mu + \mathfrak{N}_f + |\mathfrak{F}(m,0)| + \frac{(\mathfrak{M}_g\|\omega\|^\nu + \mathfrak{N}_g)}{\Gamma(\sigma+1)}\big(\psi(n)-\psi(m)\big)^\sigma.$$

Proof. To prove that Λ_1 is continuous, let us assume $\omega_l, \omega \in \mathcal{C}(\mathscr{J},\mathbb{R})$ such that $\lim_{l\to\infty}\|\omega_l - \omega\| \to 0$. Thus $\{\omega_l\}$ is a bounded subset of $\mathcal{C}(\mathscr{J},\mathbb{R})$. As a result we get that there exists a constant $r > 0$ such that $\|\omega_l\| \leq r$ for all $l \geq 1$. After taking limit we get that $\|\omega\| \leq r$. Since $\mathfrak{F}$ and $\mathcal{G}$ are continuous functions thus, $\mathfrak{F}(\tau,\omega_l(\tau)) \to \mathfrak{F}(\tau,\omega(\tau))$ and $\mathcal{G}(s,\omega_l(s)) \to \mathcal{G}(s,\omega(s))$ as $l \to \infty$. Also by using (H_2) we get that

$$\begin{aligned}
&\frac{1}{\Gamma(\sigma)}\psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\sigma-1}\|\mathcal{G}(s,\omega_l(s)) - \mathcal{G}(s,\omega(s))\| \\
&\leq \frac{2}{\Gamma(\sigma)}\psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\sigma-1}(\mathfrak{M}_g r^\nu + \mathfrak{N}_g).
\end{aligned}$$

Since the function $s \mapsto \frac{2}{\Gamma(\sigma)}\psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\sigma-1}(\mathfrak{M}_g r^\nu + \mathfrak{N}_g)$ is Lebesgue integrable over $[m,\tau]$. Thus by using Lebesgue's dominated convergence theorem together with this fact we get that

$$\int_m^\tau \frac{1}{\Gamma(\sigma)}\psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\sigma-1}\|\mathcal{G}(s,\omega_l(s)) - \mathcal{G}(s,\omega(s))\| \to 0 \quad \text{as} \quad l \to \infty.$$

Thus, we get that $\|\Lambda_1\omega_l - \Lambda_1\omega\| \to 0$ as $l \to \infty$. It implies the continuity of the operator Λ_1.

For the second proof using the hypothesis (H_2) we get

$$\begin{aligned}
|\Lambda_1\omega(\tau)| &\leq \mathfrak{M}_f\|\omega\|^\mu + \mathfrak{N}_f + |\mathfrak{F}(m,0)| + \frac{(\mathfrak{M}_g\|\omega\|^\nu + \mathfrak{N}_g)}{\Gamma(\sigma)}\int_m^\tau \psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\sigma-1}ds \\
&\leq \mathfrak{M}_f\|\omega\|^\mu + \mathfrak{N}_f + |\mathfrak{F}(m,0)| + \frac{(\mathfrak{M}_g\|\omega\|^\nu + \mathfrak{N}_g)}{\Gamma(\sigma+1)}\big(\psi(n)-\psi(m)\big)^\sigma.
\end{aligned}$$

Thus after taking the supremum we get that

$$\|\Lambda_1\omega\| \leq \mathfrak{M}_f\|\omega\|^\mu + \mathfrak{N}_f + |\mathfrak{F}(m,0)| + \frac{(\mathfrak{M}_g\|\omega\|^\nu + \mathfrak{N}_g)}{\Gamma(\sigma+1)}\big(\psi(n)-\psi(m)\big)^\sigma. \tag{7.2.7}$$

□

Lemma 7.2.4. *The operator* $\Lambda_1 : \mathcal{C}(\mathscr{J},\mathbb{R}) \to \mathcal{C}(\mathscr{J},\mathbb{R})$ *is compact. Consequently,* Λ_1 *is M-Lipschitz with zero constant.*

Proof. To show that Λ_1 is compact. Consider a bounded set $\mathfrak{A} \subset \mathcal{B}_r$. We have to show that $\Lambda_1(\mathfrak{A})$ is relatively compact in $\mathcal{C}(\mathscr{J}, \mathbb{R})$. For arbitrary $\omega \in \mathfrak{A} \subset \mathcal{B}_r$, using (7.2.7) we get the following

$$\|\Lambda_1\omega\| \leq \mathfrak{M}_f r^\mu + \mathfrak{N}_f + |\mathfrak{F}(m,0)| + \frac{(\mathfrak{M}_g r^\nu + \mathfrak{N}_g)}{\Gamma(\sigma+1)}\big(\psi(n)-\psi(m)\big)^\sigma,$$

which shows that $\Lambda_1(\mathfrak{A})$ is uniformly bounded.

Now, to prove the equi-continuity of Λ_1 let us consider for any $\omega \in \mathfrak{A}$ and $\tau_1, \tau_2 \in \mathscr{J}$ such that $\tau_1 < \tau_2$ we get

$$\begin{aligned}|\Lambda_1\omega(\tau_2) - \Lambda_1\omega(\tau_2)| &\leq |\mathfrak{F}(\tau_2,\omega(\tau_2)) - \mathfrak{F}(\tau_1,\omega(\tau_1))| \\ &\quad + \frac{(\mathfrak{M}_g r^\nu + \mathfrak{N}_g)}{\Gamma(\sigma+1)}\Big[\big(\psi(\tau_2)-\psi(m)\big)^\sigma - \big(\psi(\tau_1)-\psi(m)\big)^\sigma\Big].\end{aligned}$$

The RHS of the above inequality tends to 0 whenever $\tau_2 \to \tau_1$ without depending on $\omega \in \mathfrak{A}$. Therefore, Λ_1 is equi-continuous.

Since Λ_1 is uniformly bounded and equi-continuous thus, from Arzelá-Ascoli theorem we can say that it is a compact operator. Therefore by Proposition 1.6.14 Λ_1 is M-Lipschitz with zero constant. □

Theorem 7.2.5. *If the BVP* (7.1.1)-(7.1.3) *satisfies the assumptions* (H_1) *and* (H_2) *then* (7.1.1)-(7.1.3) *has atleast one solution* $\omega \in \mathcal{C}(\mathscr{J}, \mathbb{R})$ *provided that* $\mathfrak{L}_g\phi < 1$ *and the solution set is bounded in* $\mathcal{C}(\mathscr{J}, \mathbb{R})$*.*

Proof. Let $\Lambda_1, \Lambda_2, \Lambda : \mathcal{C}(\mathscr{J}, \mathbb{R}) \to \mathcal{C}(\mathscr{J}, \mathbb{R})$ be the operators as we defined in the beginning of this section. These are the continuous and bounded operators. From the Lemma 7.2.2 and Lemma 7.2.4 we get that Λ_2 and Λ_1 are M-Lipschitz with constants $\mathfrak{L}_g\phi$ and 0 respectively. Thus Λ is strict M-Lipschitz with constant $\mathfrak{L}_g\phi$. As $\mathfrak{L}_g\phi < 1$, hence Λ is M-condensing.

Set $\Upsilon = \big\{\omega \in \mathcal{C}(\mathscr{J}, \mathbb{R}) : \exists \;\; \gamma \in [0,1] \;\text{ such that }\; \omega = \gamma\Lambda\omega\big\}$.

We have to show that Υ is bounded in $\mathcal{C}(\mathscr{J}, \mathbb{R})$. Consider, $\omega \in \Upsilon$ and $\gamma \in [0,1]$ such that $\omega = \gamma\Lambda\omega$. From (7.2.6) and (7.2.7) we get that

$$\begin{aligned}\|\omega\| &= \|\gamma\Lambda\omega\| \leq \gamma\big(\|\Lambda_1\omega\| + \|\Lambda_2\omega\|\big) \\ &\leq \big(\mathfrak{M}_g\|\omega\|^\nu + \mathfrak{N}_g\big)\left[\phi + \frac{\big(\psi(n)-\psi(m)\big)^\sigma}{\Gamma(\sigma+1)}\right] + \frac{|\psi(n)-\psi(m)|}{|\Theta|}|(\alpha-\beta)\mathfrak{F}(m,0)| \\ &\quad + \mathfrak{M}_f\|\omega\|^\mu + \mathfrak{N}_f + |\mathfrak{F}(m,0)|,\end{aligned}$$

where ϕ is same as (7.2.4). Thus we can conclude the boundedness of Υ from the above inequality. If it is not bounded, then dividing the above inequality by considering $\|\omega\| = \mathfrak{P}$ such that $\mathfrak{P} \to \infty$ implies that

$$1 \leq \left[\phi + \frac{\big(\psi(n)-\psi(m)\big)^\sigma}{\Gamma(\sigma+1)}\right]\lim_{\mathfrak{P}\to\infty}\frac{\big(\mathfrak{M}_g\mathfrak{P}^\nu + \mathfrak{N}_g\big)}{\mathfrak{P}} + \lim_{\mathfrak{P}\to\infty}\frac{|\psi(n)-\psi(m)|}{\mathfrak{P}|\Theta|}|(\alpha-\beta)\mathfrak{F}(m,0)|$$

$$+\lim_{\mathfrak{P}\to\infty}\frac{\mathfrak{M}_f\mathfrak{P}^{\mu}+\mathfrak{N}_f+|\mathfrak{F}(m,0)|}{\mathfrak{P}}=0,$$

which is a contradiction. Thus the set Υ is bounded and by Theorem 1.6.16, Λ has at least one fixed point which represents the solution of (7.1.1)-(7.1.3) and the set of the fixed points of Λ is bounded in $\mathcal{C}(\mathscr{J},\mathbb{R})$. □

Remark 7.2.6. *Following are conclusions obtained from the Theorem 7.2.5:*

(i) If we put $\mu=1$ in the assumption (H_2) then Theorem 7.2.5 remain valid provided that, $\mathfrak{M}_f<1$.

(ii) If $\nu=1$ in the assumption (H_2) then Theorem 7.2.5 remain valid provided that, $\mathfrak{M}_g\left[\phi+\frac{\left(\psi(n)-\psi(m)\right)^{\sigma}}{\Gamma(\sigma+1)}\right]<1$.

(iii) If both $\mu=\nu=1$ in the assumption (H_2) then Theorem 7.2.5 remain valid provided that, $\mathfrak{M}_f+\mathfrak{M}_g\left[\phi+\frac{\left(\psi(n)-\psi(m)\right)^{\sigma}}{\Gamma(\sigma+1)}\right]<1$.

Theorem 7.2.7. *If the BVP* (7.1.1)-(7.1.3) *satisfies* (H_1) *then it will have a unique solution provided that*

$$\mathfrak{L}_f+\frac{\mathfrak{L}_{\mathfrak{g}}\big(\psi(\mathfrak{n})-\psi(\mathfrak{m})\big)^{\sigma}}{\Gamma(\sigma+1)}+\mathfrak{L}_g\phi<1. \tag{7.2.8}$$

Proof. Let for $\tau\in\mathscr{J}$, $\omega,\varpi\in\mathcal{C}(\mathscr{J},\mathbb{R})$ we have

$$\begin{aligned}
&|\Lambda\omega(\tau)-\Lambda\varpi(\tau)|\\
&\quad\le|\mathfrak{F}(\tau,\omega(\tau))-\mathfrak{F}(\tau,\varpi(\tau))|\\
&\quad+\frac{1}{\Gamma(\sigma)}\int_m^{\tau}\psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\sigma-1}|\mathcal{G}(s,\omega(s))-\mathcal{G}(s,\varpi(s))|ds+\frac{|\psi(\tau)-\psi(m)|}{|\Theta|}\\
&\quad\times\left[\frac{\alpha}{\Gamma(\sigma)}\int_m^{n}\psi'(s)\big(\psi(n)-\psi(s)\big)^{\sigma-1}\big|\mathcal{G}(s,\omega(s))-\mathcal{G}(s,\varpi(s))\big|ds\right.\\
&\quad\left.+\frac{\beta}{\Gamma(\sigma)}\int_m^{\zeta}\psi'(s)\big(\psi(\zeta)-\psi(s)\big)^{\sigma-1}\big|\mathcal{G}(s,\omega(s))-\mathcal{G}(s,\varpi(s))\big|ds\right]\\
&\quad\le\mathfrak{L}_f\|\omega-\varpi\|+\frac{\mathfrak{L}_g\|\omega-\varpi\|}{\Gamma(\sigma)}\int_m^{\tau}\psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\sigma-1}ds\\
&\quad+\frac{\mathfrak{L}_g|\psi(\tau)-\psi(m)|\|\omega-\varpi\|}{|\Theta|}\left[\frac{\alpha}{\Gamma(\sigma)}\int_m^{n}\psi'(s)\big(\psi(n)-\psi(s)\big)^{\sigma-1}ds\right.\\
&\quad\left.+\frac{\beta}{\Gamma(\sigma)}\int_m^{\zeta}\psi'(s)\big(\psi(\zeta)-\psi(s)\big)^{\sigma-1}ds\right]\\
&\quad\le\mathfrak{L}_f\|\omega-\varpi\|+\frac{\mathfrak{L}_g\|\omega-\varpi\|}{\Gamma(\sigma+1)}\big(\psi(\tau)-\psi(m)\big)^{\sigma}\\
&\quad+\frac{\mathfrak{L}_g\|\omega-\varpi\||\psi(n)-\psi(m)|}{|\Theta|\Gamma(\sigma+1)}\Big[\alpha\big(\psi(n)-\psi(m)\big)^{\sigma}+\beta\big(\psi(\zeta)-\psi(m)\big)^{\sigma}\Big]
\end{aligned}$$

$$\leq \left[\mathfrak{L}_f + \frac{\mathfrak{L}_g\big(\psi(n)-\psi(m)\big)^{\sigma}}{\Gamma(\sigma+1)} + \mathfrak{L}_g\phi\right]\|\omega-\varpi\|.$$

By using (7.2.8) we get that Λ is a contraction. Thus Banach contraction principle concludes that Λ has a unique fixed point which will be the unique solution of the BVP (7.1.1)-(7.1.3). This completes the proof. □

7.3 Analytical Example

Let us consider a system of hybrid BVP involving a $\psi-$Caputo fractional derivative

$$\begin{gathered}
{}_0\mathcal{D}_1^{1.5,\psi}\left[\omega(\tau) - \frac{1}{20(1+\tau)^2}\big(\omega(\tau)+1\big)\right] = \frac{1}{|\sin\tau|+100}\left(1+\frac{|\omega(\tau)|}{1+|\omega(\tau)|}\right), \quad \tau\in\mathscr{J}=[1,3],\\
\omega(1)=0,\\
2\big[\omega(\tau)-\mathfrak{F}(\tau,\omega(\tau))\big]_{\tau=3} = \big[\omega(\tau)-\mathfrak{F}(\tau,\omega(\tau))\big]_{\tau=2}.
\end{gathered} \tag{7.3.1}$$

If we compare (7.3.1) with our considered problem (7.1.1)-(7.1.3) then we get the following data:
$\sigma=1.5,\quad m=1,\quad n=3,\quad \zeta=2,\quad \alpha=2,\quad$ and $\quad\beta=1.$
Also, $\mathfrak{F}(\tau,\omega(\tau)) = \dfrac{1}{20(1+\tau)^2}\big(\omega(\tau)+1\big)$ and $\mathcal{G}(\tau,\omega(\tau)) = \dfrac{1}{|\sin\tau|+100}\left(1+\dfrac{|\omega(\tau)|}{1+|\omega(\tau)|}\right)$.

Now,

$$\begin{aligned}
|\mathfrak{F}(\tau,\omega)-\mathfrak{F}(\tau,\varpi)| &\leq \left|\frac{1}{20(1+\tau)^2}\big(\omega-\varpi\big)\right|\\
&\leq \frac{1}{80}|\omega-\varpi|.
\end{aligned}$$

And

$$\begin{aligned}
|\mathcal{G}(\tau,\omega)-\mathcal{G}(\tau,\varpi)| &\leq \left|\frac{1}{|\sin\tau|+100}\left\{\left(1+\frac{|\omega|}{1+|\omega|}\right)-\left(1+\frac{|\varpi|}{1+|\varpi|}\right)\right\}\right|\\
&\leq \frac{1}{100}\left\{\frac{|\omega-\varpi|}{\big(1+|\omega|\big)\big(1+|\varpi|\big)}\right\}\\
&\leq \frac{1}{100}|\omega-\varpi|.
\end{aligned}$$

Thus $\mathfrak{F}(\tau,\omega)$ and $\mathcal{G}(\tau,\omega)$ satisfy the assumption (H_1) with $\mathfrak{L}_f=\frac{1}{80}$ and $\mathfrak{L}_g=\frac{1}{100}$. Again

$$|\mathfrak{F}(\tau,\omega)| \leq \frac{1}{80}\big(1+|\omega|\big)$$

and

$$|\mathcal{G}(\tau,\omega)| \leq \frac{1}{100}\left(\frac{1+2|\omega|}{1+|\omega|}\right) \leq \frac{1}{100}\big(1+2|\omega|\big).$$

Therefore, $\mathfrak{F}(\tau,\omega)$ and $\mathcal{G}(\tau,\omega)$ satisfy the assumption (H_1) with $\mathfrak{M}_f = \frac{1}{80}$, $\mathfrak{N}_f = \frac{1}{80}$ and $\mathfrak{M}_g = \frac{1}{50}$, $\mathfrak{N}_g = \frac{1}{100}$. Let $\psi(\tau) = \tau$.

Now,

$$\Theta = \alpha\big(\psi(n) - \psi(m)\big) - \beta\big(\psi(\zeta) - \psi(m)\big) = 3 \neq 0,$$

$$\begin{aligned}\phi &= \frac{|\psi(n)-\psi(m)|}{|\Theta|\Gamma(\sigma+1)}\Big[\alpha\big\{\psi(n)-\psi(m)\big\}^{\sigma} + \beta\big\{\psi(\zeta)-\psi(m)\big\}^{\sigma}\Big] \\ &= \frac{2}{3\times\Gamma(2.5)}[2\times 2.82843 + 1] = 3.33778.\end{aligned}$$

In view of Theorem 7.2.5,

$$\Upsilon = \big\{\omega \in \mathcal{C}(\mathscr{J},\mathbb{R}) : \exists \ \ \gamma \in [0,1] \ \text{ such that } \ \omega = \gamma\Lambda\omega\big\}$$

is the set of solution, then

$$\begin{aligned}\|\omega\| &\leq \frac{\mathfrak{N}_g\left[\phi + \frac{\left(\psi(n)-\psi(m)\right)^{\sigma}}{\Gamma(\sigma+1)}\right] + \frac{|\psi(n)-\psi(m)|}{|\Theta|}|(\alpha-\beta)\mathfrak{F}(m,0)| + \mathfrak{N}_f + |\mathfrak{F}(m,0)|}{1 - \mathfrak{M}_g\left[\phi + \frac{\left(\psi(n)-\psi(m)\right)^{\sigma}}{\Gamma(\sigma+1)}\right] + \mathfrak{M}_f} \\ &= 0.100186.\end{aligned}$$

Since $\mu = \nu = 1$ and

$$\mathfrak{M}_f + \mathfrak{M}_g\left[\phi + \frac{\big(\psi(n)-\psi(m)\big)^{\sigma}}{\Gamma(\sigma+1)}\right] = \frac{1}{80} + \frac{1}{50}\big[3.33778 + 2.12728\big] = 0.1218 < 1.$$

Thus it satisfies the condition (iii) of Remark 7.2.6.
Also

$$\mathfrak{L}_g\phi = 0.0333778 < 1.$$

Therefore Theorem 7.2.5 guarantees the existence of atleast one solution of the BVP (7.3.1). Furthermore,

$$\mathfrak{L}_f + \frac{\mathfrak{L}_g\big(\psi(n)-\psi(m)\big)^{\sigma}}{\Gamma(\sigma+1)} + \mathfrak{L}_g\phi = 0.067078 < 1.$$

Therefore Theorem 7.2.7 concludes that the BVP (7.3.1) has an unique solution.

7.4 Example with Numerical Results

In this section we study an example for different ψ functions and different boundary points.

Example 7.4.1. *For that let us consider the following functions*

$$\mathfrak{F}(\tau, \omega(\tau)) = \frac{1}{20}\big(\sin\tau + \omega(\tau)\big),$$

$$\textit{and} \quad \mathcal{G}(\tau, \omega(\tau)) = \frac{1}{10^3}\big(\tau + \omega(\tau)\big).$$

Therefore we get that $\mathfrak{L}_f = \frac{1}{20}$ *and* $\mathfrak{L}_g = \frac{1}{10^3}$.
Let us assume the following data

$$\sigma = 1.5, \quad m = 1, \quad n = 3.8, \quad \alpha = 2 \quad \textit{and} \quad \beta = 1.$$

Also we denote the term

$$\mathfrak{L}_f + \frac{\mathfrak{L}_g\big(\psi(n) - \psi(m)\big)^{\sigma}}{\Gamma(\sigma+1)} + \mathfrak{L}_g\phi = \Omega.$$

Case 1: Let $\psi(\tau)$ is linear, i.e $\psi(\tau) = \tau$.
We get the following numerical solution by using (7.2.1) for our Example 7.4.1

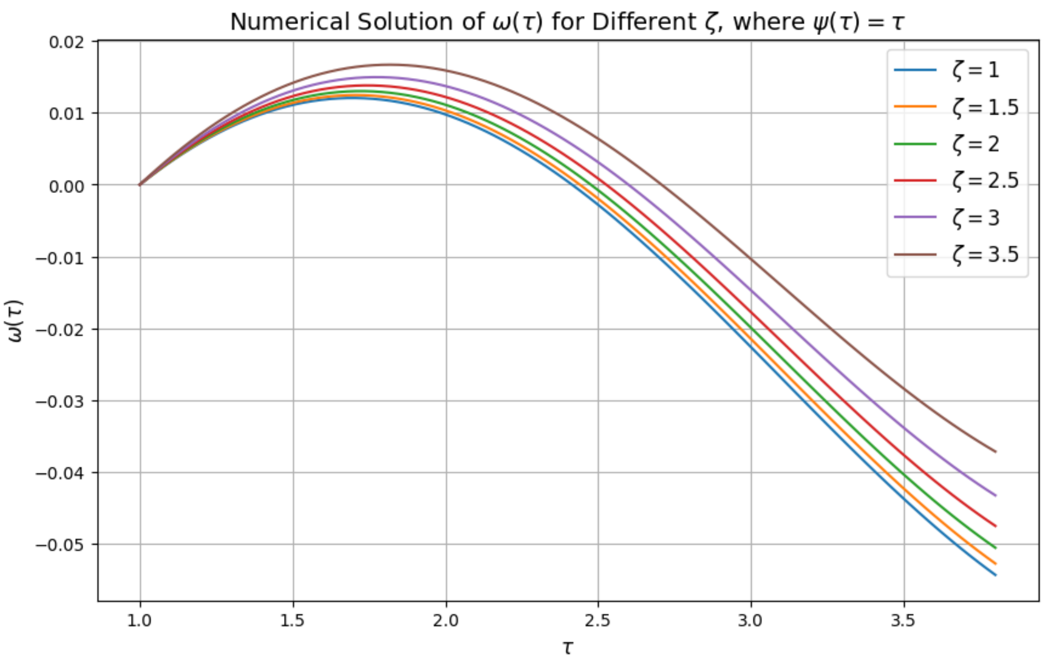

Figure 7.1: Numerical solution of $\omega(\tau)$ for various ζ values where $\psi(\tau) = \tau$.

Table 7.1: Numerical results of Θ, ϕ and Ω for $\psi(\tau) = \tau$ and $\zeta \in [1, 3.8]$

ζ	Θ	ϕ	$\Omega < 1$
1.000000	5.600000	3.524527	0.057049<1
1.147368	5.452632	3.676357	0.057201<1
1.294737	5.305263	3.880346	0.057405<1
1.442105	5.157895	4.120590	0.057645<1
1.589474	5.010526	4.391759	0.057916<1
1.736842	4.863158	4.691048	0.058216<1
1.884211	4.715789	5.016821	0.058541<1
2.031579	4.568421	5.368128	0.058893<1
2.178947	4.421053	5.744494	0.059269<1
2.326316	4.273684	6.145808	0.059670<1
2.473684	4.126316	6.572271	0.060097<1
2.621053	3.978947	7.024379	0.060549<1
2.768421	3.831579	7.502913	0.061027<1
2.915789	3.684211	8.008965	0.061533<1
3.063158	3.536842	8.543959	0.062068<1
3.210526	3.389474	9.109707	0.062634<1
3.357895	3.242105	9.708470	0.063233<1
3.505263	3.094737	10.343052	0.063868<1
3.652632	2.947368	11.016916	0.064541<1
3.800000	2.800000	11.734350	0.065259<1

Note:

These figures are obtained from the Table 7.1. It shows how Θ, ϕ and Ω varies for a linear function $\psi(\tau) = \tau$ while ζ varies in between $[1, 3.8]$. Each plot corresponds to specific parameter values discussed in the text.

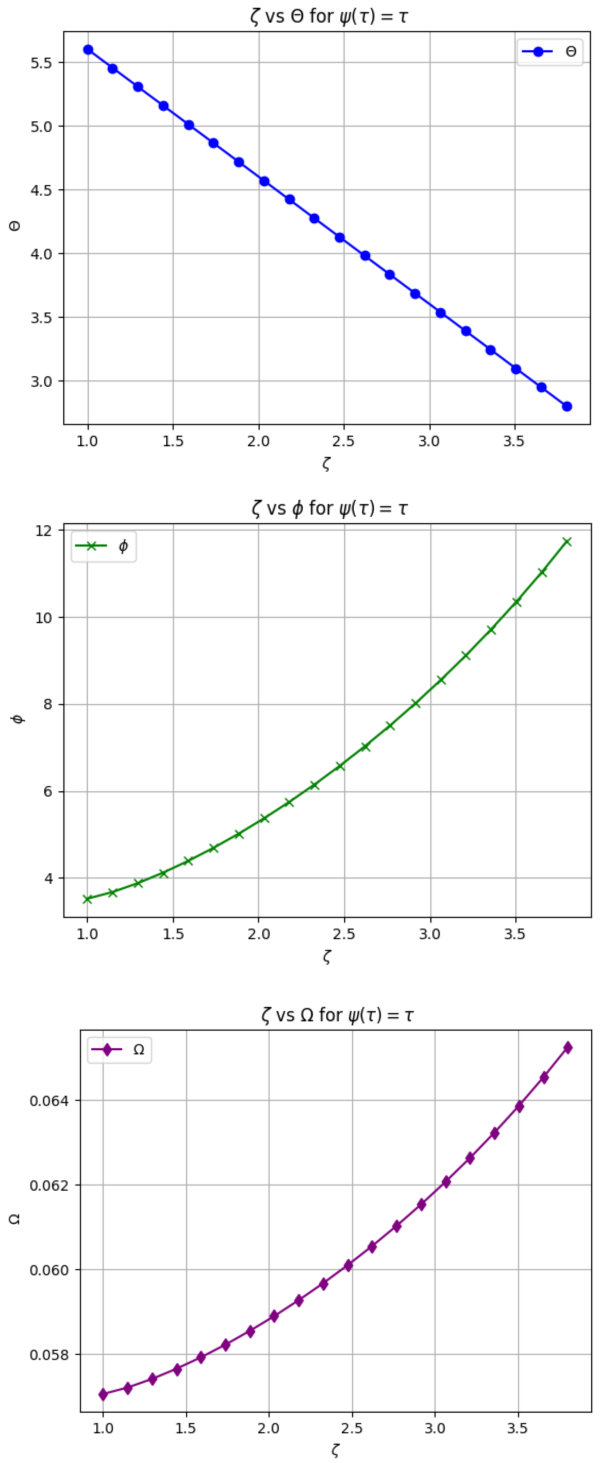

Figure 7.2: Numerical results of Θ, ϕ and Ω for $\zeta \in [1, 3.8]$ and $\psi(\tau) = \tau$. Each subfigure corresponds to different parameter values.

Case 2: Let $\psi(\tau) = 2^\tau$.
We get the following numerical solution by using (7.2.1) for our Example 7.4.1

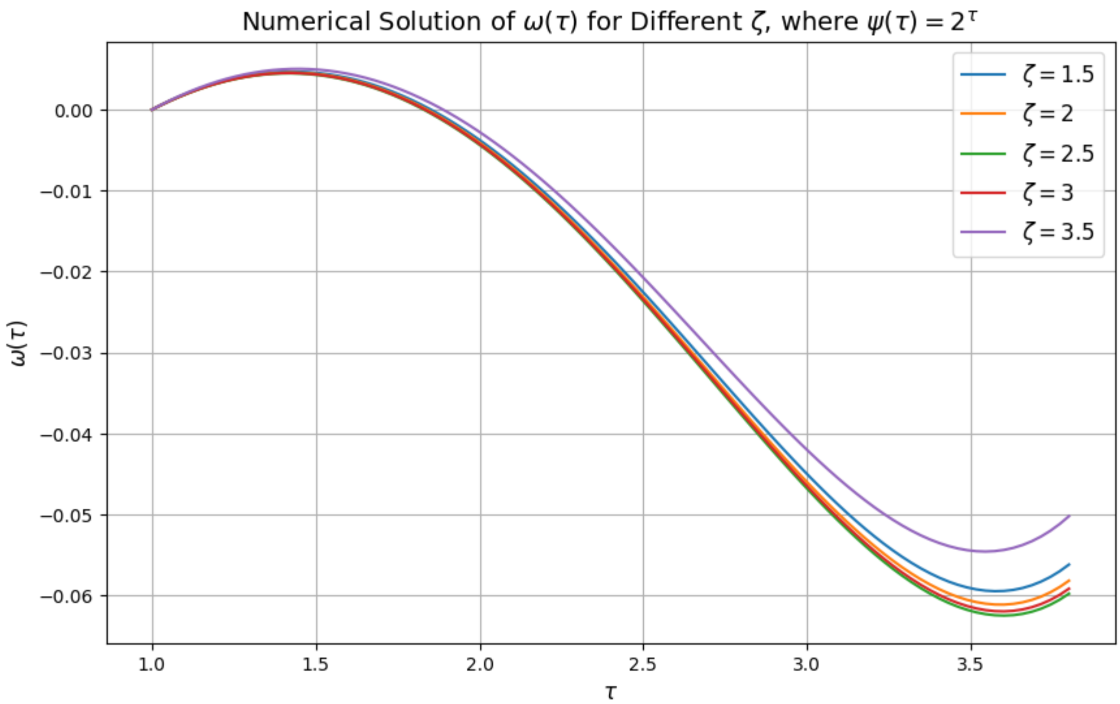

Figure 7.3: Numerical solution of $\omega(\tau)$ for various ζ values where $\psi(\tau) = 2^\tau$.

Table 7.2: Numerical results of Θ, ϕ and Ω for $\psi(\tau) = 2^\tau$ and $\zeta \in [1, 3.8]$

ζ	Θ	ϕ	$\Omega < 1$
1.000000	23.857618	30.992701	0.111985 <1
1.133333	23.663968	31.331541	0.112324<1
1.266667	23.451568	31.788065	0.112781<1
1.400000	23.218602	32.356493	0.113349<1
1.533333	22.963080	33.046092	0.114039<1
1.666667	22.682816	33.871242	0.114864<1
1.800000	22.375416	34.850251	0.115843<1
1.933333	22.038252	36.005334	0.116998<1
2.066667	21.668442	37.362990	0.118356<1
2.200000	21.262825	38.954660	0.119947<1
2.333333	20.817934	40.817627	0.121810<1
2.466667	20.329967	42.996198	0.123989<1
2.600000	19.794752	45.543281	0.126536<1
2.733333	19.207715	48.522456	0.129515<1
2.866667	18.563838	52.010816	0.133004<1
3.000000	17.857618	56.102910	0.137096<1
3.133333	17.083018	60.916424	0.141909<1
3.266667	16.233418	66.600666	0.147593<1
3.400000	15.301555	73.349799	0.154343<1
3.533333	14.279464	81.424535	0.162417<1
3.666667	13.158410	91.189845	0.172183<1
3.800000	11.928809	103.185252	0.184178<1

Note:

These figures are obtained from the Table 7.2. It shows how Θ, ϕ and Ω varies for a power function $\psi(\tau) = 2^\tau$ while ζ varies in between $[1, 3.8]$. Each plot corresponds to specific parameter values discussed in the text.

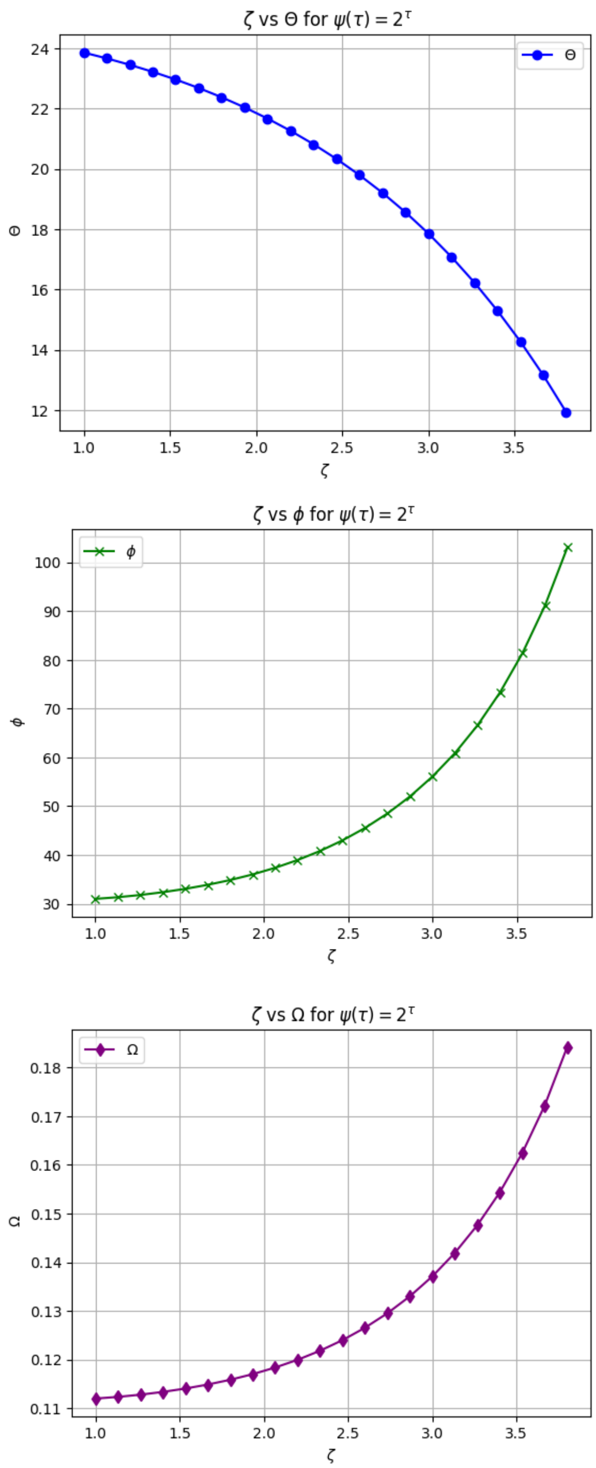

Figure 7.4: Numerical results of Θ, ϕ and Ω for $\zeta \in [1, 3.8]$ and $\psi(\tau) = 2^\tau$. Each subfigure corresponds to different parameter values.

Case 3: ψ is an exponential function i.e. $\psi(\tau) = e^{\tau}$. We get the following numerical solution by using (7.2.1) for our Example 7.4.1

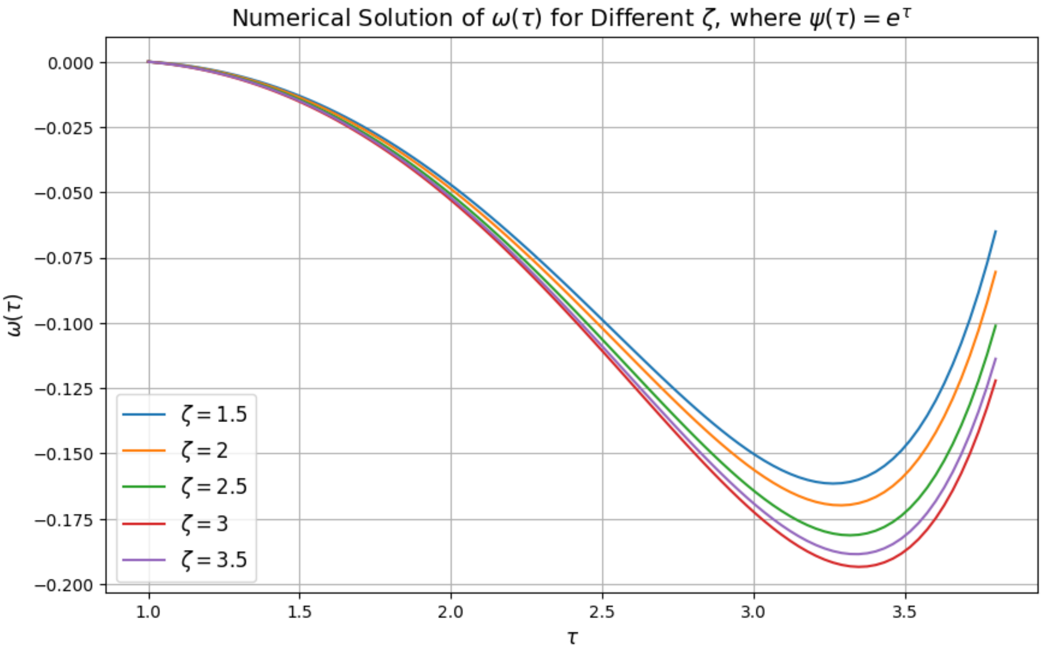

Figure 7.5: Numerical solution of $\omega(\tau)$ for various ζ values where $\psi(\tau) = e^{\tau}$.

Table 7.3: Numerical results of Θ, ϕ and Ω for $\psi(\tau) = e^{\tau}$ and $\zeta \in [1, 3.8]$

ζ	Θ	ϕ	$\Omega < 1$
1.000000	83.965805	204.631503	0.459263<1
1.133333	83.578095	205.822182	0.460454<1
1.266667	83.135084	207.433419	0.462065<1
1.400000	82.628887	209.488212	0.464120< 1
1.533333	82.050491	212.058938	0.466690< 1
1.666667	81.389597	215.243623	0.469875< 1
1.800000	80.634440	219.166154	0.473798< 1
1.933333	79.771574	223.980336	0.478612< 1
2.066667	78.785636	229.876117	0.484508< 1
2.200000	77.659074	237.087954	0.491719< 1
2.333333	76.371829	245.905737	0.500537< 1
2.466667	74.900983	256.689068	0.511321< 1
2.600000	73.220349	269.886097	0.524518< 1
2.733333	71.300005	286.058806	0.540690<1
2.866667	69.105761	305.917683	0.560549<1
3.000000	66.598550	330.370705	0.585002<1
3.133333	63.733734	360.595133	0.615227<1
3.266667	60.460306	398.147728	0.652779<1
3.400000	56.719987	445.144022	0.699776<1
3.533333	52.446183	504.571483	0.759203<1
3.666667	47.562803	580.886863	0.835518<1
3.800000	41.982903	681.287926	0.935919<1

Note:

These figures are obtained from the Table 7.3. It shows how Θ, ϕ and Ω varies for a exponential function $\psi(\tau) = e^{\tau}$ while ζ varies in between $[1, 3.8]$. Each plot corresponds to specific parameter values discussed in the text.

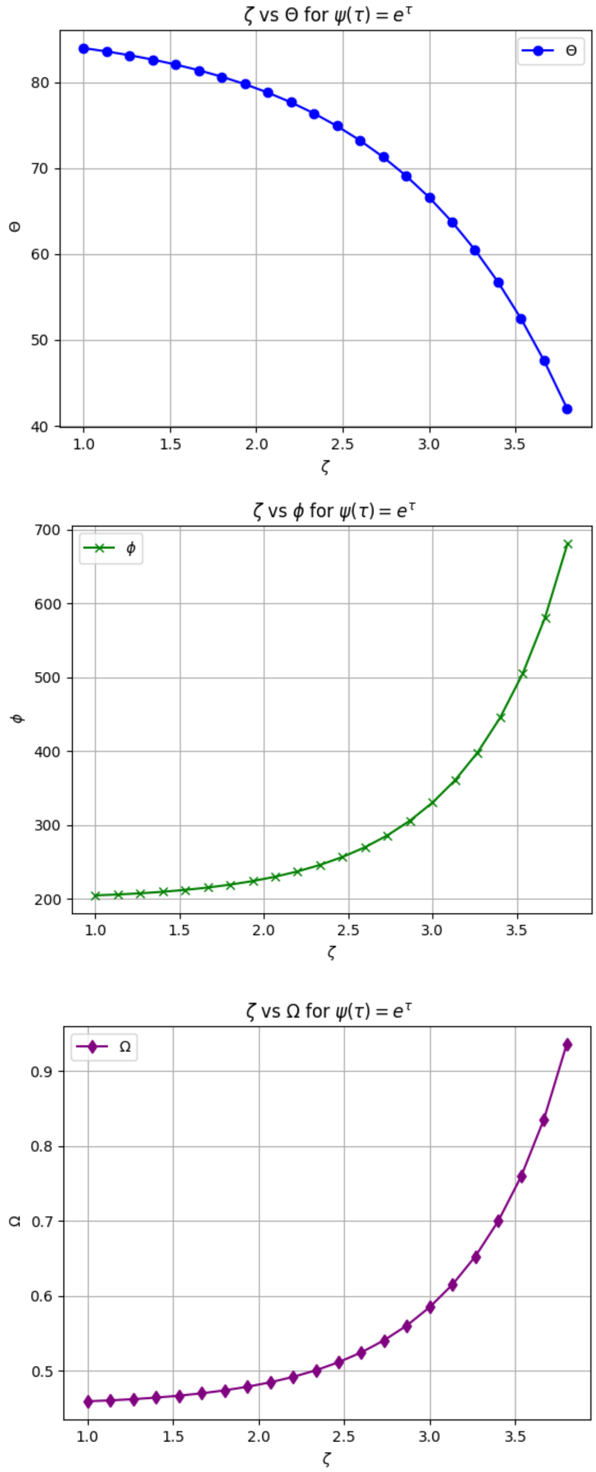

Figure 7.6: Numerical results of Θ, ϕ and Ω for $\zeta \in [1, 3.8]$ and $\psi(\tau) = e^{\tau}$. Each subfigure corresponds to different parameter values.

Case 4: ψ is an logarithmic function i.e. $\psi(\tau) = \log(\tau)$.
For $\psi(\tau) = \log(\tau)$ we consider that $m = 1, \quad n = e$ and $\zeta \in [1, e]$.
We get the following numerical solution by using (7.2.1) for our Example 7.4.1

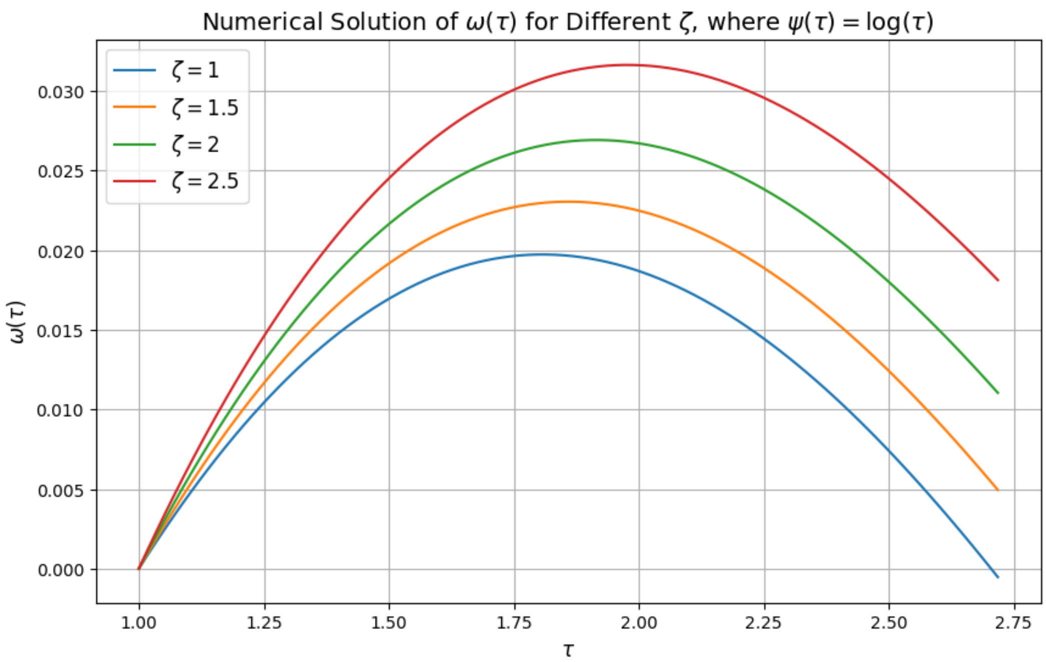

Figure 7.7: Numerical solution of $\omega(\tau)$ for various ζ values where $\psi(\tau) = \log(\tau)$.

Table 7.4: Numerical results of Θ, ϕ and Ω for $\psi(\tau) = \log(\tau)$ and $\zeta \in [1, e]$

ζ	Θ	ϕ	$\Omega < 1$
1.000000	2.000000	0.752253	0.051505<1
1.090436	1.913422	0.811765	0.051564<1
1.180872	1.833747	0.888242	0.051640<1
1.271308	1.759954	0.972464	0.051725<1
1.361744	1.691234	1.061161	0.051813<1
1.452179	1.626935	1.152614	0.051905<1
1.542615	1.566521	1.245811	0.051998<1
1.633051	1.509550	1.340131	0.052092<1
1.723487	1.455650	1.435184	0.052187<1
1.813923	1.404508	1.530727	0.052283<1
1.904359	1.355855	1.626619	0.052379<1
1.994795	1.309459	1.722784	0.052475<1
2.085231	1.265121	1.819195	0.052571<1
2.175667	1.222665	1.915863	0.052668<1
2.266102	1.181939	2.012823	0.052765<1
2.356538	1.142806	2.110132	0.052862<1
2.446974	1.105148	2.207861	0.052960<1
2.537410	1.068856	2.306099	0.053058<1
2.627846	1.033836	2.404944	0.053157<1
2.718282	1.000000	2.504506	0.053257<1

Note:

These figures are obtained from the Table 7.4. It shows how Θ, ϕ and Ω varies for a exponential function $\psi(\tau) = \log(\tau)$ while ζ varies in between $[1, e]$. Each plot corresponds to specific parameter values discussed in the text.

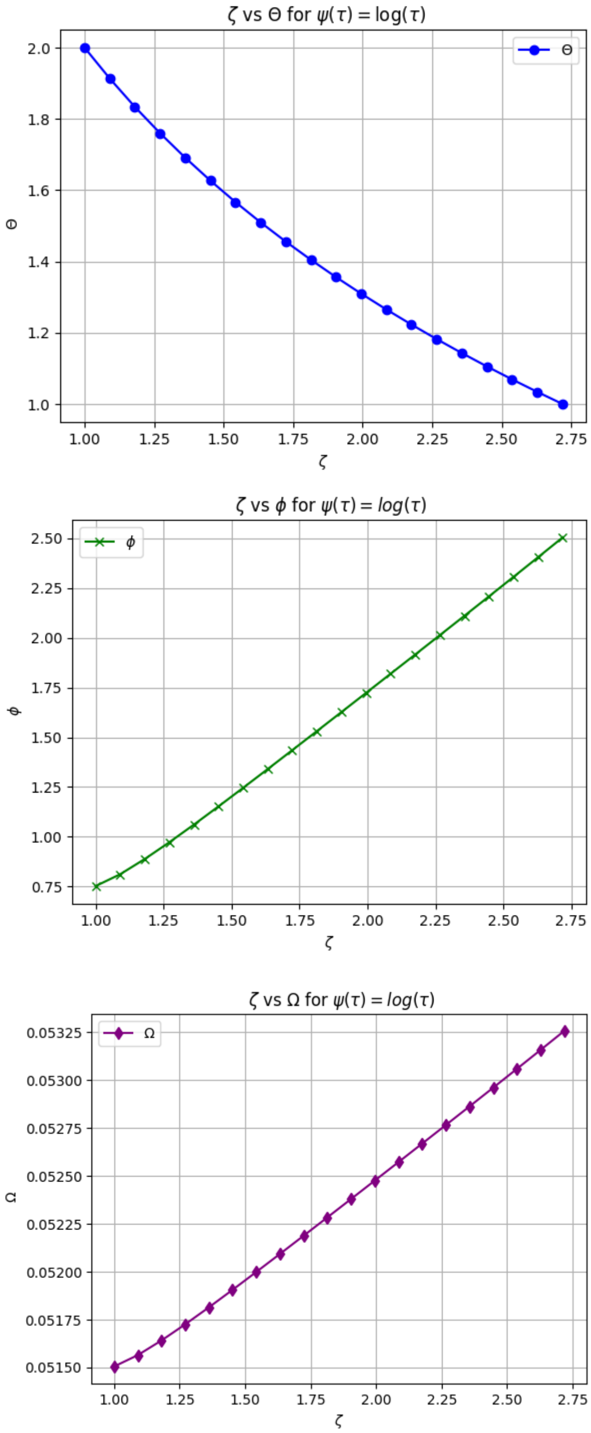

Figure 7.8: Numerical results of Θ, ϕ and Ω for $\zeta \in [1, e]$ and $\psi(\tau) = \log(\tau)$. Each subfigure corresponds to different parameter values.

7.5 Observation

A significant aspect of this study is the examination of how different $\psi-$functions influence the behaviour of key parameters, namely, $\Theta, \ \phi$, and Ω. For this purpose, we consider four distinct increasing, continuous and differentiable ψ-functions: $\psi(\tau) = \tau$, $\psi(\tau) = 2^{\tau}$, $\psi(\tau) = e^{\tau}$, and $\psi(\tau) = \log(\tau)$. Numerical solutions are computed for these functions across various values of $\zeta \in (m, n)$, providing insights into how the choice of ψ affects the values of $\Theta, \ \phi$ and Ω as ζ varies within the interval.

Furthermore, the results encompass special cases of fractional boundary value problems, such as the Caputo-type problem ($\psi(\tau) = \tau$) and the Caputo-Hadamard-type problem ($\psi(\tau) = \log(\tau)$). This work provides a framework for analysing perturbed fractional differential equations and can be extended in future research to incorporate impulsive cases or to explore the characteristics of other generalized derivatives, such as the $\psi-$Hilfer derivative, for a variety of $\psi-$functions.