CHAPTER 1

Introduction

1.1 Fractional Calculus: An Overview

Fractional calculus is a branch of mathematical analysis that extends the concept of derivatives and integrals to non-integer orders. While classical calculus deals with integer-order differentiation and integration, fractional calculus generalizes these operations, allowing for derivatives and integrals of any fractional order.

The origins of fractional calculus date back to 1695, when Leibniz exchanged letters with the French mathematician L'Hôpital. In their correspondence, Leibniz introduced the notation $\frac{d^n y}{dx^n} = D^n y$ for n^{th} order derivative where $n \in \mathbb{N}$. The term "fractional calculus" was first formally introduced by Euler in 1730, where he investigated fractional derivatives as intermediates between derivatives of integer orders. In 1812, Laplace provided a definition of fractional derivatives based on integral operations, further advancing the mathematical understanding of the topic. The development of fractional calculus progressed gradually, with significant contributions from various mathematicians over the years. Notable among them are Lacroix (1819), Fourier (1822), Abel (1823), Liouville(1832), Riemann (1832), Grünwald (1867) etc. Niels Henrik Abel became the first mathematician to apply fractional calculus to solve an integral equation, specifically addressing the Tautochrone problem. This remained the single example of fractional calculus application for a long period. In 1832 Liouville pioneered the first significant study in fractional calculus, building on the Fourier fractional integral and Abel's solution in potential theory. He was also the first mathematician to attempt solving differential equations using fractional operators.

Several approaches extend the classical derivative $\frac{d^n y}{dx^n}$ to non-integer n, including the Grünwald-Letnikov, Hadamard, Riemann-Liouville, and Caputo derivatives etc. Among these, the Riemann-Liouville and Caputo derivatives are the most prominent due to their theoretical significance and broad applications. The Riemann-Liouville derivative, introduced in the 19th century by Abel, Riemann, and Liouville, was the first formal definition of fractional derivatives and is still a key concept in fractional calculus.

The Riemann-Liouville derivative, despite its significance in fractional calculus, poses challenges in real-world applications, particularly in defining physically meaningful initial and boundary conditions. The key advantage of fractional calculus is its ability to model systems with memory and hereditary effects, making it useful for analysing long-memory processes. Unlike classical integer-order derivatives, which are local and rely only on immediate information, fractional-order derivatives exhibit nonlocal behaviour. This nonlocality means the value of a fractional derivative at a point depends on the entire history of a function, making it ideal for problems requiring a global perspective. This unique feature enables fractional calculus to better capture the complexities of natural and engineered systems.

Fractional calculus [77,93,99,138] has recently gained significant attention for its broad applications across various fields. In physics, it models anomalous diffusion, viscoelastic materials, and wave propagation in complex media. In engineering, it enhances fractional order controllers, signal and image processing, and circuit analysis with memory effects. In biology and medicine, it describes neuronal activity, disease spread, drug diffusion, and fractal-like biological structures. In economics and finance, it models stock price dynamics and long-term dependencies. Additionally, fractional calculus provides powerful mathematical tools for solving fractional differential equations and analysing nonlocal processes. By incorporating memory and hereditary effects, fractional calculus bridges the gap between theoretical mathematics and practical applications, offering deeper insights into the complexities of natural processes.

An important aspect of fractional calculus involves the study of solution characteristics, such as existence, uniqueness, and stability, which play a pivotal role in understanding and validating mathematical models. Many researchers have made significant contributions to this area, employing various mathematical techniques and fixed point theorems to establish the existence and uniqueness of solutions [22,100,109,118,129,134].

Beyond theoretical importance, these studies have important real-world applications, ensuring the well-posedness and stability of models in engineering, physics, and biology for accurate simulations. As a result, research on existence, uniqueness, and stability remains crucial in fractional calculus, supporting its growth and expanding its uses.

1.2 Preliminaries

Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}}$. Consider a finite interval $\mathscr{J} = [m, n] \subset \mathbb{R}$. The space $C(\mathscr{J}, \mathcal{X})$ consists of all continuous functions mapping $\mathscr{J}$ into $\mathcal{X}$ forms a Banach space equipped with the supremum norm

$$\|\mathscr{Y}\| = \sup_{\tau \in \mathscr{J}} \|\mathscr{Y}(\tau)\|.$$

Additionally, the space $L^p(\mathscr{J}, \mathcal{X})$ represents the collection of all measurable functions $\mathscr{Y} : \mathscr{J} \to \mathcal{X}$, with the norm defined as follows

$$\|\mathscr{Y}\|_{L^p(\mathscr{J},\mathcal{X})} = \begin{cases} \left(\int_m^n \|\mathscr{Y}(\tau)\|^p d\tau\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{\tau \in \mathscr{J}} \|\mathscr{Y}(\tau)\|, & p = \infty. \end{cases}$$

Here, $\operatorname{ess\,sup}\|\mathscr{Y}(\tau)\|$ is the essential maximum of the function $\mathscr{Y}$.

The space $AC(\mathscr{J}, \mathcal{X})$ consists of all functions $\mathscr{Y} : \mathscr{J} \to \mathcal{X}$ that are absolutely continuous on $\mathscr{J}$. A function $\mathscr{Y}$ is absolutely continuous if there exists an integrable function $g \in L^1(\mathscr{J}, \mathcal{X})$ such that

$$\mathscr{Y}(\tau) = \mathscr{Y}(m) + \int_m^\tau g(s)\, ds, \quad \forall \tau \in \mathscr{J}.$$

The space $AC^n(\mathscr{J}, \mathcal{X})$ consists of all functions $\mathscr{Y} : \mathscr{J} \to \mathcal{X}$ whose first $n-1$ derivatives exist and are absolutely continuous.

1.2.1 Special Functions

(a) Gamma Function:

The gamma function, denoted by the Greek letter Γ is a generalization of the factorial function to complex and real numbers except the non-positive integers.

For a complex number z with $\mathrm{Re}(z) > 0$, the gamma function is defined by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

For positive integers n, gamma function satisfies

$$\Gamma(n) = (n-1)!.$$

The gamma function satisfies the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad (\mathrm{Re}(z) > 0).$$

The value of gamma function at 0 and negative integers is infinite.

The gamma function is widely used in mathematics, physics, and engineering, particularly in areas involving integrals, probability distributions (e.g., the gamma distribution), and complex analysis.

(b) Beta Function:

The beta function, denoted as $\mathfrak{B}(z_1, z_2)$ is a special function closely related to the gamma function. It is defined for complex numbers z_1, z_2 with $\mathrm{Re}(z_1) > 0,\ \mathrm{Re}(z_2) > 0$. The beta function is often used in calculus, probability, and statistics, particularly in the context of integrals and distributions. The beta function is defined by the integral

$$\mathfrak{B}(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt.$$

Also

$$\mathfrak{B}(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}.$$

This relationship is often used to compute the beta function in terms of the gamma function. beta function is symmetric, i.e.

$$\mathfrak{B}(z_1, z_2) = \mathfrak{B}(z_2, z_1).$$

(c) Mittag-Leffler Function:

The Mittag-Leffler function is a special function that generalizes the exponential function and is widely used in fractional calculus, integral transforms, and the study of fractional differential equations. It is named after the Swedish mathematician Gösta Mittag-Leffler.

The Mittag-Leffler function is defined by the following series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where α is a complex parameter with $\mathrm{Re}(\alpha) > 0$, z is a complex variable, and $\Gamma(\cdot)$ is the gamma function.

For $\alpha = 1$, the Mittag-Leffler function reduces to the exponential function i.e.

$$E_1(z) = e^z.$$

A more general form of the Mittag-Leffler function includes two parameters, α and β

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$. The one-parameter Mittag-Leffler function is a special case of this, with $\beta = 1$

$$E_{\alpha,1}(z) = E_\alpha(z).$$

(d) Wright Type Function:

The Wright type function, denoted as $\mathcal{M}_\epsilon(z)$, is a special function that appears in the study of fractional calculus, asymptotic expansions, and generalized integral transforms. It is defined for $z \in \mathbb{C}$ by the following series representation

$$\mathcal{M}_\epsilon(z) = \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!\Gamma(1-\epsilon n)}, \quad 0 < \epsilon < 1.$$

Following are the properties of the Wright type function:

(i) $\mathcal{M}_\epsilon(z) \geq 0, \quad \forall z \geq 0$ and $\mathcal{M}_0(z) = e^z$,

(ii) $\int_0^\infty \mathcal{M}_\epsilon(z)dz = 1$,

(iii) for $k \geq 0$, $\int_0^\infty \mathcal{M}_\epsilon(z)z^k dz = \frac{\Gamma(1+k)}{\Gamma(1+\epsilon k)}$,

(iv) $\int_0^\infty \mathcal{M}_\epsilon(z)e^{-xz}dz = E_\epsilon(-x), \quad x \in \mathbb{C}$, where $E_\epsilon(-x)$ is the Mittag-Leffler function,

(v) $\epsilon \int_0^\infty z\mathcal{M}_\epsilon(z)e^{-xz}dz = E_{\epsilon,\epsilon}(-x), \quad x \in \mathbb{C}$, where $E_{\epsilon,\epsilon}(-x)$ is a generalized Mittag-Leffler function.

1.2.2 Some Fundamental Properties of Fractional Calculus

Now, we define some of the important fractional integrals and derivatives. Further, we also outline some basic results and properties from fractional calculus.

Definition 1.2.1. *[77] For a function $\mathscr{Y} \in C(\mathscr{J}, \mathbb{R})$ the Reimann-Liouville fractional integral of order $\sigma > 0$ is given by*

$$ {}_mI_\tau^\sigma \mathscr{Y}(\tau) = \frac{1}{\Gamma(\sigma)} \int_m^\tau (\tau - s)^{\sigma-1}\mathscr{Y}(s)ds. $$

Definition 1.2.2. *[77] For a function $\mathscr{Y} \in AC^n(\mathscr{J}, \mathbb{R})$ the Reimann-Liouville fractional derivative of order $\sigma > 0$ is given by*

$$ {}_m\mathcal{D}_\tau^\sigma \mathscr{Y}(\tau) = \frac{1}{\Gamma(n-\sigma)} \big(\mathfrak{D}_\tau\big)^n \int_m^\tau (\tau - s)^{n-\sigma-1}\mathscr{Y}(s)ds, $$

where $n = [\sigma] + 1$ and $\mathfrak{D}_\tau = \frac{d}{d\tau}$.

Definition 1.2.3. *[77] For a function $\mathscr{Y} \in AC^n(\mathscr{J}, \mathbb{R})$ the Caputo fractional derivative of order $\sigma > 0$ is given by*

$$ {}_m^C\mathcal{D}_\tau^\sigma \mathscr{Y}(\tau) = \frac{1}{\Gamma(n-\sigma)} \int_m^\tau (\tau - s)^{n-\sigma-1}\mathscr{Y}^{(n)}(s)ds, $$

where $n = [\sigma] + 1$.

Remark 1.2.4. *Unlike the classical integer-order derivative, the Riemann-Liouville fractional derivative of a constant is not necessarily zero and it depends on the order.*

The Riemann-Liouville fractional derivative of order σ for a constant $\mathcal{C}$ is given by

$$ {}_m\mathcal{D}_\tau^\sigma \mathcal{C} = \frac{\mathcal{C}}{\Gamma(1-\sigma)}(\tau - m)^{-\sigma}, $$

where $\tau > m$ and $\sigma \in (0, 1)$.

In contrast, the Caputo fractional derivative of a constant is always zero.

Remark 1.2.5. *The Caputo fractional derivative of order $\sigma > 0$ for a function $\mathscr{Y} \in L^1(\mathscr{J}, \mathbb{R}^+)$ can be expressed in terms of the Riemann-Liouville derivative as follows*

$$ {}_m^C\mathcal{D}_\tau^\sigma \mathscr{Y}(\tau) = {}_m\mathcal{D}_\tau^\sigma \left[\mathscr{Y}(\tau) - \sum_{k=0}^{n-1} \frac{\mathscr{Y}^{(k)}(m)}{k!}(\tau - m)^k \right], $$

where $\tau > m, \quad n-1 < \sigma < n$.

Lemma 1.2.6. *[77] (Semigroup property of fractional integral operators): If $\sigma > 0$ and $\sigma_1 > 0$, the following equation holds for almost every $\tau \in [m, n]$*

$$ {}_m\mathcal{D}_\tau^{-\sigma} \left({}_m\mathcal{D}_\tau^{-\sigma_1} \mathscr{Y}(\tau) \right) = {}_m\mathcal{D}_\tau^{-(\sigma+\sigma_1)} \mathscr{Y}(\tau), $$

where $\mathscr{Y} \in L^p([m, n], \mathbb{R}^N)$ and $1 \le p < \infty$.

Furthermore, if $\sigma + \sigma_1 > 1$, the relationship is valid for every $\tau \in [m, n]$.

Lemma 1.2.7. *[138] If $\sigma > 0$, $\mathscr{Y} \in L^p(\mathscr{J}, \mathbb{R}^N)$, and $1 \le p \le \infty$, then*

$$ {}_m^C\mathcal{D}_\tau^\sigma \left({}_m^C\mathcal{D}_\tau^{-\sigma} \mathscr{Y}(\tau) \right) = \mathscr{Y}(\tau). $$

Lemma 1.2.8. *[138] For $0 < \sigma \le 1$ and $\mathscr{Y} \in AC(\mathscr{J}, \mathbb{R})$, the relationship between the Caputo fractional derivative and the fractional integral is*

$$ {}_m^C\mathcal{D}_\tau^{-\sigma} \left({}_m^C\mathcal{D}_\tau^{\sigma} \mathscr{Y}(\tau) \right) = \mathscr{Y}(\tau) - \mathscr{Y}(m). $$

Definition 1.2.9. *[59] For a given function $\mathscr{Y} : [m, \infty) \to \mathbb{R}$ the $\sigma \in (0, 1)$-order and $\rho \in [0, 1]$-type Hilfer fractional derivative is defined as*

$$ {}_m\mathcal{D}_\tau^{\sigma,\rho} \mathscr{Y}(\tau) = {}_m\mathcal{I}_\tau^{\rho(1-\sigma)} \frac{d}{d\tau} {}_m\mathcal{I}_\tau^{(1-\sigma)(1-\rho)} \mathscr{Y}(\tau). $$

Remark 1.2.10. *For $\rho = 0, \quad 0 < \sigma < 1$, and $m = 0$, the Hilfer fractional derivative reduced to the classical Riemann-Liouville fractional derivative*

$$ {}_0\mathcal{D}_\tau^{0,\sigma} \mathscr{Y}(\tau) = \frac{d}{d\tau} {}_0\mathcal{I}_\tau^{1-\sigma} \mathscr{Y}(\tau) = {}_0\mathcal{D}_\tau^{\sigma} \mathscr{Y}(\tau). $$

Remark 1.2.11. *For $\rho = 1, \quad 0 < \sigma < 1$, and $m = 0$, the Hilfer fractional derivative reduced to the classical Caputo fractional derivative*

$$ {}_0\mathcal{D}_\tau^{1,\sigma} \mathscr{Y}(\tau) = {}_0\mathcal{I}_\tau^{1-\sigma} \frac{d}{d\tau} \mathscr{Y}(\tau) = {}_0^C\mathcal{D}_\tau^{\sigma} \mathscr{Y}(\tau). $$

Definition 1.2.12. *[77] The Hadamard fractional integral of order $\sigma > 0$ for the function $\mathscr{Y} \in L^1(\mathscr{J}, \mathbb{R}^+)$ is defined as*

$$ {}_mI_\tau^\sigma \mathscr{Y}(\tau) = \frac{1}{\Gamma(\sigma)} \int_m^\tau \left(\ln \frac{\tau}{s} \right)^{\sigma-1} \frac{\mathscr{Y}(s)}{s} ds. $$

Definition 1.2.13. *[77] For a function $\mathscr{Y} \in AC^n(\mathscr{J}, \mathbb{R})$, the Caputo-Hadamard fractional derivative of order σ is defined as*

$$ {}_{m}^{CH}\mathcal{D}_{\tau}^{\sigma}\mathscr{Y}(\tau) = \frac{1}{\Gamma(n-\sigma)} \int_{m}^{\tau} \left(\ln \frac{\tau}{s}\right)^{n-\sigma-1} \delta^n \frac{\mathscr{Y}(s)}{s} ds, $$

where $\delta = \tau \frac{d}{d\tau}$ and $n = [\sigma] + 1$.

Definition 1.2.14. *[72] The Katugampola fractional integral of order $\sigma \in (n-1, n)$, $n \in \mathbb{N}$ and $\rho > 0$ is defined as*

$$ {}_{m}I_{\tau}^{\sigma,\rho}\mathscr{Y}(\tau) = \frac{\rho^{1-\sigma}}{\Gamma(\sigma)} \int_{m}^{\tau} (\tau^{\rho} - s^{\rho})^{\sigma-1} s^{\rho-1} \mathscr{Y}(s) ds. $$

Definition 1.2.15. *[91] The Caputo-Katugampola fractional derivative of order $\sigma \in (n-1, n)$, $n = [\sigma] + 1$ for a function $\mathscr{Y} : \mathscr{J} \to \mathbb{R}$ such that $\mathscr{Y} \in C^n(\mathscr{J}, \mathbb{R})$ is defined as*

$$ \begin{aligned} {}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\rho}\mathscr{Y}(\tau) &= {}_{m}I_{\tau}^{n-\sigma,\rho}\big((s^{1-\rho}\mathscr{Y})^{(n)}\big)(\tau) \\ &= \frac{\rho^{1-n+\sigma}}{\Gamma(n-\sigma)} \int_{m}^{\tau} (\tau^{\rho} - s^{\rho})^{n-\sigma-1} \big(s^{1-\rho}\mathscr{Y}\big)^{(n)}(s) ds. \end{aligned} $$

Lemma 1.2.16. *[104] For $\alpha \in (n-1, n)$, $n = [\alpha] + 1$, $\sigma > 0$ and $\mathscr{Y} \in C^n(\mathcal{J}, \mathbb{R})$ we have*

$$ {}_{m}I_{\tau}^{\sigma,\rho}\,{}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\rho}\mathscr{Y}(\tau) = \mathscr{Y}(\tau) + \mathcal{C}_0 + \mathcal{C}_1\left(\frac{\tau^{\rho} - m^{\rho}}{\rho}\right) + \mathcal{C}_2\left(\frac{\tau^{\rho} - m^{\rho}}{\rho}\right)^2 + \cdots + \mathcal{C}_{n-1}\left(\frac{\tau^{\rho} - m^{\rho}}{\rho}\right)^{n-1}, $$

where, for all $i = 1, 2, \ldots, n-1$, $\mathcal{C}_i \in \mathbb{R}$.

Consider a function $\psi \in C^1(\mathscr{J}, \mathbb{R})$ that is continuous, differentiable, increasing, and satisfies $\psi'(\tau) \neq 0$ for all $\tau \in \mathscr{J}$. With this, we introduce the definitions of the ψ-operators.

Definition 1.2.17. *[13] The σ-th order ψ-Riemann-Liouville fractional integral of a function $\mathscr{Y} : \mathscr{J} \to \mathbb{R}$ is expressed as*

$$ {}_{m}I_{\tau}^{\sigma,\psi}\mathscr{Y}(\tau) = \frac{1}{\Gamma(\sigma)} \int_{m}^{\tau} \psi'(s)\,(\psi(\tau) - \psi(s))^{\sigma-1}\,\mathscr{Y}(s)\,ds, \quad \tau \in \mathscr{J},\ \sigma > 0. $$

Note: If we put $\psi(\tau) = \tau$ and $\psi(\tau) = \ln \tau$, then we will get Riemann-Liouville and Hadamard fractional integrals, respectively.

Definition 1.2.18. *[13] Let $n-1 < \sigma < n$, and $\psi \in C^n(\mathscr{J}, \mathbb{R})$. The σ-th order ψ-Riemann-Liouville fractional derivative of an integrable function $\mathscr{Y} : \mathscr{J} \to \mathbb{R}$ is defined by*

$$ {}_{m}\mathcal{D}_{\tau}^{\sigma,\psi}\mathscr{Y}(\tau) = \left(\frac{1}{\psi'(\tau)}\mathfrak{D}_{\tau}\right)^{n} {}_{m}I_{\tau}^{n-\sigma,\psi}\mathscr{Y}(\tau), $$

where $n = [\sigma] + 1$ and $\mathfrak{D}_{\tau} = \frac{d}{d\tau}$.

Definition 1.2.19. *[13] Let $n-1<\sigma<n$, and $\psi, \mathscr{Y} \in C^n(\mathscr{J},\mathbb{R})$. The σ-th order ψ-Caputo fractional derivative is defined by*

$$ {}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\psi}\mathscr{Y}(\tau) = {}_{m}I_{\tau}^{n-\sigma,\psi}\mathscr{Y}_{\psi}^{[n]}(\tau), $$

where $n=[\sigma]+1$ for $\sigma \notin \mathbb{N}$, $n=\sigma$ for $\sigma \in \mathbb{N}$ and

$$ \mathscr{Y}_{\psi}^{[n]}(\tau) = \left(\frac{1}{\psi'(\tau)}\mathfrak{D}_{\tau}\right)^{n}\mathscr{Y}(\tau). $$

From the above, it is clear that

$$ {}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\psi}\mathscr{Y}(\tau) = \begin{cases} \frac{1}{\Gamma(n-\sigma)}\int_{m}^{\tau}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{n-\sigma-1}\mathscr{Y}_{\psi}^{[n]}(s)ds, & \text{if } \sigma \notin \mathbb{N}, \\ \mathscr{Y}_{\psi}^{[n]}(s)ds, & \text{if } \sigma \in \mathbb{N}. \end{cases} \tag{1.2.1} $$

Note: When $\psi(\tau)=\tau$ and $\psi(\tau)=\ln\tau$, Equation (1.2.1) reduces to the Caputo fractional derivative and Caputo-Hadamard fractional derivative, respectively.

If $\mathscr{Y} \in C^n(\mathscr{J},\mathbb{R})$ and $\sigma>0$, then

$$ {}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\psi}\mathscr{Y}(\tau) = {}_{m}\mathcal{D}_{\tau}^{\sigma,\psi}\left[\mathscr{Y}(\tau) - \sum_{k=0}^{n-1}\frac{1}{k!}\left(\psi(\tau)-\psi(m)\right)^{k}\mathscr{Y}_{\psi}^{[k]}(m)\right]. $$

Lemma 1.2.20. *[14] Let $\mathscr{Y} \in L^1(\mathscr{J},\mathbb{R})$ and $\sigma,\rho>0$, then*

$$ {}_{m}I_{\tau}^{\sigma,\psi}{}_{m}I_{\tau}^{\rho,\psi}\mathscr{Y}(\tau) = {}_{m}I_{\tau}^{\sigma+\rho,\psi}\mathscr{Y}(\tau), \quad a.e. \quad \tau \in \mathscr{J}. $$

In particular, if $\mathscr{Y} \in C(\mathscr{J},\mathbb{R})$ then

$$ {}_{m}I_{\tau}^{\sigma,\psi}{}_{m}I_{\tau}^{\rho,\psi}\mathscr{Y}(\tau) = {}_{m}I_{\tau}^{\sigma+\rho,\psi}\mathscr{Y}(\tau), \quad \tau \in \mathscr{J}. $$

Lemma 1.2.21. *[14] For $\sigma>0$, the following are true.*

(i) If $\mathscr{Y} \in C(\mathscr{J},\mathbb{R})$ then

$$ {}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\psi}{}_{m}I_{\tau}^{\sigma,\psi}\mathscr{Y}(\tau) = \mathscr{Y}(\tau), \quad \tau \in \mathscr{J}. $$

(ii) If $\mathscr{Y} \in C^n(\mathscr{J},\mathbb{R})$, $\quad n-1<\sigma<n$ then

$$ {}_{m}I_{\tau}^{\sigma,\psi}\,{}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\psi}\mathscr{Y}(\tau) = \mathscr{Y}(\tau) - \sum_{k=0}^{n-1}\frac{1}{k!}\left(\psi(\tau)-\psi(m)\right)^{k}\mathscr{Y}_{\psi}^{[k]}(m), \quad \tau \in \mathscr{J}. $$

Lemma 1.2.22. *[17, 23] Let $\tau>m$, $\quad \sigma \geq 0$, and $\rho>0$, then*

(i) ${}_{m}I_{\tau}^{\sigma,\psi}(\psi(\tau)-\psi(m))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)}(\psi(\tau)-\psi(m))^{\rho+\sigma-1}$,

(ii) ${}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\psi}(\psi(\tau)-\psi(m))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho-\sigma)}(\psi(\tau)-\psi(m))^{\rho-\sigma-1}$,

(iii) ${}_{m}^{C}\mathcal{D}_{\tau}^{\sigma,\psi}(\psi(\tau)-\psi(m))^{k} = 0$, *for all* $k \in \{0,\ldots,n-1\}$, $\quad n \in \mathbb{N}$.

1.3 Perturbed Differential Equations

Perturbation methods are fundamental tools in the field of nonlinear analysis, offering valuable insights into the behaviour of dynamical systems characterized by nonlinear differential and integral equations. In many cases, such equations are complex and cannot be directly solved or analysed using conventional methods. However, by introducing perturbations in a structured manner, it becomes possible to reformulate the problem, enabling the application of established analytical and computational techniques. This approach helps explore different aspects of solutions, like stability, bifurcations, and dynamic behaviour, making perturbation methods a useful tool for studying nonlinear dynamical systems.

More specifically, let us consider the following initial value problem for a nonlinear first-order ordinary differential equation defined over a closed and bounded interval $\mathscr{J} = [0, T]$ of the real line $\mathbb{R}$

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad \text{a.e.} \quad t \in \mathscr{J}, \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned} \tag{1.3.1}$$

where $x(t)$ represents the unknown function, $f : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ is a given nonlinear function, and the initial condition x_0 specifies the value of $x(t)$ at $t = 0$.

The initial value problem (IVP) presented in equation (1.3.1) forms a fundamental pillar of nonlinear analysis and has been extensively studied in the literature over the years, addressing various aspects of its solutions. It could be argued that the field of nonlinear analysis generally begins with the study of such nonlinear differential equations. However, if the nonlinearity f in equation (1.3.1) lacks sufficient smoothness or regularity, it can make analysing the existence of solutions and other properties more challenging. In such cases, one effective approach is to decompose the function f into the sum of two functions, f_1 and f_2, such that $f = f_1 + f_2$. This leads to the reformulated equation

$$\begin{aligned} x'(t) &= f_1(t, x(t)) + f_2(t, x(t)), \quad \text{a.e.} \quad t \in \mathscr{J}, \\ x(0) &= x_0 \in \mathbb{R}. \end{aligned} \tag{1.3.2}$$

By doing so, these functions may possess more desirable properties, which is more tractable and solvable using the analysis of the nonlinear differential equation. The approach of splitting f into f_1 and f_2 is known as the perturbation method, and equation (1.3.2) is referred to as a perturbation of the differential equation

$$\begin{aligned} x'(t) &= f_1(t, x(t)), \quad \text{a. e.} \quad t \in \mathscr{J}, \\ x(0) &= x_0 \in \mathbb{R}. \end{aligned} \tag{1.3.3}$$

The differential equation in (1.3.2) is obtained by applying a perturbation to the nonlinearity f_1 from equation (1.3.3), and is commonly referred to as the perturbed differential equation.

1.3.1 Classifications of Perturbations

Perturbed differential equations are generally categorized into two types depending on how the perturbation is introduced into the equation

(*i*) Perturbations of the first type: When the free, unknown function in the equation is perturbed in a specific manner, the resulting equation is referred to as a perturbed differential equation of the first type. The nonlinear differential equation (1.3.2) itself can be regarded as an implicit perturbation of the first type, corresponding to the following well-known initial value problem for a first-order linear differential equation

$$x'(t) = x(t), \quad \text{a.e.} \quad t \in \mathscr{J},$$
$$x(0) = x_0 \in \mathbb{R}.$$

(*ii*) Perturbations of the second type: When the perturbation is applied to the unknown function appearing under the derivative operator, the resulting equation is categorized as a perturbed differential equation of the second type.

$$\begin{aligned} \frac{d}{dt}\left[x(t) - f_2(t, x(t))\right] = f_1(t, x(t)), \quad \text{a.e.} \quad t \in \mathscr{J}, \\ x(0) = x_0 \in \mathbb{R}. \end{aligned} \tag{1.3.4}$$

In this perturbation of the differential equation, the term under the derivative is modified, and this form of perturbation is referred to as a perturbation of the second type, resulting in what is called a hybrid differential equation.

Again perturbations can also be classified based on the nature of the modification.

A perturbation of a nonlinear equation that involves the addition or subtraction of a term is referred to as a linear perturbation, while a perturbation involving the multiplication or division by a term is known as a quadratic perturbation. Furthermore, if the unknown function in the differential equation is modified by introducing another function, the perturbation is classified as an implicit perturbation of the differential equation.

For example, the equation

$$\frac{d}{dt}\left[\frac{x(t)}{f_2(t, x(t))}\right] = f_1(t, x(t)), \quad \text{a.e.} \quad t \in \mathscr{J},$$
$$x(0) = x_0 \in \mathbb{R},$$

is a quadratic perturbation of second type for the equation (1.3.3). This class of equations is referred to as the first type of hybrid differential equations.

Likewise, (1.3.4) is a linear perturbation of the second type for equation (1.3.3) and is also categorized as a second type of hybrid differential equation.

Similarly, the equation

$$\begin{aligned} x'(t) &= f_1\left(t, \int_0^t x(s)\, ds\right), \quad \text{a.e.} \quad t \in \mathscr{J}, \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}$$

represents an implicit perturbation of the first type for the equation (1.3.3).

The following equation

$$\begin{gathered} \frac{d}{dt}[f(t, x(t))] = f_1(t, x(t)), \quad \text{a.e.} \quad t \in \mathscr{J}, \\ x(0) = x_0 \in \mathbb{R}, \end{gathered}$$

is an implicit perturbation of the second type for the equation (1.3.3).

These classifications illustrate the versatility of perturbation methods in analysing and solving nonlinear differential equations by tailoring the problem to available mathematical techniques.

In a similar manner, nonlinear differential equations, integral equations, or integro-differential equations may also undergo other forms of perturbations that involve a combination of linear and quadratic perturbations of both the first and second types. Such perturbations are referred to as mixed-type perturbations for nonlinear differential and integral equations.

Most perturbed nonlinear differential or integral equations are commonly addressed using hybrid fixed point theory. Conversely, the study of nonlinear perturbed equations has often served as the foundation or motivation for the development of hybrid fixed point theory in abstract spaces. It is well-established that the inversion of linearly perturbed differential equations leads to operator equations involving the sum of two operators, such as $Ax + Bx = x$. Similarly, the inversion of quadratically perturbed differential equations results in operator equations involving the product of two operators, such as $AxBx = x$ and for the mixed type of perturbation it will be $AxBx + Cx = x$, within the appropriate function spaces. Consequently, linear perturbations are typically resolved using hybrid fixed point theory based on Krasnoselskii's fixed point theorem [29], whereas quadratic perturbations are generally addressed using hybrid fixed point theory inspired by Dhage's fixed point theorem [47].

1.4 Functional Differential Equations

Functional differential equations (FDEs) are a class of differential equations where the evolution of a system depends not only on its current state but also on its past states or history. Unlike ordinary differential equations (ODEs), which depend solely on the present state, FDEs incorporate delays, memory effects, or distributed dependencies, making them suitable for modelling systems with time lags or hereditary influences. As a significant branch of dynamical systems theory, FDEs involve deviating arguments, meaning the behaviour of the system is influenced by its history over a time interval rather than just its instantaneous state. These equations are useful for modelling some real-world phenomena, where past states play a crucial role in determining the current and future behaviour of the system. For example, the rate of change of glucose levels in the human body at a given time is more accurately modelled by considering glucose levels over a past interval rather than just the current level. This leads to differential equations that incorporate delay terms, capturing the system's dependence on its history and providing a more realistic representation of complex, real-world dynamics.

FDEs are categorized based on the nature of the delay. Delay differential equations involve finite delays, where the derivative depends on the past state of the system. Integro-differential equations include integral terms that depend on the state over a range of times, representing distributed delays. Neutral differential equations involve delays in both the state and its derivative. The inclusion of delays or memory effects makes FDEs more complex than ODEs, often leading to oscillatory behaviour, stability challenges, or sensitivity to initial conditions. Analytical methods, such as Laplace transforms, characteristic equations, or the method of steps, are commonly used to solve FDEs.

FDEs have a wide range of applications in various fields. In biology, they model population dynamics, where growth rates depend on past population levels, or the spread of diseases with incubation periods. In engineering, FDEs describe control systems with time delays, such as those in aerospace or robotics, where the system's response depends on past inputs. In economics, they model systems with time lags, such as supply chains or financial markets, where decisions depend on historical data. In physics, FDEs are used to study systems with memory effects, such as viscoelastic materials or heat conduction with time delays. The ability of FDEs to capture the influence of past states on current behaviour makes them indispensable for understanding and predicting the dynamics of complex systems.

Despite their complexity, FDEs provide a powerful framework for modelling real-world systems with memory or delays, offering insights that cannot be captured by ODEs. Ongoing research continues to develop new analytical methods and stability criteria for FDEs, ensuring their relevance in both theoretical and applied contexts. By studying

FDEs, we can improve system modelling and control in science and engineering, leading to better predictions and new solutions.

1.5 Nonlocal Condition

In certain systems of differential equations, having more detailed initial information can be beneficial. This is often accomplished by replacing local initial conditions with nonlocal ones. Unlike local conditions, which depend on a single measurement, nonlocal conditions utilize information from multiple points, offering greater precision. Nonlocal conditions are particularly significant due to their practical applications in physics and applied mathematics. For instance, in the theory of elasticity, nonlocal conditions often yield more accurate and reliable results compared to traditional local conditions.

Byszewski [30] initiated the study of nonlocal problems in 1991, focusing on semilinear nonlocal evolution problems. He demonstrated the existence and uniqueness of mild, strong, and classical solutions. Notably, in the same year, Byszewski and Lakshmikantham [31] highlighted that nonlocal conditions often provide a more effective framework than traditional initial conditions for describing certain physical phenomena.

The nonlocal conditions ${}^{0}D_t^{q-1}x(0) + g(x) = x_0$ and $x(0) + g(x) = x_0$ provide a more versatile and effective framework for applications in physics compared to the classical initial conditions ${}^{0}D_t^{q-1}x(0) = x_0$ and $x(0) = x_0$, respectively. These nonlocal conditions incorporate additional information from the system, enabling a more comprehensive description of physical phenomena. For example, $g(x)$ can be defined as

$$g(x) = \sum_{i=1}^{n} c_i x(t_i), \tag{1.5.1}$$

where c_i $(i = 1, 2, \ldots, n)$ are predefined constants, and $0 < t_1 < t_2 < \ldots < t_n \leq a$. This generalization allows nonlocal conditions to consider interactions over a longer time, improving accuracy in modelling and prediction. Deng [39] employed a nonlocal condition of the form (1.5.1).

Recently Dhawan et al. [51] studied the existence of solution of a Hilfer implicit fractional differential equations (HIFDE) of order in between 1 and 2 involving nonlocal boundary conditions using fixed point theory. In addition to existence and uniqueness results, the Hyers-Ulam and Generalized Hyers-Ulam stability analysis for the solutions of HIFDE is also provided. Also, Chen and Feng [36] studied a class of fractional evolution equations with nonlocal initial conditions.

1.6 Methods

The development of fractional differential equations relies on tools from functional analysis due to their complex behaviour. These essential tools are needed for further analysis, as outlined below.

Let $\mathscr{P}$ be a class of functions mapping from $\mathscr{J}$ to a normed space $\mathcal{X}$.

Definition 1.6.1. *$\mathscr{P}$ is called equicontinuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\|\mathfrak{f}(t_1) - \mathfrak{f}(t_2)\| < \epsilon \quad \text{whenever} \quad |t_1 - t_2| < \delta, \quad \text{for all} \ \ \mathfrak{f} \in \mathscr{P}.$$

Here, δ is independent of $\mathfrak{f} \in \mathscr{P}$.

Definition 1.6.2. *$\mathscr{P}$ is uniformly bounded if there exists a constant $\mathscr{M} > 0$ that is independent on $\mathfrak{f} \in \mathscr{P}$, such that*

$$\|\mathfrak{f}(t)\| \leq \mathscr{M}, \quad \text{for all} \ \ t \in \mathscr{J}, \quad \mathfrak{f} \in \mathscr{P}.$$

Lemma 1.6.3. *(Hölder's inequality) [105] Let p and q be conjugate exponents, meaning $\frac{1}{p} + \frac{1}{q} = 1$. Then for $1 \leq p \leq \infty$, $\mathfrak{f} \in L^p(\mathscr{J}, \mathcal{X})$ and $\mathfrak{g} \in L^q(\mathscr{J}, \mathcal{X})$, with $\mathfrak{f}\mathfrak{g} \in L^1(\mathscr{J}, \mathcal{X})$, the following inequality holds*

$$\|\mathfrak{f}\mathfrak{g}\|_{L^1(\mathscr{J})} \leq \|\mathfrak{f}\|_{L^p(\mathscr{J})} \|\mathfrak{g}\|_{L^q(\mathscr{J})}.$$

Lemma 1.6.4. *(Arzela-Ascoli theorem) [105] If a family $\mathscr{F} = \{\mathfrak{f}(t)\}$ in $C(\mathscr{J}, \mathcal{X})$ satisfies that $\mathscr{F}$ is uniformly bounded and equicontinuous on $\mathscr{J}$, also for each $t \in \mathscr{J}$, the set $\{\mathfrak{f}(t)\}$ is relatively compact in $\mathcal{X}$, then the family $\mathscr{F}$ is relatively compact in $C(\mathscr{J}, \mathcal{X})$.*

Lemma 1.6.5. *(Lebesgue's dominated convergence theorem) [105] Let $\{\mathfrak{f}_n\}$ be a sequence of measurable functions such that $\mathfrak{f}_n \to \mathfrak{f}$ a.e. on a measurable set $\mathcal{E}$. Suppose there exists an integrable function $\mathfrak{g}$ on $\mathcal{E}$ such that $|\mathfrak{f}_n(u)| \leq \mathfrak{g}(u)$ a.e., $n \in \mathbb{N}$. Then*

$$\lim_{n \to \infty} \int_{\mathcal{E}} \mathfrak{f}_n(u)\, du = \int_{\mathcal{E}} \mathfrak{f}(u)\, du.$$

Lemma 1.6.6. *(Closed graph theorem) [105] If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $\mathcal{T} : D(\mathcal{T}) \subseteq \mathcal{X} \to \mathcal{Y}$ is a linear operator, then $\mathcal{T}$ is continuous if and only if the graph of $\mathcal{T}$ is closed.*

Definition 1.6.7. *If for every $\omega_0 \in \mathcal{X}$, $\mathcal{K}(\omega_0)$ is a nonempty closed subset of $\mathcal{X}$, and if there is an open neighbourhood $\mathcal{U}$ of ω_0 such that $\mathcal{K}(\mathcal{U}) \subseteq \mathcal{O}$ for every open set $\mathcal{O}$ of $\mathcal{X}$ containing $\mathcal{K}(\omega_0)$, then $\mathcal{K}$ is termed upper semicontinuous on $\mathcal{X}$.*

Definition 1.6.8. *$\mathscr{K}$ is said to be completely continuous if for any bounded subset $\mathscr{Q}$ of $\mathcal{X}$, $\mathscr{K}(\mathscr{Q})$ is relatively compact.*

The multivalued map $\mathscr{U}$ is called upper semi continuous if and only if it has a closed graph and is completely continuous with non-empty values.

1.6.1 Measure of Noncompactness

In the Banach space $\big(\mathcal{X}, \|\cdot\|\big)$, a closed ball with center η and radius r is denoted by $\mathcal{B}(\eta, r)$. For $\mathfrak{E} \neq \emptyset$ and $\mathfrak{E} \subseteq \mathcal{X}$, $\bar{\mathfrak{E}}$ represents the closure of $\mathfrak{E}$ and Conv$\mathfrak{E}$ denotes its convex hull. If $\mathfrak{E}$ is a bounded subset, diam$(\mathfrak{E})$ represents it's diameter. Denote, $\mathfrak{M}_{\mathcal{X}}$ as the class of non-empty and bounded subsets of $\mathcal{X}$.

The results outlined below are taken from references [138].

Definition 1.6.9. *The mapping $M : \mathfrak{M}_{\mathcal{X}} \to [0, \infty)$ is referred to as the Kuratowski measure of noncompactness, defined on each bounded subset $\mathfrak{B}$ of $\mathcal{X}$ by*

$$M(\mathfrak{B}) = \inf \Big\{ \epsilon > 0 : \mathfrak{B} \subseteq \cup_{i=1}^{n} \mathfrak{B}_i \quad and \quad \text{diam}(\mathfrak{B}_i) \leq \epsilon \Big\}.$$

Lemma 1.6.10. *[138] The following properties are satisfied by the Kuratowski measure of noncompactness.*

(i) $M(\mathfrak{B}) = 0$ *if and only if* $\mathfrak{B}$ *is relatively compact,*

(ii) if $\mathfrak{B} \subset \mathfrak{R}$ *then* $M(\mathfrak{B}) \leq M(\mathfrak{R})$,

(iii) $M(\mathfrak{B}) = M(\bar{\mathfrak{B}}) = M(\text{Conv}\mathfrak{B})$,

(iv) $M(\mathfrak{B} + \mathfrak{R}) \leq M(\mathfrak{B}) + M(\mathfrak{R})$,

(v) $M(\xi\mathfrak{B}) = |\xi| M(\mathfrak{B})$, *for all* $\xi \in \mathbb{R}$.

1.6.2 Topological Degree Theory

Definition 1.6.11. *[138] Consider a continuous and bounded operator $\mathcal{Q} : \mathfrak{E} \to \mathcal{X}$. The operator $\mathcal{Q}$ is referred to as M-Lipschitz if there exists a constant $\lambda \geq 0$ that satisfies the following condition:*

$$M\big(\mathcal{Q}(\mathfrak{B})\big) \leq \lambda M(\mathfrak{B}), \quad \textit{for every} \quad \mathfrak{B} \subset \mathfrak{E}.$$

Furthermore, $\mathcal{Q}$ is termed a strict M-contraction if $\lambda < 1$.

Definition 1.6.12. *[138] The function $\mathcal{Q}$ is referred to as M-condensing if*

$$M\big(\mathcal{Q}(\mathfrak{B})\big) \leq M(\mathfrak{B}),$$

for all bounded $\mathfrak{B} \subset \mathfrak{E}$ with $M(\mathfrak{B}) > 0$.

In other words, if $M\big(\mathcal{Q}(\mathfrak{B})\big) \geq M(\mathfrak{B})$, then it follows that $M(\mathfrak{B}) = 0$.

Furthermore, $\mathcal{Q} : \mathfrak{E} \to \mathcal{X}$ is said to be Lipschitz if there exists a constant $\lambda > 0$ such that

$$|\mathcal{Q}(u) - \mathcal{Q}(v)| \leq \lambda |u - v| \quad \textit{for all} \quad u, v \in \mathfrak{E}.$$

If $\lambda < 1$, $\mathcal{Q}$ is referred to as a strict contraction.

Proposition 1.6.13. *If $\mathcal{P}, \mathcal{Q} : \mathfrak{E} \to \mathcal{X}$ are M-Lipschitz mappings with respective constants λ and λ', then the combined mapping $\mathcal{P} + \mathcal{Q} : \mathfrak{E} \to \mathcal{X}$ is also M-Lipschitz, with a Lipschitz constant equal to $\lambda + \lambda'$.*

Proposition 1.6.14. *If $\mathcal{Q} : \mathfrak{E} \to \mathcal{X}$ is a compact operator, then it satisfies the M-Lipschitz condition with the constant $\lambda = 0$.*

Proposition 1.6.15. *If $\mathcal{Q} : \mathfrak{E} \to \mathcal{X}$ is Lipschitz with constant λ, then it is also M-Lipschitz with the same constant λ.*

The following results are presented by Isaia [67] using topological degree theory.

In $(\mathcal{X}, \|\cdot\|)$, $\mathcal{B}(\eta, r)$ remains the same as before. When $\eta = 0$, the notation $\mathcal{B}_r$ is used instead of $\mathcal{B}(0, r)$.

Theorem 1.6.16. *Let $\Lambda : \mathcal{X} \to \mathcal{X}$ be M-condensing and*

$$\Upsilon = \big\{\omega \in \mathcal{X} : \exists \;\; \gamma \in [0,1] \;\; \text{such that} \;\; \omega = \gamma\Lambda\omega\big\}.$$

If Υ is a bounded set in $\mathcal{X}$, then there exists a radius $r > 0$ such that $\Upsilon \subset \mathcal{B}_r$, then the degree

$$\mathfrak{D}\big(I - \gamma\Lambda, \mathcal{B}_r, 0\big) = 1, \quad \text{for all} \quad \gamma \in [0,1].$$

As a result, Λ has at least one fixed point, and the set of fixed points of Λ is contained within $\mathcal{B}_r$.

1.6.3 Fixed Point Theorems

Fixed point theorems are among the most powerful tools for studying the existence, uniqueness, and controllability of fractional dynamical systems (FDS). By appropriately defining operators, an existence problem can often be transformed into a fixed point problem. Numerous fixed point theorems from functional analysis are available, and some of those utilized in this thesis are outlined below.

Theorem 1.6.17. *(Banach contraction principle) [78] Consider a Banach space $(\mathcal{X}, \|\cdot\|)$ and a operator $\mathcal{P} : \mathcal{X} \to \mathcal{X}$. If $\mathcal{P}$ is a contraction operator, i.e. for $u, v \in \mathcal{X}$ there exists a constant λ, $0 < \lambda < 1$, such that*

$$\|\mathcal{P}(u) - \mathcal{P}(v)\| \leq \lambda\|u - v\|,$$

then it will have a unique fixed point.

Theorem 1.6.18. *(Schauder's fixed point theorem) [138] Consider a closed-convex subset S of a Banach space $\mathcal{X}$ and let the operator $\mathcal{P} : S \to S$ is continuous and compact. Thus, $\mathcal{P}$ has a fixed point.*

Lemma 1.6.19. *(Krasnoselskii's fixed point theorem) [29] Consider a nonempty, closed, convex and bounded subset* $\mathcal{S}$ *of a Banach space* $\mathcal{X}$*. Let the two operators* $\mathcal{P}:\mathcal{X}\to\mathcal{X}$ *and* $\mathcal{Q}:\mathcal{S}\to\mathcal{X}$ *such that the following are hold*

(i) $\mathcal{P}$ *is a contraction,*

(ii) $\mathcal{Q}$ *is completely continuous, and*

(iii) $u=\mathcal{P}(u)+\mathcal{Q}(v)$ *for all* $v\in S \implies u\in S$.

Then the operator equation $\mathcal{P}(u)+\mathcal{Q}(u)=u$ *has a solution.*

Theorem 1.6.20. *(Dhage's fixed point theorem for two operators) [50] Let* S *be a non-empty, closed convex and bounded subset of the Banach algebra* $\mathcal{X}$ *and let two operators* $A:\mathcal{X}\to\mathcal{X}$ *and* $B:S\to X$ *be such that*

(i) A *is Lipschitzian with a Lipschitz constant* α,

(ii) B *is completely continuous,*

(iii) $x=AxBy \implies x\in S$ *for all* $y\in S$*, and*

(iv) $\alpha\mathcal{M}<1$*, where* $\mathcal{M}=\|B(S)\|=\sup\{\|Bx\|:x\in S\}$.

Then the operator equation $AxBx=x$ *has a solution in* S.

Theorem 1.6.21. *(Dhage's fixed point theorem for three operators) [46] Assume* S *be a non-empty, closed convex and bounded subset of the Banach algebra* $\mathcal{X}$*. Let the three operators* $A:\mathcal{X}\to\mathcal{X}$, $B:S\to\mathcal{X}$ *and* $C:\mathcal{X}\to\mathcal{X}$ *be such that*

(i) A *and* C *are Lipschitzian with a Lipschitz constant* α *and* β *respectively,*

(ii) B *is compact and continuous,*

(iii) $u=AuBv+Cu \implies u\in S$ *for all* $v\in S$*, and*

(iv) $\alpha\mathcal{M}+\beta<1$*, where* $\mathcal{M}=\|B(S)\|=\sup\{\|Bx\|:x\in S\}$.

Then the operator equation $u=AuBu+Cu$ *has a solution in* S.

Theorem 1.6.22. *(Dhage's fixed point theorem for multivalued operator) [48] Suppose that* $\mathcal{X}$ *is a Banach algebra. Consider a single-valued operator* $\Lambda_1:\mathcal{X}\to\mathcal{X}$ *and a multivalued operator* $\Lambda_2:\mathcal{X}\to P_{cp,cv}(\mathcal{X})$*, where* $P_{cp,cv}(\mathcal{X})=\{Y\in P(\mathcal{X}):Y \text{ is compact and convex}\}$ *with the following requirements*

(i) With a Lipschitz constant α, Λ_1 *is single-valued Lipschitz,*

(ii) Λ_2 *is upper semi continuous and compact,*

(iii) $\alpha\Upsilon < 1$, *where* $\Upsilon = \|\Lambda_2(\mathcal{X})\|$.

Then, one of the following statements must hold

(a) *There is a solution for the operator inclusion* $\omega \in \Lambda_1\omega\Lambda_2\omega$, *or*

(b) *The set* $\xi = \{\bar{\chi} \in \mathcal{X} : \sigma\bar{\chi} \in \Lambda_1\bar{\chi}\Lambda_2\bar{\chi}, \sigma > 1\}$ *is unbounded.*

1.6.4 Semigroup of Bounded Linear Operator

The exploration of solutions to semi linear fractional evolution equations has been significantly enriched by the application of semigroup theory. The insights and results derived from this theory have provided a strong framework for analysing such equations. For a deeper understanding, we present some foundational definitions.

Definition 1.6.23. *[98] A one parameter family* $\{\mathcal{T}(t)\}_{t\geq 0}$ *of bounded linear operators from a Banach space* $\mathcal{X}$ *into* $\mathcal{X}$ *is said to be a semigroup of bounded linear operators on* $\mathcal{X}$ *if*

(i) $\mathcal{T}(0) = I$, *(I is the identity operator on* $\mathcal{X}$*),*

(ii) $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$ *for every* $t, s \geq 0$ *(the semigroup property).*

Definition 1.6.24. *[98] Let* $\{\mathcal{T}(t)\}_{t\geq 0}$ *be a semigroup of bounded linear operators on* $\mathcal{X}$. *The infinitesimal generator* $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ *of* $\mathcal{T}(t)$ *is defined by*

$$Ax = \lim_{t\to 0^+} \frac{\mathcal{T}(t)x - x}{t}, \quad x \in D(A),$$

where $D(A)$ *is the domain of* A *and it is defined as*

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{\mathcal{T}(t)x - x}{t} \quad \textit{exists} \right\}.$$

Definition 1.6.25. *[98] A semigroup* $\{\mathcal{T}(t)\}_{t\geq 0}$ *is said to be a strongly continuous semigroup if*

$$\lim_{t\to 0^+} \mathcal{T}(t)x = x, \quad \textit{for every} \quad x \in \mathcal{X}.$$

A strongly continuous semigroup of bounded linear operators on $\mathcal{X}$ *is also known as semigroup of class* C_0 *or simply* C_0 *semigroup.*

Definition 1.6.26. *[98] A semigroup* $\{\mathcal{T}(t)\}_{t\geq 0}$ *is said to be uniformly continuous if*

$$\lim_{t\to 0^+} \|\mathcal{T}(t) - I\| = 0.$$

Remark 1.6.27. *[98] Let $\{\mathcal{T}(t)\}_{t\geq 0}$ denote a C_0 semigroup of bounded linear operators on $\mathcal{X}$. Then there exist constants $w \geq 0$ and $\mathcal{M} \geq 1$ such that*

$$\|\mathcal{T}(t)\| \leq \mathcal{M}e^{wt} \quad \textit{for} \quad 0 \leq t < \infty.$$

The following classifications apply:

(i) If $w = 0$, the semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ is referred to as uniformly bounded.

(ii) If $w = 0$ and $\mathcal{M} = 1$, the semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ is called a C_0-semigroup of contractions.

1.7 Stability

The stability of solutions to fractional differential equations is a key area in fractional calculus. It addresses the question posed by Stanislaw Ulam in 1940: "When is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?" This idea led to the development of various stability concepts, including Hyers-Ulam and Hyers-Ulam-Rassias stability. These concepts are particularly relevant in fractional calculus, as they ensure that small deviations in solutions do not lead to significant errors. In our study, we focus on analysing these two types of stability for the problems under consideration.

1.7.1 Hyers-Ulam Stability

Hyers-Ulam stability is a concept that originated in the study of functional equations and has since become an important topic in mathematical analysis, particularly in the context of differential equations, integral equations, and dynamical systems. It addresses the question of how small perturbations in the inputs or parameters of a system affect its solutions. This type of stability is named after Stanislaw Ulam and Donald Hyers, who first introduced the idea in the context of functional equations.

Definition 1.7.1. *A functional equation or system is said to have Hyers-Ulam stability if, for every approximate solution that satisfies the equation to within a small error, there exists an exact solution that is close to the approximate solution. Formally, consider a functional equation*

$$F(x, y) = 0,$$

where F is a functional operator. If for every $\epsilon > 0$ and every approximate solution y satisfying

$$\|F(x, y)\| \leq \epsilon,$$

there exists an exact solution y_0 *such that*

$$\|y - y_0\| \leq K\epsilon,$$

where $K > 0$ *is a constant, then the equation is said to have Ulam-Hyers stability.*

Hyers-Ulam stability ensures that small deviations from the exact solution do not lead to large errors, keeping the system stable under disturbances.

1.7.2 Hyers-Ulam-Rassias Stability

Hyers-Ulam-Rassias stability is a generalization of Hyers-Ulam stability, introduced by Themistocles Rassias. It incorporates a control function that allows for a more flexible and generalized analysis of stability.

Definition 1.7.2. *A functional equation or system has Hyers-Ulam-Rassias stability if, for every approximate solution y satisfying*

$$\|F(x, y)\| \leq \phi(x),$$

where $\phi(x)$ *is a non-negative control function, there exists an exact solution* y_0 *such that*

$$\|y - y_0\| \leq K\phi(x),$$

where $K > 0$ *is a constant.*

The control function $\phi(x)$ allows for a more detailed analysis of stability, as it can depend on the input x and capture varying levels of perturbation.

1.8 Controllability

Controllability stands as one of the most fundamental and widely studied concepts in mathematical control theory. It addresses the critical question of whether a dynamical system can be guided from an arbitrary initial state to a desired final state using appropriate control inputs within a specified time frame. Rudolf Kalman first introduced this concept in 1965, and it has since become a key idea in control systems research, with wide applications in engineering, applied mathematics, and technology. Controllability ensures that different systems, like aircraft autopilots and building temperature controls, work properly, stay stable, and adjust to changing conditions. Controllability can be categorized into several types, depending on the characteristics of the system.

Exact Controllability, where the system can be steered exactly to the desired final state. This is the strongest form of controllability and is often achievable in finite-dimensional systems. Approximate Controllability, which allows the system to be driven arbitrarily close to the desired final state. This is more natural in infinite-dimensional systems, where exact controllability may not be feasible. Null Controllability, where the system can be steered to the zero state (origin) from any initial state. And trajectory controllability, which ensures the system can follow a specific path in the state space. In finite-dimensional systems, such as those described by ordinary differential equations, exact and approximate controllability often coincide, meaning the system can achieve both precise and near-precise control. However, in infinite-dimensional systems, such as those modelled by partial differential equations, approximate controllability is more natural and practical, as exact controllability may not always be feasible due to the complexity and infinite nature of the state space.

Despite the significant advancements made in the study of controllability, there remain many open questions and challenges, particularly in the context of fractional-order systems. Fractional differential equations, which model systems with memory and hereditary effects, present unique challenges for controllability analysis. These systems are increasingly relevant in fields such as biology, physics, and engineering, where traditional integer-order models may not adequately capture the behaviour of the system. As a result, controllability in fractional systems is an active and growing area of research, offering numerous opportunities for innovation and exploration.

1.9 Literature Review

Over the past three decades, the theory of fractional differential equations has gained significant importance due to its wide range of applications in various fields, including control technology, communications, electrical engineering, impact mechanics, and medicine. The mathematical framework of fractional calculus, particularly fractional differential equations, serves as a powerful tool for modelling and analysing the behaviour of numerous processes across diverse disciplines. As a result, extensive research has been conducted on fractional differential equations under different conditions [53, 77, 99, 138].

The study of hybrid differential and integral equations was pioneered by Dhage [45] in 1988. Following this foundational work, numerous researchers have contributed to the field. In 2010, Dhage and Lakshmikantham [50] investigated the following first-order hybrid differential equation

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in [t_0, t_0 + a) = \mathscr{J},$$
$$x(t_0) = x_0 \in \mathbb{R},$$

where $f \in C(\mathscr{J} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(\mathscr{J} \times \mathbb{R}, \mathbb{R})$. They established fundamental results on the existence of solutions under mixed Lipschitz and Carathéodory conditions.

Additionally, they developed differential inequalities to prove the existence of extremal solutions and provided a comparison result.

Subsequently, Zhao et al. [137] extended the work of Dhage and Lakshmikantham by examining the following Riemann-Liouville type fractional hybrid differential equation

$$
{}_0D_t^\sigma \left[\frac{x(t)}{f(t,x(t))} \right] = g(t,x(t)), \quad t \in [0,T) = \mathscr{J},
$$
$$
x(0) = 0,
$$

where $0 < \sigma < 1$, $f \in C(\mathscr{J} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $g \in C(\mathscr{J} \times \mathbb{R}, \mathbb{R})$. They demonstrated the existence of solutions and extremal solutions, along with a comparison result.

Further contributions were made by Sun et al. [125], who studied the existence of solutions for Riemann-Liouville type fractional hybrid differential equations with boundary conditions. Their work expanded the theoretical framework and provided additional insights into the behaviour of such systems.

These studies collectively advance the understanding of hybrid differential equations, particularly in the context of fractional calculus, and highlight the importance of advanced mathematical tools in analysing complex systems. Recent advancements in hybrid fractional differential equations can be found in [21, 79, 87, 115, 124].

In 2013, Dhage and Jadhav [49] investigated the existence of solutions for a second type hybrid differential equation

$$
\frac{d}{dt}[x(t) - f(t,x(t))] = g(t,x(t)), \quad t \in [0,T) = \mathscr{J}
$$
$$
x(0) = 0,
$$

where $f, g \in C(\mathscr{J} \times \mathbb{R}, \mathbb{R})$. They established existence and uniqueness results along with fundamental differential inequalities.

In the same year, Lu et al. [88] proved an existence theorem for a fractional hybrid differential equation of the second type, involving Riemann-Liouville differential operators of order between 0 and 1. The articles [3,19,63,76,111] explored the second type of hybrid differential equations, contributing to the understanding and analysis of this specific class of equations.

Stability analysis of a fractional differential equation examines the long-term behaviour of its solutions under small perturbations. It often involves techniques like Lyapunov functions, Mittag-Leffler stability, and fixed point methods.

Samina et al. [111] studied the existence of solutions for a coupled system of fractional hybrid differential equations under mixed-type Lipschitz and Carathéodory conditions. Using nonlinear analysis and hybrid fixed point theory, they established existence results and explore stability concepts such as Hyers-Ulam, generalized Hyers-Ulam, Hyers-Ulam-Rassias, and generalized Hyers-Ulam-Rassias stability. Similarly, Ahmad et al. [10] explored Hyers-Ulam stability for a coupled system of fractional hybrid boundary value problems (BVPs) with finite delays. Boutiara et al. [28] examined the existence and uniqueness of solutions for a coupled system of hybrid fractional integro-differential equations with ψ-Caputo operators. Using Dhage's hybrid fixed point theorem for three operators and Banach contraction principle, they established existence and uniqueness results. Additionally, they analysed Hyers-Ulam and generalized Hyers-Ulam stability for the system. Recently, Ali et al. [17] studied hybrid fractional Langevin equations involving the ψ-Caputo fractional operator. They applied Schauder and Banach fixed point theorems to establish existence and uniqueness results. Furthermore, they examined Hyers-Ulam stability.

A fractional evolution equation is a type of differential equation that involves fractional derivatives and describes the evolution of a system over time. El-Borai [54] studied the existence and uniqueness of the solution for the following Cauchy problem

$$\begin{aligned} {}_0\mathcal{D}_t^{\sigma} x(t) &= Ax(t) + f(t, x(t)), \quad 0 < \sigma < 1, \\ x(0) &= x_0. \end{aligned}$$

A is a linear closed operator defined on a Banach space $\mathcal{X}$.

Balachandran and Park [25] studied the existence of solutions for fractional semilinear evolution equations in Banach spaces. They also examined the nonlocal Cauchy problem for these equations. The results were established using fractional calculus and fixed point theorems. Kavitha et al. [74] studied the following Hilfer-fractional evolution equation with infinite delay

$$\begin{aligned} {}_0\mathcal{D}_t^{\sigma,\rho} x(t) &= Ax(t) + f(t, x_t), \quad t \in N \\ {}_0I_t^{(1-\rho)(1-\sigma)} x(t) &= \phi(t), \end{aligned}$$

where ${}_0\mathcal{D}_t^{\sigma,\rho}$ is the Hilfer fractional derivative of order $\sigma \in [0,1]$ and type $\rho \in (1/2, 1)$. The linear operator A is the infinitesimal generator of an analytic semigroup. They established the existence of mild solutions for the system via measures of noncompactness.

In case of hybrid fractional differential equations, Yang and Wang [133] investigated a linear category of hybrid evolution equations involving the Hilfer fractional derivative. By utilizing the fixed point theorem and the noncompact measure technique, they developed several novel criteria to ensure the existence and uniqueness of mild solutions. Their analysis addressed both scenarios where the associated semigroup is compact and where it is not.

Zerbib et al. [135] studied the existence of mild solutions for initial value problems involving second kind and quadratic perturbation of fractional semilinear evolution equations with Caputo fractional derivative. They established the existence results using

Dhage's fixed point theorem. Additionally, they proved four different types of Hyers-Ulam stability for mild solutions.

Controllability is a fundamental concept in control theory that examines whether a dynamical system can be steered from an initial state to a desired final state using a suitable control input. The problems of controllability in fractional evolution systems has been extensively explored by numerous researchers across various contexts [85, 102, 106, 108, 110].

Wang and Zhou [131] studied the controllability results for the following fractional semilinear differential inclusions with Caputo fractional derivative

$$
\begin{aligned}
&{}^{C}_{0}\mathcal{D}^{\sigma}_{t} x(t) \in Ax(t) + f(t, x(t)) + Bu(t), \quad t \in \mathscr{J} = [0, b], \\
&x(0) = x_0,
\end{aligned}
$$

where A is the infinitesimal generator of a strongly continuous semigroup, the compactness of the operator semigroups is not considered. B is a bounded linear operator. They established existence and controllability results for fractional semilinear differential inclusions with the Caputo derivative in Banach spaces. The findings were derived using fractional calculus, operator semigroup theory, and Bohnenblust-Karlin's fixed point theorem. Recently Mohan Raja [94] studied the existence and approximate controllability of impulsive fractional differential systems of order in between 1 and 2 with infinite delay. Using the Leray-Schauder fixed point theorem, sequence methods, and impulsive systems, they first established the existence of mild solutions and then proved approximate controllability for nonlocal fractional delay equations.

There is no existing literature on the controllability of hybrid fractional evolution systems.

1.10 Motivation

The motivation behind this thesis comes from the growing importance and applicability of fractional differential equations in modelling complex real-world phenomena. Unlike classical integer-order differential equations, fractional differential equations are capable of capturing memory effects, hereditary properties, and long-range dependencies, making them indispensable in fields such as physics, biology, engineering, and economics. However, the study of hybrid functional fractional differential equations and inclusions, the systems that combine fractional derivatives with impulsive effects, nonlocal boundary conditions, and nonlinear operators remains relatively under explored. These hybrid systems are particularly relevant in modelling processes with sudden changes and memory-dependent dynamics, such as viscoelastic materials, anomalous diffusion, and biological systems with intermittent external influences.

The study of nonlocal boundary conditions, impulsive effects, and generalized fractional derivatives (e.g., Caputo, Riemann-Liouville, Hilfer, Caputo-Katugampola, and $\psi-$Caputo) presents significant challenges due to their nonlinear and complex nature. Advanced mathematical tools, including Dhage's fixed point theorem, Krasnoselskii's theorem, Schauder's theorem, and the Banach contraction principle, are employed to effectively address these challenges. These tools, along with results from functional analysis, semigroup theory and topological degree theory, provide a strong framework for studying the complex interactions between fractional derivatives, nonlocal conditions, and impulsive effects, allowing a thorough analysis of such systems.

1.11 Overview of the Thesis

In this thesis we investigate hybrid functional fractional differential equations and inclusions, focusing on existence, uniqueness, stability, and controllability.

Chapter 1: In this chapter we introduce foundational concepts, including key definitions in fractional calculus, essential fixed point theorems, other methodologies and a comprehensive literature review.

Chapter 2: We devote this chapter to investigate Caputo type hybrid fractional differential equations with nonlocal boundary conditions, employing Dhage's fixed point theorem for two operators in Banach algebras.

Chapter 3: We focus on hybrid Caputo fractional Volterra-Fredholm equations with nonlocal conditions in this chapter, Dhage's fixed point theorem for three operators is used to study the existence of solution. An analytical example is also provided.

Chapter 4: In this chapter we analyse a nonlinear p-Laplacian hybrid fractional differential equations involving Caputo and Riemann-Liouville derivatives. Existence, uniqueness, and stability are established using Krasnoselskii's fixed point theorem and the Banach contraction principle. An illustrative example is added at the end of this chapter.

Chapter 5: In this chapter we examine a hybrid non-instantaneous impulsive fractional differential equations incorporating the p-Laplacian and Caputo-Katugampola derivatives. Existence, uniqueness and stability results are examined via Schauder's fixed point theorem and Banach contraction principle respectively. We also have added an analytical example.

Chapter 6: We focus on a semilinear hybrid evolution inclusion with Hilfer fractional derivatives in this chapter, establishing existence and exact controllability through Dhage's fixed point theorem for multivalued operator. An example is provided to illustrate the main findings.

Chapter 7: We study a hybrid fractional differential equations with multi-point boundary conditions under the ψ-Caputo derivative. Existence and uniqueness results are derived using topological degree theory and the Banach contraction principle. Analytical and numerical examples are added to illustrate the main results.

Chapter 8: In the final chapter we summarize the findings and suggests potential directions for future research.