CHAPTER 2

Study of a Nonlinear Hybrid Functional Fractional Differential Equation with Nonlocal Conditions

2.1 Introduction

Differential equations with deviating arguments are fundamental in modelling dynamical systems where the evolution of a state variable is influenced by its past or future values. These equations extend classical differential equations by incorporating delays or advanced arguments, thereby effectively capturing memory effects and time-dependent feedback mechanisms. Their applicability spans various disciplines, including heat transfer, epidemic modelling, signal processing, species evolution, traffic flow, automatic control, and economic planning.

The introduction of deviating arguments [55] enhances the mathematical formulation of dynamical systems, making them particularly effective for modelling self-oscillating systems, automatic control processes, biophysical models, and combustion dynamics in rocket engines. Systems with finite delays, where information is transmitted after a specified time, are well-characterized by these equations. The presence of deviating arguments [35, 38, 58, 69, 82] establishes a crucial link between a function and its derivative at different time instances, offering a refined mathematical framework for modelling complex systems. By incorporating these arguments, such equations provide a powerful approach to translating real-world problems into predictive mathematical models, contributing to advancements in both theoretical research and practical applications across science and engineering.

In this chapter we examine the existence of solution for the following nonlinear non-

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local hybrid functional fractional differential equation with deviating arguments

$$
\begin{gathered}
{}^{C}_{0}\mathcal{D}^{q}_{t}\left[\frac{x(t)}{\mathfrak{F}(t,x(t),x(\eta(x(t),t)))}\right]=\mathcal{G}(t,x(t),x(\xi(x(t),t))),\quad t\in\mathscr{J}=[0,T],\\
x(0)=\phi(x),\quad x(T)=a.
\end{gathered}
\tag{2.1.1}
$$

Here, ${}^{C}\mathcal{D}^{q}_{t}$ denotes the Caputo fractional derivative of order $1<q\leq 2$, the functions $\mathfrak{F}\in C(\mathscr{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $\mathcal{G}\in\mathfrak{C}(\mathscr{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\phi: C(\mathscr{J},\mathbb{R})\to\mathbb{R}$ be a continuous and bounded function. Also, η and ξ are functions from $\mathbb{R}\times\mathscr{J}$ into $\mathscr{J}$.

This chapter is organized as follows: In Section 2.2, we derive the integral solution which is used to find the main result. Next, by using a fixed point theorem in Banach algebra the existence of solution for the boundary value problem (2.1.1) under mixed Lipschitz and Carathéodory conditions is discussed in Section 2.3.

2.2 Preliminaries

Let $\mathscr{J}=[0,T]$ be a bounded interval in $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Consider the class of continuous functions $\mathfrak{F}:\mathscr{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\mathcal{G}:\mathscr{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, denoted by $C(\mathscr{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\mathfrak{C}(\mathscr{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, respectively, satisfying the following conditions

(i) For each $x,\tilde{x}\in\mathbb{R}$, the mapping $t\mapsto\mathcal{G}(t,x,\tilde{x})$ is measurable.

(ii) For each $t\in\mathscr{J}$, the mapping $x\mapsto\mathcal{G}(t,x,\tilde{x})$ is continuous.

The class of functions $\mathfrak{C}(\mathscr{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ is referred to as the Carathéodory class, consisting of functions defined on $\mathscr{J}\times\mathbb{R}\times\mathbb{R}$ that are Lebesgue integrable whenever they are bounded by a Lebesgue integrable function on $\mathscr{J}$.

Lemma 2.2.1. *[77] Let $q>0$, $n=[q]+1$ then for some $c_i\in\mathbb{R}$,*

$$
{}_{0}I^{q}_{t}\{{}^{C}_{0}\mathcal{D}^{q}_{t}y(t)\}=y(t)-\sum_{i=0}^{n-1}c_i t^i.
$$

Lemma 2.2.2. *Let $u\in C(\mathscr{J},\mathbb{R})$ then the following hybrid differential equation*

$$
\begin{gathered}
{}^{C}_{0}\mathcal{D}^{q}_{t}\left[\frac{x(t)}{\mathfrak{F}(t,x(t),x(\eta(x(t),t)))}\right]=u(t),\quad t\in[0,T],\\
x(0)=\phi(x),\qquad x(T)=a
\end{gathered}
\tag{2.2.1}
$$

has the unique integral solution for $t \in \mathscr{J}$

$$x(t) = \mathfrak{F}(t, x(t), x(\eta(x(t), t)))\Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}u(\tau)d\tau + \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))}$$
$$+ \frac{1}{T}\Bigg\{\frac{a}{\mathfrak{F}(T, a, x(\eta(a, T)))} - \frac{1}{\Gamma(q)}\int_0^T (T-\tau)^{q-1}u(\tau)d\tau - \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))}\Bigg\}t\Bigg]. \tag{2.2.2}$$

Proof. By employing Lemma 2.2.1 and Definition 1.2.1, we establish the following result

$$x(t) = \mathfrak{F}(t, x(t), x(\eta(x(t), t)))\Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}u(\tau)d\tau + c_0 + c_1 t\Bigg], \tag{2.2.3}$$

where c_0 and $c_1 \in \mathbb{R}$.
Using the condition $x(0) = \phi(x)$ in (2.2.3) we get

$$c_0 = \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))}.$$

By applying the second condition $x(T) = a$ and substituting the value of c_0 into (2.2.3), we obtain

$$c_1 = \frac{1}{T}\Bigg\{\frac{a}{\mathfrak{F}(T, a, x(\eta(a, T)))} - \frac{1}{\Gamma(q)}\int_0^T (T-\tau)^{q-1}u(\tau)d\tau - \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))}\Bigg\}.$$

Now, substituting the values of c_0 and c_1 into (2.2.3), we obtain

$$x(t) = \mathfrak{F}(t, x(t), x(\eta(x(t), t)))\Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}u(\tau)d\tau + \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))}$$
$$+ \frac{1}{T}\Bigg\{\frac{a}{\mathfrak{F}(T, a, x(\eta(a, T)))} - \frac{1}{\Gamma(q)}\int_0^T (T-\tau)^{q-1}u(\tau)d\tau - \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))}\Bigg\}t\Bigg].$$

□

2.3 Existence Result

For the closed and bounded interval $\mathscr{J} = [0, T]$, we define the supremum norm $\|\cdot\|$ on the space $C(\mathscr{J}, \mathbb{R})$ as

$$\|y\| = \sup_{t\in\mathscr{J}}|y(t)|.$$

In proving our main results, we employ a fixed point theorem in Banach algebra, established by Dhage for two operators (Theorem 1.6.20).

2.3.1 Hypotheses

Next, we introduce the following hypotheses, which will be utilized to establish the main result.

(H_1) For all $t \in \mathscr{J}$ and $x,\ y,\ \tilde{x},\ \tilde{y} \in \mathbb{R}$, there exists a constant $L_{\mathfrak{F}} > 0$ such that the function $\mathfrak{F} : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ satisfies the following condition

$$|\mathfrak{F}(t, x, \tilde{x}) - \mathfrak{F}(t, y, \tilde{y})| \leq L_{\mathfrak{F}} \left(|x - y| + |\tilde{x} - \tilde{y}|\right).$$

(H_2) For all $t \in \mathscr{J}$ and $x,\ y \in \mathbb{R}$, there exists a constant $L_\eta > 0$ such that the function $\eta : \mathbb{R} \times \mathscr{J} \to \mathscr{J}$ satisfies the following condition

$$|\eta(x, t) - \eta(y, t)| \leq L_\eta |x - y|.$$

(H_3) There exists a constant $\mathcal{N}_1 > 0$ such that

$$\max\left\{\left|\frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))}\right|, \left|\frac{a}{\mathfrak{F}(T, a, x(\eta(a, T)))}\right|\right\} \leq \mathcal{N}_1.$$

(H_4) For all $t \in \mathscr{J}$ and $x \in \mathbb{R}$ there exist continuous functions $\beta : \mathscr{J} \to (0, +\infty)$ and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$|\mathcal{G}(t, x(t), x(\xi(x(t), t)))| \leq \beta(t)\gamma(\|x\|) \ \text{ and } \ {}_0D_t^{-q}\beta \in C(\mathscr{J}, \mathbb{R}^+).$$

2.3.2 Main Result

Theorem 2.3.1. *Assume that the boundary value problem* (2.1.1) *satisfies the hypotheses* (H_1)-(H_4)*. Further, if*

$$\frac{F_0\left[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)}T^q\right]}{1 - L_A\left[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)}T^q\right]} \leq \rho, \tag{2.3.1}$$

where,

$$L_A\left[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)}T^q\right] < 1, \tag{2.3.2}$$

also $\rho > 0$, $L_A = L_{\mathfrak{F}}(2 + LL_\eta)$ *and* $F_0 = \sup_{t\in\mathscr{J}} \mathfrak{F}(t, 0, x(\eta(0, t)))$, *then* (2.1.1) *has a solution defined on* $\mathscr{J}$.

Proof. Let, $\mathcal{X} = C(\mathscr{J}, \mathbb{R})$ and define the subset $S \subset \mathcal{X}$ is given by

$$S = \{x \in \mathcal{X} | \ \ \|x\| \leq \rho\}$$

and ρ satisfies (2.3.1). It is straightforward to verify that S is a closed, convex and bounded subset of the Banach algebra $\mathcal{X}$.

By applying Lemma 2.2.2, we derive the following equivalent integral equation for the considered problem (2.1.1)

$$x(t) = \mathfrak{F}(t, x(t), x(\eta(x(t), t))) \Bigg[\frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \mathcal{G}(\tau, x(\tau), x(\xi(x(\tau), \tau))) d\tau + \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))} + \frac{1}{T} \Bigg\{ \frac{a}{\mathfrak{F}(T, a, x(\eta(a, T)))} - \frac{1}{\Gamma(q)} \int_0^T (T-\tau)^{q-1} \mathcal{G}(\tau, x(\tau), x(\xi(x(\tau), \tau))) d\tau - \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))} \Bigg\} t \Bigg], \quad t \in \mathscr{J}. \tag{2.3.3}$$

Consider two operators $A : \mathcal{X} \to \mathcal{X}$ and $B : S \to \mathcal{X}$ such that,

$$Ax(t) = \mathfrak{F}(t, x(t), x(\eta(x(t), t))), \quad t \in \mathscr{J}, \tag{2.3.4}$$

and

$$Bx(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \mathcal{G}(\tau, x(\tau), x(\xi(x(\tau), \tau))) d\tau + \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))} + \frac{1}{T} \Bigg\{ \frac{a}{\mathfrak{F}(T, a, x(\eta(a, T)))} - \frac{1}{\Gamma(q)} \int_0^T (T-\tau)^{q-1} \mathcal{G}(\tau, x(\tau), x(\xi(x(\tau), \tau))) d\tau - \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))} \Bigg\} t, \quad t \in \mathscr{J}. \tag{2.3.5}$$

Therefore, from (2.3.3) we get,

$$x(t) = Ax(t)Bx(t), \quad t \in \mathscr{J}.$$

Our next objective is to demonstrate that the operators A and B meet the criteria of the Theorem 1.6.20.

Step 1: The first step is to establish condition (i) of Theorem 1.6.20. For that we have to show that with a specific Lipschitz constant the operator A is Lipschitz on $\mathcal{X}$.

Let $x, y \in \mathcal{X}$. Using (2.3.4) and hypothesis (H_1), (H_2) we get,

$$|Ax(t) - Ay(t)| = |\mathfrak{F}(t, x(t), x(\eta(x(t), t))) - \mathfrak{F}(t, y(t), y(\eta(y(t), t)))| \leq L_A \|x - y\|, \tag{2.3.6}$$

for all $t \in \mathscr{J}$, where, $L_A = L_{\mathfrak{F}}(2 + LL_\eta)$, and x has to satisfy the following criteria

$$|x(\eta(x, t)) - x(\eta(y, t))| \leq L|\eta(x, t) - \eta(y, t)|.$$

After taking supremum over the interval $\mathscr{J}$ in (2.3.6) we get,

$$\|Ax - Ay\| \leq L_A \|x - y\|.$$

Thus, the operator A is Lipschitz on $\mathcal{X}$ with a Lipschitz constant L_A.

Step 2: To prove condition (ii) of Theorem 1.6.20, i.e. the operator B is completely continuous on S, we first establish the continuity of B on S. For that, let us consider $\{x_n\} \in S$ be a sequence converging to $x \in S$. By applying the Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \Bigg[\frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \mathcal{G}(\tau, x_n(\tau), x_n(\xi(x_n(\tau), \tau))) d\tau \\
&+ \frac{\phi(x_n)}{\mathfrak{F}(0, \phi(x_n), x_n(\eta(\phi(x_n), 0)))} + \frac{1}{T} \Bigg\{ \frac{a}{\mathfrak{F}(T, a, x_n(\eta(a, T)))} \\
&- \frac{1}{\Gamma(q)} \int_0^T (T-\tau)^{q-1} \mathcal{G}(\tau, x_n(\tau), x_n(\xi(x_n(\tau), \tau))) d\tau \\
&- \frac{\phi(x_n)}{\mathfrak{F}(0, \phi(x_n), x_n(\eta(\phi(x_n), 0)))} \Bigg\} t \Bigg] \\
&= \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \lim_{n\to\infty} \mathcal{G}(\tau, x_n(\tau), x_n(\xi(x_n(\tau), \tau))) d\tau \\
&+ \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))} + \frac{1}{T} \Bigg\{ \frac{a}{\mathfrak{F}(T, a, x(\eta(a, T)))} \\
&- \frac{1}{\Gamma(q)} \int_0^T (T-\tau)^{q-1} \lim_{n\to\infty} \mathcal{G}(\tau, x_n(\tau), x_n(\xi(x_n(\tau), \tau))) d\tau \\
&- \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))} \Bigg\} t \\
&= \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \mathcal{G}(\tau, x(\tau), x(\xi(x(\tau), \tau))) d\tau + \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))} \\
&+ \frac{1}{T} \Bigg\{ \frac{a}{\mathfrak{F}(T, a, x(\eta(a, T)))} - \frac{1}{\Gamma(q)} \int_0^T (T-\tau)^{q-1} \mathcal{G}(\tau, x(\tau), x(\xi(x(\tau), \tau))) d\tau \\
&- \frac{\phi(x)}{\mathfrak{F}(0, \phi(x), x(\eta(\phi(x), 0)))} \Bigg\} t \\
&= Bx(t),
\end{aligned}$$

for all $t \in \mathscr{J}$. Thus, the operator B is continuous on S.

Next, to establish that the operator B is compact on S, it suffices to show that $B(S)$

is uniformly bounded and equi-continuous in $\mathcal{X}$. Using (H_3) and (H_4) we get,

$$\begin{aligned}|Bx(t)| &= \bigg|\frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}\mathcal{G}(\tau,x(\tau),x(\xi(x(\tau),\tau)))d\tau + \frac{\phi(x)}{\mathfrak{F}(0,\phi(x),x(\eta(\phi(x),0)))} \\ &+\frac{1}{T}\bigg\{\frac{a}{\mathfrak{F}(T,a,x(\eta(a,T)))} - \frac{1}{\Gamma(q)}\int_0^T (T-\tau)^{q-1}\mathcal{G}(\tau,x(\tau),x(\xi(x(\tau),\tau)))d\tau \\ &- \frac{\phi(x)}{\mathfrak{F}(0,\phi(x),x(\eta(\phi(x),0)))}\bigg\}t\bigg| \\ &\leq 3\mathcal{N}_1 + \frac{\|\beta\|\gamma(\rho)}{\Gamma(q)}\bigg|\int_0^t (t-\tau)^{q-1}d\tau\bigg| + \frac{\|\beta\|\gamma(\rho)}{\Gamma(q)}\bigg|\int_0^T (T-\tau)^{q-1}d\tau\bigg| \\ &\leq 3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)}T^q,\end{aligned}$$

for all $t \in \mathscr{J}$. Taking supremum of the above inequality over the interval $\mathscr{J}$ we get,

$$\|Bx\| \leq 3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)}T^q,$$

for all $x \in S$. Hence, B is uniformly bounded on S.

Next, for any $x \in S$ and $t_1 < t_2$, where $t_1, t_2 \in \mathscr{J}$ we have,

$$\begin{aligned}|Bx(t_2)-Bx(t_1)| &= \bigg|\frac{1}{\Gamma(q)}\int_0^{t_2} (t_2-\tau)^{q-1}\mathcal{G}(\tau,x(\tau),x(\xi(x(\tau),\tau)))d\tau \\ &- \frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-\tau)^{q-1}\mathcal{G}(\tau,x(\tau),x(\xi(x(\tau),\tau)))d\tau + \frac{1}{T}\bigg\{\frac{a}{\mathfrak{F}(T,a,x(\eta(a,T)))} \\ &- \frac{1}{\Gamma(q)}\int_0^T (T-\tau)^{q-1}\mathcal{G}(\tau,x(\tau),x(\xi(x(\tau),\tau)))d\tau \\ &- \frac{\phi(x)}{\mathfrak{F}(0,\phi(x),x(\eta(\phi(x),0)))}\bigg\}(t_2-t_1)\bigg| \\ &\leq \frac{\|\beta\|\gamma(\rho)}{\Gamma(q)}\bigg|\int_0^{t_1} [(t_2-\tau)^{q-1}-(t_1-\tau)^{q-1}]d\tau\bigg| \\ &+ \frac{\|\beta\|\gamma(\rho)}{\Gamma(q)}\bigg|\int_{t_1}^{t_2} (t_2-\tau)^{q-1}d\tau\bigg| + \bigg|\frac{1}{T}\bigg\{\frac{a}{\mathfrak{F}(T,a,x(\eta(a,T)))} \\ &- \frac{1}{\Gamma(q)}\int_0^T (T-\tau)^{q-1}\mathcal{G}(\tau,x(\tau),x(\xi(x(\tau),\tau)))d\tau \\ &- \frac{\phi(x)}{\mathfrak{F}(0,\phi(x),x(\eta(\phi(x),0)))}\bigg\}(t_2-t_1)\bigg|.\end{aligned}$$

As $t_2 - t_1 \to 0$, the RHS of the above inequality converges to 0 independently of $x \in S$. Therefore, $B(S)$ is an equi-continuous set in $\mathcal{X}$.

Since B is uniformly bounded and equi-continuous, it follows from the Arzelá-Ascoli theorem that B is a compact operator on S.

Step 3: Now we prove the condition (iii) of Theorem 1.6.20. Let two arbitrary elements $x \in \mathcal{X}$ and $y \in S$ such that $x = AxBy$. Then by using the hypotheses we get,

$$
\begin{aligned}
|x(t)| &= |Ax(t)||By(t)| \\
&= |\mathfrak{F}(t, x(t), x(\eta(x(t), t)))| \Bigg| \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \mathcal{G}(\tau, y(\tau), y(\xi(y(\tau), \tau))) d\tau \\
&+ \frac{\phi(y)}{\mathfrak{F}(0, \phi(y), y(\eta(\phi(y), 0)))} + \frac{1}{T} \Bigg\{ \frac{a}{\mathfrak{F}(T, a, y(\eta(a, T)))} \\
&- \frac{1}{\Gamma(q)} \int_0^T (T-\tau)^{q-1} \mathcal{G}(\tau, y(\tau), y(\xi(y(\tau), \tau))) d\tau - \frac{\phi(y)}{\mathfrak{F}(0, \phi(y), y(\eta(\phi(y), 0)))} \Bigg\} t \Bigg| \\
&\le \Big[|\mathfrak{F}(t, x(t), x(\eta(x(t), t))) - \mathfrak{F}(t, 0, x(\eta(0, t)))| + |\mathfrak{F}(t, 0, x(\eta(0, t)))| \Big] \\
&\times \left[3\mathcal{N}_1 + \frac{\|\beta\|\gamma(\rho)}{\Gamma(q)} \left| \int_0^t (t-\tau)^{q-1} d\tau \right| + \frac{\|\beta\|\gamma(\rho)}{\Gamma(q)} \left| \int_0^T (T-\tau)^{q-1} d\tau \right| \right] \\
&\le \Big[L_A |x(t)| + F_0 \Big] \left[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q)} T^q \right].
\end{aligned}
$$

Hence,

$$
|x(t)| \left\{ 1 - L_A \left[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)} T^q \right] \right\} \le F_0 \left[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)} T^q \right].
$$

Therefore,

$$
|x(t)| \le \frac{F_0 \Big[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)} T^q \Big]}{1 - L_A \Big[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)} T^q \Big]}.
$$

Taking supremum over t we get,

$$
\|x\| \le \frac{F_0 \Big[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)} T^q \Big]}{1 - L_A \Big[3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)} T^q \Big]} \le \rho.
$$

Hence, $x \in S$. This completes the proof of (iii) of Theorem 1.6.20.

Step 4: For the final steps of the theorem we have,

$$
\mathcal{M} = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \le 3\mathcal{N}_1 + \frac{2\|\beta\|\gamma(\rho)}{\Gamma(q+1)} T^q.
$$

Thus, $\alpha\mathcal{M} < 1$, where $\alpha = L_A$. Therefore, the condition (iv) of Theorem 1.6.20 is satisfied.

Thus, all the conditions of Theorem 1.6.20 are satisfied. Hence, we can conclude that the operator equation $AxBx = x$ has a solution in S. Thus, the considered problem (2.1.1) has a solution defined on the interval $[0, T]$. This completes the proof.

$\square$