CHAPTER 3

Study of a Nonlinear Volterra-Fredholm Type Hybrid Fractional Differential Equation

3.1 Introduction

Volterra-Fredholm differential equations are a special class of integral-differential equations that combine characteristics of both Volterra and Fredholm integral operators. These equations are named after the mathematicians Vito Volterra and Erik Ivar Fredholm, who made significant contributions to the study of integral equations. These equations typically involve an unknown function that appears both under a derivative and within integral terms of different types [61, 68, 95, 97, 103]. The Volterra integral is usually defined with variable upper limits of integration, reflecting causal dependencies in physical and biological systems. In contrast, the Fredholm integral term involves fixed limits, capturing global interactions over a given domain. Such equations arise in various applications, including viscoelasticity, fluid dynamics, population dynamics, and thermal diffusion processes etc.

In this chapter we study the existence of solution for the following nonlinear nonlocal hybrid Volterra-Fredholm pantograph type fractional differential equation

$$\begin{aligned}
&{}_0^C\mathcal{D}_t^q\left[\frac{u(t)-\mathcal{G}(t,u(t),u(\lambda t))}{\mathfrak{F}(t,u(t),\int_0^t k_1(t,\tau)h_1(\tau,u(\tau))d\tau,\int_0^T k_2(t,\tau)h_2(\tau,u(\tau))d\tau)}\right]=\mathcal{W}(t,u(t),u(\lambda t)),\\
&\qquad\qquad\qquad\qquad t\in\mathscr{J}=[0,T],\\
&u(0)=\phi(u),\quad u(T)=a.
\end{aligned}\tag{3.1.1}$$

Here, ${}_0^C\mathcal{D}_t^q$ denotes the Caputo fractional derivative of order $1<q<2$ and $0<\lambda<1$. Assume, $\mathfrak{F}\in C(\mathscr{J}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$; $\mathcal{G},\mathcal{W}\in C(\mathscr{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and for $i=1,2$, the functions $h_i:\mathscr{J}\times\mathbb{R}\to\mathbb{R}$ are continuous, also $\phi:C(\mathscr{J},\mathbb{R})\to\mathbb{R}$ be a continuous

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and bounded function.

The pantograph equation is a functional differential equation with a proportional delay, where the delay varies with the current time instead of remaining constant. This key feature differentiates it from traditional delay differential equations, introducing nonlinear behaviour and added complexity, which makes finding analytical solutions more challenging. Over time, its applications have expanded to diverse fields, including electrodynamics, quantum dot lasers, material modelling, and control systems. In mathematics and physics, pantograph equations play a crucial role in areas such as number theory, probability, and quantum mechanics. Due to their significance, researchers have generalized these equations into various forms, exploring their solvability using both theoretical and numerical methods. While extensive work has been conducted on classical pantograph equations, fractional versions remain relatively unexplored, with only a few contributions addressing their properties and solutions. In 2013, Balachandran et al. [24] initiated a comprehensive overview of various types of pantograph equations and explored their existence by using fractional calculus and fixed point theorems. Nisar [96] examined the existence and uniqueness of integral solutions for a Hilfer pantograph model with a nonlocal integral condition. He employed the Leray-Schauder fixed point theorem to establish the existence of solutions and the Banach contraction principle to to study the uniqueness. Numerous researchers have contributed to the study of pantograph equations, exploring various aspects and methodologies [4, 5, 18, 116, 122].

We organise the chapter as follows: In Section 3.2, we focuse on deriving the integral solution. In Section 3.3, the existence of a solution for the boundary value problem (3.1.1) is examined using a fixed point theorem under mixed Lipschitz and Carathéodory conditions. Finally, in Section 3.4, we present an illustrative example to support the main result.

3.2 Preliminaries

Lemma 3.2.1. *Let $\mathcal{W} \in C(\mathscr{J}, \mathbb{R})$ then for $1 < q < 2$ the fractional hybrid differential equation*

$$\begin{gathered} {}_0^C D_t^q \left[\frac{u(t) - \mathcal{G}((t, u(t), u(\lambda t))}{\mathfrak{F}\big(t, u(t), \int_0^t k_1(t,\tau) h_1(\tau, u(\tau)) d\tau, \int_0^T k_2(t,\tau) h_2(\tau, u(\tau)) d\tau\big)} \right] = \mathcal{W}(t), \quad t \in \mathscr{J}, \\ u(0) = \phi(u), \qquad u(T) = a \end{gathered} \tag{3.2.1}$$

has an integral solution

$$
\begin{aligned}
u(t) &= \mathfrak{F}\Big(t, u(t), \int_0^t k_1(t,\tau)h_1(\tau,u(\tau))d\tau, \int_0^T k_2(t,\tau)h_2(\tau,u(\tau))d\tau\Big) \\
&\times \Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathcal{W}(s)ds + \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}\big(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau\big)} \\
&+ \frac{1}{T}\Bigg\{\frac{a-\mathcal{G}((T,a,u(\lambda T))}{\mathfrak{F}\big(T,a,\int_0^T k_1(T,\tau)h_1(\tau,u(\tau))d\tau, \int_0^T k_2(T,\tau)h_2(\tau,u(\tau))d\tau\big)} \\
&- \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}\big(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau\big)} - \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\mathcal{W}(s)ds\Bigg\}t\Bigg] \\
&+ \mathcal{G}((t,u(t),u(\lambda t)), \quad t \in \mathscr{J}.
\end{aligned} \tag{3.2.2}
$$

Proof. The equivalent integral form of (3.2.1) is obtained by applying Lemma 2.2.1.

$$
\begin{aligned}
u(t) &= \mathfrak{F}\Big(t, u(t), \int_0^t k_1(t,\tau)h_1(\tau,u(\tau))d\tau, \int_0^T k_2(t,\tau)h_2(\tau,u(\tau))d\tau\Big) \\
&\times \Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathcal{W}(s)ds + c_1 + c_2 t\Bigg] + \mathcal{G}(t,u(t),u(\lambda t)),
\end{aligned} \tag{3.2.3}
$$

where c_1 and $c_2 \in \mathbb{R}$.

By applying the conditions of the problem (3.2.1) into (3.2.3), we obtain

$$
c_1 = \frac{\phi(u)-\mathcal{G}(0,\phi(u),\phi(u))}{\mathfrak{F}\big(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau\big)}
$$

and

$$
\begin{aligned}
c_2 = \frac{1}{T}\Bigg\{&\frac{a-\mathcal{G}((T,a,u(\lambda T))}{\mathfrak{F}\big(T,a,\int_0^T k_1(T,\tau)h_1(\tau,u(\tau))d\tau, \int_0^T k_2(T,\tau)h_2(\tau,u(\tau))d\tau\big)} \\
&- \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}\big(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau\big)} - \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\mathcal{W}(s)ds\Bigg\}.
\end{aligned}
$$

Substituting the values of c_1 and c_2 into (3.2.3), we obtain

$$
\begin{aligned}
u(t) &= \mathfrak{F}\Big(t, u(t), \int_0^t k_1(t,\tau)h_1(\tau,u(\tau))d\tau, \int_0^T k_2(t,\tau)h_2(\tau,u(\tau))d\tau\Big) \\
&\times \Bigg[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathcal{W}(s)ds + \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau)} \\
&+ \frac{1}{T}\Bigg\{\frac{a-\mathcal{G}((T,a,u(\lambda T))}{\mathfrak{F}(T,a,\int_0^T k_1(T,\tau)h_1(\tau,u(\tau))d\tau, \int_0^T k_2(T,\tau)h_2(\tau,u(\tau))d\tau)} \\
&- \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau)} - \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}\mathcal{W}(s)ds\Bigg\}t\Bigg] \\
&+ \mathcal{G}((t,u(t),u(\lambda t)), \quad t \in \mathscr{J}.
\end{aligned}
$$

□

3.3 Existence Result

Consider the space of continuous real-valued functions $\mathcal{X} = C(\mathscr{J}, \mathbb{R})$ defined on $\mathscr{J} = [0, T]$, equipped with the supremum norm

$$\|y\| = \sup_{t\in\mathscr{J}}|y(t)|.$$

Additionally, a multiplication operation in $\mathcal{X}$ is defined as

$$(xy)(t) = x(t)y(t), \quad \forall t \in \mathscr{J}.$$

It is clear that $\mathcal{X}$, under the given norm and multiplication, forms a Banach algebra.

In the proof of our main result, we employ Dhage's fixed point theorem for three operators (Theorem 1.6.21) within this Banach algebra framework.

3.3.1 Hypotheses

Now, we introduce a set of hypotheses that will be utilized in proving the main result.

(H_1) For all $t \in \mathscr{J}$ and $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$ there exists a constant $L_{\mathfrak{F}} > 0$ such that the function $\mathfrak{F} : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ satisfies

$$|\mathfrak{F}(t, x_1, y_1, z_1) - \mathfrak{F}(t, x_2, y_2, z_2)| \leq L_{\mathfrak{F}}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|).$$

(H_2) For each $i = 1, 2$, there exists a constant $L_{h_i} > 0$ such that, for all $t \in \mathscr{J}$ and $x, y \in \mathbb{R}$ the function $h_i : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|h_i(t, x) - h_i(t, y)| \leq L_{h_i}|x - y|.$$

(H_3) There exists a constant $\mathcal{K} > 0$ such that

$$\max_{t,\tau\in\mathscr{J}} \big\{|k_1(t, \tau)|, |k_2(t, \tau)|\big\} \leq \mathcal{K}.$$

(H_4) There exists a constant $L_{\mathcal{G}} > 0$ such that, for all $t \in \mathscr{J}$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}$ the function $\mathcal{G} : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|\mathcal{G}((t, x_1, y_1) - \mathcal{G}((t, x_2, y_2)| \leq L_{\mathcal{G}}\big(|x_1 - x_2| + |y_1 - y_2|\big).$$

(H_5) There exist constants $\mathcal{X}_0, \mathcal{X}_1 > 0$ such that

$$\left|\frac{\phi(u) - \mathcal{G}((0, \phi(u), \phi(u))}{\mathfrak{F}(0, \phi(u), 0, \int_0^T k_2(0, \tau)h_2(\tau, u(\tau))d\tau)}\right| \leq \mathcal{X}_0,$$

$$\left|\frac{a - \mathcal{G}((T, a, u(\lambda T))}{\mathfrak{F}(T, a, \int_0^T k_1(T, \tau)h_1(\tau, u(\tau))d\tau, \int_0^T k_2(T, \tau)h_2(\tau, u(\tau))d\tau)}\right| \leq \mathcal{X}_1.$$

(H_6) For all $t \in \mathscr{J}$ and $x \in \mathbb{R}$ there exist a continuous function $\eta : \mathscr{J} \to (0, +\infty)$ and a non-decreasing continuous function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|\mathcal{W}(t, x(t), x(\lambda t))| \leq \eta(t)\Psi(\|x\|) \text{ and } {}_0I_t^q\eta \in C(J, \mathbb{R}^+).$$

3.3.2 Main Result

Theorem 3.3.1. *Assume that the considered problem* (3.1.1) *satisfies the hypotheses* (H_1)-(H_6). *Further, if*

$$\frac{F_0\Big[2\mathcal{X}_0+\mathcal{X}_1+\frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\Big]+G_0}{1-L_{\mathfrak{F}}(1+\mathcal{K}L_{h_1}T+\mathcal{K}L_{h_2}T)\Big[2\mathcal{X}_0+\mathcal{X}_1+\frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\Big]-2L_{\mathcal{G}}}\leq\mathfrak{r}, \tag{3.3.1}$$

where,

$$L_{\mathfrak{F}}(1+\mathcal{K}L_{h_1}T+\mathcal{K}L_{h_2}T)\Big[2\mathcal{X}_0+\mathcal{X}_1+\frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\Big]+2L_{\mathcal{G}}<1, \tag{3.3.2}$$

also $\mathfrak{r}>0$ *and* $F_0=\sup_{t\in\mathscr{J}}\Big|\mathfrak{F}(t,0,\int_0^t k_1(t,\tau)h_1(\tau,0)d\tau,\int_0^T k_2(t,\tau)h_2(\tau,0)d\tau)\Big|$, $G_0=\sup_{t\in\mathscr{J}}|\mathcal{G}((t,0,0)|$ *then* (3.1.1) *has a solution defined on* $\mathscr{J}$.

Proof. Define the subset S of $\mathcal{X}$ as

$$S=\{x\in\mathcal{X}\mid\|x\|\leq\mathfrak{r}\},$$

where $\mathfrak{r}$ satisfies condition (3.3.1).

It is clear that S is a closed, convex, and bounded subset of the Banach algebra $\mathcal{X}$.

Applying Lemma 3.2.1, we obtain the following equivalent integral equation corresponding to (3.1.1)

$$\begin{aligned}
u(t)=\,&\mathfrak{F}\Big(t,u(t),\int_0^t k_1(t,\tau)h_1(\tau,u(\tau))d\tau,\int_0^T k_2(t,\tau)h_2(\tau,u(\tau))d\tau\Big)\\
&\times\Bigg[\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\mathcal{W}(s,u(s),u(\lambda s))ds+\frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau)}\\
&+\frac{1}{T}\Bigg\{\frac{a-\mathcal{G}((T,a,u(\lambda T))}{\mathfrak{F}(T,a,\int_0^T k_1(T,\tau)h_1(\tau,u(\tau))d\tau,\int_0^T k_2(T,\tau)h_2(\tau,u(\tau))d\tau)}\\
&-\frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau)}-\frac{1}{\Gamma(q)}\int_0^T(T-s)^{q-1}\mathcal{W}(s,u(s),u(\lambda s))ds\Bigg\}t\Bigg]\\
&+\mathcal{G}((t,u(t),u(\lambda t)),\quad t\in\mathscr{J}.
\end{aligned} \tag{3.3.3}$$

Now we consider three operators $A:\mathcal{X}\to\mathcal{X}$, $B:S\to\mathcal{X}$ and $C:\mathcal{X}\to\mathcal{X}$ such that,

$$Au(t)=\mathfrak{F}\Big(t,u(t),\int_0^t k_1(t,\tau)h_1(\tau,u(\tau))d\tau,\int_0^T k_2(t,\tau)h_2(\tau,u(\tau))d\tau\Big),\quad t\in\mathscr{J}, \tag{3.3.4}$$

$$
\begin{aligned}
Bu(t) = {} & \frac{1}{\Gamma(q)} \int\limits_0^t (t-s)^{q-1} \mathcal{W}(s, u(s), u(\lambda s)) ds + \frac{\phi(u) - \mathcal{G}((0, \phi(u), \phi(u))}{\mathfrak{F}(0, \phi(u), 0, \int_0^T k_2(0,\tau) h_2(\tau, u(\tau)) d\tau)} \\
& + \frac{1}{T} \Bigg\{ \frac{a - \mathcal{G}((T, a, u(\lambda T))}{\mathfrak{F}(T, a, \int_0^T k_1(T,\tau) h_1(\tau, u(\tau)) d\tau, \int_0^T k_2(T,\tau) h_2(\tau, u(\tau)) d\tau)} \\
& - \frac{\phi(u) - \mathcal{G}((0, \phi(u), \phi(u))}{\mathfrak{F}(0, \phi(u), 0, \int_0^T k_2(0,\tau) h_2(\tau, u(\tau)) d\tau)} - \frac{1}{\Gamma(q)} \int\limits_0^T (T-s)^{q-1} \mathcal{W}(s, u(s), u(\lambda s)) ds \Bigg\} t, \\
& t \in \mathscr{J},
\end{aligned} \tag{3.3.5}
$$

and

$$
Cu(t) = \mathcal{G}((t, u(t), u(\lambda t)), \quad t \in \mathscr{J}. \tag{3.3.6}
$$

Therefore, (3.3.3) can be written as,

$$
u(t) = Au(t) Bu(t) + Cu(t), \quad t \in \mathscr{J}.
$$

The next objective is to prove that the operators A, B, and C fulfill the requirements of Theorem 1.6.21.

Step 1: We begin by establishing that the operators A and C are Lipschitzian on $\mathcal{X}$ with a specific Lipschitz constant.

Consider $u, v \in \mathcal{X}$, for all $t \in \mathscr{J}$ from (3.3.4) and (H_1)-(H_3) we get,

$$
\begin{aligned}
|Au(t) - Av(t)| &= \Big| \mathfrak{F}\Big(t, u(t), \int_0^t k_1(t,\tau) h_1(\tau, u(\tau)) d\tau, \int_0^T k_2(t,\tau) h_2(\tau, u(\tau)) d\tau\Big) \\
&\quad - \mathfrak{F}\Big(t, v(t), \int_0^t k_1(t,\tau) h_1(\tau, v(\tau)) d\tau, \int_0^T k_2(t,\tau) h_2(\tau, v(\tau)) d\tau\Big) \Big| \\
&\leq L_{\mathfrak{F}} \{ \|u - v\| + \mathcal{K} L_{h_1} \|u - v\| T + \mathcal{K} L_{h_2} \|u - v\| T \} \\
&= L_{\mathfrak{F}} (1 + \mathcal{K} L_{h_1} T + \mathcal{K} L_{h_2} T) \|u - v\|.
\end{aligned}
$$

After taking supremum over $\mathscr{J}$ we obtain,

$$
\|Au - Av\| \leq L_{\mathfrak{F}} (1 + \mathcal{K} L_{h_1} T + \mathcal{K} L_{h_2} T) \|u - v\|.
$$

Thus, A is Lipschitzian on $\mathcal{X}$ with a Lipschitz constant $L_{\mathfrak{F}}(1 + \mathcal{K} L_{h_1} T + \mathcal{K} L_{h_2} T)$.

Likewise, for any $u, v \in \mathcal{X}$ and $t \in \mathscr{J}$, applying (3.3.6) along with assumption (H_4), we obtain

$$
\begin{aligned}
|Cu(t) - Cv(t)| &= |\mathcal{G}((t, u(t), u(\lambda t)) - \mathcal{G}((t, v(t), v(\lambda t))| \\
&\leq L_{\mathcal{G}} \{ |u(t) - v(t)| + |u(\lambda t) - v(\lambda t)| \}.
\end{aligned}
$$

After taking supremum over $\mathscr{J}$ we get,

$$
\|Cu - Cv\| \leq 2 L_{\mathcal{G}} \|u - v\|.
$$

Therefore, C is Lipschitzian on $\mathcal{X}$ with a Lipschitz constant $2L_{\mathcal{G}}$.

Step 2: Next, we aim to show that B is completely continuous on S. To achieve this, we first establish the continuity of the operator B on S. Let $\{u_n\}$ be a sequence in S that converges to a point $u \in S$. Then, for all $t \in \mathscr{J}$, applying the Lebesgue's dominated convergence theorem, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} Bu_n(t)\\
&= \lim_{n\to\infty}\Bigg[\frac{1}{\Gamma(q)}\int\limits_0^t (t-s)^{q-1}\mathcal{W}(s,u_n(s),u_n(\lambda s))ds\\
&+ \frac{\phi(u_n)-\mathcal{G}((0,\phi(u_n),\phi(u_n))}{\mathfrak{F}(0,\phi(u_n),0,\int_0^T k_2(0,\tau)h_2(\tau,u_n(\tau))d\tau)}\\
&+ \frac{1}{T}\Bigg\{\frac{a-\mathcal{G}((T,a,u_n(\lambda T))}{\mathfrak{F}(T,a,\int_0^T k_1(T,\tau)h_1(\tau,u_n(\tau))d\tau,\int_0^T k_2(T,\tau)h_2(\tau,u_n(\tau))d\tau)}\\
&- \frac{\phi(u_n)-\mathcal{G}((0,\phi(u_n),\phi(u_n))}{\mathfrak{F}(0,\phi(u_n),0,\int_0^T k_2(0,\tau)h_2(\tau,u_n(\tau))d\tau)} - \frac{1}{\Gamma(q)}\int\limits_0^T (T-s)^{q-1}\mathcal{W}(s,u_n(s),u_n(\lambda s))ds\Bigg\}t\Bigg]\\
&= \frac{1}{\Gamma(q)}\int\limits_0^t (t-s)^{q-1}\lim_{n\to\infty}\mathcal{W}(s,u_n(s),u_n(\lambda s))ds\\
&+ \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}(0,\phi(u),0,\int_0^T k_2(0,\tau)\lim\limits_{n\to\infty} h_2(\tau,u_n(\tau))d\tau)}\\
&+ \frac{1}{T}\Bigg\{\frac{a-\mathcal{G}((T,a,u(\lambda T))}{\mathfrak{F}(T,a,\int_0^T k_1(T,\tau)\lim\limits_{n\to\infty} h_1(\tau,u_n(\tau))d\tau,\displaystyle\int_0^T k_2(T,\tau)\lim\limits_{n\to\infty} h_2(\tau,u_n(\tau))d\tau)}\\
&- \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}(0,\phi(u),0,\int_0^T k_2(0,\tau)\lim\limits_{n\to\infty} h_2(\tau,u_n(\tau))d\tau)}\\
&- \frac{1}{\Gamma(q)}\int\limits_0^T (T-s)^{q-1}\lim_{n\to\infty}\mathcal{W}(s,u_n(s),u_n(\lambda s))ds\Bigg\}t\\
&= \frac{1}{\Gamma(q)}\int\limits_0^t (t-s)^{q-1}\mathcal{W}(s,u(s),u(\lambda s))ds + \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau)}\\
&+ \frac{1}{T}\Bigg\{\frac{a-\mathcal{G}((T,a,u(\lambda T))}{\mathfrak{F}(T,a,\int_0^T k_1(T,\tau)h_1(\tau,u(\tau))d\tau,\int_0^T k_2(T,\tau)h_2(\tau,u(\tau))d\tau)}\\
&- \frac{\phi(u)-\mathcal{G}((0,\phi(u),\phi(u))}{\mathfrak{F}(0,\phi(u),0,\int_0^T k_2(0,\tau)h_2(\tau,u(\tau))d\tau)} - \frac{1}{\Gamma(q)}\int\limits_0^T (T-s)^{q-1}\mathcal{W}(s,u(s),u(\lambda s))ds\Bigg\}t\\
&= Bu(t).
\end{aligned}
$$

Thus, the operator B is continuous on S.

To establish the compactness of the operator B on S, it is necessary to demonstrate that $B(S)$ is both uniformly bounded and equicontinuous in $\mathcal{X}$. By applying assumptions (H_5) and (H_6), we obtain the following for all $t \in \mathscr{J}$.

$$
\begin{aligned}
|Bu(t)| = \Bigg| &\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathcal{W}(s, u(s), u(\lambda s)) ds + \frac{\phi(u) - \mathcal{G}((0, \phi(u), \phi(u))}{\mathfrak{F}(0, \phi(u), 0, \int_0^T k_2(0,\tau) h_2(\tau, u(\tau)) d\tau)} \\
&+ \frac{1}{T} \Bigg\{ \frac{a - \mathcal{G}((T, a, u(\lambda T))}{\mathfrak{F}(T, a, \int_0^T k_1(T,\tau) h_1(\tau, u(\tau)) d\tau, \int_0^T k_2(T,\tau) h_2(\tau, u(\tau)) d\tau)} \\
&- \frac{\phi(u) - \mathcal{G}((0, \phi(u), \phi(u))}{\mathfrak{F}(0, \phi(u), 0, \int_0^T k_2(0,\tau) h_2(\tau, u(\tau)) d\tau)} \\
&- \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \mathcal{W}(s, u(s), u(\lambda s)) ds \Bigg\} t \Bigg| \\
&\leq 2\mathcal{X}_0 + \mathcal{X}_1 + \frac{\|\eta\| \Psi(\mathfrak{r})}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} ds \right| + \frac{\|\eta\| \Psi(\mathfrak{r})}{\Gamma(q)} \left| \int_0^T (T-s)^{q-1} ds \right| \\
&\leq 2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\| \Psi(\mathfrak{r})}{\Gamma(q+1)} T^q.
\end{aligned}
$$

For all $u \in S$ taking supremum over $\mathscr{J}$ we get,

$$
\|Bu\| \leq 2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\| \Psi(\mathfrak{r})}{\Gamma(q+1)} T^q.
$$

Hence, B is uniformly bounded on S.

Next, for any $u \in S$ and $t_1, t_2 \in \mathscr{J}$ such that $t_1 < t_2$ we get,

$$
\begin{aligned}
|Bu(t_2) - Bu(t_1)| = \Bigg| &\frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} \mathcal{W}(s, u(s), u(\lambda s)) ds \\
&- \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} \mathcal{W}(s, u(s), u(\lambda s)) ds \\
&+ \frac{1}{T} \Bigg\{ \frac{a - \mathcal{G}((T, a, u(\lambda T))}{\mathfrak{F}(T, a, \int_0^T k_1(T,\tau) h_1(\tau, u(\tau)) d\tau, \int_0^T k_2(T,\tau) h_2(\tau, u(\tau)) d\tau)} \\
&- \frac{\phi(u) - \mathcal{G}((0, \phi(u), \phi(u))}{\mathfrak{F}(0, \phi(u), 0, \int_0^T k_2(0,\tau) h_2(\tau, u(\tau)) d\tau)} \\
&- \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \mathcal{W}(s, u(s), u(\lambda s)) ds \Bigg\} (t_2 - t_1) \Bigg| \\
&\leq \frac{\|\eta\| \Psi(\mathfrak{r})}{\Gamma(q)} \left| \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] ds \right| + \frac{\|\eta\| \Psi(\mathfrak{r})}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_2-s)^{q-1} ds \right| \\
&+ \Bigg| \frac{1}{T} \Bigg\{ \frac{a - \mathcal{G}((T, a, u(\lambda T))}{\mathfrak{F}(T, a, \int_0^T k_1(T,\tau) h_1(\tau, u(\tau)) d\tau, \int_0^T k_2(T,\tau) h_2(\tau, u(\tau)) d\tau)}
\end{aligned}
$$

$$- \frac{\phi(u) - \mathcal{G}((0, \phi(u), \phi(u))}{\mathfrak{F}(0, \phi(u), 0, \int_0^T k_2(0,\tau)h_2(\tau, u(\tau))d\tau)}$$
$$- \frac{1}{\Gamma(q)} \int\limits_0^T (T-s)^{q-1}\mathcal{W}(s, u(s), u(\lambda s))ds \Big\}(t_2 - t_1)\Big|.$$

The RHS of the above inequality tends to 0 whenever $t_2 - t_1 \to 0$ without depending on $u \in S$. This confirms that, $B(S)$ is an equicontinuous set in $\mathcal{X}$.

Since B is both uniformly bounded and equicontinuous, the Arzelá-Ascoli theorem ensures that B is a compact operator on S.

Step 3: Next, we establish condition (iii) of Theorem 1.6.21. Let $u \in \mathcal{X}$ and $v \in S$ be arbitrary elements such that $u = AuBv + Cu$. By applying the given hypotheses, we obtain the following result.

$$\begin{aligned}
|u(t)| &= |Au(t)||Bv(t)| + |Cu(t)| \\
&= \left|\mathfrak{F}\Big(t, u(t), \int_0^t k_1(t,\tau)h_1(\tau, u(\tau))d\tau, \int_0^T k_2(t,\tau)h_2(\tau, u(\tau))d\tau\Big)\right| \\
&\times \Bigg|\frac{1}{\Gamma(q)} \int\limits_0^t (t-s)^{q-1}\mathcal{W}(s, v(s), v(\lambda s))ds + \frac{\phi(v) - \mathcal{G}((0, \phi(v), \phi(v))}{\mathfrak{F}(0, \phi(v), 0, \int_0^T k_2(0,\tau)h_2(\tau, v(\tau))d\tau)} \\
&+ \frac{1}{T}\Bigg\{\frac{a - \mathcal{G}((T, a, v(\lambda T))}{\mathfrak{F}(T, a, \int_0^T k_1(T,\tau)h_1(\tau, u(\tau))d\tau, \int_0^T k_2(T,\tau)h_2(\tau, v(\tau))d\tau)} \\
&- \frac{1}{\Gamma(q)} \int\limits_0^T (T-s)^{q-1}\mathcal{W}(s, v(s), v(\lambda s))ds \\
&- \frac{\phi(v) - \mathcal{G}((0, \phi(v), \phi(v))}{\mathfrak{F}(0, \phi(v), 0, \int_0^T k_2(0,\tau)h_2(\tau, v(\tau))d\tau)}\Bigg\}t\Bigg| + |\mathcal{G}((t, u(t), u(\lambda t))| \\
&\le \Bigg[\Bigg|\mathfrak{F}\Big(t, u(t), \int_0^t k_1(t,\tau)h_1(\tau, u(\tau))d\tau, \int_0^T k_2(t,\tau)h_2(\tau, u(\tau))d\tau\Big) \\
&- \mathfrak{F}\Big(t, 0, \int_0^t k_1(t,\tau)h_1(\tau, 0)d\tau, \int_0^T k_2(t,\tau)h_2(\tau, 0)d\tau\Big)\Bigg| \\
&+ \left|\mathfrak{F}\Big(t, 0, \int_0^t k_1(t,\tau)h_1(\tau, 0)d\tau, \int_0^T k_2(t,\tau)h_2(\tau, 0)d\tau\Big)\right|\Bigg] \\
&\times \left[2\mathcal{X}_0 + \mathcal{X}_1 + \frac{\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q)}\Big|\int\limits_0^t (t-s)^{q-1}ds\Big| + \frac{\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q)}\Big|\int\limits_0^T (T-s)^{q-1}ds\Big|\right] \\
&+ \big[|\mathcal{G}((t, u(t), u(\lambda t)) - \mathcal{G}((t,0,0) + \mathcal{G}((t,0,0)|\big] \\
&\le [L_{\mathfrak{F}}(1 + \mathcal{K}L_{h_1}T + \mathcal{K}L_{h_2}T)\|u\| + F_0]\left[2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\right] + 2L_{\mathcal{G}}\|u\| + G_0.
\end{aligned}$$

Thus taking supremum over t we get,

$$\|u\|\left\{1 - L_{\mathfrak{F}}(1 + \mathcal{K}L_{h_1}T + \mathcal{K}L_{h_2}T)\Big[2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\Big] - 2L_{\mathcal{G}}\right\}$$

$$\leq F_0\Big[2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\Big] + G_0.$$

Therefore,

$$\|u\| \leq \frac{F_0\Big[2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\Big] + G_0}{1 - L_{\mathfrak{F}}(1 + \mathcal{K}L_{h_1}T + \mathcal{K}L_{h_2}T)\Big[2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\Big] - 2L_{\mathcal{G}}} \leq \mathfrak{r}.$$

Hence, $u \in S$. This completes the proof.

Step 4: To prove the last condition of the Theorem 1.6.21 we have,

$$\mathcal{M} = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \leq 2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q.$$

By (3.3.2) we have, $\alpha\mathcal{M} + \beta < 1$ where, $\alpha = L_{\mathfrak{F}}(1 + \mathcal{K}L_{h_1}T + \mathcal{K}L_{h_2}T)$ and $\beta = 2L_{\mathcal{G}}$. This satisfies the last condition of the Theorem 1.6.21.

Therefore, all the conditions of Theorem 1.6.21 are fulfilled. Consequently, it follows that the operator equation $u = AuBu + Cu$ has a solution in S. Hence, the problem (3.1.1) has a solution defined on $\mathscr{J}$, thereby completing the proof.

□

3.4 Example

In this section, we consider the following example to illustrate the main result

$$\begin{aligned}
&{}_0^C\mathcal{D}_t^{3/2}\left[\frac{u(t) - \frac{1}{10^4}\{\sin u(t) + \sin u(\lambda t)\}}{2t + \frac{u(t)}{10^2} + \int_0^t \frac{e^{t+\tau}}{10^2(2 + |\sin u(\tau)|)}d\tau + \int_0^1 \frac{e^{t-\tau}}{10^2(2 + |\cos u(\tau)|)}d\tau}\right] \\
&= \frac{1}{10^3}\left(\cos u(t) + \cos u(\lambda t)\right), \quad t \in [0,1], \\
&u(0) = \frac{1}{10^4}\left(1 + \cos u(\gamma)\right), \quad \gamma \in (0,1), \\
&u(1) = \frac{1}{10^3}.
\end{aligned} \tag{3.4.1}$$

Assume $q = \frac{3}{2} \in (1,2)$ and $\lambda \in (0,1)$.
Here,

$$\begin{aligned}
&\mathfrak{F}\left(t, u(t), \int_0^t k_1(t,\tau)h_1(\tau, u(\tau))d\tau, \int_0^1 k_2(t,\tau)h_2(\tau, u(\tau))d\tau\right) \\
&= 2t + \frac{u(t)}{10^2} + \int_0^t \frac{e^{t+\tau}}{10^2(2 + |\sin u(\tau)|)}d\tau + \int_0^1 \frac{e^{t-\tau}}{10^2(2 + |\cos u(\tau)|)}d\tau.
\end{aligned}$$

And from hypothesis (H_3)

$$\max_{t,\tau\in[0,1]}\{|k_1(t,\tau)|, |k_2(t,\tau)|\} = \max_{t,\tau\in[0,1]}\left\{\left|\frac{e^{t+\tau}}{10^2}\right|, \left|\frac{e^{t-\tau}}{10^2}\right|\right\} \leq \frac{e^2}{10^2} = \mathcal{K}.$$

Now,

$$\begin{aligned}
&\Bigg|\mathfrak{F}\left(t, u(t), \int_0^t k_1(t,\tau)h_1(\tau, u(\tau))d\tau, \int_0^1 k_2(t,\tau)h_2(\tau, u(\tau))d\tau\right) \\
&- \mathfrak{F}\left(t, v(t), \int_0^t k_1(t,\tau)h_1(\tau, v(\tau))d\tau, \int_0^1 k_2(t,\tau)h_2(\tau, v(\tau))d\tau\right)\Bigg| \\
&\leq \Bigg|\frac{u(t)-v(t)}{10^2} + \frac{e^2}{10^2}\int_0^t \left\{\frac{1}{(2+|\sin u(\tau)|)} - \frac{1}{(2+|\sin v(\tau)|)}\right\} d\tau \\
&+ \frac{e^2}{10^2}\int_0^1 \left\{\frac{1}{(2+|\cos u(\tau)|)} - \frac{1}{(2+|\cos v(\tau)|)}\right\} d\tau\Bigg| \\
&\leq \frac{1}{10^2}\|u-v\| + \frac{e^2}{10^2}\int_0^t \left|\frac{|\sin v(\tau)| - |\sin u(\tau)|}{(2+|\sin u(\tau)|)(2+|\sin v(\tau)|)}\right| d\tau \\
&+ \frac{e^2}{10^2}\int_0^1 \left|\frac{|\cos v(\tau)| - |\cos u(\tau)|}{(2+|\cos u(\tau)|)(2+|\cos v(\tau)|)}\right| d\tau \\
&\leq \frac{1}{10^2}\|u-v\| + \frac{e^2}{10^2\times 2}\|u-v\| \\
&= \left(\frac{1}{10^2} + \frac{e^2}{10^2\times 2}\right)\|u-v\| = 0.0469\|u-v\|.
\end{aligned}$$

Hence the hypotheses (H_1) and (H_2) are satisfied.
Next, in (3.4.1), $\mathcal{G}((t, u(t), u(\lambda t)) = \dfrac{1}{10^4}\{\sin u(t) + \sin u(\lambda t)\}$. This implies,

$$\begin{aligned}
&|\mathcal{G}((t, u(t), u(\lambda t)) - \mathcal{G}((t, v(t), v(\lambda t))| \\
&= \left|\frac{1}{10^4}\{\sin u(t) + \sin u(\lambda t) - \sin v(t) - \sin v(\lambda t)\}\right| \leq \frac{2}{10^4}\|u-v\| = 0.0002\|u-v\|.
\end{aligned}$$

Thus, the hypothesis (H_4) is satisfied. Next we have to do the following steps to show that the hypothesis (H_5) holds.

$$\text{Now,}\quad |u(0) - \mathcal{G}((0, u(0), u(0))| \leq \frac{2}{10^4} + \frac{2}{10^4} = \frac{4}{10^4} = 0.0004,$$

$$\begin{aligned}
&\left|\mathfrak{F}(0, u(0), 0, \int_0^1 k_2(0,\tau)h_2(\tau, u(\tau))d\tau\right| \\
&= \left|\frac{u(0)}{10^2} + \int_0^1 \frac{e^{-\tau}}{10^2(2+|\cos u(\tau)|)}d\tau\right| \\
&\geq \left|\frac{1}{10^6}(1+\cos u(\alpha)) + \int_0^1 \frac{1}{3\times 10^2}e^{-\tau}d\tau\right| \geq \frac{1}{3\times 10^2}\left(1-\frac{1}{e}\right) = 0.0021.
\end{aligned}$$

This implies,

$$\frac{1}{|\mathfrak{F}(0, u(0), 0, \int_0^1 k_2(0,\tau)h_2(\tau, u(\tau))d\tau|} \leq \frac{1}{0.0021}.$$

Hence,

$$\left|\frac{u(0) - \mathcal{G}((0, u(0), u(0))}{\mathfrak{F}(0, u(0), 0, \int_0^1 k_2(0,\tau)h_2(\tau, u(\tau))d\tau}\right| \leq \frac{0.0004}{0.0021} = 0.1905 = \mathcal{X}_0.$$

Similarly,

$$|u(1) - \mathcal{G}((1, u(1), u(\lambda))| \leq \frac{1}{10^3} + \frac{2}{10^4} = 0.0012,$$

$$\begin{aligned}
&\left|\mathfrak{F}\Big(1, u(1), \int_0^1 k_1(1,\tau)h_1(\tau, u(\tau))d\tau, \int_0^1 k_2(1,\tau)h_2(\tau, u(\tau)\Big)d\tau\right| \\
&\geq 1 + \frac{1}{10^5} + \frac{1}{10^2 \times 3}\left|\int_0^1 e^{1+\tau}d\tau + \int_0^1 e^{1-\tau}d\tau\right| = 2.0213.
\end{aligned}$$

Hence,

$$\left|\frac{u(1) - \mathcal{G}((1, u(1), u(\lambda))}{\mathfrak{F}(1, u(1), \int_0^1 k_1(1,\tau)h_1(\tau, u(\tau))d\tau, \int_0^1 k_2(1,\tau)h_2(\tau, u(\tau))d\tau}\right| \leq \frac{0.0012}{2.0213} = 0.0006 = \mathcal{X}_1.$$

Therefore, hypothesis (H_5) is satisfied.
Again in (3.4.1), $\mathcal{W}(t, u(t), u(\lambda t)) = \frac{1}{10^3}(\cos u(t) + \cos u(\lambda t))$.
So, $|\mathcal{W}(t, u(t), u(\lambda t))| \leq \frac{2}{10^3} = 0.002 = \|\eta\|\Psi(\mathfrak{r})$.
Thus, hypothesis (H_6) is satisfied.
So from the above calculations we get,

$$L_{\mathfrak{F}}(1 + \mathcal{K}L_{h_1}T + \mathcal{K}L_{h_2}T)\Big[2\mathcal{X}_0 + \mathcal{X}_1 + \frac{2\|\eta\|\Psi(\mathfrak{r})}{\Gamma(q+1)}T^q\Big] + 2L_{\mathcal{G}} = 0.0184 < 1.$$

As (3.4.1) satisfies all the hypotheses, we can conclude from the Theorem 3.3.1 that the boundary value problem (3.4.1) has a solution.