CHAPTER 4

A Study on an Implicit Hybrid Fractional Differential Equation with $p-$Laplacian Operator

4.1 Introduction

Implicit type fractional differential equations naturally arise in various applications, including physics, control theory, bioengineering, and finance. These equations play a crucial role in describing, analysing, and simulating physical and mathematical systems with nonlinear or complex behaviour. Implicit differential equations form a class where the relationship between the unknown function and its derivatives is not explicitly solved for the highest-order derivative. When fractional derivatives are involved, the dependent variable appears in an implicit form, making their analysis more intricate than explicit fractional differential equations. On the other hand, the study of the existence and stability of solutions to implicit fractional differential equations has become a significant area of research within fractional calculus, attracting rigorous mathematical attention [2, 26, 32, 121, 128]. Gul et al. [60] studied the existence, uniqueness, and stability of a class of implicit fractional differential equations involving the Caputo-Fabrizio fractional derivative under Dirichlet boundary conditions using classical fixed point theory techniques. Recently, Rahman et al. investigated multi-term fractional differential equations with variable type delay, employing fixed point theorems to establish existence, uniqueness, and stability results [126].

In this chapter, we study the following hybrid fractional differential equations of implicit form with the $p-$Laplacian operator

$$
\begin{aligned}
&{}_0\mathcal{D}_t^{\alpha}\left[\phi_p\left({}_0^C\mathcal{D}_t^{\beta}\frac{u(t)-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)\right]\\
&\quad=\mathcal{G}\left(t,u(t),{}_0\mathcal{D}_t^{\alpha}\left[\phi_p\left({}_0^C\mathcal{D}_t^{\beta}\frac{u(t)-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)\right]\right), \qquad t\in\mathscr{J}=[0,T],\quad T>1,\\
&\phi_p\left({}_0^C\mathcal{D}_t^{\beta}\frac{u(t)-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)_{t=0}=0,\\
&\phi_p\left({}_0^C\mathcal{D}_t^{\beta}\frac{u(t)-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)_{t=1}=0,\\
&\left.\frac{u(t)-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right|_{t=0}=-\left.\frac{u(t)-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right|_{t=1},\\
&\left.\frac{u(t)-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}'\right|_{t=0}=-\left.\frac{u(t)-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}'\right|_{t=1}.
\end{aligned}
\tag{4.1.1}
$$

Here, ${}_0\mathcal{D}_t^{\alpha}$ denotes the Riemann-Liouville fractional derivative and ${}_0^C\mathcal{D}_t^{\beta}$ is the Caputo fractional derivative of order $1<\alpha,\beta<2$ and the functions $\mathfrak{F}\in C(\mathscr{J}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $\mathcal{G}\in C(\mathscr{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. Furthermore I^{ω_i} represents the Riemann-Liouville fractional integral. Additionally, let $\chi_i: C(\mathscr{J},\mathbb{R})\to\mathbb{R},\quad i=1,\cdots,m$ be continuous and bounded functions. The $p-$Laplacian operator, $\phi_p(x)$ is defined as $\phi_p(x)=|x|^{p-2}x,\quad p>1$ and $\phi_p^{-1}=\phi_q$, where $\frac{1}{p}+\frac{1}{q}=1$.

The concept of the $p-$Laplacian differential equations, pioneered by Leibenson [84], has become a key in the analysis of complex physical phenomena, particularly in modelling turbulent flow in porous media. Its applications span a wide range of fields, including viscoelasticity, heat conduction, biology, finance, and control systems. Additionally, the $p-$Laplacian is extensively used in modelling physical phenomena, image processing, and material science, demonstrating its versatility and importance. A critical aspect of studying $p-$Laplacian equations lies in understanding how the solutions' behaviour evolves with changes in the parameter p, providing valuable insights into the dynamics of complex systems. Recent advancements in the theory of fractional differential equations involving the $p-$Laplacian operator have focused on the existence and uniqueness and stability of solutions. For instance, Devi and Kumar [41] investigated the existence of solutions for a first type hybrid fractional differential equations with the $p-$Laplacian operator using Dhage's fixed point theorem and Green's function, while Khan et al. [76] explored the second type hybrid fractional differential equation by applying the Leray-Schauder fixed point theorem. In [75] authors investigated the existence and uniqueness of solutions using the Guo-Krasnoselskii fixed point theorem, as well as the Hyers-Ulam stability, for an Atangana-Baleanu-Caputo (ABC) fractional differential equation involving the $p-$Laplacian operator. These studies show the importance of advanced analytical methods in solving nonlinear fractional differential equations and emphasize the increasing role of the $p-$Laplacian in both theory and applications.

The study of implicit hybrid fractional differential equations with $p-$Laplacian operator has not been explored yet. Motivated by the existing literature, we formulate and investigate our own problem.

This chapter is organised as follows: In Section 4.2, we present the necessary preliminaries required for the subsequent analysis. In Section 4.3, we derive the equivalent integral formulation of (4.1.1) and establish the existence and uniqueness of its solutions. Section 4.4 is dedicated to analysing the Hyers-Ulam stability of (4.1.1). Finally, in Section 4.5, we provide an illustrative example to demonstrate the applicability of our theoretical findings.

4.2 Preliminaries

Lemma 4.2.1. *[77] Let* $\alpha > 0$, $n = [\alpha] + 1$ *and let* $u_{n-\alpha}(t) = {}_0I_t^{(n-\alpha)}u(t)$. *Then for* $u \in L^1([0,T], \mathbb{R}^N)$ *and* $u_{n-\alpha} \in AC^n([0,T], \mathbb{R}^N)$

$$ {}_0I_t^{\alpha}\left[{}_0\mathcal{D}_t^{\alpha}u(t)\right] = u(t) - \sum_{i=1}^{n} c_i t^{\alpha-i}, $$

for some $c_i \in \mathbb{R}$.

Lemma 4.2.2. *[41] The p−Laplacian operator* ϕ_p *satisfy the following conditions.*

(i) *If* $|x|, |y| \geq \Lambda(>0)$ *and* $1 < p \leq 2$ *then*

$$ |\phi_p(x) - \phi_p(y)| \leq (p-1)\Lambda^{p-2}|x-y|. $$

(ii) *If* $|x|, |y| \leq \lambda(>0)$ *and* $p > 2$ *then*

$$ |\phi_p(x) - \phi_p(y)| \leq (p-1)\lambda^{p-2}|x-y|. $$

4.3 Existence and Uniqueness Result

The space of continuous functions from $\mathscr{J}$ into $\mathbb{R}$ denoted by $C(\mathscr{J},\mathbb{R})$ is a Banach space equipped with the following norm

$$\|y\| = \sup_{t\in\mathscr{J}}|y(t)|.$$

Lemma 4.3.1. *Assume that* $\mathcal{G} \in C(\mathscr{J},\mathbb{R})$ *then the hybrid fractional differential equation*

$$ {}_0\mathcal{D}_t^{\alpha}\left[\phi_p\left({}_0^C\mathcal{D}_t^{\beta}\frac{u(t)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)\right] = \mathcal{G}(t), \tag{4.3.1}$$

$$\phi_p\left({}_0^C\mathcal{D}_t^{\beta}\frac{u(t)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)_{t=0} = 0, \tag{4.3.2}$$

$$\phi_p\left({}_0^C\mathcal{D}_t^{\beta}\frac{u(t)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)_{t=1} = 0, \tag{4.3.3}$$

$$\left.\frac{u(t)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right|_{t=0} = -\left.\frac{u(t)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right|_{t=1}, \tag{4.3.4}$$

$$\left.\frac{u(t)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right|_{t=0}^{\prime} = -\left.\frac{u(t)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right|_{t=1}^{\prime}, \tag{4.3.5}$$

has an integral solution of the form

$$\begin{aligned} u(t) = \mathfrak{F}(t,u(t))\Big[&\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q\big(\mathcal{A}\big)ds - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\phi_q\big(\mathcal{A}\big)ds \\ &+ \frac{1}{4\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}\phi_q\big(\mathcal{A}\big)ds - \frac{1}{2\Gamma(\beta-1)}\int_0^1 t(1-s)^{\beta-2}\phi_p\big(\mathcal{A}\big)ds\Big] \\ &+ \sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t)), \end{aligned}$$

where

$$\mathcal{A} = \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau - \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau.$$

Proof. Using Lemma 4.2.1 in (4.3.1) we have

$$\phi_p\left({}_0^C\mathcal{D}_t^{\beta}\frac{u(t)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{G}(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2}. \tag{4.3.6}$$

Applying the condition (4.3.2) in (4.3.6) we get $c_2 = 0$.
Also using the condition (4.3.3) in (4.3.6) we get

$$c_1 = -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mathcal{G}(s)ds.$$

Substituting the values of c_1 and c_2, obtained from the computations above, into (4.3.6), we obtain

$$\phi_p\left({}_0^C\mathcal{D}_t^\beta \frac{u(t) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{G}(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}\mathcal{G}(s)ds.$$

This implies

$$\begin{aligned} {}_0^C\mathcal{D}_t^\beta \frac{u(t) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))} &= \phi_q\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{G}(s)ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}\mathcal{G}(s)ds\Big]. \end{aligned} \tag{4.3.7}$$

Again from (4.3.7) we get

$$\begin{aligned} \frac{u(t) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))} &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big[\phi_q\Big\{\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau \\ &- \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau\Big\}\Big]ds + c_3 + c_4 t. \end{aligned} \tag{4.3.8}$$

Applying the condition (4.3.4) in (4.3.6) we get

$$\begin{aligned} c_3 = -\frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\Big[\phi_q\Big\{&\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau \\ &- \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau\Big\}\Big]ds - \frac{1}{2}c_4, \end{aligned}$$

and by using the condition (4.3.5) in (4.3.6) we get

$$\begin{aligned} c_4 = -\frac{1}{2\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}\Big[\phi_q\Big\{&\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau \\ &- \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau\Big\}\Big]ds. \end{aligned}$$

Substituting the values of c_3 and c_4, obtained from the above calculations, into (4.3.8), we obtain

$$\frac{u(t) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))} = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q\big(\mathcal{A}\big)ds - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\phi_q\big(\mathcal{A}\big)ds$$

$$+\frac{1}{4\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}\phi_q\big(\mathcal{A}\big)ds-\frac{1}{2\Gamma(\beta-1)}\times\int_0^1 t(1-s)^{\beta-2}\phi_p\big(\mathcal{A}\big)ds.$$

This implies that

$$u(t)=\mathfrak{F}(t,u(t))\Big[\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\phi_q\big(\mathcal{A}\big)ds-\frac{1}{2\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\phi_q\big(\mathcal{A}\big)ds+\frac{1}{4\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}\phi_q\big(\mathcal{A}\big)ds-\frac{1}{2\Gamma(\beta-1)}\int_0^1 t(1-s)^{\beta-2}\phi_p\big(\mathcal{A}\big)ds\Big]+\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t)),$$

where

$$\mathcal{A}=\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau-\frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1}\mathcal{G}(\tau)d\tau.$$

□

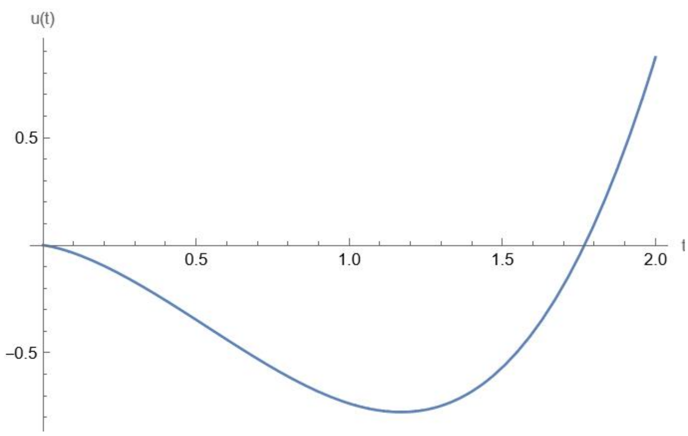
It is not possible to find the exact solution for every function. In this context, we construct two simple examples by setting $p=2$, and derive the exact solutions using Lemma 4.3.1.

Example 1. Let us consider an example of type (4.3.1)-(4.3.5) with $\mathfrak{F}=\sin t$, $\mathcal{G}=t$, $\chi_1=\log t$, $\chi_2=e^t$, $\alpha=\frac{3}{2}$, $\beta=\frac{4}{3}$, $\omega_1=2$, $\omega_2=3$. Lemma 4.3.1 implies that

$$\mathcal{A}=\frac{1}{\Gamma\Big(\frac{7}{2}\Big)}(s^{\frac{5}{2}}-s^{\frac{1}{2}}).$$

By doing some calculations, from Lemma 4.3.1 we get the following exact solution of the considered problem

$$u(t)=\sin t\big[0.053389\times t^{\frac{23}{6}}-0.15463\times t^{\frac{11}{6}}+0.039415\times t-0.181288\big]+\frac{t^2(2\log t-3)}{4}+\frac{1}{2}(2e^t-t^2-2t-2).$$



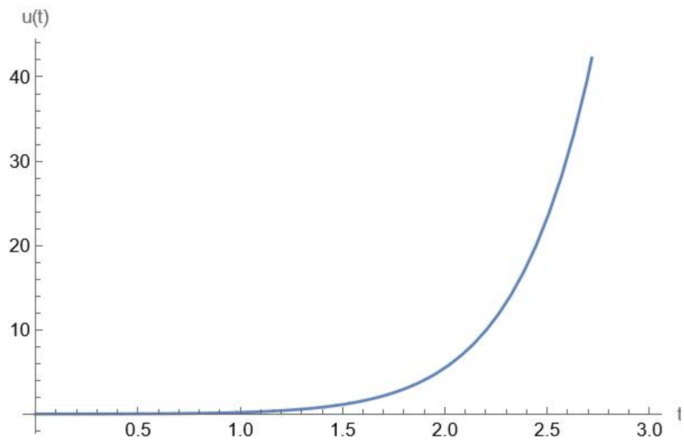
This t vs $u(t)$ graph gives the trajectory of the solution for $t \in [0, 2]$.

Example 2. Let us consider another example of type (4.3.1)-(4.3.5) such that $\mathfrak{F} = e^t$, $\mathcal{G} = t^2$, $\chi_1 = t$, $\chi_2 = \sin t$, $\chi_3 = \cos t$, $\alpha = \frac{5}{3}$, $\beta = \frac{6}{5}$, $\omega_1 = 2$, $\omega_2 = 3$, $\omega_3 = 4$. From Lemma 4.3.1 we obtain

$$\mathcal{A} = \frac{2}{\Gamma(\frac{14}{3})}(s^{\frac{11}{3}} - s^{\frac{2}{3}}).$$

The exact solution of the considered problem can be obtained from Lemma 4.3.1

$$u(t) = e^t\big[0.020889 \times t^{\frac{73}{15}} - 0.069145 \times t^{\frac{28}{15}} + 0.0137072 \times t + 0.017276\big] + \frac{t^3}{6} + \frac{1}{2}(2\cos t + t^2 - 2) + \frac{1}{6}(6\cos t + 3t^2 - 6).$$



This t vs $u(t)$ graph gives the trajectory of the solution for $t \in [0, 3]$.

4.3.1 Hypotheses

We introduce the following hypotheses, which will be used to prove the main results.

(H_1) For all $t \in \mathscr{J}$ and $u, v \in \mathbb{R}$, there exists a constant $\mathfrak{L} > 0$ such that $\mathfrak{F} : \mathscr{J} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ satisfies

$$|\mathfrak{F}(t, u) - \mathfrak{F}(t, v)| \leq \mathfrak{L}|u - v|.$$

(H_2) For all $t \in \mathscr{J}$ and $u_1, v_1, u_2, v_2 \in \mathbb{R}$, there exist constants $\mathfrak{L}_1 > 0$, $0 < \mathfrak{L}_2 < 1$ such that $\mathcal{G} : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|\mathcal{G}(t, u_1, v_1) - \mathcal{G}(t, u_2, v_2)| \leq \mathfrak{L}_1|u_1 - u_2| + \mathfrak{L}_2|v_1 - v_2|.$$

(H_3) For all $t \in \mathscr{J}$ and $u, v \in \mathbb{R}$ there exists a constant $\kappa_i > 0$ such that $\chi_i : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$ satisfies

$$|\chi_i(t, u) - \chi_i(t, v)| \leq \kappa_i|u - v|.$$

(H_4) For all $t \in \mathscr{J}$ and $u, v \in \mathbb{R}$ there exist non-negative functions $\xi,\ \mu,\ \nu\ \in C(\mathscr{J}, \mathbb{R})$ such that

$$|\mathcal{G}(t,u,v)| \leq \xi(t) + \mu(t)|u| + \nu(t)|v|.$$

(H_5) For all $t \in \mathscr{J}$ and $u \in \mathbb{R}$ there exists a function $\eta \in C(\mathscr{J}, \mathbb{R}^+)$ such that

$$|\mathfrak{F}(t,u)| \leq \eta(t).$$

(H_6) For all $t \in \mathscr{J}$ and $u \in \mathbb{R}$ there exists a function $\lambda_i \in C(\mathscr{J}, \mathbb{R}^+)$ for $i = 1, 2, \ldots, m$ such that

$$|\chi_i(t,u)| \leq \lambda_i(t).$$

4.3.2 Main Results

In this section, we establish and prove the main results of our study.

To prove the existence and uniqueness of (4.1.1) assume that

$$\Theta = \mathfrak{L}\big(\Phi_1 + \Phi_2\|u\|\big) + \|\eta\|(q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta + \sum_{i=1}^{m}\frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\kappa_i < 1. \tag{4.3.9}$$

In this chapter, from now onwards let us consider

$$\Phi_1 = (q-1)\Lambda^{(q-2)}\left(\frac{\|\xi\|}{1-\|\nu\|}\right)\Delta,$$
$$\Phi_2 = (q-1)\Lambda^{(q-2)}\left(\frac{\|\mu\|}{1-\|\nu\|}\right)\Delta,$$

and

$$\Delta = \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha+\beta-1}}{\alpha\Gamma(\alpha+\beta)} + \frac{1}{2\Gamma(\alpha+\beta+1)} + \frac{1}{2\alpha\Gamma(\alpha+\beta)} + \frac{1}{4\Gamma(\alpha+\beta)} + \frac{1}{4\alpha\Gamma(\alpha+\beta-1)} + \frac{T}{2\Gamma(\alpha+\beta)} + \frac{T}{2\alpha\Gamma(\alpha+\beta-1)}.$$

Theorem 4.3.2. *Let* (4.1.1) *satisfies the assumptions* (H_1)-(H_6) *and* (4.3.9). *Then the problem* (4.1.1) *has at least one solution on* $\mathscr{J}$ *provided that,*

$$\frac{\|\eta\|\Phi_1 + \sum_{i=1}^{m}\frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\|\lambda_i\|}{1-\|\eta\|\Phi_2} \leq \mathfrak{R}, \quad \|\eta\|\Phi_2 < 1.$$

Proof. Set, $\sup_{t\in\mathscr{J}}|\xi(t)| = \|\xi\|$, $\sup_{t\in\mathscr{J}}|\mu(t)| = \|\mu\|$, $\sup_{t\in\mathscr{J}}|\nu(t)| = \|\nu\|$, $\sup_{t\in\mathscr{J}}|\eta(t)| = \|\eta\|$, $\sup_{t\in\mathscr{J}}|\lambda_i(t)| = \|\lambda_i\|$, $i = 1, 2, \ldots, m$.

Let us consider $\mathcal{X} = \mathcal{C}(\mathscr{J}, \mathbb{R})$ and $\mathcal{S} = \{x \in \mathcal{X} : \|x\| \leq \mathfrak{R}\}$ be a subset of $\mathcal{X}$. From Lemma 4.3.1, equation (4.1.1) can be written as

$$\begin{aligned} u(t) = \mathfrak{F}(t,u(t))\Big[&\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q\big(\mathcal{A}_{u(s)}\big)ds - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\phi_q\big(\mathcal{A}_{u(s)}\big)ds \\ &+ \frac{1}{4\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}\phi_q\big(\mathcal{A}_{u(s)}\big)ds - \frac{1}{2\Gamma(\beta-1)}\int_0^1 t(1-s)^{\beta-2}\phi_p\big(\mathcal{A}_{u(s)}\big)ds\Big] \\ &+ \sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t)), \end{aligned} \tag{4.3.10}$$

where

$$\begin{aligned} \mathcal{A}_{u(s)} = &\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\mathcal{G}\left(\tau, u(\tau), \mathcal{D}^{\alpha}_{0+}\left[\phi_p\left({}^{C}\mathcal{D}^{\beta}_{0+}\frac{u(\tau)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau,u(\tau))}{\mathfrak{F}(\tau,u(\tau))}\right)\right]\right)d\tau \\ &- \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1}\mathcal{G}\left(\tau, u(\tau), \mathcal{D}^{\alpha}_{0+}\left[\phi_p\left({}^{C}\mathcal{D}^{\beta}_{0+}\frac{u(\tau)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau,u(\tau))}{\mathfrak{F}(\tau,u(\tau))}\right)\right]\right)d\tau. \end{aligned}$$

For $t \in \mathscr{J}$, we define the operators $\mathcal{R} : \mathcal{X} \to \mathcal{X}$, $\mathcal{T} : \mathcal{S} \to \mathcal{X}$ as

$$\mathcal{R}u(t) = \mathfrak{F}(t,u(t)),$$

and

$$\begin{aligned} \mathcal{T}u(t) = &\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q\big(\mathcal{A}_{u(s)}\big)ds - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\phi_q\big(\mathcal{A}_{u(s)}\big)ds \\ &+ \frac{1}{4\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}\phi_q\big(\mathcal{A}_{u(s)}\big)ds - \frac{1}{2\Gamma(\beta-1)}\int_0^1 t(1-s)^{\beta-2}\phi_q\big(\mathcal{A}_{u(s)}\big)ds. \end{aligned}$$

Now we define two operators $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ and $\mathcal{Q} : \mathcal{S} \to \mathcal{X}$ as follows

$$\mathcal{P}u(t) = \mathcal{R}u(t).\mathcal{T}u(t), \quad t \in \mathscr{J},$$

and

$$\mathcal{Q}u(t) = \sum_{i=1}^{m} I^{\omega_i}\chi_i(t,u(t)), \quad t \in \mathscr{J}.$$

So, (4.3.10) becomes

$$u(t) = \mathcal{P}u(t) + \mathcal{Q}u(t).$$

Next we have to show that the operator $\mathcal{P}$ and $\mathcal{Q}$ satisfy the conditions of Lemma 1.6.19.

Step 1:

The following inequality is essential for subsequent calculations

$$|\phi_q(\mathcal{A}_{u(s)})| \leq (q-1)\Lambda^{(q-2)}\left[\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\right.$$

$$\times \left|\mathcal{G}\left(\tau, u(\tau), \mathcal{D}_{0+}^{\alpha}\left\{\phi_p\left({}^C\mathcal{D}_{0+}^{\beta}\frac{u(\tau)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau,u(\tau))}{\mathfrak{F}(\tau,u(\tau))}\right)\right\}\right)\right|d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1}$$
$$\times \left|\mathcal{G}\left(\tau, u(\tau), \mathcal{D}_{0+}^{\alpha}\left\{\phi_p\left({}^C\mathcal{D}_{0+}^{\beta}\frac{u(\tau)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau,u(\tau))}{\mathfrak{F}(\tau,u(\tau))}\right)\right\}\right)\right|d\tau\Bigg].$$

By (H_4) we have,

$$\left|\mathcal{G}\left(\tau, u(\tau), \mathcal{D}_{0+}^{\alpha}\left\{\phi_p\left({}^C\mathcal{D}_{0+}^{\beta}\frac{u(\tau)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau,u(\tau))}{\mathfrak{F}(\tau,u(\tau))}\right)\right\}\right)\right| \le \frac{\|\xi\|+\|\mu\|\|u\|}{1-\|\nu\|}, \quad \|\nu\|<1.$$

Hence,

$$|\phi_q(\mathcal{A}_{u(s)})| \le (q-1)\Lambda^{(q-2)}\left(\frac{\|\xi\|+\|\mu\|\|u\|}{1-\|\nu\|}\right)\frac{1}{\alpha\Gamma(\alpha)}(s^{\alpha}+s^{\alpha-1}).$$

Therefore,

$$\begin{aligned}|\mathcal{T}u(t)| &\le (q-1)\Lambda^{(q-2)}\left(\frac{\|\xi\|+\|\mu\|\|u\|}{1-\|\nu\|}\right)\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^{\alpha+\beta-1}}{\alpha\Gamma(\alpha+\beta)}+\frac{1}{2\Gamma(\alpha+\beta+1)}\right.\\ &\quad\left.+\frac{1}{2\alpha\Gamma(\alpha+\beta)}+\frac{1}{4\Gamma(\alpha+\beta)}+\frac{1}{4\alpha\Gamma(\alpha+\beta-1)}+\frac{T}{2\Gamma(\alpha+\beta)}+\frac{T}{2\alpha\Gamma(\alpha+\beta-1)}\right)\\ &= \Phi_1+\Phi_2\|u\|.\end{aligned}$$

Now for any $v\in\mathcal{S}$ we have,

$$\begin{aligned}|u(t)| = |\mathcal{P}u(t)+\mathcal{Q}v(t)| &\le |\mathcal{R}u(t)||\mathcal{T}u(t)| + \sum_{i=1}^{m}\frac{1}{\Gamma(\omega_i)}\int_0^t (t-s)^{\omega_i-1}|\chi_i(s,v(s))|ds\\ &\le \|\eta\|(\Phi_1+\Phi_2\|u\|)+\sum_{i=1}^{m}\frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\|\lambda_i\|.\end{aligned}$$

Taking supremum over t we have,

$$\|u\| \le \frac{\|\eta\|\Phi_1+\sum_{i=1}^{m}\frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\|\lambda_i\|}{1-\|\eta\|\Phi_2} \le \mathfrak{R}.$$

Hence, $u\in\mathcal{S}$. It gives the proof of (*iii*) of Lemma 1.6.19.

Step 2:

Next, we establish that the operator $\mathcal{Q}$ satisfies the condition (ii) of Lemma 1.6.19. The operator $\mathcal{Q}$ is continuous and $\|\mathcal{Q}u\| \leq \sum_{i=1}^{m} \frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\|\lambda_i\|$. Hence $\mathcal{Q}$ is uniformly bounded on $\mathcal{S}$.

Let us assume for any $u \in \mathcal{S}$ and $t_1, t_2 \in \mathscr{J}$ such that $t_2 < t_1$ we have,

$$\begin{aligned}|\mathcal{Q}u(t_1) - \mathcal{Q}u(t_2)| &= \sum_{i=1}^{m} \frac{1}{\Gamma(\omega_i)}\Big| \int_0^{t_1} (t_1 - s)^{\omega_i - 1}\chi_i(s, u(s))ds \\ &\qquad - \int_0^{t_2} (t_2 - s)^{\omega_i - 1}\chi_i(s, u(s))ds\Big| \\ &\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\omega_i+1)}\|\lambda_i\||t_1^{\omega_i} - t_2^{\omega_i}|,\end{aligned}$$

which tends to 0 without depending on $u \in \mathcal{S}$ as $t_1 \to t_2$. Thus, $\mathcal{Q}$ is equi-continuous.

Since $\mathcal{Q}$ is both uniformly bounded and equi-continuous, it follows from the Arzelá-Ascoli theorem that $\mathcal{Q}$ is a compact operator on $\mathcal{S}$.

Step 3:

For this step, the following inequality is required

$$\begin{aligned}&|\phi_q(\mathcal{A}_{u(s)}) - \phi_q(\mathcal{A}_{v(s)})| \leq (q-1)\Lambda^{(q-2)} \\ &\times \Bigg[\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Bigg|\mathcal{G}\Bigg(\tau, u(\tau), \mathcal{D}_{0+}^{\alpha}\Bigg\{\phi_p\Bigg({}^{C}\mathcal{D}_{0+}^{\beta}\frac{u(\tau) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau, u(\tau))}{\mathfrak{F}(\tau, u(\tau))}\Bigg)\Bigg\}\Bigg) \\ &- \mathcal{G}\Bigg(\tau, v(\tau), \mathcal{D}_{0+}^{\alpha}\Bigg\{\phi_p\Bigg({}^{C}\mathcal{D}_{0+}^{\beta}\frac{v(\tau) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau, v(\tau))}{\mathfrak{F}(\tau, v(\tau))}\Bigg)\Bigg\}\Bigg)\Bigg|d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1}\Bigg|\mathcal{G}\Bigg(\tau, u(\tau), \mathcal{D}_{0+}^{\alpha}\Bigg\{\phi_p\Bigg({}^{C}\mathcal{D}_{0+}^{\beta}\frac{u(\tau) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau, u(\tau))}{\mathfrak{F}(\tau, u(\tau))}\Bigg)\Bigg\}\Bigg) \\ &- \mathcal{G}\Bigg(\tau, v(\tau), \mathcal{D}_{0+}^{\alpha}\Bigg\{\phi_p\Bigg({}^{C}\mathcal{D}_{0+}^{\beta}\frac{v(\tau) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau, v(\tau))}{\mathfrak{F}(\tau, v(\tau))}\Bigg)\Bigg\}\Bigg)\Bigg|d\tau\Bigg].\end{aligned}$$

From (H_2)

$$\Bigg|\mathcal{G}\Bigg(\tau, u(\tau), \mathcal{D}_{0+}^{\alpha}\Bigg\{\phi_p\Bigg({}^{C}\mathcal{D}_{0+}^{\beta}\frac{u(\tau) - \sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau, u(\tau))}{\mathfrak{F}(\tau, u(\tau))}\Bigg)\Bigg\}\Bigg)$$

$$
-\mathcal{G}\left(\tau, v(\tau), \mathcal{D}_{0+}^{\alpha}\left\{\phi_p\left({}^C\mathcal{D}_{0+}^{\beta}\frac{v(\tau)-\sum_{i=1}^{m} I^{\omega_i}\chi_i(\tau, v(\tau))}{\mathfrak{F}(\tau, v(\tau))}\right)\right\}\right)\Bigg| \leq \frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}|u(t)-v(t)|.
$$

Hence,

$$
|\phi_q(\mathcal{A}_{u(s)})-\phi_q(\mathcal{A}_{v(s)})| \leq (q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\frac{1}{\alpha\Gamma(\alpha)}(s^{\alpha}+s^{\alpha+1})\|u-v\|.
$$

Therefore,

$$
|\mathcal{T}u(t)-\mathcal{T}v(t)| \leq (q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta\|u-v\|.
$$

Now we have to show that $\mathcal{P}$ is a contraction mapping. Assume $u, v \in \mathcal{S}$ for all $t \in \mathscr{J}$ we have

$$
\begin{aligned}
|\mathcal{P}u(t)-\mathcal{P}v(t)| &= |\mathcal{R}u(t).\mathcal{T}u(t)-\mathcal{R}v(t).\mathcal{T}v(t)| \\
&= |\mathcal{R}u(t).\mathcal{T}u(t)-\mathcal{R}v(t).\mathcal{T}u(t)+\mathcal{R}v(t).\mathcal{T}u(t)-\mathcal{R}v(t).\mathcal{T}v(t)| \\
&\leq |\mathcal{T}u(t)||\mathcal{R}u(t)-\mathcal{R}v(t)|+|\mathcal{R}v(t)||\mathcal{T}u(t)-\mathcal{T}v(t)| \\
&\leq \big(\Phi_1+\Phi_2\|u\|\big)\mathfrak{L}\|u-v\|+\|\eta\|(q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta\|u-v\| \\
&= \left\{\mathfrak{L}\big(\Phi_1+\Phi_2\|u\|\big)+\|\eta\|(q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta\right\}\|u-v\|.
\end{aligned}
$$

Thus by (4.3.9) $\mathcal{P}$ is a contraction mapping. Hence condition (i) of Lemma 1.6.19 is satisfied.

Thus, all the conditions of Lemma 1.6.19 are satisfied. Consequently, we can conclude that the considered implicit hybrid fractional differential equation (4.1.1) has at least one solution on $\mathscr{J}$. This completes the proof.

□

Theorem 4.3.3. *If the considered problem* (4.1.1) *satisfies the conditions* (H_1)-(H_6) *and* (4.3.9) *then* (4.1.1) *has a unique solution in* $C(\mathscr{J}, \mathbb{R})$.

Proof. Define, $\mathscr{D}u = \mathcal{P}u + \mathcal{Q}u : \mathcal{X} \to \mathcal{X}$.

For any $u, v \in \mathcal{S}$ and for all $t \in \mathscr{J}$ we have

$$
\begin{aligned}
|\mathscr{D}u(t)-\mathscr{D}v(t)| =& |\mathcal{P}u(t)+\mathcal{Q}u(t)-\mathcal{P}v(t)-\mathcal{Q}v(t)| \leq |\mathcal{P}u(t)-\mathcal{P}v(t)|+|\mathcal{Q}u(t)-\mathcal{Q}v(t)| \\
\leq& \left\{\mathfrak{L}\big(\Phi_1+\Phi_2\|u\|\big)+\|\eta\|(q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta\right\}\|u-v\|
\end{aligned}
$$

$$+\sum_{i=1}^{m}\frac{1}{\Gamma(\omega_i)}\int_0^t (t-s)^{\omega_i-1}\big|\chi_i(s,u(s))-\chi_i(s,v(s))\big|ds.$$

By using (H_3) we get,

$$\begin{aligned}&|\mathscr{D}u(t)-\mathscr{D}v(t)|\\ &\le\left[\mathfrak{L}\big(\Phi_1+\Phi_2\|u\|\big)+\|\eta\|(q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta+\sum_{i=1}^{m}\frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\kappa_i\right]\|u-v\|.\end{aligned}$$

By taking supremum over t we get,

$$\|\mathscr{D}u-\mathscr{D}v\|\le\left[\mathfrak{L}\big(\Phi_1+\Phi_2\|u\|\big)+\|\eta\|(q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta+\sum_{i=1}^{m}\frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\kappa_i\right]\|u-v\|.$$

Thus, by (4.3.9), $\mathscr{D}$ is a contraction. Thus, with the help of Banach contraction mapping principle, we conclude that $\mathscr{D}$ has a unique fixed point, which corresponds to the unique solution of the considered problem (4.1.1). This completes the proof. □

4.4 Hyers-Ulam Stability

Definition 4.4.1. *The integral equation* (4.3.10) *is said to be Hyers-Ulam stable if there exists a constant* $\Theta>0$ *and for every* $\epsilon>0$ *if the following inequality*

$$\begin{aligned}&\Bigg|u(t)-\mathfrak{F}(t,u(t))\Big[\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\phi_q\big(\mathcal{A}_{u(s)}\big)ds-\frac{1}{2\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\phi_q\big(\mathcal{A}_{u(s)}\big)ds\\ &+\frac{1}{4\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}\phi_q\big(\mathcal{A}_{u(s)}\big)ds-\frac{1}{2\Gamma(\beta-1)}\int_0^1 t(1-s)^{\beta-2}\phi_p\big(\mathcal{A}_{u(s)}\big)ds\Big]\\ &-\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,u(t))\Bigg|\le\epsilon,\end{aligned}$$

holds, then there exists a continuous function $\bar{u}(t)$ *satisfying the following*

$$\begin{aligned}\bar{u}(t)=\mathfrak{F}(t,\bar{u}(t))\times&\Big[\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\phi_q\big(\mathcal{A}_{\bar{u}(s)}\big)ds-\frac{1}{2\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\phi_q\big(\mathcal{A}_{\bar{u}(s)}\big)ds\\ &+\frac{1}{4\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}\phi_q\big(\mathcal{A}_{\bar{u}(s)}\big)ds-\frac{1}{2\Gamma(\beta-1)}\int_0^1 t(1-s)^{\beta-2}\phi_p\big(\mathcal{A}_{\bar{u}(s)}\big)ds\Big]\\ &+\sum_{i=1}^{m}I^{\omega_i}\chi_i(t,\bar{u}(t)),\end{aligned}\tag{4.4.1}$$

such that

$$|u(t)-\bar{u}(t)|\le\Theta\epsilon,\quad t\in\mathscr{J}.$$

Theorem 4.4.2. *Let the considered problem* (4.1.1) *satisfies the assumptions* (H_1)-(H_6), *then* (4.1.1) *is Hyers-Ulam stable.*

Proof. Let $u(t)$ be the unique solution of (4.1.1) and $\bar{u}(t)$ is the approximate solution of (4.1.1) satisfying (4.4.1), then we get the following by using considered assumptions,

$$\begin{aligned}
&|u(t)-\bar{u}(t)| \\
&= \Bigg| \mathfrak{F}(t,u(t))\Big[\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q\big(\mathcal{A}_{u(s)}\big)ds - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\phi_q\big(\mathcal{A}_{u(s)}\big)ds \\
&+ \frac{1}{4\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}\phi_q\big(\mathcal{A}_{u(s)}\big)ds - \frac{1}{2\Gamma(\beta-1)}\int_0^1 t(1-s)^{\beta-2}\phi_p\big(\mathcal{A}_{u(s)}\big)ds\Big] \\
&- \mathfrak{F}(t,\bar{u}(t))\Big[\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q\big(\mathcal{A}_{\bar{u}(s)}\big)ds - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\phi_q\big(\mathcal{A}_{\bar{u}(s)}\big)ds \\
&+ \frac{1}{4\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}\phi_q\big(\mathcal{A}_{\bar{u}(s)}\big)ds - \frac{1}{2\Gamma(\beta-1)}\int_0^1 t(1-s)^{\beta-2}\phi_p\big(\mathcal{A}_{\bar{u}(s)}\big)ds\Big] \\
&+ \sum_{i=1}^m I^{\omega_i}\chi_i(t,u(t)) - \sum_{i=1}^m I^{\omega_i}\chi_i(t,\bar{u}(t)) \Bigg| \\
&\le \left[\mathfrak{L}\big(\Phi_1+\Phi_2\|u\|\big) + \|\eta\|(q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta + \sum_{i=1}^m \frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\kappa_i\right]\|u-\bar{u}\| \\
&= \Theta\|u-\bar{u}\|,
\end{aligned}$$

where

$$\Theta = \mathfrak{L}\big(\Phi_1+\Phi_2\|u\|\big) + \|\eta\|(q-1)\Lambda^{(q-2)}\left(\frac{\mathfrak{L}_1}{1-\mathfrak{L}_2}\right)\Delta + \sum_{i=1}^m \frac{T^{\omega_i}}{\Gamma(\omega_i+1)}\kappa_i.$$

Thus the implicit hybrid fractional differential equation with $p-$Laplacian operator (4.1.1) is Hyres-Ulam stable.

□

4.5 Example

Let us consider the following functions. For $t \in [0,1]$,

$$\begin{aligned}
\mathfrak{F}(t,u(t)) &= \frac{t}{10^2}\sin u(t), \\
\mathcal{G}\left(t,u(t),\mathcal{D}_{0+}^{\alpha}\left[\phi_2\left({}^{C}\mathcal{D}_{0+}^{\beta}\frac{u(t)-\sum_{i=1}^{2}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)\right]\right) \qquad (4.5.1) \\
= \frac{\cos t}{10^5} + \frac{tu(t)}{10^4} + \frac{1}{10}\left(\mathcal{D}_{0+}^{\alpha}\left[\phi_2\left({}^{C}\mathcal{D}_{0+}^{\beta}\frac{u(t)-\sum_{i=1}^{2}I^{\omega_i}\chi_i(t,u(t))}{\mathfrak{F}(t,u(t))}\right)\right]\right),
\end{aligned}$$

$$\chi_1 = \frac{1}{10^3}(t^2 + \sin u(t)),$$
$$\chi_2 = \frac{1}{10^2}\sin u(t).$$

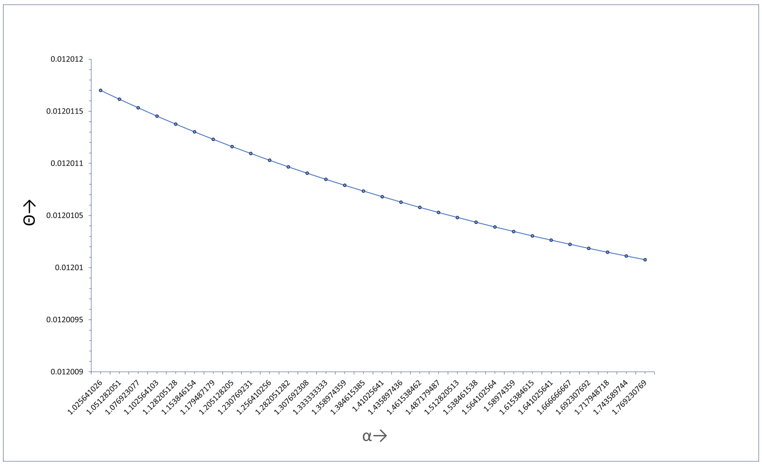
Here, assume $\omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{3}{4}, \quad m = 2, \quad p = 2.$
Thus from the assumptions (H_1)-(H_6) we have,
$\mathfrak{L} = \frac{1}{10^2}, \ \mathfrak{L}_1 = \frac{1}{10^4}, \ \mathfrak{L}_2 = \frac{1}{10}, \ \kappa_1 = \frac{1}{10^3}, \ \kappa_2 = \frac{1}{10^2}, \ \xi(t) = \frac{\cos t}{10^5}, \ \mu(t) = \frac{t}{10^4}, \ \nu(t) = \frac{1}{10}, \ \eta(t) = \frac{t}{10^2}, \ \lambda_1(t) = \frac{1+t^2}{10^3}, \ \lambda_2(t) = \frac{1}{10^2}.$
Hence, $\|\xi\| = \frac{1}{10^5}, \ \|\mu\| = \frac{1}{10^4}, \ \|\nu\| = \frac{1}{10}, \ \|\eta\| = \frac{1}{10^2}, \ \|\lambda_1\| = \frac{2}{10^3}, \ \|\lambda_2\| = \frac{1}{10^2}.$

Now let us check the values of Θ for some specific values of α and β.

Case 1: Let β is fixed and say $\beta = \frac{4}{3}$ and for arbitrary $\alpha (\in (1,2))$ we have the following values

Table 4.1: Values of Θ changing on α

α	Δ	Φ_1	Φ_2	Θ
1.025641026	2.157828133	2.39759E-05	0.000239759	0.012011701
1.051282051	2.088799373	2.32089E-05	0.000232089	0.012011615
1.076923077	2.022233564	2.24693E-05	0.000224693	0.012011533
1.102564103	1.957997129	2.17555E-05	0.000217555	0.012011453
1.128205128	1.895968245	2.10663E-05	0.000210663	0.012011377
1.153846154	1.836035509	2.04004E-05	0.000204004	0.012011303
1.179487179	1.778096785	1.97566E-05	0.000197566	0.012011231
⋮	⋮	⋮	⋮	⋮
1.615384615	1.028878106	1.1432E-05	0.00011432	0.012010304
1.641025641	0.995749728	1.10639E-05	0.000110639	0.012010263
1.666666667	0.963597859	1.07066E-05	0.000107066	0.012010223
1.692307692	0.932393715	1.03599E-05	0.000103599	0.012010185
1.717948718	0.902109723	1.00234E-05	0.000100234	0.012010147
1.743589744	0.872719435	9.69688E-06	9.69688E-05	0.012010111
1.769230769	0.844197436	9.37997E-06	9.37997E-05	0.012010076

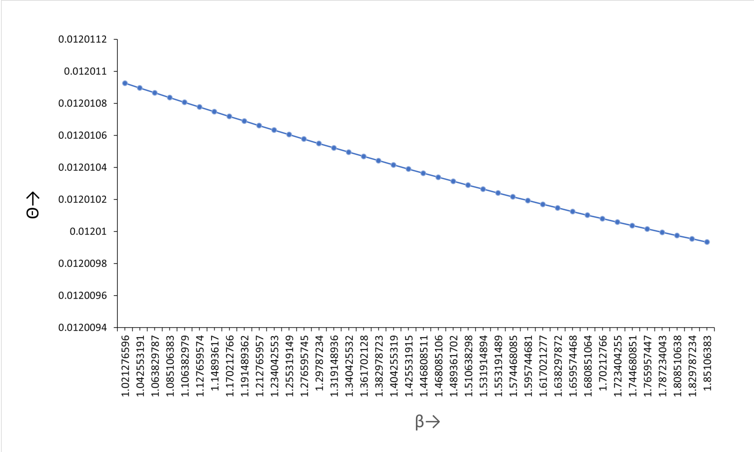


So from the above graph obtained from the Table 4.1, we can conclude that as α increases the values of Θ decreases gradually not exceeding 1.

Case 2: Next, assume that α is fixed and let $\alpha = \frac{3}{2}$. For arbitrary $\beta(\in (1, 2))$ we have the following

Table 4.2: Values of Θ changing on β

β	Δ	Φ_1	Φ_2	Θ
1.021276596	1.53187406	1.70208E-05	0.000170208	0.012010926
1.042553191	1.50745967	1.67496E-05	0.000167496	0.012010896
1.063829787	1.483187514	1.64799E-05	0.000164799	0.012010866
1.085106383	1.459065167	1.62118E-05	0.000162118	0.012010836
1.106382979	1.435099858	1.59456E-05	0.000159456	0.012010807
1.127659574	1.411298475	1.56811E-05	0.000156811	0.012010777
1.14893617	1.387667575	1.54185E-05	0.000154185	0.012010748
1.170212766	1.364213389	1.51579E-05	0.000151579	0.012010719
⋮	⋮	⋮	⋮	⋮
1.723404255	0.830192374	9.22436E-06	9.22436E-05	0.012010058
1.744680851	0.812881194	9.03201E-06	9.03201E-05	0.012010037
1.765957447	0.795819037	8.84243E-06	8.84243E-05	0.012010016
1.787234043	0.779005802	8.65562E-06	8.65562E-05	0.012009995
1.808510638	0.762441258	8.47157E-06	8.47157E-05	0.012009975
1.829787234	0.746125048	8.29028E-06	8.29028E-05	0.012009955
1.85106383	0.730056697	8.11174E-06	8.11174E-05	0.012009935

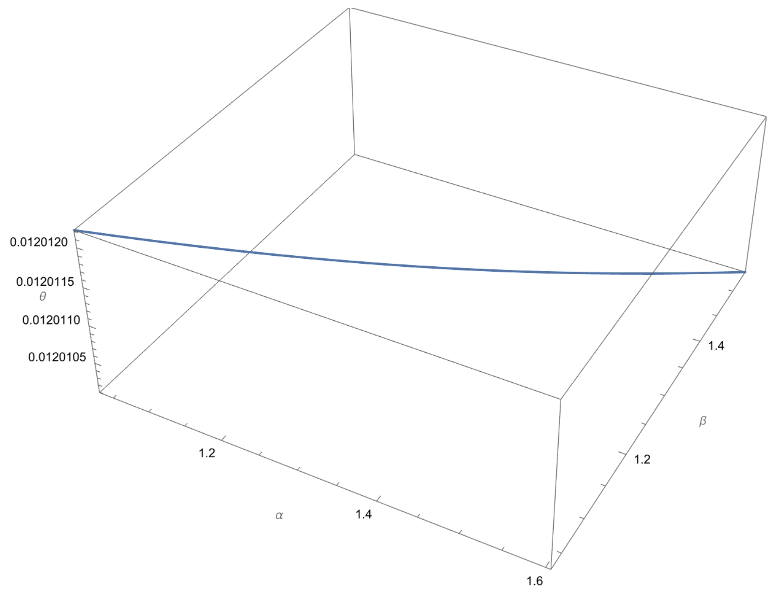


From the above graph obtained from the Table 4.2, we can conclude that as β increases the values of Θ decreases gradually not exceeding 1.

Case 3: Next we assume that both α and $\beta \in (1, 2)$ are arbitrary. We get that

Table 4.3: Values of Θ changing on both α and β

α	β	Δ	Φ_1	Φ_2	Θ
1.03030303	1.027027027	2.574770691	2.86086E-05	0.000286086	0.012012216
1.060606061	1.054054054	2.452572849	2.72508E-05	0.000272508	0.012012065
1.090909091	1.081081081	2.33359215	2.59288E-05	0.000259288	0.012011918
1.121212121	1.108108108	2.21797931	2.46442E-05	0.000246442	0.012011775
1.151515152	1.135135135	2.105853319	2.33984E-05	0.000233984	0.012011636
1.181818182	1.162162162	1.997304544	2.21923E-05	0.000221923	0.012011502
1.212121212	1.189189189	1.892397691	2.10266E-05	0.000210266	0.012011372
⋮	⋮	⋮	⋮	⋮	⋮
1.696969697	1.621621622	0.698170497	7.75745E-06	7.75745E-05	0.012009895
1.727272727	1.648648649	0.650707377	7.23008E-06	7.23008E-05	0.012009836
1.757575758	1.675675676	0.605918897	6.73243E-06	6.73243E-05	0.012009781
1.787878788	1.702702703	0.563705024	6.26339E-06	6.26339E-05	0.012009729
1.818181818	1.72972973	0.523965035	5.82183E-06	5.82183E-05	0.01200968
1.848484848	1.756756757	0.486597933	5.40664E-06	5.40664E-05	0.012009634
1.878787879	1.783783784	0.451502847	5.0167E-06	5.0167E-05	0.01200959
1.909090909	1.810810811	0.418579421	4.65088E-06	4.65088E-05	0.012009549



In this case also, for arbitrary α and β we get from the above graph and the Table 4.3 that, as α and β increases then Θ decreases. So the values of Θ can't exceed 1.

As, (4.5.1) satisfies all the assumptions (H_1)-(H_6) and (4.3.9) in all the above mentioned cases. Thus, from Theorem 4.3.3 we can conclude that the implicit hybrid fractional differential equation with $p-$Laplacian (4.5.1) has a unique solution on $[0, 1]$.