CHAPTER 5

Study of an Impulsive Hybrid Fractional Differential Equation with $p-$Laplacian Operator

5.1 Introduction

The purpose of this chapter is to examine the following hybrid fractional differential equation of second type with $p-$Laplacian operator and non-instantaneous impulses

$$ {}^{C}\mathcal{D}^{\alpha,\sigma}\left[\phi_p\left\{{}^{C}\mathcal{D}^{\beta,\sigma}\big(\omega(\mathfrak{t})-\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))\big)\right\}\right]=\mathcal{G}(\mathfrak{t},\omega(\mathfrak{t})),\quad \mathfrak{t}\in(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}],\quad \kappa=0,1,\ldots,\mathfrak{m}, \tag{5.1.1}$$

$$\omega(\mathfrak{t})-\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))=\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t})),\quad \mathfrak{t}\in(\mathfrak{t}_\kappa,\mathfrak{s}_\kappa],\quad \kappa=1,2,\ldots,\mathfrak{m}, \tag{5.1.2}$$

$$\omega(\mathfrak{t})-\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))\Big|_{\mathfrak{t}=0}=0, \tag{5.1.3}$$

$$\phi_p\left\{{}^{C}\mathcal{D}^{\beta,\sigma}\big(\omega(\mathfrak{s}_\kappa)-\mathfrak{F}(\mathfrak{s}_\kappa,\omega(\mathfrak{s}_\kappa))\big)\right\}=0,\quad \kappa=0,1,\ldots,\mathfrak{m}. \tag{5.1.4}$$

In this problem, ${}^{C}\mathcal{D}^{\alpha,\sigma}$ and ${}^{C}\mathcal{D}^{\beta,\sigma}$ denotes the Caputo-Katugampola fractional derivatives of order α and β ranging from 0 to 1, $\sigma>0$ and the $p-$Laplacian operator $\phi_p(\mathfrak{r})=|\mathfrak{r}|^{p-2}\mathfrak{r},\ \ p>1$ satisfies $\phi_p^{-1}=\phi_q$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that $[0,\mathcal{T}]=\mathscr{J}$ and some pre-fixed numbers $0=\mathfrak{s}_0<\mathfrak{t}_1<\mathfrak{s}_1<\mathfrak{t}_2<\cdots<\mathfrak{t}_\mathfrak{m}<\mathfrak{s}_\mathfrak{m}<\mathfrak{t}_{\mathfrak{m}+1}=\mathcal{T}$. Let $\mathfrak{F}$, $\mathcal{G}\in C(\mathscr{J}\times\mathbb{R},\mathbb{R})$ and for each $\kappa=1,2,\ldots,\mathfrak{m}$, let $\mathfrak{H}_\kappa\in C\big([\mathfrak{t}_\kappa,\mathfrak{s}_\kappa]\times\mathbb{R},\mathbb{R}\big)$.

Katugampola [73] introduced a novel fractional derivative that generalizes the well-established Riemann-Liouville and Hadamard fractional derivatives into a single framework. Further extended this concept [15] by generalizing the Caputo and Caputo-Hadamard fractional derivatives through the introduction of a new fractional operator, known as the Caputo-Katugampola derivative [11, 12, 89, 132]. The latter operator is fundamentally defined as the left inverse of the Katugampola fractional integral and it preserves several key properties of the Caputo and Caputo-Hadamard derivatives.

This chapter is based on the published work in *Mathematical Methods in the Applied Sciences* [44].

Over the years, the theory of impulsive differential equations [62, 70, 107, 116] has evolved significantly, becoming an fundamental tool in applied mathematics for modelling real-world phenomena in fields such as population dynamics, economics, physics, and chemistry. The article [83] and [112] provided a fundamental theory of impulsive differential equations. Depending on their effect on the systems, impulses are categorized as instantaneous or non-instantaneous. Instantaneous impulses induce sudden state transitions at discrete moments, whereas non-instantaneous impulses generate changes that unfold over a finite time interval. Non-instantaneous impulses are particularly useful for modelling systems with time delays and memory effects, where the response to an impulse is distributed over time rather than occurring immediately. Now-a-days the study of the existence of solutions for impulsive differential equations has gained considerable attention. Recently, Hilal et al. [64] studied the existence of solution for a hybrid differential equation of first kind with impulses by using Dhage's fixed point theorem. Zhang [136] investigated an initial value problem for an instantaneous impulsive differential equation involving the Caputo-Katugampola fractional derivative, with a primary focus on the non-uniqueness of solutions.

The study of hybrid fractional differential equations with $p-$Laplacian operator with non-instantaneous impulse has not been explored yet. So motivated from the above mentioned literature we consider our own problem.

This chapter is designed as follows: In Section 5.2, we derive the equivalent integral equation for (5.1.1)-(5.1.4). In Section 5.3, we establish the existence and uniqueness results. Stability analysis of (5.1.1)-(5.1.4) is carried out in Section 5.4. Finally, an example is presented in Section 5.5 to illustrate the main results.

5.2 Preliminaries

Lemma 5.2.1. *[1] For any $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}$ such that $\mathfrak{b} > \mathfrak{a}$ and $\mu, \nu > 0$ we have the following integral value*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} (\mathfrak{b}^\sigma - \mathfrak{s}^\sigma)^{\nu-1}(\mathfrak{s}^\sigma - \mathfrak{a}^\sigma)^{\mu-1}\mathfrak{s}^{\sigma-1}d\mathfrak{s} = \frac{(\mathfrak{b}^\sigma - \mathfrak{a}^\sigma)^{\mu+\nu-1}}{\sigma}\mathfrak{B}(\mu,\nu),$$

where $\mathfrak{B}(\mu,\nu)$ is the beta function and is equal to $\dfrac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$.

Lemma 5.2.2. *The considered problem (5.1.1)-(5.1.4) has a solution $\omega \in \mathcal{PC}(\mathscr{J}, \mathbb{R})$ of the following integral form*
For $\mathfrak{t} \in [0, \mathfrak{t}_1]$,

$$\begin{aligned}\omega(\mathfrak{t}) &= \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) \\ &+ \frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau, \omega(\tau))d\tau\Big]d\mathfrak{s}.\end{aligned}$$

For $\mathfrak{t} \in (\mathfrak{t}_\kappa, \mathfrak{s}_\kappa], \quad \kappa = 1, 2, \ldots, \mathfrak{m}$,

$$\omega(\mathfrak{t}) = \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) + \mathfrak{H}_\kappa(\mathfrak{t}, \omega(\mathfrak{t})).$$

For $\mathfrak{t} \in (\mathfrak{s}_\kappa, \mathfrak{t}_{\kappa+1}], \quad \kappa = 1, 2, \ldots, \mathfrak{m}$,

$$\begin{aligned}\omega(\mathfrak{t}) &= \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) + \mathfrak{H}_\kappa(\mathfrak{s}_\kappa, \omega(\mathfrak{s}_\kappa)) \\ &+ \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau\Big] d\mathfrak{s} \\ &- \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{s}_\kappa} ({\mathfrak{s}_\kappa}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau\Big] d\mathfrak{s}.\end{aligned}$$

Proof. For $\mathfrak{t} \in [0, \mathfrak{t}_1]$, from (5.1.1) we get

$$\phi_p\Big\{{}^C\mathcal{D}^{\beta,\sigma}\big(\omega(\mathfrak{t}) - \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t}))\big)\Big\} = \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\alpha-1} \mathfrak{s}^{\sigma-1} \mathcal{G}(\mathfrak{s}, \omega(\mathfrak{s})) d\mathfrak{s} + \mathcal{C}_0. \tag{5.2.1}$$

From (5.1.4) we obtain that $\mathcal{C}_0 = 0$. Therefore, (5.2.1) gives

$$\phi_p\Big\{{}^C\mathcal{D}^{\beta,\sigma}\big(\omega(\mathfrak{t}) - \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t}))\big)\Big\} = \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\alpha-1} \mathfrak{s}^{\sigma-1} \mathcal{G}(\mathfrak{s}, \omega(\mathfrak{s})) d\mathfrak{s}.$$

That implies

$$\begin{aligned}\omega(\mathfrak{t}) - \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) &= \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \\ &\times \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau\Big] d\mathfrak{s} + \mathcal{C}_1.\end{aligned}$$

Hence, from (5.1.3), it follows that $\mathcal{C}_1 = 0$.
Thus for $\mathfrak{t} \in [0, \mathfrak{t}_1]$ we have

$$\begin{aligned}\omega(\mathfrak{t}) &= \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) \\ &+ \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau\Big] d\mathfrak{s}.\end{aligned}$$

For $\kappa = 1, 2, \ldots, \mathfrak{m}$, $\mathfrak{t} \in (\mathfrak{t}_\kappa, \mathfrak{s}_\kappa]$, (5.1.2) implies that

$$\omega(\mathfrak{t}) = \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) + \mathfrak{H}_\kappa(\mathfrak{t}, \omega(\mathfrak{t})).$$

For $\kappa = 1, 2, \ldots, \mathfrak{m}$, $\mathfrak{t} \in (\mathfrak{s}_\kappa, \mathfrak{t}_{\kappa+1}]$, from (5.1.1) we get

$$\phi_p\Big\{{}^C\mathcal{D}^{\beta,\sigma}\big(\omega(\mathfrak{t}) - \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t}))\big)\Big\} = \frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\alpha-1} \mathfrak{s}^{\sigma-1} \mathcal{G}(\mathfrak{s}, \omega(\mathfrak{s})) d\mathfrak{s} + \mathcal{C}_2. \tag{5.2.2}$$

From (5.1.4) we get $\mathcal{C}_2 = 0$. Thus using (5.2.2) we obtain

$$\omega(\mathfrak{t}) - \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) = \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1}$$

$$\times \phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}+\mathcal{C}_3.$$

From (5.1.2) we get

$$\begin{aligned}\mathfrak{H}_\kappa(\mathfrak{s}_\kappa,\omega(\mathfrak{s}_\kappa)) &= \frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}(\mathfrak{s}_\kappa^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\\ &\times \phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}+\mathcal{C}_3.\end{aligned}$$

Thus, we have

$$\begin{aligned}\omega(\mathfrak{t}) &= \mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))+\mathfrak{H}_\kappa(\mathfrak{s}_\kappa,\omega(\mathfrak{s}_\kappa))\\ &+\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\\ &-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}(\mathfrak{s}_\kappa^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}.\end{aligned}$$

□

Example. Consider a specific case of the problem (5.1.1)-(5.1.4) by setting $\mathcal{G}(\mathfrak{t},\omega(\mathfrak{t}))=0$, which corresponds to the homogeneous version of the original problem with $0=\mathfrak{s}_0<\mathfrak{t}_1=1<\mathfrak{s}_1=2<\mathfrak{t}_3=3$.

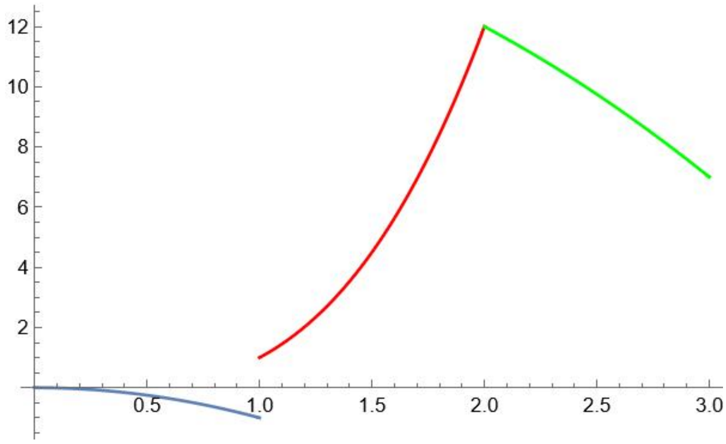
By applying Lemma 5.2.2, the exact solution can be obtained by substituting the values of the functions $\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))$ and $\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t}))$. In this example, we analyse the trajectory of the solution for three distinct values of $\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))$. A similar approach can be employed to investigate the behaviour of the solution for different values of $\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t}))$.

Set, $\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t}))=-2\mathfrak{t}^3$.

(i) For the function $\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))=\mathfrak{t}^2+2\omega(t)$ we get the solution

$$\omega(t)=\begin{cases}-\mathfrak{t}^2, & \text{for } \mathfrak{t}\in[0,1],\\ -\mathfrak{t}^2+2\mathfrak{t}^3, & \text{for } \mathfrak{t}\in(1,2],\\ -\mathfrak{t}^2+16, & \text{for } \mathfrak{t}\in(2,3].\end{cases}$$

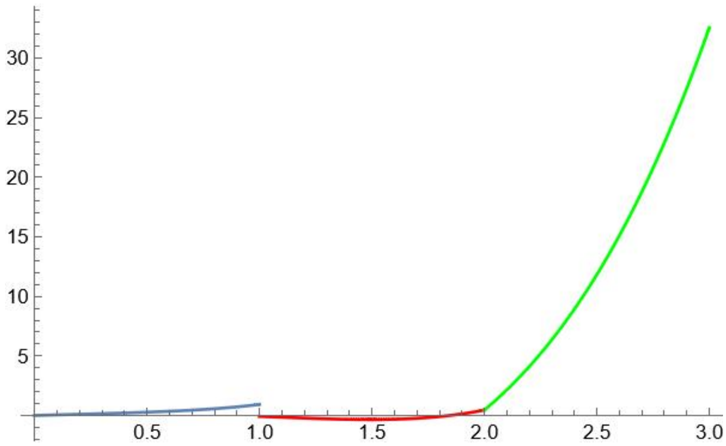
Hence, the trajectory of the solution can be observed from the graph presented below



(ii) For $\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t})) = \sin\mathfrak{t} + \mathfrak{t}^4 - \omega(\mathfrak{t})$ we get the solution

$$\omega(t) = \begin{cases} \frac{1}{2}(\sin(\mathfrak{t}) + \mathfrak{t}^4), & \text{for } \mathfrak{t} \in [0,1], \\ \frac{1}{2}(\sin(\mathfrak{t}) + \mathfrak{t}^4 - 2\mathfrak{t}^3), & \text{for } \mathfrak{t} \in (1,2], \\ \frac{1}{2}(\sin(\mathfrak{t}) + \mathfrak{t}^4 - 16), & \text{for } \mathfrak{t} \in (2,3]. \end{cases}$$

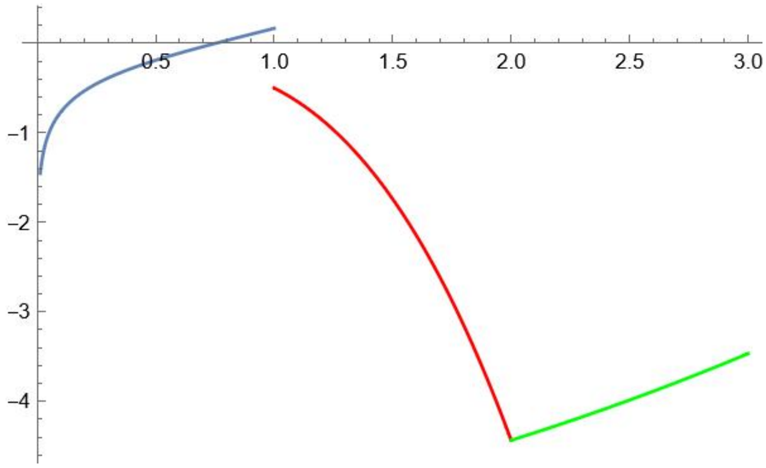
Thus, we get the trajectory of the solution from the following graph



(iii) Let $\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t})) = \log(\mathfrak{t}) + \frac{1}{2}\mathfrak{t}^2 - 2\omega(\mathfrak{t})$, then the solution is

$$\omega(t) = \begin{cases} \frac{1}{3}(\log(\mathfrak{t}) + \frac{1}{2}\mathfrak{t}^2), & \text{for } \mathfrak{t} \in [0,1], \\ \frac{1}{3}(\log(\mathfrak{t}) + \frac{1}{2}\mathfrak{t}^2 - 2\mathfrak{t}^3), & \text{for } \mathfrak{t} \in (1,2], \\ \frac{1}{3}(\log(\mathfrak{t}) + \frac{1}{2}\mathfrak{t}^2 - 16), & \text{for } \mathfrak{t} \in (2,3]. \end{cases}$$

The trajectory of the solution is given by the following graph



5.3 Main Results

The space of piecewise continuous functions is defined as $\mathcal{PC}(\mathscr{J},\mathbb{R}) = \big\{\omega : \mathscr{J} \to \mathbb{R} |\, \omega \in C\big((\mathfrak{t}_\kappa, \mathfrak{t}_{\kappa+1}], \mathbb{R}\big), \kappa = 0, 1, \ldots, \mathfrak{m}$ also for $\kappa = 1, 2, \ldots, \mathfrak{m},\ \omega(\mathfrak{t}_\kappa^+)$ and $\omega(\mathfrak{t}_\kappa^-)$ exist with $\omega(\mathfrak{t}_\kappa) = \omega(\mathfrak{t}_\kappa^-)\big\}$.

The space $\mathcal{PC}(\mathscr{J},\mathbb{R})$ forms a Banach space when equipped with the norm

$$\|\omega\|_{\mathcal{PC}} = \sup_{\mathfrak{t}\in\mathscr{J}} |\omega(\mathfrak{t})|.$$

Following are the hypotheses required to establish the existence result.

(H_1) For all $\mathfrak{t} \in \mathscr{J}$ and $\omega \in \mathcal{PC}(\mathcal{J},\mathbb{R})$ there exist a function $\xi \in C\big(\mathscr{J}, (0, +\infty)\big)$ and a non-decreasing continuous function $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ such that

$$|\mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t}))| \leq \xi(\mathfrak{t})\psi(\|\omega\|_{\mathcal{PC}}).$$

(H_2) For all $\mathfrak{t} \in (\mathfrak{s}_\kappa, \mathfrak{t}_{\kappa+1}]$, $\kappa = 0, 1, \ldots, \mathfrak{m}$ and $\omega \in \mathcal{PC}(\mathscr{J},\mathbb{R})$ there exist a function $\mu \in L^1\big((\mathfrak{s}_\kappa, \mathfrak{t}_{\kappa+1}], (0, +\infty)\big)$ and a non-decreasing continuous function $\eta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ such that

$$|\mathcal{G}(\mathfrak{t}, \omega(\mathfrak{t}))| \leq \mu(\mathfrak{t})\eta(\|\omega\|_{\mathcal{PC}}).$$

(H_3) For all $\mathfrak{t} \in (\mathfrak{t}_\kappa, \mathfrak{s}_\kappa]$, $\kappa = 1, 2, \ldots, m$ and $\omega \in \mathcal{PC}(\mathscr{J},\mathbb{R})$ there exists a function $\gamma \in C\big((\mathfrak{t}_\kappa, \mathfrak{s}_\kappa], (0, \infty)\big)$ such that

$$|\mathfrak{H}_\kappa(\mathfrak{t}, \omega(\mathfrak{t}))| \leq \gamma(\mathfrak{t}).$$

Let us consider an operator $\mathcal{Q}:\mathcal{PC}(\mathscr{J},\mathbb{R})\to\mathcal{PC}(\mathscr{J},\mathbb{R})$ defined by

$$\mathcal{Q}\omega(t)=\begin{cases}\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))\\ +\dfrac{\sigma^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s},\\ \qquad\text{for } \mathfrak{t}\in[0,\mathfrak{t}_1],\\ \mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))+\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t})), \qquad \text{for } \mathfrak{t}\in(\mathfrak{t}_\kappa,\mathfrak{s}_\kappa],\quad \kappa=1,2,\dots,\mathfrak{m},\\ \mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))+\mathfrak{H}_\kappa(\mathfrak{s}_\kappa,\omega(\mathfrak{s}_\kappa))\\ +\dfrac{\sigma^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\\ -\dfrac{\sigma^{1-\beta}}{\Gamma(\beta)}\displaystyle\int_0^{\mathfrak{s}_\kappa}({\mathfrak{s}_\kappa}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\dfrac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s},\\ \qquad\text{for } \mathfrak{t}\in(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}],\quad \kappa=1,2,\dots,\mathfrak{m}.\end{cases}$$

Existence Result

Theorem 5.3.1. *Let the problem* (5.1.1)-(5.1.4) *satisfies the assumptions* (H_1)-(H_3). *Further if,*

$$\rho\geq\max\Bigg\{\|\xi\|\psi(\rho)+\|\mu\|\eta(\rho)(q-1)\Lambda^{(q-2)}(\mathcal{T}^{\sigma})^{\alpha+\beta}\frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha+\beta+1)},$$
$$\|\xi\|\psi(\rho)+\|\gamma\|+\|\mu\|\eta(\rho)(q-1)\Lambda^{(q-2)}\frac{\Gamma(\alpha+1)}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\alpha+\beta+1)} \tag{5.3.1}$$
$$\times\Big[(\mathcal{T}^{\sigma})^{\alpha+\beta}+\max_{0\leq\kappa\leq\mathfrak{m}}(\mathfrak{s}_\kappa^{\sigma})^{\alpha+\beta}\Big]\Bigg\},$$

then the considered problem has a solution.

Proof. Set, $\mathcal{B}_\rho=\{\omega\in\mathcal{PC}(\mathscr{J},\mathbb{R}):\|\omega\|_{\mathcal{PC}}\leq\rho\}$ and assume that ρ satisfies (5.3.1).

Step 1: We need to establish the continuity of the operator $\mathcal{Q}$. For this purpose, let $\{\omega_n\}$ be a sequence in the space $\mathcal{PC}(\mathscr{J},\mathbb{R})$ such that $\omega_n\to\omega$ as $n\to\infty$. For $\mathfrak{t}\in[0,\mathfrak{t}_1]$, we obtain

$$\begin{aligned}&|\mathcal{Q}\omega_n(\mathfrak{t})-\mathcal{Q}\omega(\mathfrak{t})|\\ &\leq|\mathfrak{F}(\mathfrak{t},\omega_n(\mathfrak{t}))-\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))|\\ &+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_n(\tau))d\tau\Big]d\mathfrak{s}\\ &-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|\\ &\leq\|\mathfrak{F}(\cdot,\omega_n(\cdot))-\mathfrak{F}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}+\|\mathcal{G}(\cdot,\omega_n(\cdot))-\mathcal{G}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}\end{aligned}$$

$$\times \Big|(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|$$
$$\leq \|\mathfrak{F}(\cdot,\omega_n(\cdot))-\mathfrak{F}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}$$
$$+(q-1)\Lambda^{q-2}(\mathcal{T}^\sigma)^{\alpha+\beta}\frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\|\mathcal{G}(\cdot,\omega_n(\cdot))-\mathcal{G}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}.$$

For $\mathfrak{t}\in(\mathfrak{t}_\kappa,\mathfrak{s}_k]$, $\kappa=1,2,\ldots,\mathfrak{m}$, we obtain

$$|\mathcal{Q}\omega_n(\mathfrak{t})-\mathcal{Q}\omega(\mathfrak{t})|\leq\|\mathfrak{F}(\cdot,\omega_n(\cdot))-\mathfrak{F}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}+\|\mathfrak{H}_\kappa(\cdot,\omega_n(\cdot))-\mathfrak{H}_\kappa(\cdot,\omega(\cdot))\|_{\mathcal{PC}}.$$

For $\mathfrak{t}\in(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}]$, $\kappa=1,2,\ldots,\mathfrak{m}$, we obtain

$$|\mathcal{Q}\omega_n(\mathfrak{t})-\mathcal{Q}\omega(\mathfrak{t})|$$
$$\leq|\mathfrak{F}(\mathfrak{t},\omega_n(\mathfrak{t}))-\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))|+|\mathfrak{H}_\kappa(\mathfrak{s}_\kappa,\omega_n(\mathfrak{s}_\kappa))-\mathfrak{H}_\kappa(\mathfrak{s}_\kappa,\omega(\mathfrak{s}_\kappa))|$$
$$+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_n(\tau))d\tau\Big]d\mathfrak{s}$$
$$-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|$$
$$+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}({\mathfrak{s}_\kappa}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_n(\tau))d\tau\Big]d\mathfrak{s}$$
$$-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}({\mathfrak{s}_\kappa}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|$$
$$\leq\|\mathfrak{F}(\cdot,\omega_n(\cdot))-\mathfrak{F}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}+\|\mathfrak{H}_\kappa(\cdot,\omega_n(\cdot))-\mathfrak{H}_\kappa(\cdot,\omega(\cdot))\|_{\mathcal{PC}}$$
$$+\|\mathcal{G}(\cdot,\omega_n(\cdot))-\mathcal{G}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}$$
$$\times\Bigg[\Big|(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|$$
$$+\Big|(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}({\mathfrak{s}_\kappa}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\Bigg]$$
$$\leq\|\mathfrak{F}(\cdot,\omega_n(\cdot))-\mathfrak{F}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}+\|\mathfrak{H}_\kappa(\cdot,\omega_n(\cdot))-\mathfrak{H}_\kappa(\cdot,\omega(\cdot))\|_{\mathcal{PC}}$$
$$+(q-1)\Lambda^{q-2}\frac{\Gamma(\alpha+1)}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\Big[(\mathcal{T}^\sigma)^{\alpha+\beta}+(\mathfrak{s}_\kappa^\sigma)^{\alpha+\beta}\Big]\|\mathcal{G}(\cdot,\omega_n(\cdot))-\mathcal{G}(\cdot,\omega(\cdot))\|_{\mathcal{PC}}.$$

Consequently, from the above analysis it follows that, $\|\mathcal{Q}\omega_n-\mathcal{Q}\omega\|_{\mathcal{PC}}\to 0$ as $n\to\infty$. This demonstrates that the operator $\mathcal{Q}$ is continuous.

Step 2: Next, we will show that the operator $\mathcal{Q}(\mathcal{B}_\rho)$ is uniformly bounded.

For $\omega\in\mathcal{B}_\rho$ and $\mathfrak{t}\in[0,\mathfrak{t}_1]$, we have

$$|\mathcal{Q}\omega(\mathfrak{t})|$$
$$\leq|\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))|+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|$$

$$
\begin{aligned}
&\leq \|\xi\|\psi(\rho) + \|\mu\|_{L^1_{[0,\mathfrak{t}_1]}}\eta(\rho)\\
&\times\Big|\frac{(q-1)\Lambda^{q-2}\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\\
&\leq \|\xi\|\psi(\rho) + \|\mu\|_{L^1_{[0,\mathfrak{t}_1]}}\eta(\rho)(q-1)\Lambda^{q-2}(\mathcal{T}^\sigma)^{\alpha+\beta}\frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\\
&\leq \rho.
\end{aligned}
$$

For $\omega\in\mathcal{B}_\rho$ and $\mathfrak{t}\in(\mathfrak{t}_\kappa,\mathfrak{s}_\kappa]$, $\kappa=1,2,\ldots,\mathfrak{m}$, we get

$$
|\mathcal{Q}\omega(\mathfrak{t})|\leq|\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))|+|\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t}))|\leq\|\xi\|\phi(\rho)+\|\gamma\|\leq\rho.
$$

For $\omega\in\mathcal{B}_\rho$ and $\mathfrak{t}\in(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}]$, $\kappa=1,2,\ldots,\mathfrak{m}$, we obtain

$$
\begin{aligned}
&|\mathcal{Q}\omega(\mathfrak{t})|\\
&\leq|\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))|+|\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t}))|\\
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}({\mathfrak{s}_\kappa}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&\leq\|\xi\|\phi(\rho)+\|\gamma\|+\|\mu\|_{L^1_{(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}]}}\eta(\rho)\\
&\times\Bigg[\Big|(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\\
&+\Big|(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}({\mathfrak{s}_\kappa}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\Bigg]\\
&\leq\|\xi\|\psi(\rho)+\|\gamma\|+\|\mu\|_{L^1_{(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}]}}\eta(\rho)(q-1)\Lambda^{(q-2)}\frac{\Gamma(\alpha+1)}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\\
&\times\Big[(\mathcal{T}^\sigma)^{\alpha+\beta}+(\mathfrak{s}_\kappa^\sigma)^{\alpha+\beta}\Big]\\
&\leq\rho.
\end{aligned}
$$

Thus, $\|\mathcal{Q}\omega\|_{\mathcal{PC}}\leq\rho$. This implies that $\mathcal{Q}$ is uniformly bounded on $\mathcal{B}_\rho$.
Step 3: Next, we need to show that $\mathcal{Q}(\mathcal{B}_\rho)$ is equicontinuous.
For $\omega\in\mathcal{B}_\rho$ and ζ_1, $\zeta_2\in[0,\mathfrak{t}_1]$ with $\zeta_1<\zeta_2$, we have

$$
\begin{aligned}
&|\mathcal{Q}\omega(\zeta_2)-\mathcal{Q}\omega(\zeta_1)|\\
&\leq|\mathfrak{F}(\zeta_2,\omega(\zeta_2))-\mathfrak{F}(\zeta_1,\omega(\zeta_1))|\\
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\zeta_2}(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\\
&-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\zeta_1}(\zeta_1^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq |\mathfrak{F}(\zeta_2,\omega(\zeta_2))-\mathfrak{F}(\zeta_1,\omega(\zeta_1))| + \Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\zeta_1}\big\{(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}-(\zeta_1^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\big\}\mathfrak{s}^{\sigma-1}\\
&\times \phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_{\zeta_1}^{\zeta_2}(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&\leq |\mathfrak{F}(\zeta_2,\omega(\zeta_2))-\mathfrak{F}(\zeta_1,\omega(\zeta_1))| + \|\mu\|_{L^1_{[0,\mathfrak{t}_1]}}\eta(\rho)(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\\
&\times\Bigg[\Big|\int_0^{\zeta_1}\big\{(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}-(\zeta_1^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\big\}\mathfrak{s}^{\sigma-1}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\\
&+\Big|\int_{\zeta_1}^{\zeta_2}(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\Bigg]\\
&\leq |\mathfrak{F}(\zeta_2,\omega(\zeta_2))-\mathfrak{F}(\zeta_1,\omega(\zeta_1))| + \|\mu\|_{L^1_{[0,\mathfrak{t}_1]}}\eta(\rho)(q-1)\Lambda^{q-2}\frac{1}{\alpha\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\\
&\times\Big[B(\alpha+1,\beta)\big\{(\zeta_2^\sigma)^{\alpha+\beta}-(\zeta_1^\sigma)^{\alpha+\beta}\big\}+\frac{1}{1+\alpha}(\zeta_2^\sigma-\zeta_1^\sigma)^{\beta-1}\big\{\zeta_2^{\sigma(1+\alpha)}-\zeta_1^{\sigma(1+\alpha)}\big\}\Big].
\end{aligned}
$$

For $\omega\in\mathcal{B}_\rho$ and $\zeta_1,\zeta_2\in(\mathfrak{t}_\kappa,\mathfrak{s}_\kappa], \kappa=1,2,\ldots,\mathfrak{m}$ with $\zeta_1<\zeta_2$, we get

$$
|\mathcal{Q}\omega(\zeta_2)-\mathcal{Q}\omega(\zeta_1)|\leq|\mathfrak{F}(\zeta_2,\omega(\zeta_2))-\mathfrak{F}(\zeta_1,\omega(\zeta_1))|+|\mathfrak{H}_\kappa(\zeta_2,\omega(\zeta_2))-\mathfrak{H}_\kappa(\zeta_1,\omega(\zeta_1))|.
$$

For $\omega\in\mathcal{B}_\rho$ and $\zeta_1,\zeta_2\in(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}], \kappa=1,2,\ldots,\mathfrak{m}$ with $\zeta_1<\zeta_2$, we obtain

$$
\begin{aligned}
&|\mathcal{Q}\omega(\zeta_2)-\mathcal{Q}\omega(\zeta_1)|\\
&\leq |\mathfrak{F}(\zeta_2,\omega(\zeta_2))-\mathfrak{F}(\zeta_1,\omega(\zeta_1))|\\
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\zeta_2}(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\\
&-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\zeta_1}(\zeta_1^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&\leq |\mathfrak{F}(\zeta_2,\omega(\zeta_2))-\mathfrak{F}(\zeta_1,\omega(\zeta_1))| + \Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\zeta_1}\big\{(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}-(\zeta_1^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\big\}\mathfrak{s}^{\sigma-1}\\
&\times \phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_{\zeta_1}^{\zeta_2}(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&\leq |\mathfrak{F}(\zeta_2,\omega(\zeta_2))-\mathfrak{F}(\zeta_1,\omega(\zeta_1))| + \|\mu\|_{L^1_{(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}]}}\eta(\rho)(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\\
&\times\Bigg[\Big|\int_0^{\zeta_1}\big\{(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}-(\zeta_1^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\big\}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\\
&+\Big|\int_{\zeta_1}^{\zeta_2}(\zeta_2^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\Bigg]
\end{aligned}
$$

$$\leq |\mathfrak{F}(\zeta_2,\omega(\zeta_2)) - \mathfrak{F}(\zeta_1,\omega(\zeta_1))| + \|\mu\|L^1_{(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}]}\eta(\rho)(q-1)\Lambda^{q-2}\frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}$$
$$\times\Big[B(\alpha+1,\beta)\big\{(\zeta_2^\sigma)^{\alpha+\beta} - (\zeta_1^\sigma)^{\alpha+\beta}\big\} + \frac{1}{1+\alpha}(\zeta_2^\sigma-\zeta_1^\sigma)^{\beta-1}\big\{\zeta_2^{\sigma(1+\alpha)} - \zeta_1^{\sigma(1+\alpha)}\big\}\Big].$$

The RHS of the above inequality tends to 0 as $\zeta_2 \to \zeta_1$ without depending on ω. Thus $\mathcal{Q}(\mathcal{B}_\rho)$ is equicontinuous. As, $\mathcal{Q}$ is uniformly bounded and equicontinuous, it follows from the Arzelá-Ascoli theorem $\mathcal{Q}$ is a compact operator.

Hence by Schauder's fixed point theorem, we can say that $\mathcal{Q}$ has at least one fixed point $\omega \in \mathcal{B}_\rho$ which is a solution of the problem (5.1.1)-(5.1.4). □

Uniqueness Result

To establish the further results we impose the following hypotheses.

(A_1) For $\mathfrak{t} \in \mathscr{J}$ there exists a constant $\mathcal{L}_{\mathfrak{F}} > 0$ such that $\mathfrak{F} : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ satisfies the following
$$|\mathfrak{F}(\mathfrak{t},\omega_1) - \mathfrak{F}(\mathfrak{t},\omega_2)| \leq \mathcal{L}_{\mathfrak{F}}|\omega_1-\omega_2|, \qquad \forall \omega_1,\omega_2 \in \mathbb{R}.$$

(A_2) For $\mathfrak{t} \in (\mathfrak{s}_\kappa, \mathfrak{t}_{\kappa+1}]$ there exists a constant $\mathcal{L}_{\mathcal{G}} > 0$ such that $\mathcal{G} : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ satisfies
$$|\mathcal{G}(\mathfrak{t},\omega_1) - \mathcal{G}(\mathfrak{t},\omega_2)| \leq \mathcal{L}_{\mathcal{G}}|\omega_1-\omega_2|, \qquad \forall \omega_1,\omega_2 \in \mathbb{R}.$$

(A_3) For $\mathfrak{t} \in [\mathfrak{t}_\kappa, \mathfrak{s}_\kappa]$ there exists a constant $\mathcal{L}_\kappa$ such that for all $\kappa = 1,2,\ldots,\mathfrak{m}$, $\mathfrak{H}_\kappa : [\mathfrak{t}_\kappa,\mathfrak{s}_\kappa] \times \mathbb{R} \to \mathbb{R}$ satisfies
$$|\mathfrak{H}_\kappa(\mathfrak{t},\omega_1) - \mathfrak{H}_\kappa(\mathfrak{t},\omega_2)| \leq \mathcal{L}_\kappa|\omega_1-\omega_2|, \qquad \forall \omega_1,\omega_2 \in \mathbb{R}.$$

Theorem 5.3.2. *If the system* (5.1.1)-(5.1.4) *satisfies* (A_1)-(A_3) *then it will have a unique solution provided that,*

$$0 < \Theta = \max\Bigg\{\mathcal{L}_{\mathfrak{F}} + \mathcal{L}_{\mathcal{G}}(q-1)\Lambda^{(q-2)}(\mathcal{T}^\sigma)^{\alpha+\beta}\frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha+\beta+1)},$$
$$\mathcal{L}_{\mathfrak{F}} + \mathcal{L}_\kappa + \mathcal{L}_{\mathcal{G}}(q-1)\Lambda^{(q-2)}\frac{\Gamma(\alpha+1)}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\alpha+\beta+1)}$$
$$\times\Big[(\mathcal{T}^\sigma)^{\alpha+\beta} + \max_{0\leq\kappa\leq\mathfrak{m}}(\mathfrak{s}_\kappa^\sigma)^{\alpha+\beta}\Big]\Bigg\} \leq 1. \tag{5.3.2}$$

Proof. We have to show that the operator $\mathcal{Q}$ is contraction.

For $\omega_1,\omega_2 \in \mathcal{B}_\rho$ and $\mathfrak{t} \in [0,\mathfrak{t}_1]$, we have

$$|\mathcal{Q}\omega_1(\mathfrak{t}) - \mathcal{Q}\omega_2(\mathfrak{t})|$$
$$\leq |\mathfrak{F}(\mathfrak{t},\omega_1(\mathfrak{t})) - \mathfrak{F}(\mathfrak{t},\omega_2(\mathfrak{t}))|$$

$$
\begin{aligned}
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_1(\tau))d\tau\Big]d\mathfrak{s}\\
&-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_2(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&\le \mathcal{L}_{\mathfrak{F}}\|\omega_1-\omega_2\|_{\mathcal{PC}}+\frac{(q-1)\Lambda^{q-2}\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\mathcal{L}_{\mathcal{G}}\|\omega_1-\omega_2\|_{\mathcal{PC}}\\
&\times\Big|\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\int_0^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\\
&\le\Big\{\mathcal{L}_{\mathfrak{F}}+\mathcal{L}_{\mathcal{G}}(q-1)\Lambda^{q-2}(\mathcal{T}^{\sigma})^{\alpha+\beta}\frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Big\}\|\omega_1-\omega_2\|_{\mathcal{PC}}.
\end{aligned}
$$

For $\omega_1,\omega_2\in\mathcal{B}_{\rho}$ and $\mathfrak{t}\in(\mathfrak{t}_{\kappa},\mathfrak{s}_{\kappa}]$, $\kappa=1,2,\dots,\mathfrak{m}$, we get

$$
\begin{aligned}
|\mathcal{Q}\omega_1(\mathfrak{t})-\mathcal{Q}\omega_2(\mathfrak{t})| &\le|\mathfrak{F}(\mathfrak{t},\omega_1(\mathfrak{t}))-\mathfrak{F}(\mathfrak{t},\omega_2(\mathfrak{t}))|+|\mathfrak{H}_{\kappa}(\mathfrak{t},\omega_1(\mathfrak{t}))-\mathfrak{H}_{\kappa}(\mathfrak{t},\omega_2(\mathfrak{t}))|\\
&\le(\mathcal{L}_{\mathfrak{F}}+\mathcal{L}_{\kappa})\|\omega_1-\omega_2\|_{\mathcal{PC}}.
\end{aligned}
$$

For $\omega_1,\omega_2\in\mathcal{B}_{\rho}$ and $\mathfrak{t}\in(\mathfrak{s}_{\kappa},\mathfrak{t}_{\kappa+1}]$, $\kappa=1,2,\dots,\mathfrak{m}$, we get

$$
\begin{aligned}
&|\mathcal{Q}\omega_1(\mathfrak{t})-\mathcal{Q}\omega_2(\mathfrak{t})|\\
&\le|\mathfrak{F}(\mathfrak{t},\omega_1(\mathfrak{t}))-\mathfrak{F}(\mathfrak{t},\omega_2(\mathfrak{t}))|+|\mathfrak{H}_{\kappa}(\mathfrak{t},\omega_1(\mathfrak{t}))-\mathfrak{H}_{\kappa}(\mathfrak{t},\omega_2(\mathfrak{t}))|\\
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_{\kappa}}^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_1(\tau))d\tau\Big]d\mathfrak{s}\\
&-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_{\kappa}}^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_2(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&+\Big|\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{s}_{\kappa}}({\mathfrak{s}_{\kappa}}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_{\kappa}}^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_1(\tau))d\tau\Big]d\mathfrak{s}\\
&-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{s}_{\kappa}}({\mathfrak{s}_{\kappa}}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_{\kappa}}^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega_2(\tau))d\tau\Big]d\mathfrak{s}\Big|\\
&\le(\mathcal{L}_{\mathfrak{F}}+\mathcal{L}_{\kappa})\|\omega_1-\omega_2\|_{\mathcal{PC}}+\mathcal{L}_{\mathcal{G}}\|\omega_1-\omega_2\|_{\mathcal{PC}}\\
&\times\Bigg[\Big|(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_{\kappa}}^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\\
&+\Big|(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{s}_{\kappa}}({\mathfrak{s}_{\kappa}}^{\sigma}-\mathfrak{s}^{\sigma})^{\beta-1}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_{\kappa}}^{\mathfrak{s}}(\mathfrak{s}^{\sigma}-\tau^{\sigma})^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\Big|\Bigg]\\
&\le\Big[\mathcal{L}_{\mathfrak{F}}+\mathcal{L}_{\kappa}+\mathcal{L}_{\mathcal{G}}(q-1)\Lambda^{(q-2)}\frac{\Gamma(\alpha+1)}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\Big\{(\mathcal{T}^{\sigma})^{\alpha+\beta}+(\mathfrak{s}_{\kappa}^{\sigma})^{\alpha+\beta}\Big\}\Big]\|\omega_1-\omega_2\|_{\mathcal{PC}}.
\end{aligned}
$$

Thus, we get

$$\|\mathcal{Q}\omega_1-\mathcal{Q}\omega_2\|_{\mathcal{PC}}\le\Theta\|\omega_1-\omega_2\|_{\mathcal{PC}}.$$

From (5.3.2), it follows that $\mathcal{Q}$ is a contraction. Therefore, by the Banach contraction principle, we conclude that $\mathcal{Q}$ has a unique fixed point, which corresponds to the unique solution of the system (5.1.1)-(5.1.4). This completes the proof.

□

5.4 Stability Results

Here, we investigate two types of stability results:

5.4.1 Hyers-Ulam Stability

Definition 5.4.1. *The integral equation of* (5.1.1)-(5.1.4) *is said to be Hyers-Ulam stable, if for given $\epsilon > 0$ the following:*
for $\mathfrak{t} \in [0, \mathfrak{t}_1]$,

$$\Big|\omega(\mathfrak{t}) - \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) - \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau\Big] d\mathfrak{s}\Big| \leq \epsilon,$$

for $\mathfrak{t} \in (\mathfrak{t}_\kappa, \mathfrak{s}_\kappa]$, $\quad \kappa = 1, 2, \ldots, \mathfrak{m}$,

$$\big|\omega(\mathfrak{t}) - \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) - \mathfrak{H}_\kappa(\mathfrak{t}, \omega(\mathfrak{t}))\big| \leq \epsilon,$$

for $\mathfrak{t} \in (\mathfrak{s}_\kappa, \mathfrak{t}_{\kappa+1}]$, $\quad \kappa = 1, 2, \ldots, \mathfrak{m}$,

$$\begin{aligned}\Big|&\omega(\mathfrak{t}) - \mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) - \mathfrak{H}_\kappa(\mathfrak{s}_\kappa, \omega(\mathfrak{s}_\kappa)) \\ &- \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau\Big] d\mathfrak{s} \\ &+ \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{s}_\kappa} ({\mathfrak{s}_\kappa}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau\Big] d\mathfrak{s}\Big| \leq \epsilon\end{aligned}$$

hold. There is another function $\bar{\omega}(\mathfrak{t}) \in \mathcal{PC}(\mathscr{J}, \mathbb{R})$ satisfying

$$\bar{\omega}(\mathfrak{t}) = \begin{cases} \mathfrak{F}(\mathfrak{t}, \bar{\omega}(\mathfrak{t})) \\ +\frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \bar{\omega}(\tau)) d\tau\Big] d\mathfrak{s}, \\ \qquad\qquad \text{for } \mathfrak{t} \in [0, \mathfrak{t}_1], \\ \mathfrak{F}(\mathfrak{t}, \bar{\omega}(\mathfrak{t})) + \mathfrak{H}_\kappa(\mathfrak{t}, \bar{\omega}(\mathfrak{t})), \qquad \text{for } \mathfrak{t} \in (\mathfrak{t}_\kappa, \mathfrak{s}_\kappa], \quad \kappa = 1, 2, \ldots, \mathfrak{m}, \\ \mathfrak{F}(\mathfrak{t}, \bar{\omega}(\mathfrak{t})) + \mathfrak{H}_\kappa(\mathfrak{s}_\kappa, \bar{\omega}(\mathfrak{s}_\kappa)) \\ +\frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \bar{\omega}(\tau)) d\tau\Big] d\mathfrak{s} \\ -\frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{s}_\kappa} ({\mathfrak{s}_\kappa}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \bar{\omega}(\tau)) d\tau\Big] d\mathfrak{s}, \\ \qquad\qquad \text{for } \mathfrak{t} \in (\mathfrak{s}_\kappa, \mathfrak{t}_{\kappa+1}], \quad \kappa = 1, 2, \ldots, \mathfrak{m}, \end{cases} \tag{5.4.1}$$

and a constant $\Theta > 0$ independent of $\omega(\mathfrak{t})$ and $\bar{\omega}(\mathfrak{t})$ such that

$$|\omega(\mathfrak{t}) - \bar{\omega}(\mathfrak{t})| \leq \Theta\epsilon, \quad \mathfrak{t} \in \mathscr{J}.$$

Theorem 5.4.2. *Assume that* (A_1)-(A_3) *hold. Then the considered problem* (5.1.1)-(5.1.4) *is Hyers-Ulam stable.*

Proof. Let $\omega(\mathfrak{t})$ be the unique solution of (5.1.1)-(5.1.4) and let $\bar{\omega}(\mathfrak{t})$ is the approximate solution of (5.1.1)-(5.1.4) satisfying (5.4.1). Then, by arguing similar way in the proof of the Theorem 5.3.2, we obtain
For $\mathfrak{t} \in [0, \mathfrak{t}_1]$, we have

$$\begin{aligned}
&|\omega(\mathfrak{t}) - \bar{\omega}(\mathfrak{t})| \\
&\leq |\mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) - \mathfrak{F}(\mathfrak{t}, \bar{\omega}(\mathfrak{t}))| \\
&+ \Big| \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau \Big] d\mathfrak{s} \\
&- \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \bar{\omega}(\tau)) d\tau \Big] d\mathfrak{s} \Big| \\
&\leq \mathcal{L}_{\mathfrak{F}} \|\omega - \bar{\omega}\|_{\mathcal{PC}} + \frac{(q-1)\Lambda^{q-2}\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \mathcal{L}_{\mathcal{G}} \|\omega - \bar{\omega}\|_{\mathcal{PC}} \\
&\times \Big| \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \int_0^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} d\tau d\mathfrak{s} \Big| \\
&\leq \Big\{ \mathcal{L}_{\mathfrak{F}} + \mathcal{L}_{\mathcal{G}} (q-1)\Lambda^{q-2} (\mathcal{T}^\sigma)^{\alpha+\beta} \frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \Big\} \|\omega - \bar{\omega}\|_{\mathcal{PC}}.
\end{aligned}$$

For $\mathfrak{t} \in (\mathfrak{t}_\kappa, \mathfrak{s}_\kappa]$, $\kappa = 1, 2, \ldots, \mathfrak{m}$, we get

$$\begin{aligned}
|\omega(\mathfrak{t}) - \bar{\omega}(\mathfrak{t})| &\leq |\mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) - \mathfrak{F}(\mathfrak{t}, \bar{\omega}(\mathfrak{t}))| + |\mathfrak{H}_\kappa(\mathfrak{t}, \omega(\mathfrak{t})) - \mathfrak{H}_\kappa(\mathfrak{t}, \bar{\omega}(\mathfrak{t}))| \\
&\leq (\mathcal{L}_{\mathfrak{F}} + \mathcal{L}_\kappa) \|\omega - \bar{\omega}\|_{\mathcal{PC}}.
\end{aligned}$$

For $\mathfrak{t} \in (\mathfrak{s}_\kappa, \mathfrak{t}_{\kappa+1}]$, $\kappa = 1, 2, \ldots, \mathfrak{m}$, we have

$$\begin{aligned}
&|\omega(\mathfrak{t}) - \bar{\omega}(\mathfrak{t})| \\
&\leq |\mathfrak{F}(\mathfrak{t}, \omega(\mathfrak{t})) - \mathfrak{F}(\mathfrak{t}, \bar{\omega}(\mathfrak{t}))| + |\mathfrak{H}_\kappa(\mathfrak{t}, \omega(\mathfrak{t})) - \mathfrak{H}_\kappa(\mathfrak{t}, \bar{\omega}(\mathfrak{t}))| \\
&+ \Big| \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau \Big] d\mathfrak{s} \\
&- \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \bar{\omega}(\tau)) d\tau \Big] d\mathfrak{s} \Big| \\
&+ \Big| \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{s}_\kappa} ({\mathfrak{s}_\kappa}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \omega(\tau)) d\tau \Big] d\mathfrak{s} \\
&- \frac{\sigma^{1-\beta}}{\Gamma(\beta)} \int_0^{\mathfrak{s}_\kappa} ({\mathfrak{s}_\kappa}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \phi_q \Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} \mathcal{G}(\tau, \bar{\omega}(\tau)) d\tau \Big] d\mathfrak{s} \Big| \\
&\leq (\mathcal{L}_{\mathfrak{F}} + \mathcal{L}_\kappa) \|\omega - \bar{\omega}\|_{\mathcal{PC}} + \mathcal{L}_{\mathcal{G}} \|\omega - \bar{\omega}\|_{\mathcal{PC}} \\
&\times \Bigg[\Big| (q-1)\Lambda^{q-2} \frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\mathfrak{t}} (\mathfrak{t}^\sigma - \mathfrak{s}^\sigma)^{\beta-1} \mathfrak{s}^{\sigma-1} \int_{\mathfrak{s}_\kappa}^{\mathfrak{s}} (\mathfrak{s}^\sigma - \tau^\sigma)^{\alpha-1} \tau^{\sigma-1} d\tau d\mathfrak{s} \Big|
\end{aligned}$$

$$+\left|(q-1)\Lambda^{q-2}\frac{\sigma^{2-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}({\mathfrak{s}_\kappa}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}d\tau d\mathfrak{s}\right|\Bigg]$$
$$\leq\Bigg[\mathcal{L}_{\mathfrak{F}}+\mathcal{L}_\kappa+\mathcal{L}_{\mathcal{G}}(q-1)\Lambda^{(q-2)}\frac{\Gamma(\alpha+1)}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\Big\{(\mathcal{T}^\sigma)^{\alpha+\beta}+(\mathfrak{s}_\kappa^\sigma)^{\alpha+\beta}\Big\}\Bigg]\|\omega-\bar{\omega}\|_{\mathcal{PC}}.$$

Thus from (5.3.2) we conclude that, $|\omega(\mathfrak{t})-\bar{\omega}(\mathfrak{t})|\leq\Theta\|\omega-\bar{\omega}\|_{\mathcal{PC}}$. Hence the considered problem (5.1.1)-(5.1.4) is Hyers-Ulam stable. □

5.4.2 Hyers-Ulam-Rassias Stability

Definition 5.4.3. *The integral equation of* (5.1.1)-(5.1.4) *is said to be Hyers-Ulam-Rassias stable if for a non-decreasing function χ the following:*
for $\mathfrak{t}\in[0,\mathfrak{t}_1]$,

$$\Bigg|\omega(\mathfrak{t})-\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Bigg|\leq\chi(\mathfrak{t}),$$

for $\mathfrak{t}\in(\mathfrak{t}_\kappa,\mathfrak{s}_\kappa],\quad\kappa=1,2,\ldots,\mathfrak{m}$,

$$\Big|\omega(\mathfrak{t})-\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))-\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t}))\Big|\leq\chi(\mathfrak{t}),$$

for $\mathfrak{t}\in(\mathfrak{s}_\kappa,\mathfrak{t}_{\kappa+1}],\quad\kappa=1,2,\ldots,\mathfrak{m}$,

$$\Bigg|\omega(\mathfrak{t})-\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))-\mathfrak{H}_\kappa(\mathfrak{s}_\kappa,\omega(\mathfrak{s}_\kappa))-\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}+\frac{\sigma^{1-\beta}}{\Gamma(\beta)}\int_0^{\mathfrak{s}_\kappa}({\mathfrak{s}_\kappa}^\sigma-\mathfrak{s}^\sigma)^{\beta-1}\mathfrak{s}^{\sigma-1}\phi_q\Big[\frac{\sigma^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathfrak{s}_\kappa}^{\mathfrak{s}}(\mathfrak{s}^\sigma-\tau^\sigma)^{\alpha-1}\tau^{\sigma-1}\mathcal{G}(\tau,\omega(\tau))d\tau\Big]d\mathfrak{s}\Bigg|\leq\chi(\mathfrak{t})$$

hold. There is an another solution $\bar{\omega}(\mathfrak{t})\in\mathcal{PC}(\mathscr{J},\mathbb{R})$ satisfying (5.4.1) and a constant $\Theta>0$ independent of both $\omega(\mathfrak{t})$ and $\bar{\omega}(\mathfrak{t})$ such that

$$|\omega(\mathfrak{t})-\bar{\omega}(\mathfrak{t})|\leq\Theta\chi(\mathfrak{t}),\qquad\forall t\in\mathscr{J}.$$

Theorem 5.4.4. *The considered problem* (5.1.1)-(5.1.4) *is Hyers-Ulam-Rassias stable under the assumptions* (A_1)-(A_3).

Proof. The proof proceeds similarly to the proof of Theorem 5.4.2. After following all the steps of the Theorem 5.4.2, with the help of the Definition 5.4.3 we will get that $|\omega(\mathfrak{t})-\bar{\omega}(\mathfrak{t})|\leq\Theta\chi(\mathfrak{t})$, where Θ shares the same value as in (5.3.2). Thus the considered problem (5.1.1)-(5.1.4) is Hyers-Ulam-Rassias stable. □

5.5 Example

Consider the following impulsive fractional differential equation with $p-$Laplacian operator

$$\begin{aligned}
&{}^{C}\mathcal{D}^{\frac{1}{2},\frac{5}{2}}\left[\phi_2\left\{{}^{C}\mathcal{D}^{\frac{1}{3},\frac{5}{2}}\left(\omega(\mathfrak{t})-\frac{|\cos\mathfrak{t}|}{10^3}\omega(\mathfrak{t})\right)\right\}\right]=\frac{1}{\mathfrak{t}+10^4}\omega(\mathfrak{t}),\quad \mathfrak{t}\in\left(0,\frac{1}{2}\right]\cup\left(1,\frac{3}{2}\right],\\
&\omega(\mathfrak{t})-\frac{|\cos\mathfrak{t}|}{10^3}\omega(\mathfrak{t})=\frac{1}{10^2}\big(\mathfrak{t}^2+|\sin\omega(\mathfrak{t})|\big),\quad \mathfrak{t}\in\left(\frac{1}{2},1\right],\\
&\omega(\mathfrak{t})-\frac{|\cos\mathfrak{t}|}{10^3}\omega(\mathfrak{t})\Big|_{\mathfrak{t}=0}=0,\\
&\phi_2\left\{{}^{C}\mathcal{D}^{\frac{1}{3},\frac{5}{2}}\left(\omega(\mathfrak{s}_\kappa)-\frac{|\cos\mathfrak{s}_\kappa|}{10^3}\omega(\mathfrak{s}_\kappa)\right)\right\}=0,\quad \kappa=0,1.
\end{aligned}\tag{5.5.1}$$

Here we have,

$$\alpha=\frac{1}{2},\quad \beta=\frac{1}{3},\quad \sigma=\frac{5}{2},\quad p=q=2,\quad \mathscr{J}=\left[0,\frac{3}{2}\right],\quad 0=\mathfrak{s}_0<\mathfrak{t}_1=\frac{1}{2}<\mathfrak{s}_1=1<\mathfrak{t}_2=\frac{3}{2}.$$

Compared with (5.1.1)-(5.1.4), we get

$$\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))=\frac{|\cos\mathfrak{t}|}{10^3}\omega(\mathfrak{t}),\quad \mathcal{G}(\mathfrak{t},\omega(\mathfrak{t}))=\frac{1}{\mathfrak{t}+10^4}\omega(\mathfrak{t}),\quad \mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t}))=\frac{1}{10^2}\big(\mathfrak{t}^2+|\sin\omega(\mathfrak{t})|\big).$$

Thus, we have

$$|\mathfrak{F}(\mathfrak{t},\omega(\mathfrak{t}))|\le\frac{1}{10^3}\|\omega\|_{\mathcal{PC}},\quad |\mathcal{G}(\mathfrak{t},\omega(\mathfrak{t}))|\le\frac{1}{10^4}\|\omega\|_{\mathcal{PC}},\quad |\mathfrak{H}_\kappa(\mathfrak{t},\omega(\mathfrak{t}))|\le\frac{2}{10^2}.$$

Now, we observe that

$$\begin{aligned}
\max\Bigg\{&\|\xi\|\psi(\rho)+\|\mu\|\eta(\rho)(q-1)\Lambda^{(q-2)}(\mathcal{T}^\sigma)^{\alpha+\beta}\frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha+\beta+1)},\\
&\|\xi\|\psi(\rho)+\|\gamma\|+\|\mu\|\eta(\rho)(q-1)\Lambda^{(q-2)}\frac{\Gamma(\alpha+1)}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\\
&\times\left[(\mathcal{T}^\sigma)^{\alpha+\beta}+\max_{0\le\kappa\le\mathfrak{m}}(\mathfrak{s}_\kappa^\sigma)^{\alpha+\beta}\right]\Bigg\}\\
&=\max\{0.00111\rho,0.0210824\rho\}=0.0210824\rho<\rho.
\end{aligned}$$

Hence all the assumptions (H_1)-(H_3) hold and Theorem 5.3.1 is satisfied. Therefore (5.5.1) has a solution. And

$$|\mathfrak{F}(\mathfrak{t},\omega_1(\mathfrak{t}))-\mathfrak{F}(\mathfrak{t},\omega_2(\mathfrak{t}))|\le\frac{1}{10^3}\|\omega_1-\omega_2\|_{\mathcal{PC}},$$

$$|\mathcal{G}(\mathfrak{t},\omega_1(\mathfrak{t}))-\mathfrak{G}(\mathfrak{t},\omega_2(\mathfrak{t}))|\le\frac{1}{10^4}\|\omega_1-\omega_2\|_{\mathcal{PC}},$$

$$|\mathfrak{H}_\kappa(\mathfrak{t},\omega_1(\mathfrak{t})) - \mathfrak{H}_\kappa(\mathfrak{t},\omega_2(\mathfrak{t}))| \leq \frac{1}{10^2}\|\omega_1 - \omega_2\|_{\mathcal{PC}}.$$

Hence the assumptions (A_1)-(A_3) hold. Further we have,

$$\max\Bigg\{\mathcal{L}_{\mathfrak{F}} + \mathcal{L}_{\mathcal{G}}(q-1)\Lambda^{(q-2)}(\mathcal{T}^\sigma)^{\alpha+\beta}\frac{1}{\sigma^{\alpha+\beta}\Gamma(\alpha+\beta+1)},$$

$$\mathcal{L}_{\mathfrak{F}} + \mathcal{L}_\kappa + \mathcal{L}_{\mathcal{G}}(q-1)\Lambda^{(q-2)}\frac{\Gamma(\alpha+1)}{\sigma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\Big[(\mathcal{T}^\sigma)^{\alpha+\beta} + \max_{0\leq\kappa\leq\mathfrak{m}}(\mathfrak{s}_\kappa^\sigma)^{\alpha+\beta}\Big]\Bigg\}$$

$$= \max\{0.00111, 0.0110824\} = 0.0110824 < 1.$$

Thus, from Theorem 5.3.2 we can say that (5.5.1) has a unique solution. Also the conditions of Theorem 5.4.2 and Theorem 5.4.4 can be examined easily in the similar way. Therefore the equation (5.5.1) is Hyers-Ulam stable.