Chapter 6

Arithmetic properties of overcubic partition triples

6.1 Introduction

From Section 1.7, we recall that a cubic partition (denoted by $a(n)$) of n is a partition of n in which the even parts can appear in two colors. Assuming $a(0) = 1$, the generating function is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{f_1 f_2}.$$

Shortly after Chan introduced the cubic partition, Kim [97] studied the overpartition analogue of the cubic partition, called the *overcubic partition* function. The number of overcubic partitions of n, denoted by $\overline{a}(n)$ counts the number of overlined version of the cubic partitions counted by $a(n)$: that is, the cubic partitions where the first instance of each part is allowed to be overlined. The generating function is given by

$$\sum_{n=0}^{\infty} \overline{a}(n)q^n = \frac{f_4}{f_1^2 f_2}. \tag{6.1}$$

Zhao and Zhong [166] subsequently studied the number of *cubic partition pairs*,

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denoted by $b(n)$ with the following generating function

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{f_1^2 f_2^2}.$$

The overpartitions version of this function was studied by Kim [98], who denoted by $\overline{b}(n)$ the number of overcubic partition pairs of n. The generating function for this function is given by

$$\sum_{n=0}^{\infty} \overline{b}(n)q^n = \frac{f_4^2}{f_1^4 f_2^2}.$$

As can be seen, we can extend this definition even further, as was done recently by Nayaka, Dharmendra, and Kumar [111]. They denoted by $\overline{bt}(n)$, the number of overcubic partition triples of n, and gave the following generating function

$$\sum_{n=0}^{\infty} \overline{bt}(n)q^n = \frac{f_4^3}{f_1^6 f_2^3}. \tag{6.2}$$

As mentioned in Section 1.4, Ramanujan-type congruences are widely studied for many subsets of the partition function. Such studies have also been done for $a(n), \overline{a}(n), b(n), \overline{b}(n)$, and as mentioned in Section 1.7, very recently for $\overline{bt}(n)$ by Nayaka, Dharmendra, and Kumar [111]. The main goal of this chapter is to extend the list of congruences given by Nayaka, Dharmendra, and Kumar [111]. We also extend the definition of overcubic partition triples to overcubic partition k-tuples and explore arithmetic properties of this class of partitions.

Before stating our main results, we note that Nayaka, Dharmendra, and Kumar [111, Eq. (46)] showed that for all $n \geq 1$, we have

$$\overline{bt}(2n+1) \equiv 0 \pmod{2}.$$

In fact, we have for all $n \geq 1$

$$\overline{bt}(n) \equiv 0 \pmod{2}. \tag{6.3}$$

This follows easily from the binomial theorem by observing that

$$\sum_{n=0}^{\infty} \overline{bt}(n)q^n = \prod_{i=1}^{\infty} \left(\frac{1+q^{2i}}{1+q^{2i}-2q^i}\right)^3 = \prod_{i=1}^{\infty} \left(1 + 2\frac{q^i}{1+q^{2i}-2q^i}\right)^3 \equiv 1 \pmod{2}.$$

We now state our first result.

Theorem 6.1. *For all $n \geq 0$, we have*

$$\overline{bt}(4n+3) \equiv 0 \pmod{4}, \tag{6.4}$$

$$\overline{bt}(8n+5) \equiv 0 \pmod{32}, \tag{6.5}$$

$$\overline{bt}(8n+6) \equiv 0 \pmod{4}, \tag{6.6}$$

$$\overline{bt}(8n+7) \equiv 0 \pmod{64}, \tag{6.7}$$

$$\overline{bt}(16n+10) \equiv 0 \pmod{32}, \tag{6.8}$$

$$\overline{bt}(16n+12) \equiv 0 \pmod{4}, \tag{6.9}$$

$$\overline{bt}(16n+14) \equiv 0 \pmod{64}, \tag{6.10}$$

$$\overline{bt}(32n+20) \equiv 0 \pmod{32}, \tag{6.11}$$

$$\overline{bt}(32n+24) \equiv 0 \pmod{4}, \tag{6.12}$$

$$\overline{bt}(32n+28) \equiv 0 \pmod{64}. \tag{6.13}$$

Remark 6.2. *Some of our congruences are better than those found by Nayaka, Dharmendra, and Kumar. For instance, they had proved [111, Theorem 1] for all $n \geq 0$*

$$\overline{bt}(8n+5) \equiv 0 \pmod{8} \quad \textit{and} \quad \overline{bt}(8n+7) \equiv 0 \pmod{32}.$$

They also proved [111, Theorem 4], for all $n \geq 0$

$$\overline{bt}(16n+10) \equiv 0 \pmod{16} \quad \textit{and} \quad \overline{bt}(16n+14) \equiv 0 \pmod{16}.$$

Finally, they had also proved [111, Theorem 5]

$$\overline{bt}(32n+20) \equiv 0 \pmod{16} \quad \textit{and} \quad \overline{bt}(32n+28) \equiv 0 \pmod{16}.$$

There are numerous other congruences modulo powers of 2 and multiples of 3 that the $\overline{bt}(n)$ function satisfies. We prove the following two such congruences here to give a flavour.

Theorem 6.3. *For all $n \geq 0$, we have*

$$\overline{bt}(72n+21) \equiv 0 \pmod{128}, \tag{6.14}$$

$$\overline{bt}(72n+69) \equiv 0 \pmod{384}. \tag{6.15}$$

We prove Theorems 6.1 and 6.3 in Section 6.2, using Smoot's implementation [148] of Radu's algorithm [122, 123] coming from the theory of modular forms.

The main purpose of proving Theorem 6.1 was to 'guess' the following theorem.

Theorem 6.4. *For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{bt}(2^\alpha(4n+3)) \equiv 0 \pmod{4}, \tag{6.16}$$

$$\overline{bt}(2^\alpha(8n+5)) \equiv 0 \pmod{32}. \tag{6.17}$$

We prove Theorem 6.4 in Section 6.3 using elementary means.

Likewise, we can define $\overline{b}_k(n)$ to be the number of overcubic partition k-tuples, with the following generating function

$$\sum_{n=0}^{\infty} \overline{b}_k(n) q^n = \frac{f_4^k}{f_1^{2k} f_2^k}. \tag{6.18}$$

Then $\overline{b}_1(n) = \overline{a}(n), \overline{b}_2(n) = \overline{b}(n)$ and $\overline{b}_3(n) = \overline{bt}(n)$. Similar to the $\overline{bt}(n)$ function, there seems to be many congruences that the $\overline{b}_k(n)$ function satisfies for powers of 2. We just state a few of them here, in the results below.

We begin with the following easy to prove result.

Theorem 6.5. *For all $n \geq 0$ and $k \geq 1$ with $n, k \in \mathbb{Z}$ we have*

$$\overline{b}_{2k+1}(n) \equiv \overline{a}(n) \pmod{4}.$$

Proof. We have

$$\sum_{n=0}^{\infty} \overline{b}_{2k+1}(n) q^n = \frac{f_4^{2k+1}}{f_1^{2(2k+1)} f_2^{2k+1}} = \frac{f_4^{2k} f_4}{f_1^{4k} f_2^{2k} f_1^2 f_2} \equiv \frac{f_4}{f_1^2 f_2} \pmod{4}.$$

This completes the proof, via (6.1). □

Our next theorem gives a general modulo 4 congruence for the overcubic partition k-tuples function.

Theorem 6.6. *For all $n \geq 0$, $k \geq 0$ with $n, k \in \mathbb{Z}$ and $p \geq 3$ prime, and all quadratic nonresidues r modulo p with $1 \leq r \leq p-1$ we have*

$$\overline{b}_{2k+1}(2pn + R) \equiv 0 \pmod{4},$$

where

$$R = \begin{cases} r, & \text{if } r \text{ is odd,} \\ p + r, & \text{if } r \text{ is even.} \end{cases}$$

Again the proof is not difficult, so we complete it here.

Proof of Theorem 6.6. From [142, Theorem 2.5], we know that, for all $n \geq 1$, $\overline{a}(n) \equiv 0 \pmod 4$ if and only if n is neither a square nor twice a square. So it is enough for us to show that $2pn + R$ as defined above is never a square and never twice a square, thanks to Theorem 6.5. Clearly, $2pn + R$ is always odd by definition, so it cannot be twice a square. Next, from the definition of R we see that $2pn + R \equiv r \pmod p$. Since r is defined to be a quadratic nonresidue modulo p, we know that r cannot be congruent to a square modulo p. Thus, $2pn + R$ cannot equal a square. This concludes the proof. □

Our final congruence result is the following result.

Theorem 6.7. *For all $n \geq 0$ and $k \geq 0$, we have*

$$\begin{aligned}
\overline{b}_{2k+1}(8n+1) &\equiv 0 \pmod 2, \\
\overline{b}_{2k+1}(8n+2) &\equiv 0 \pmod 2, \\
\overline{b}_{2k+1}(8n+3) &\equiv 0 \pmod 4, \\
\overline{b}_{2k+1}(8n+4) &\equiv 0 \pmod 2, \\
\overline{b}_{2k+1}(8n+5) &\equiv 0 \pmod 8, \\
\overline{b}_{2k+1}(8n+6) &\equiv 0 \pmod 4, \\
\overline{b}_{2k+1}(8n+7) &\equiv 0 \pmod{16}.
\end{aligned}$$

We give a sketch proof of this theorem in Section 6.5.

Finally, it turns out that the partition function $\overline{bt}(n)$ is almost always divisible by 2^k for $k \geq 1$. Specifically, we prove the following result in Section 6.4.

Theorem 6.8. *For $\ell \geq 1$, let $G(q) = \sum_{n=0}^{\infty} \overline{b}_\ell(n) q^n$. Then for every positive integer k,*

$$\lim_{X \to \infty} \delta_0(G, 2^k; X) = 1.$$

This chapter is organized as follows: Sections 6.2 – 6.4 contain the proofs of our results, and we end this chapter with some concluding remarks in Section 6.6.

6.2 Proofs of Theorems 6.1 and 6.3

In this section we prove Theorems 6.1 and 6.3 using an algorithmic approach. More specifically, we use Smoot's [148] implementation of Radu's algorithm [122, 123], which can be used to prove Ramanujan type congruences of the form stated in Section 6.1. The algorithm takes as an input the generating function

$$\sum_{n=0}^{\infty} a_r(n)q^n = \prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta},$$

and positive integers m and N, with M another positive integer and $(r_\delta)_{\delta|M}$ is a sequence indexed by the positive divisors δ of M. With this input, Radu's algorithm tries to produce a set $P_{m,j}(j) \subseteq \{0,1,\ldots,m-1\}$ which contains j and is uniquely defined by $m, (r_\delta)_{\delta|M}$ and j. Then, it decides if there exists a sequence $(s_\delta)_{\delta|N}$ such that

$$q^\alpha \prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{s_\delta} \cdot \prod_{j'\in P_{m,j}(j)} \sum_{n=0}^{\infty} a(mn+j')q^n,$$

is a modular function with certain restrictions on its behaviour on the boundary of $\mathbb{H}$.

Smoot [148] implemented this algorithm in Mathematica and we use his `RaduRK` package, which requires the software package `4ti2`. Documentation on how to intall and use these packages are available from Smoot [148]. We use this implemented `RaduRK` algorithm to prove Theorem 6.1 in the next section.

It is natural to guess that $N = m$ (which corresponds to the congruence subgroup $\Gamma_0(N)$), but this is not always the case, although they are usually closely related to one another. The determination of the correct value of N is an important problem for the usage of `RaduRK` and it depends on the Δ^* criterion described in the previous subsection. It is easy to check the minimum N which satisfies this criterion by running `minN[M, r, m, j]`. The generating function of $\overline{bt}(n)$ given in (6.2) can be described by setting $M = 4$ and $r = \{-6, -3, 3\}$.

Proof of Theorem 6.1. We only prove (6.7) in detail. We use (6.2) and calculate `minN[4,{-6,-3,3},8,7]`, which gives $N = 8$, which is easily handled in a modest laptop. Radu's algorithm now gives a straight proof of (6.7). Here we give the output of RK.

```
In[1] := RK[8,4,{-6,-3,3},8,7]
```

$$\prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty} a(n)\,q^n$$

$$\boxed{f_1(q)\cdot \prod_{j'\in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn+j')\,q^n = \sum_{g\in AB} g\cdot p_g(t)}$$

Modular Curve: $X_0(N)$

```
Out[2] =
```

N:	8
$\{M,(r_\delta)_{\delta\|M}\}$:	$\{4,\{-6,-3,3\}\}$
m:	8
$P_{m,r}(j)$:	$\{7\}$
$f_1(q)$:	$\frac{(q;q)_\infty^{69}(q^4;q^4)_\infty^{30}}{q^{15}(q^2;q^2)_\infty^{29}(q^8;q^8)_\infty^{64}}$
t:	$\frac{(q^4;q^4)_\infty^{12}}{q(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}$
AB:	$\{1\}$
$\{p_g(t)\text{: } g\in AB\}$	$\{9792t^{15}+6606336t^{14}+905825280t^{13}$ $+46058225664t^{12}+1124900028416t^{11}$ $+15177685794816t^{10}+122507156520960t^9$ $+616578030764032t^8+1960114504335360t^7$ $+3885487563472896t^6+4607590516391936t^5$ $+3018471877115904t^4+949826648801280t^3$ $+110835926040576t^2+2628519985152t\}$
Common Factor:	64

The interpretation of this output is as follows.

The first entry in the procedure call `RK[8, 4, {-6, -3, 3}, 8, 7]` corresponds to specifying $N = 8$, which fixes the space of modular functions

$$M(\Gamma_0(N)) := \text{the algebra of modular functions for } \Gamma_0(N).$$

The second and third entry of the procedure call `RK[8, 4, {-6, -3, 3}, 8, 7]` gives the assignment $\{M, (r_\delta)_{\delta|M}\} = \{4, (-6, -3, 3)\}$, which corresponds to specifying $(r_\delta)_{\delta|M} = (r_1, r_2, r_4) = (-6, -3, 3)$, so that

$$\sum_{n=0}^{\infty} \overline{bt}(n)q^n = \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \frac{f_4^3}{f_1^6 f_2^3}.$$

The last two entries of the procedure call `RK[8, 4, {-6, -3, 3}, 8, 7]` corresponds to the assignment $m = 8$ and $j = 7$, which means that we want the generating function

$$\sum_{n=0}^{\infty} \overline{bt}(n)(mn + j)q^n = \sum_{n=0}^{\infty} \overline{bt}(n)(8n + 7)q^n.$$

So, $P_{m,r}(j) = P_{8,r}(7)$ with $r = (-6, -3, 3)$.

The output $P_{m,r}(j) := P_{8,(-6,3,3)}(7) = \{7\}$ means that there exists an infinite product

$$f_1(q) = \frac{(q;q)_\infty^{69} (q^4;q^4)_\infty^{30}}{q^{15} (q^2;q^2)_\infty^{29} (q^8;q^8)_\infty^{64}},$$

such that

$$f_1(q) \sum_{n=0}^{\infty} \overline{bt}(n)(8n + 7)q^n \in M(\Gamma_0(8)).$$

Finally, the output

$$t = \frac{(q^4;q^4)_\infty^{12}}{q (q^2;q^2)_\infty^4 (q^8;q^8)_\infty^8}, \quad AB = \{1\}, \quad \text{and} \quad \{p_g(\text{t})\text{: g} \in AB\},$$

presents a solution to the question of finding a modular function $t \in M(\Gamma_0(8))$ and polynomials $p_g(t)$ such that

$$f_1(q) \sum_{n=0}^{\infty} \overline{bt}(8n + 7)q^n = \sum_{g \in AB} p_g(t) \cdot g$$

In this specific case, we see that the singleton entry in the set $\{p_g(\text{t})\text{: g} \in AB\}$ has the common factor 64, thus proving (6.7).

Since the proofs of the remaining congruences listed in Theorem 6.1 are similar, we record only the values of the input and the corresponding common factors in the output below.

Congruence	Input=RK[N,M,r,m,j]	Common factor in the output
(6.4)	RK[8,4,{-6,-3,3},4,3]	4
(6.5)	RK[8,4,{-6,-3,3},8,5]	32
(6.6)	RK[8,4,{-6,-3,3},8,6]	4
(6.8)	RK[8,4,{-6,-3,3},16,10]	32
(6.9)	RK[8,4,{-6,-3,3},16,12]	4
(6.10)	RK[8,4,{-6,-3,3},16,14]	64
(6.11)	RK[8,4,{-6,-3,3},32,20]	32
(6.12)	RK[8,4,{-6,-3,3},32,24]	4
(6.13)	RK[8,4,{-6,-3,3},32,28]	64

□

Remark 6.9. *One can refer to [11] or [137] for some more recent applications of the method.*

Proof of Theorem 6.3. Since the proof of Theorem 6.3 is similar to the proof of Theorem 6.1, we omit the details and record only the input and the corresponding common factors in the output below.

Congruence	Input=RK[N,M,r,m,j]	Common factor in the output
(6.14)	RK[12,4,{-6,-3,3},72,21]	128
(6.15)	RK[12,4,{-6,-3,3},72,69]	384

□

6.3 Proof of Theorem 6.4

Proof of (6.16). Nayaka, Dharmendra, and Kumar had found [111, Eq. (45)]

$$\sum_{n=0}^{\infty} \overline{bt}(n)q^n = \frac{f_4^3 f_8^{15}}{f_2^{18} f_{16}^6} + 6q\frac{f_4^5 f_8^9}{f_2^{18} f_{16}^2} + 12q^2\frac{f_4^7 f_8^3 f_{16}^2}{f_2^{18}} + 8q^3\frac{f_4^9 f_{16}^6}{f_2^{18} f_8^3}. \tag{6.19}$$

From (6.19) we have

$$\sum_{n=0}^{\infty} \overline{bt}(2n)q^n \equiv \frac{f_2^3 f_4^{15}}{f_1^{18} f_8^6} \pmod{4}. \tag{6.20}$$

Using (6.2) and (6.20) we obtain, for all $n \geq 0$

$$\overline{bt}(2n) \equiv \overline{bt}(n) \pmod{4}. \tag{6.21}$$

Combining (6.21) with (6.4) we obtain (6.16). □

Remark 6.10. *We note here, it can be proved that* $\overline{bt}(n) \equiv \overline{a}(n) \pmod{4}$*. With this and a result of Sellers [142, Corollary 2.6] for* $\overline{a}(n)$*, we get an alternate proof of* (6.16)*.*

Proof of (6.17)*.* We re-write (6.2) as

$$\sum_{n=0}^{\infty} \overline{bt}(n)q^n = \frac{f_4^3}{f_2^3} \cdot \frac{1}{(f_1^4)^2} \cdot f_1^2.$$

Employing (4.17), (2.33) and then extracting the even powered terms of q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{bt}(2n)q^n &= \frac{f_2^{29}}{f_1^{30} f_4^3 f_8^2} + 16q\frac{f_4^{13} f_2^5}{f_1^{22} f_8^2} - 16q\frac{f_8^2 f_2^{19}}{f_1^{26} f_4} \\ &\equiv \frac{f_1^2 f_2^{13}}{f_4^3 f_8^2} + 16qf_1^{24} - 16qf_1^{24} \equiv \frac{f_1^2 f_2^{13}}{f_4^3 f_8^2} \pmod{32}.\end{aligned}$$

Extracting the even powered terms of q by using (4.17), we have

$$\sum_{n=0}^{\infty} \overline{bt}(4n)q^n \equiv \frac{f_1^{14} f_4^3}{f_2^5 f_8^2} \pmod{32}. \tag{6.22}$$

Again, re-writing (6.2) as

$$\sum_{n=0}^{\infty} \overline{bt}(n)q^n = \frac{f_4^3}{f_2^3} \cdot \frac{1}{f_1^4} \cdot \frac{1}{f_1^2}.$$

Employing (2.32), (2.33) and then extracting the even powered terms of q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{bt}(2n)q^n &= \frac{f_4 f_2^{17}}{f_1^{22} f_8^2} + 8q\frac{f_4^3 f_8^2 f_2^7}{f_1^{18}} \equiv \frac{f_2 f_4 f_1^{10}}{f_8^2} + 8qf_2 f_4 f_1^{18} \pmod{32} \\ &\equiv \frac{f_2 f_4}{f_8^2} \cdot (f_1^4)^2 \cdot f_1^2 + 8qf_2 f_4 \cdot (f_1^4)^4 \cdot f_1^2 \pmod{32}.\end{aligned}$$

Employing (4.17), (2.31) and further extracting the even powered terms of q, we have

$$\sum_{n=0}^{\infty} \overline{bt}(4n)q^n \equiv \frac{f_2^{19}}{f_1^2 f_4^5 f_8^2} + 16q\frac{f_1^2 f_8^2 f_2^9}{f_4^3} + 16q\frac{f_1^6 f_4^{11}}{f_2^5 f_8^2} - 16q\frac{f_8^2 f_2^{41}}{f_1^6 f_4^{17}}$$

$$\equiv \frac{f_2^{19}}{f_1^2 f_4^5 f_8^2} + 16q f_1^{24} + 16q f_1^{24} - 16q f_1^{24}$$
$$\equiv \frac{f_2^{19}}{f_1^2 f_4^5 f_8^2} + 16q f_{16} f_8 \pmod{32}.$$

Finally, employing (2.32) and extracting the terms involving q^{2n}, we have

$$\sum_{n=0}^{\infty} \overline{bt}(8n) q^n \equiv \frac{f_1^{14} f_4^3}{f_2^5 f_8^2} \pmod{32}. \tag{6.23}$$

From (6.22) and (6.23), we have

$$\overline{bt}(4n) \equiv \overline{bt}(8n) \pmod{32}. \tag{6.24}$$

Combining (6.5), (6.8), (6.11) and (6.24), we conclude the proof. □

6.4 Proof of Theorem 6.8

Before going into the proof of Theorem 6.8, we recall some fundamental ideas from the theory of modular forms. We recall that the Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an *eta*-quotient if it can be expressed as a finite product of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and each r_δ is an integer.

Define

$$G(\tau) := \frac{\eta(\delta_1\tau)^{r_1}\eta(\delta_2\tau)^{r_2}\cdots\eta(\delta_u\tau)^{r_u}}{\eta(\gamma_1\tau)^{s_1}\eta(\gamma_2\tau)^{s_2}\cdots\eta(\gamma_t\tau)^{s_t}} = q^{\frac{E_G}{24}} \sum_{n=0}^{\infty} \rho(n) q^n, \tag{6.25}$$

where r_i, s_i, δ_i, and γ_i are positive integers with $\delta_1, ..., \delta_u, \gamma_1, ..., \gamma_t$ distinct, $u, t \geq 0$ and

$$E_G := \sum_{i=1}^{u} \delta_i r_i - \sum_{i=1}^{t} \gamma_i s_i.$$

We say that $G(\tau)$ is lacunary modulo M whenever $\sum_{n=0}^{\infty} \rho(n) q^n$ has that property.

The weight of $G(\tau)$ is given by

$$\frac{1}{2}\left(\sum_{i=1}^{u} r_i - \sum_{i=1}^{t} s_i\right).$$

Also define $\mathcal{D}_{\mathcal{G}} := \gcd(\delta_1, \delta_2, \ldots, \delta_u)$. The following result by Cotron *et al.* [63] will be useful in proving Theorem 6.8.

Theorem 6.11. [63, Theorem 1.1] *Suppose $G(\tau)$ is an eta-quotient of the form* (6.25) *with integer weight. If p is a prime such that p^a divides $\mathcal{D}_{\mathcal{G}}$ and*

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t} \gamma_i s_i}{\sum_{i=1}^{u} \frac{r_i}{\delta_i}}}, \tag{6.26}$$

then $G(\tau)$ is lacunary modulo p^j for any positive integer j. Moreover, there exists a positive constant α, depending on p and j, such that the number of integers $n \leq X$ with p^j not dividing $b(n)$ is $O\left(\frac{X}{\log^{\alpha} X}\right)$.

Proof of Theorem 6.8. From (6.18), we recall

$$\sum_{n=0}^{\infty} \overline{b}_{\ell}(n) q^n = \frac{f_4^{\ell}}{f_1^{2\ell} f_2^{\ell}} = \frac{\eta^{\ell}(4z)}{\eta^{2\ell}(z)\eta^{\ell}(2z)}.$$

Following the notations used in (6.25) and the paragraph succeeding it, we have

$$\delta_1 = 4, r_1 = \ell, \quad \gamma_1 = 1, \quad s_1 = 2\ell, \quad \gamma_2 = 2, \quad \text{and} \quad s_2 = \ell.$$

Also $\mathcal{D}_{\mathcal{G}} = 4$ and the weight is $-\ell \in \mathbb{Z}$. Next, we see that $2^2 | 4$ and

$$2^2 \geq \sqrt{\frac{2\ell + 2\ell}{\frac{\ell}{4}}} = \sqrt{16} = 4.$$

Choosing $p = 2$ and $a = 2$ in Theorem 6.11, we complete the proof. □

6.5 Sketch Proof of Theorem 6.7

From the work of Sellers [142, Corollary 2.3] we can write

$$\sum_{n=0}^{\infty} \overline{b}_t(n) q^n = \left(\varphi(q) \prod_{i \geq 1} \varphi(q^{2^i})^{3 \cdot 2^{i-1}}\right)^t,$$

where

$$\varphi(q) := 1 + 2\sum_{n=0}^{\infty} q^{n^2} = \varphi(q^4) + 2q\psi(q^8),$$

with $\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}$. Since $\left(\prod_{i\geq 3}\varphi(q^{2^i})\right)^{3\cdot 2^{i-1}\cdot t}$ is a function of q^8, it is enough to do the 8-dissection of the first three terms. Also, as the highest modulus involved in the theorem is 16 and the other moduli are divisors of 16, we will prove our result modulo if we consider only modulo 16 this 8-dissection. Re-writing

$$\sum_{n=0}^{\infty}\overline{b}_t(n)q^n = \left(\sum_{j=0}^{7} a_{t,j}q^j F_{t,j}(q^8)\right)\left(\prod_{i\geq 3}\varphi(q^{2^i})\right)^{3\cdot 2^{i-1}\cdot t},$$

where $F_{t,j}(q^8)$ is a function of q^8 whose power series representation has integer coefficients. It suffices to just prove the following congruences

$$a_{t,1} \equiv 0 \pmod{2},$$
$$a_{t,2} \equiv 0 \pmod{2},$$
$$a_{t,3} \equiv 0 \pmod{4},$$
$$a_{t,4} \equiv 0 \pmod{2},$$
$$a_{t,5} \equiv 0 \pmod{8},$$
$$a_{t,6} \equiv 0 \pmod{4},$$
$$a_{t,7} \equiv 0 \pmod{16}.$$

We can do this using induction, and follows exactly the same pattern as the proofs of [143, Theorem 2.2] and [139, Lemma 6.1], so we omit the details here.

6.6 Concluding Remarks

1. Based on numerical calculations, we speculated the following congruences similar to (6.16) and (6.17): For all $n \geq 0$ and $\alpha \geq 0$, we have

$$\overline{bt}(2^{\alpha}(8n+7)) \equiv 0 \pmod{64},$$
$$\overline{bt}(144n+42) \equiv 0 \pmod{384},$$
$$\overline{bt}(2^{\alpha}(72n+21)) \equiv 0 \pmod{128},$$
$$\overline{bt}(2^{\alpha}(72n+69)) \equiv 0 \pmod{128}.$$

These congruences appeared as conjectures in a published paper [138] based on this chapter. However, Chen *et al.* [51] proved the congruences.

2. There is a closely related function to the $\overline{b}_k(n)$ function, namely overpartition k-tuples with odd parts. We denote by $\overline{OPT}_k(n)$ the number of overpartition k-tuples with odd parts of n. The generating function is given by
$$\sum_{n=0}^{\infty} \overline{OPT}_k(n) = \frac{f_2^{3k}}{f_1^{2k} f_4^{k}}.$$
In a series of papers [72, 139], the authors and their collaborators have studied several arithmetic properties that this function satisfies. It seems that, some of the techniques used in these papers translate directly for the $\overline{b}_k(n)$ function. In particular, the authors in [72] have explored congruences modulo powers of 2 and 3 for the $\overline{OPT}_{2k+1}(n)$ function (with $k \geq 1$), which we believe may lead to similar results for the $\overline{b}_k(n)$ function. Additionally, in [139] the authors have also found inifnite family of congruences modulo small powers of 2 for $\overline{OPT}_{2k+1}(n)$ (with $k \geq 1$) and $\overline{OPT}_4(n)$, we believe similar results also hold for $\overline{b}_k(n)$.

3. We have just scratched the surface for congruences modulo powers of 2, a systematic study will unearth several more. For instance, we believe that Theorem 6.7 can be strengthened further by taking appropriate values of k; however, the techniques that we have used in this chapter seem unsuitable to find a more general result. We again leave this as a future direction of research.