Abstract

In this thesis, we study sign patterns and congruences of certain infinite products involving the Rogers-Ramanujan continued fraction, arithmetic properties of certain partition functions, and parity biases in integer partitions.

We study the behavior of the signs of the coefficients of certain infinite products involving the Rogers-Ramanujan continued fraction. For example, if

$$\sum_{n=0}^{\infty} A(n)q^n := \frac{(q^2;q^5)_\infty^5 (q^3;q^5)_\infty^5}{(q;q^5)_\infty^5 (q^4;q^5)_\infty^5},$$

then $A(5n+1) > 0$, $A(5n+2) > 0$, $A(5n+3) > 0$, and $A(5n+4) < 0$. We also find a few congruences satisfied by some coefficients. For example, for all nonnegative integers n, $A(9n+4) \equiv 0 \pmod 3$, $A(16n+13) \equiv 0 \pmod 4$, and $A(15n+r) \equiv 0 \pmod{15}$, where $r \in \{4, 8, 13, 14\}$.

Partitions wherein the even parts appear in two different colors are known as cubic partitions. Recently, Merca introduced and studied the function $\Lambda(n)$, which is defined as the difference between the number of cubic partitions of n into an even number of parts and the number of cubic partitions of n into an odd number of parts. In particular, using Smoot's RaduRK Mathematica package, Merca proved the following congruences by finding the exact generating functions of the respective functions. For all $n \geq 0$,

$$\Lambda(9n+5) \equiv 0 \pmod 3,$$

$$\Lambda(27n+26) \equiv 0 \pmod 3.$$

By using generating function manipulations and dissections, da Silva and Sellers proved these congruences and two infinite families of congruences modulo 3 arising from these congruences. By employing Ramanujan's theta function identities, we present simplified formulas of the generating functions from which proofs of the

congruences of Merca as well as those of da Silva and Sellers follow quite naturally. We also study analogous partition functions wherein multiples of k appear in two different colors, where $k \in \{3, 5, 7, 23\}$.

A partition is said to be ℓ-regular if none of its parts is a multiple of ℓ. Let $b_5'(n)$ denote the number of 5-regular partitions into distinct parts (equivalently, into odd parts) of n. This function also has close connections to representation theory and combinatorics. We study arithmetic properties of $b_5'(n)$. We provide full characterization of the parity of $b_5'(2n+1)$, present several congruences modulo 4, and prove that the generating function of the sequence $(b_5'(5n+1))$ is lacunary modulo any arbitrary positive powers of 5.

In 2018, Andrews introduced the partition function $\mathcal{EO}(n)$, which counts the number of partitions of a positive integer n where each even part is less than each odd part. Furthermore, he defined another class of partitions, namely $\overline{\mathcal{EO}}(n)$ which counts the number of partitions counted by $\mathcal{EO}(n)$ where only the largest even part appears an odd number of times. Thereafter, many mathematicians proved numerous results for $\overline{\mathcal{EO}}(n)$. We continue the study and prove several congruences modulo $16, 10$ and 40 for $\overline{\mathcal{EO}}(n)$. We further prove that $\overline{\mathcal{EO}}(10n+8)$ and $\overline{\mathcal{EO}}(40n+38)$ are almost always divisible by 10 and 40, respectively.

We prove several new congruences for the overcubic partition triples function, using both elementary techniques and the theory of modular forms. We also generalize overcubic partition triples to overcubic partition k-tuples and prove a few arithmetic properties for these type of partitions.

Recently, the concept of parity bias in integer partitions has been studied by several authors. We continue this study here, but for non-unitary partitions (namely, partitions with parts greater than 1). We prove analogous results for these restricted partitions as those that have been obtained by B. Kim, E. Kim, and Lovejoy (2020) and B. Kim and E. Kim (2021). We also look at inequalities between two classes of partitions studied by Andrews (2019) where the parts are separated by parity (either all odd parts are smaller than all even parts or vice versa).