Chapter 1

Introduction

The thesis consists of seven chapters which includes this introductory chapter. This chapter aims to briefly explain the topics of the thesis. It mainly contains three topics. In Chapter 2, we study sign patterns and congruences of certain infinite products related to the Rogers-Ramanujan continued fraction. In Chapters 3–6, we study arithmetic properties of some partition functions which we shortly explain in this chapter. And finally, in Chapter 7, we study parity biases in non-unitary partitions (namely, the partitions where 1 is not allowed to be a part).

Throughout this thesis, for complex numbers a and q with $\mid q \mid < 1$ and integers $n \geq 0$, we define

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right), \qquad (a;q)_\infty := \prod_{j=0}^{\infty} \left(1 - aq^j\right).$$

For convenience, we adopt the notations:

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n$$

and

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty .$$

For $k \geq 1$, we also adopt

$$f_k := \left(q^k; q^k\right)_\infty .$$

In the following sections, we present a few useful definitions as well as some background material.

1.1 Ramanujan's theta functions

Ramanujan's general theta function $f(a,b)$ [34, Eq. 1.2.1] is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a,-b,ab;ab)_\infty, \quad | ab |< 1, \tag{1.1}$$

where the last equality is Jacobi's famous triple product identity [33, p. 35, Entry 19]. Three special cases of $f(a,b)$ are

$$f(-q) := f(-q,-q^2) = \sum_{j=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1, \tag{1.2}$$

$$\varphi(q) := f(q,q) = \sum_{j=-\infty}^{\infty} q^{j^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.3}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{j=0}^{\infty} q^{j(j+1)/2} = \frac{f_2^2}{f_1}. \tag{1.4}$$

Replacing q by $-q$ in (1.3) and (1.4), we have

$$\varphi(-q) = \frac{f_1^2}{f_2}, \tag{1.5}$$

$$\psi(-q) = \frac{f_1 f_4}{f_2}. \tag{1.6}$$

We also define

$$\chi(q) := (-q;q^2)_\infty. \tag{1.7}$$

The Euler Pentagonal Number Theorem [34, Corollary 1.3.5] is given by

$$f_1 = \sum_{k=-\infty}^{\infty} (-1)^n q^{\frac{k(3k-1)}{2}} \equiv \sum_{k=-\infty}^{\infty} q^{\frac{k(3k-1)}{2}} \pmod 2. \tag{1.8}$$

1.2 The Rogers-Ramanujan continued fraction

The celebrated Rogers-Ramanujan continued fraction $\mathcal{R}(q)$ is defined as

$$\mathcal{R}(q) := \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cdots}}}, \quad |q|<1,$$

and the Rogers-Ramanujan identities are given by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty},$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty},$$

where $G(q)$ and $H(q)$ are called the Rogers-Ramanujan functions.

Ramanujan [131] and Rogers [135] (see [33, Corollary, p. 30]) proved that

$$R(q) = \frac{H(q)}{G(q)} = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.9}$$

where we have defined $R(q) := q^{-1/5}\mathcal{R}(q)$.

1.3 n-Dissection of power series

For a power series $P(q)$ in q and a positive integer $n > 1$, the n-dissection of $P(q)$ is given by

$$P(q) = \sum_{j=0}^{n-1} q^j P_j\left(q^n\right),$$

where P_j's are power series in q. As an example, a 5-dissection of f_1 [34, p. 165] is

$$f_1 = f_{25}\left(\frac{1}{R\left(q^5\right)} - q - q^2 R\left(q^5\right)\right).$$

1.4 Partitions of positive integers

An integer partition of a positive integer n is a sequence of non-increasing parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ such that $\sum_{i=1}^{k} \lambda_k = n$. Let $p(n)$ count the number of partitions of any nonnegative integer n. For instance, there are 5 partitions of 4, namely

$$4, 3+1, 2+2, 2+1+1, \text{ and } 1+1+1+1,$$

and hence $p(4) = 5$. The generating function of $p(n)$ (due to Euler) is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)} = \frac{1}{f_1},$$

where we take the convention, $p(0)$=1. For a general survey of the theory of partitions, we refer to the books of Andrews [6] and Johnson [93].

One of the fundamental questions that arises in the theory of integer partitions is whether $p(n)$ or any interesting subclass of partitions satisfy any nice arithmetic properties. For $p(n)$, this answer is found in the following celebrated congruences of Ramanujan [128], [129], [130]: For all $n \geq 0$, we have

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

In 1919, Ramanujan [128] also conjectured that if $\delta = 5^a 7^b 11^c$ and γ is an integer such that $24\gamma \equiv 1 \pmod{\delta}$, then for all $n \geq 0$,

$$p(\delta n + \gamma) \equiv 0 \pmod{\delta}. \tag{1.10}$$

In [36], Ramanujan proved (1.10) for the case when a is arbitrary and $b = c = 0$ but the case when b is arbitrary and $a = c = 0$ of (1.10) turned out to be incorrect. The corrected version of (1.10) is stated below.

If $\delta' = 5^a 7^{b'} 11^c$, where $b' = b$ when $b = 0, 1, 2$ and $b' = \lfloor \frac{b+2}{2} \rfloor$ when $b > 2$, and γ is an integer such that $24\gamma \equiv 1 \pmod{\delta}$, then for all $n \geq 0$,

$$p(\delta n + \gamma) \equiv 0 \pmod{\delta'}.$$

In the above corrected version, Watson [154] proved the case when b is arbitrary and $a = c = 0$. In 1967, Atkin [15] provided a proof of (1.10) for arbitrary c and $a = b = 0$. In [87], Hirschhorn and Hunt found the exact generating function of $p(\delta n + \gamma)$ for arbitrary a and $b = c = 0$.

1.5 Arithmetic density

Given an integral power series $A(q) := \sum_{n=0}^{\infty} a(n) q^n$ and $0 \leq r \leq M$, the arithmetic density $\delta_r(A, M; X)$ is defined as

$$\delta_r(A, M; X) = \frac{\#\{n \leq X : a(n) \equiv r \pmod M\}}{X}.$$

An integral power series A is called *lacunary modulo M* if

$$\lim_{X \to \infty} \delta_0(A, M; X) = 1,$$

which means that almost all the coefficients of A are divisible by M.

Along with the study of Ramanujan-type congruences, the study of distribution of the coefficients of a power series modulo M is also an interesting arithmetic property to explore. In Chapters 4, 5 and 6, we study the arithmetic density of some partition functions. For example, in Chapter 5, we prove that

Theorem 1.1. *We have*

$$\lim_{X\to\infty} \frac{\#\{0 \le n \le X : \overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod{10}\}}{X} = 1,$$

where the function $\overline{\mathcal{EO}}(n)$ will be explained shortly in the paragraph succeeding Theorem 1.3.

In the remaining part of this chapter, we briefly present the topics of the thesis.

1.6 Some infinite products related to the Rogers-Ramanujan Identities

In his second notebook [131, p. 289] and the lost notebook [132, p. 365], Ramanujan recorded the following elegant identity:

$$R^5(q) = R(q^5) \cdot \frac{1 - 2qR(q^5) + 4q^2R^2(q^5) - 3q^3R^3(q^5) + q^4R^4(q^5)}{1 + 3qR(q^5) + 4q^2R^2(q^5) + 2q^3R^3(q^5) + q^4R^4(q^5)}. \tag{1.11}$$

The identity can also be found in his first letter to Hardy written on January 16, 1913. Various proofs of (1.11) can be found in the literature. For example, see the papers by Rogers [136], Watson [152], Ramanathan [126], Yi [165], and Gugg [82].

In Chapter 2, we study the sign patterns of $\frac{1}{R^5(q)}$, $R^5(q)$, $\frac{R^5(q)}{R(q^5)}$ and $\frac{R(q^5)}{R^5(q)}$. For instance, we prove that if $A(n)$ is defined by

$$\frac{1}{R^5(q)} := \sum_{n=0}^{\infty} A(n)q^n,$$

then for all nonnegative integers n, we have

$$A(5n+1) > 0.$$

We also prove some congruences for the infinite products mentioned above. For example, we prove that for all $n \geq 0$, we have

$$A(9n+4) \equiv 0 \pmod{3}.$$

1.7 Some partition functions

Since the time of Euler, the study of various special subsets as well as generalizations of the set of partitions has been an active topic of study. In this section, we take a look at some partition functions that we study in this thesis.

The cubic partition function $a(n)$ counts the number of partitions of a positive integer n in which even parts can appear in two colors. This function was introduced by Hei-Chi Chan [47], [48] in 2010 in connection to the so-called Ramanujan's cubic continued fraction. With the convention of $a(0) = 1$, the generating function of $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{f_1 f_2}.$$

Motivated by Ramanujan's so-called "most beautiful identity"

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6},$$

Chan [47] proved the following analogous identity:

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{f_3^3 f_6^3}{f_1^4 f_2^4}.$$

This clearly gives

$$a(3n+2) \equiv 0 \pmod{3},$$

which is analogous to

$$p(5n+4) \equiv 0 \pmod{5},$$

one of the three famous congruences for the partition function discovered by Ramanujan.

In Chapter 3, we find exact generating functions and congruences for some partition functions related to the cubic partition function. For instance, we prove the following theorem.

Theorem 1.2. *We have*

$$\sum_{n=1}^{\infty} \Lambda(9n+5)q^n = -3q\frac{f_1 f_2^4 f_{12}^6}{f_4^{11}},$$

where the partition function $\Lambda(n)$ will be defined in Section 3.1.

Next, a partition is said to be ℓ-regular if none of its parts is a multiple of ℓ. For example, 3+3+2+1 is a 4-regular partition of 9. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. Then, with the convention, $b_\ell(0) = 1$, the generating function of $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}.$$

The ℓ-regular partition function $b_\ell(n)$ has been studied quite extensively in the recent past by various authors. It is to be noted that the function $b_\ell(n)$ for prime ℓ gives the number of irreducible ℓ-modular representation of the symmetric group S_n [92]. One can see the following non-exhaustive list of papers for various works related to $b_\ell(n)$ for $\ell \geq 3$ (note that $b_2(n)$ counts the number of partitions of n into odd parts, which is, by Euler's theorem, equal to the number of partitions of n into distinct parts); arranged in alphabetical order of the first authors:

Ahmed and Baruah [1], Alladi [4], Andrews, Hirschhorn, and Sellers [14], Ballantine and Merca [17], Barman *et al.* [21], Baruah and Das [27], Calkin *et al.* [45], Carlson and Webb [46], Cui and Gu [64], [65], [66], [67] Dai [69], Dai *et al.* [70], Dandurand and Penniston [71], Furcy and Penniston [75], Gordon and Ono [78], Granville and Ono [80], Hirschhorn and Sellers [89], Hou, Sun, and Zhang [90], Iwata [91], Keith [94], Keith and Zanello [95, 96], Lin and Wang [104], Lovejoy [106], Lovejoy and Penniston [107], Mestrige [120], Ono and Penniston [114, 115], Penniston [117, 118, 119], Singh and Barman [144, 145], Singh, Singh, and Barman [146], Wang [151], Webb [155], Xia [157], Xia and Yao [158, 159], Yao [163, 164], Zhao, Jin, and Yao [167].

In Chapter 4, we extend this study and find arithmetic properties of 5-regular partitions into distinct parts (denoted by $b'_5(n)$). For example, we show that

Theorem 1.3. *For all $n \geq 0$,*

$$b'_5(2n+1) \equiv \begin{cases} 1 \pmod 2, & \textit{if } n = 15k^2 - 5k \textit{ for } k \in \mathbb{Z}, \\ 0 \pmod 2, & \textit{Otherwise.} \end{cases}$$

Studies related to the partition function have been an active topic of study since the last century. Studies related to restricted partition functions have also been of interest. One such function was first introduced by Andrews [8, 9]. Let $\mathcal{EO}(n)$ count the number of partitions of n in which each even part is less than each odd part. Also, let $\overline{\mathcal{EO}}(n)$ count the number of partitions of n counted by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. Andrews [8, Corollary 3.2] proved that the generating function of $\overline{\mathcal{EO}}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^n = \frac{f_4^3}{f_2^2}. \tag{1.12}$$

Since then, many mathematicians have found several arithmetic properties for $\overline{\mathcal{EO}}(n)$. In Chapter 5, we extend this study and find new congruences and density results for the $\overline{\mathcal{EO}}(n)$ function. For example, we prove that

Theorem 1.4. *For all $n \geq 0$ and $t \in \{1, 2, 3, 4\}$, we have*

$$\overline{\mathcal{EO}}(1250n + 250t + 208) \equiv 6\overline{\mathcal{EO}}(50n + 10t + 8) \pmod{16},$$
$$\overline{\mathcal{EO}}(10n) \equiv 13\overline{\mathcal{EO}}(250n + 8) \equiv 5\overline{\mathcal{EO}}(6250n + 208) \pmod{16},$$
$$\overline{\mathcal{EO}}(10n + 6) \equiv 13\overline{\mathcal{EO}}(250n + 158) \equiv 5\overline{\mathcal{EO}}(6250n + 3958) \pmod{16}.$$

An overpartition [62] of n is a partition of n where the first occurence of a part may be overlined. For example, the eight overpartitions of 3 are

$$3, \bar{3}, 2+1, \bar{2}+1, 2+\bar{1}, \bar{2}+\bar{1}, 1+1+1, \text{ and } \bar{1}+1+1.$$

In 2024, Nayaka, Dharmendra, and Kumar [111] proved various arithmetic properties for overcubic partition triples (denoted by $\overline{bt}$) which is the overpartition version of cubic partition triples. In Chapter 6, we extend this study and find new arithmetic properties for overcubic partition triples. For example, we prove that

Theorem 1.5. *For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{bt}(2^\alpha(4n+3)) \equiv 0 \pmod 4,$$

$$\overline{bt}(2^{\alpha}(8n+5)) \equiv 0 \pmod{32}.$$

1.8 Parity biases in integer partitions

In the theory of partitions, inequalities arising between two classes of partitions have a long tradition of study, see for instance work in this direction by Alder [3], Andrews [7], McLaughlin [109], Chern, Fu, and Tang [59] and Berkovich and Uncu [32]. In 2020, B. Kim, E. Kim and Lovejoy [102] introduced a phenomenon in integer partitions called *parity bias*, wherein the number of partitions of n with more odd parts (denoted by $p_o(n)$) are more in number than the number of partitions of n with more even parts (denoted by $p_e(n)$). To be specific, they proved the following theorem.

Theorem 1.6. *For $n \neq 2$, we have*

$$p_o(n) > p_e(n). \tag{1.13}$$

Further generalizations of the results of B. Kim, E. Kim, and Lovejoy [102] have been found by B. Kim and E. Kim [99] and Chern [57].

In Chapter 7, we study parity biases in non-unitary partitions, i.e., partitions where the part 1 is not allowed to be a part of the partition. We also study parity biases for another class of partitions defined by Andrews [8, 9].