Chapter 2

Sign patterns and congruences of certain infinite products involving the Rogers - Ramanujan continued fraction

2.1 Introduction

In 1978, Richmond and Szekeres [134] examined asymptotically the power series coefficients of a large class of infinite products including the product given in (1.9) and its reciprocal. In particular, they [134, Eq. (3.9)] proved that, if

$$\frac{1}{R(q)} = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} := \sum_{n=0}^{\infty} c(n) q^n,$$

then

$$c(n) = \frac{\sqrt{2}}{(5n)^{3/4}} \exp\left(\frac{4\pi}{25}\sqrt{5n}\right) \times \left\{ \cos\left(\frac{2\pi}{5}\left(n - \frac{2}{5}\right)\right) + \mathcal{O}(n^{-1/2}) \right\},$$

which implies that, for n sufficiently large,

$$c(5n) > 0, \ c(5n+1) > 0, \ c(5n+2) < 0, \ c(5n+3) < 0, \ \text{and } c(5n+4) < 0. \quad (2.1)$$

The contents of this chapter have appeared in *The Ramanujan Journal* [30].

They also gave a similar result for the power series coefficients of $R(q)$ from which it follows that, for n sufficiently large,

$$d(5n) > 0,\ d(5n+1) < 0,\ d(5n+2) > 0,\ d(5n+3) < 0,\ \text{and}\ d(5n+4) < 0, \tag{2.2}$$

where

$$R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} := \sum_{n=0}^{\infty} d(n) q^n.$$

Ramanujan, in his lost notebook [132, p. 50] recorded formulas for $\sum_{n=0}^{\infty} c(5n+j)q^n$ and $\sum_{n=0}^{\infty} d(5n+j)q^n$, $0 \le j \le 4$, which were proved by Andrews [5] (Also see Andrews and Berndt [10, Chapter 4]). Andrews used the formulas and a theorem of Gordon [77] to give partition-theoretic interpretations of these coefficients, and hence proved that (2.1) and (2.2) hold for all n except $c(2) = c(4) = c(9) = 0$, $d(3) = d(8) = 0$. Using the quintuple product identity [60], Hirschhorn [84] found exact q-product representations of $\sum_{n=0}^{\infty} c(5n+j)q^n$ and $\sum_{n=0}^{\infty} d(5n+j)q^n$, $0 \le j \le 4$, and concluded the periodicity of the signs of the coefficients $c(n)$ and $d(n)$, with two more exceptions, namely $d(13) = d(23) = 0$.

There are results on periodicity of the signs of the coefficients of certain infinite products in more general settings. For $1 \le r, s < m$, define

$$F_{m,r,s}(q) := \frac{(q^r;q^m)_\infty (q^{m-r};q^m)_\infty}{(q^s;q^m)_\infty (q^{m-s};q^m)_\infty}.$$

Note that $F_{5,1,2}(q) = R(q)$. In 1988, Ramanathan [127] proved the following result.

Theorem 2.1. *Suppose* $\gcd(m,r) = 1$. *Let*

$$F_{m,2r,r}(q) = \sum_{n=0}^{\infty} b(n) q^n.$$

If $\gcd(m,6) = 1$, *the signs of the* $b(n)$*'s are periodic with period* m.

Chan and Yesilyurt [49] proved Theorem 2.1 using the ideas of Hirschhorn [84] in a more general setting and without the condition $\gcd(m,6) = 1$. They also studied the periodicity of infinite products for some other values of r and s. Chern and Tang [58] studied the sign patterns of certain q-products related to the Rogers-Ramanujan continued fraction, namely, $R(q)R^2(q^2)$, $R^2(q)/R(q^2)$ and their reciprocals. For work on the periodicity of coefficients of similar infinite products, one can look at the following non-exhaustive list of papers: Dou and Xiao [73], Hirschhorn [85], Lin [105], Tang [149], Tang and Xia [150], Xia and Yao [160], and Xia and Zhou [161].

In this chapter, we investigate the behavior of the signs of the coefficients of the infinite products $R^5(q)$, $R^5(q)/R(q^5)$, and their reciprocals appearing in (1.11). We also find some interesting congruences satisfied by some coefficients. We state our results in the following theorems.

Theorem 2.2. *If $A(n)$ is defined by*

$$\frac{1}{R^5(q)} := \sum_{n=0}^{\infty} A(n)q^n,$$

then for all nonnegative integers n, we have

$$A(5n+1) > 0, \tag{2.3}$$

$$A(5n+2) > 0, \tag{2.4}$$

$$A(5n+3) > 0, \tag{2.5}$$

$$A(5n+4) < 0. \tag{2.6}$$

Theorem 2.3. *If $B(n)$ is defined by*

$$R^5(q) := \sum_{n=0}^{\infty} B(n)q^n,$$

then for all nonnegative integers n, we have

$$B(5n+1) < 0, \tag{2.7}$$

$$B(5n+2) > 0, \tag{2.8}$$

$$B(5n+3) < 0, \tag{2.9}$$

$$B(5n+4) > 0. \tag{2.10}$$

Theorem 2.4. *If $C(n)$ is defined by*

$$\frac{R^5(q)}{R(q^5)} := \sum_{n=0}^{\infty} C(n)q^n,$$

then for all nonnegative integers n, we have

$$C(5n) < 0, \tag{2.11}$$

$$C(5n+1) < 0, \tag{2.12}$$

$$C(5n+2) > 0, \tag{2.13}$$

$$C(5n+3) < 0, \tag{2.14}$$

$$C(5n+4) > 0, \tag{2.15}$$

except $C(0) = 1$.

Remark 2.5. *Theorem 2.4 shows that the signs of* $C(n)$ *are periodic with period 5.*

Theorem 2.6. *If* $D(n)$ *is defined by*

$$\frac{R(q^5)}{R^5(q)} := \sum_{n=0}^{\infty} D(n) q^n,$$

then for all nonnegative integers n*, we have*

$$D(5n) < 0, \tag{2.16}$$

$$D(5n+2) > 0, \tag{2.17}$$

$$D(5n+3) > 0, \tag{2.18}$$

$$D(5n+4) < 0, \tag{2.19}$$

except $D(0) = 1$.

We state some congruences satisfied by $A(n)$, $B(n)$, $C(n)$, and $D(n)$ in the next theorem.

Theorem 2.7. *For all* $n \geq 0$*, we have*

$$A(9n+4) \equiv 0 \pmod{3}, \tag{2.20}$$

$$B(9n+2) \equiv 0 \pmod{3}, \tag{2.21}$$

$$A(16n+13) \equiv 0 \pmod{4}, \tag{2.22}$$

$$B(16n+11) \equiv 0 \pmod{4}, \tag{2.23}$$

$$A(15n+r) \equiv 0 \pmod{15}, \text{ where } r \in \{4, 8, 13, 14\}, \tag{2.24}$$

$$B(15n+r) \equiv 0 \pmod{15}, \text{ where } r \in \{2, 6, 11, 12\}, \tag{2.25}$$

$$C(15n+r) \equiv 0 \pmod{30}, \text{ where } r \in \{3, 13\}, \tag{2.26}$$

$$D(15n+r) \equiv 0 \pmod{30}, \text{ where } r \in \{7, 12\}. \tag{2.27}$$

We organize this chapter as follows: In the next section, we give some preliminary results on Ramanujan's theta functions and t-dissections. In Sections 2.3–2.7, we prove Theorems 2.2–2.7. Some concluding remarks and conjectures are presented in the final section of this chapter.

2.2 Preliminary lemmas

In the following lemma, we recall two useful identities.

Lemma 2.8. [33, p. 51 and p. 350] *We have*

$$f(q,q^5)=\psi(-q^3)\chi(q), \tag{2.28}$$

$$f(q,q^2)=\frac{\varphi(-q^3)}{\chi(-q)}, \tag{2.29}$$

where

$$\chi(-q)=(q;q^2)_\infty=\frac{f_1}{f_2}. \tag{2.30}$$

In the next three lemmas, we recall some known 2-, 3-, and 5-dissection formulas.

Lemma 2.9. [26, Lemma 2] *We have*

$$f_1^4=\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}, \tag{2.31}$$

$$\frac{1}{f_1^2}=\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}, \tag{2.32}$$

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}, \tag{2.33}$$

$$\frac{f_1}{f_5}=\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}, \tag{2.34}$$

$$\frac{f_5}{f_1}=\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}. \tag{2.35}$$

Lemma 2.10. [33, p. 49] *We have*

$$\varphi(q)=\varphi(q^9)+2qf(q^3,q^{15}), \tag{2.36}$$

$$\psi(q)=f(q^3,q^6)+q\psi(q^9). \tag{2.37}$$

Lemma 2.11. [34, p. 165] *We have*

$$f_1=f_{25}\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right) \tag{2.38}$$

and

$$\frac{1}{f_1}=\frac{f_{25}^5}{f_5^6}\left(\frac{1}{R^4(q^5)}+\frac{q}{R^3(q^5)}+\frac{2q^2}{R^2(q^5)}+\frac{3q^3}{R(q^5)}+5q^4-3q^5R(q^5)\right.$$

$$+ 2q^6R^2(q^5) - q^7R^3(q^5) + q^8R^4(q^5)\Bigg). \tag{2.39}$$

In the next two lemmas, we recall some useful identities involving $R(q)$, $G(q)$, $H(q)$, and f_k.

Lemma 2.12. [82, Lemma 7] *We have*

$$R^2(q^5)\frac{f_1^2H^5(q)}{f_{25}^2H(q^5)} = 1 - 2qR(q^5) + 4q^2R^2(q^5) - 3q^3R^3(q^5) + q^4R^4(q^5), \tag{2.40}$$

$$R^2(q^5)\frac{f_1^2G^5(q)}{f_{25}^2G(q^5)} = 1 + 3qR(q^5) + 4q^2R^2(q^5) + 2q^3R^3(q^5) + q^4R^4(q^5). \tag{2.41}$$

Lemma 2.13. [34, Theorem 7.4.4], [22, Eq. (1.22) and (2.14)], [26, Eq. (66) and p. 532] *We have*

$$\frac{1}{R^5(q)} - q^2R^5(q) = 11q + \frac{f_1^6}{f_5^6} \tag{2.42}$$

$$= 4q + \frac{f_2f_5^5}{f_1f_{10}^5} + 8q^2\frac{f_1f_{10}^5}{f_2f_5^5} + 16q^3\frac{f_1^2f_{10}^{10}}{f_2^2f_5^{10}}, \tag{2.43}$$

$$\frac{1}{R^3(q)R(q^2)} + q^2R^3(q)R(q^2) = 2q + \frac{f_2f_5^5}{f_1f_{10}^5} + 4q^2\frac{f_1f_{10}^5}{f_2f_5^5}, \tag{2.44}$$

$$\frac{R(q^2)}{R^2(q)} + \frac{R^2(q)}{R(q^2)} = 2\frac{f_2^2f_{10}^{10}}{f_4f_5^8f_{20}^3} + 8q^2\frac{f_1f_4f_{10}f_{20}^3}{f_2f_5^5}, \tag{2.45}$$

$$\frac{1}{R(q)R^2(q^2)} + q^2R(q)R^2(q^2) = \frac{f_2^3f_{10}^5}{f_1f_4f_5^3f_{20}^3} + 4q^2\frac{f_4f_{20}^3}{f_{10}^4}. \tag{2.46}$$

2.3 Proof of Theorem 2.2

Proof of (2.3). Setting

$$N := 1 - 2qR(q^5) + 4q^2R^2(q^5) - 3q^3R^3(q^5) + q^4R^4(q^5) \tag{2.47}$$

and

$$D := 1 + 3qR(q^5) + 4q^2R^2(q^5) + 2q^3R^3(q^5) + q^4R^4(q^5), \tag{2.48}$$

we rewrite (1.11), (2.40) and (2.41) as

$$\frac{R^5(q)}{R(q^5)} = \frac{N}{D}, \tag{2.49}$$

$$N = R^2(q^5)\frac{f_1^2H^5(q)}{f_{25}^2H(q^5)}, \tag{2.50}$$

$$D = R^2(q^5)\frac{f_1^2G^5(q)}{f_{25}^2G(q^5)}. \tag{2.51}$$

Multiplying (2.50) and (2.51), we have

$$ND = R^4(q^5)\frac{f_1^4}{f_{25}^4}\frac{G^5(q)H^5(q)}{G(q^5)H(q^5)}. \tag{2.52}$$

Now,

$$H(q)G(q) = \frac{1}{(q;q^5)_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty(q^4;q^5)_\infty} = \frac{f_5}{f_1}. \tag{2.53}$$

Employing (2.53) in (2.52), we have

$$ND = \frac{f_5^6 R^4(q^5)}{f_1 f_{25}^5}. \tag{2.54}$$

From (2.49) and (2.54), we have

$$\frac{1}{R^5(q)} = \frac{D^2}{R(q^5)ND} = \frac{f_1 f_{25}^5 D^2}{f_5^6 R^5(q^5)}.$$

By (2.38) and (2.48), we rewrite the above identity as

$$\frac{1}{R^5(q)} = \sum_{n=0}^{\infty} A(n)q^n = \frac{f_{25}^6}{f_5^6 R^5(q^5)}(1+3qR(q^5)+4q^2R^2(q^5)+2q^3R^3(q^5)+q^4R^4(q^5))^2$$
$$\times\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right). \tag{2.55}$$

Extracting the terms of the form q^{5n+1} from (2.55), dividing both sides by q, and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} A(5n+1)q^n = \frac{f_5^6}{f_1^6}\left(\frac{5}{R^5(q)} - 40q\right).$$

Employing (2.42) in the above, we have

$$\sum_{n=0}^{\infty} A(5n+1)q^n = 5 + 15q\frac{f_5^6}{f_1^6} + 5q^2\frac{f_5^6 R^5(q)}{f_1^6}$$
$$= 5 + 15q\frac{f_5^6}{f_1^6} + \frac{5q^2}{(q;q^5)_\infty(q^2;q^5)_\infty^{11}(q^3;q^5)_\infty^{11}(q^4;q^5)_\infty},$$

from which it easily follows that, for $n \geq 0$, $A(5n+1) > 0$. This proves (2.3).

Proof of (2.4). Extracting the terms of the form q^{5n+2} from (2.55), dividing both sides by q^2, and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} A(5n+2)q^n = 10\frac{f_5^6}{f_1^6}\left(\frac{1}{R^4(q)} - 3qR(q)\right). \tag{2.56}$$

Now, let $b_{25}(n)$ denote the number of 25-regular partitions of n, that is, the number of partitions in which parts are not divisible by 25. The generating function of $b_{25}(n)$ is given by

$$\sum_{n=0}^{\infty} b_{25}(n)q^n = \frac{f_{25}}{f_1}.$$

Employing (2.39) in the above and then extracting the terms involving q^{5n}, we find that

$$\sum_{n=0}^{\infty} b_{25}(5n)q^n = \frac{f_5^6}{f_1^6}\left(\frac{1}{R^4(q)} - 3qR(q)\right). \tag{2.57}$$

It follows from (2.56) and (2.57) that

$$\sum_{n=0}^{\infty} A(5n+2)q^n = 10\sum_{n=0}^{\infty} b_{25}(5n)q^n.$$

As $b_{25}(5n) > 0$ for all $n \geq 0$, it readily follows from the above that $A(5n+2) > 0$ for all $n \geq 0$, which is (2.4).

Proof of (2.5). Extracting the terms of the form q^{5n+3} from (2.55), dividing both sides by q^3, and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} A(5n+3)q^n = 5\frac{f_5^6}{f_1^6}\left(\frac{1}{R^3(q)} - 3qR^2(q)\right). \tag{2.58}$$

Now, define $F(n)$ by

$$\sum_{n=0}^{\infty} F(n)q^n = \frac{f_{25}R(q^5)}{f_1} \tag{2.59}$$

$$= \frac{1}{(q^{1,2,3,4,6,7,8,9,10,10,11,12,13,14,15,15,16,17,18,19,21,22,23,24}; q^{25})_\infty},$$

where $(q^{a_1,a_2,\ldots,a_j}; q^k)_\infty := (q^{a_1}; q^k)_\infty (q^{a_2}; q^k)_\infty \cdots (q^{a_j}; q^k)_\infty$. Combinatorially, $F(n)$ counts the number of partitions of n into parts not congruent to 0 or ± 5 modulo 25 and parts congruent to ± 10 modulo 25 have two colors. Clearly, $F(n) > 0$ for all $n \geq 0$.

Now, employing (2.39) in (2.59) and then extracting the terms involving q^{5n}, we find that

$$\sum_{n=0}^{\infty} F(5n)q^n = \frac{f_5^6}{f_1^6}\left(\frac{1}{R^3(q)} - 3qR^2(q)\right). \tag{2.60}$$

From (2.58) and (2.60), we have

$$\sum_{n=0}^{\infty} A(5n+3)q^n = 5\sum_{n=0}^{\infty} F(5n)q^n,$$

from which it follows that $A(5n+3) = 5F(5n)$. Now, the positivity of $F(n)$ implies that $A(5n+3)>0$ for all $n\geq 0$, which is (2.5).

Proof of (2.6). Extracting the terms of the form q^{5n+4} from (2.55), dividing both sides by q^4, and then replacing q^5 by q, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} A(5n+4)q^n \\
&= -5\frac{f_5^6}{f_1^6}\left(\frac{3}{R^2(q)}+qR^3(q)\right) \qquad (2.61)\\
&= -5\left(\frac{3}{(q,q^4;q^5)_\infty^8(q^2,q^3;q^5)_\infty^4}+\frac{q}{(q,q^4;q^5)_\infty^3(q^2,q^3;q^5)_\infty^9}\right),
\end{aligned}$$

which clearly implies that $A(5n+4)<0$ for all $n\geq 0$.

2.4 Proof of Theorem 2.3

Proof of (2.7). From (2.49) and (2.54), we have

$$R^5(q)=\frac{R(q^5)N^2}{ND}=\frac{f_1f_{25}^5N^2}{f_5^6R^3(q^5)}.$$

With the aid of (2.38), (2.47) and (2.54), we rewrite the above as

$$\begin{aligned}
R^5(q)=\sum_{n=0}^{\infty}B(n)q^n=\frac{f_{25}^6}{f_5^6R^3(q^5)}&\left(1-2qR(q^5)+4q^2R^2(q^5)-3q^3R^3(q^5)+q^4R^4(q^5)\right)^2\\
&\times\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right). \qquad (2.62)
\end{aligned}$$

Extracting the terms of the form q^{5n+1} from both sides of the above, dividing both sides by q, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty}B(5n+1)q^n=-5\frac{f_5^6}{f_1^6}\left(\frac{1}{R^3(q)}-3qR^2(q)\right). \qquad (2.63)$$

From (2.58) and (2.63), we have

$$\sum_{n=0}^{\infty}B(5n+1)q^n=-\sum_{n=0}^{\infty}A(5n+3)q^n, \qquad (2.64)$$

which implies that for all n, $B(5n+1)=-A(5n+3)$. Therefore, (2.5) implies (2.7).

Proof of (2.8). Extracting the terms of the form q^{5n+2} from both sides of (2.62), dividing both sides by q^2, and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty}B(5n+2)q^n=5\frac{f_5^6}{f_1^6}\left(\frac{3}{R^2(q)}+qR^3(q)\right), \qquad (2.65)$$

which, by (2.61), implies that

$$\sum_{n=0}^{\infty} B(5n+2)q^n = -\sum_{n=0}^{\infty} A(5n+4)q^n. \tag{2.66}$$

It follows that $B(5n+2) = -A(5n+4)$. Thus, (2.6) implies (2.8).

Proof of (2.9). Extracting the terms of the form q^{5n+3} from both sides of (2.62), dividing both sides by q^3, and then replacing q^5 by q, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} B(5n+3)q^n &= -10\frac{f_5^6}{f_1^6}\left(\frac{3}{R(q)} + qR^4(q)\right) \qquad (2.67)\\ &= -10\left(\frac{3}{(q,q^4;q^5)_\infty^7(q^2,q^3;q^5)_\infty^5} + \frac{q}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^{10}}\right),\end{aligned}$$

which readily implies that $B(5n+3) < 0$ for all $n \geq 0$, which is (2.9).

Proof of (2.10). Extracting the terms of the form q^{5n+4} from both sides of (2.62), dividing both sides by q^4, and then replacing q^5 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} B(5n+4)q^n &= \frac{f_5^6}{f_1^6}\left(40 + 5qR^5(q)\right)\\ &= 40\frac{f_5^6}{f_1^6} + \frac{5q}{(q,q^4;q^5)_\infty(q^2,q^3;q^5)_\infty^{11}},\end{aligned}$$

which implies that $B(5n+4) > 0$ for all $n \geq 0$.

2.5 Proof of Theorem 2.4

Proof of (2.11). From (2.49), we have

$$\frac{R^5(q)}{R(q^5)} = \frac{N^2}{ND} = \frac{f_1 f_{25}^5 N^2}{f_5^6 R^4(q^5)}.$$

Invoking (2.38), (2.47) and (2.54) in the above, we obtain

$$\begin{aligned}\frac{R^5(q)}{R(q^5)} = \sum_{n=0}^{\infty} C(n)q^n &= \frac{f_{25}^6}{f_5^6 R^4(q^5)}(1 - 2qR(q^5) + 4q^2R^2(q^5) - 3q^3R^3(q^5) + q^4R^4(q^5))^2\\ &\quad \times \left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right). \qquad (2.68)\end{aligned}$$

Extracting the terms of the form q^{5n} and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} C(5n)q^n = \frac{f_5^6}{f_1^6}\left(\frac{1}{R^5(q)} - 36q - q^2R^5(q)\right). \tag{2.69}$$

With the aid of (2.42), the last equation can be recast as

$$\sum_{n=0}^{\infty} C(5n)q^n = 1 - 25q\frac{f_5^6}{f_1^6}$$

which clearly implies $C(5n) < 0$ for all $n \geq 1$, which is (2.11).

Proof of (2.12). Extracting the terms of the form q^{5n+1} from (2.68), dividing both sides by q and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} C(5n+1)q^n = -5\frac{f_5^6}{f_1^6}\left(\frac{1}{R^4(q)} - 3qR(q)\right). \tag{2.70}$$

It follows from (2.56) and (2.70) that, for all $n \geq 0$, we have $2C(5n+1) = -A(5n+2)$. Thus, (2.12) follows by (2.4).

Proof of (2.13). Extracting the terms of the form q^{5n+2} from (2.68), dividing both sides by q^2 and then replacing q^5 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} C(5n+2)q^n &= \frac{f_5^6}{f_1^6}\left(\frac{15}{R^3(q)} + 5qR^2(q)\right)\\ &= \frac{15}{(q,q^4;q^5)_\infty^9(q^2,q^3;q^5)_\infty^3} + \frac{5q}{(q,q^4;q^5)_\infty^4(q^2,q^3;q^5)_\infty^8},\end{aligned}$$

which readily yields $C(5n+2) > 0$ for $n \geq 0$, which is (2.13).

Proof of (2.14). Extracting the terms of the form q^{5n+3} from (2.68), dividing both sides by q^3 and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} C(5n+3)q^n = -10\frac{f_5^6}{f_1^6}\left(\frac{3}{R^2(q)} + qR^3(q)\right). \tag{2.71}$$

From (2.61) and (2.71), we have $C(5n+3) = 2A(5n+4)$ for all $n \geq 0$. Thus, (2.6) implies (2.14).

Proof of (2.15). Extracting the terms of the form q^{5n+4} from (2.68), dividing both sides by q^4, and then replacing q^5 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} C(5n+4)q^n &= \frac{f_5^6}{f_1^6}\left(\frac{40}{R(q)} + 5qR^4(q)\right)\\ &= \frac{40}{(q,q^4;q^5)_\infty^7(q^2,q^3;q^5)_\infty^5} + \frac{5q}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^{10}},\end{aligned}$$

which yields $C(5n+4) > 0$ for all $n \geq 0$, which is (2.15).

2.6 Proof of Theorem 2.6

Proof of (2.16). From (2.49), we have

$$\frac{R(q^5)}{R^5(q)} = \frac{D^2}{ND} = \frac{f_1 f_{25}^5 D^2}{f_5^6 R^4(q^5)}.$$

Invoking (2.38), (2.48) and (2.54) in the last equation, we arrive at

$$\frac{R(q^5)}{R^5(q)} = \sum_{n=0}^{\infty} D(n)q^n = \frac{f_{25}^6}{f_5^6 R^4(q^5)}(1 + 3qR(q^5) + 4q^2R^2(q^5) + 2q^3R^3(q^5) + q^4R^4(q^5))^2$$
$$\times \left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right). \tag{2.72}$$

Extracting the terms of the form q^{5n} and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} D(5n)q^n = \frac{f_5^6}{f_1^6}\left(\frac{1}{R^5(q)} - 36q - q^2R^5(q)\right).$$

Comparing the above identity with (2.69), we see that $D(5n) = C(5n)$ for all $n \geq 0$. Thus, (2.11) implies (2.16).

Proof of (2.17). Extracting the terms of the form q^{5n+2} from (2.72), dividing both sides by q^2 and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} D(5n+2)q^n = 10\frac{f_5^6}{f_1^6}\left(\frac{1}{R^3(q)} - 3qR^2(q)\right). \tag{2.73}$$

It follows from the above identity and (2.58) that $D(5n+2) = 2A(5n+3)$ for all $n \geq 0$. Therefore, (2.17) follows by (2.5).

Proof of (2.18). Extracting the terms of the form q^{5n+3} from (2.72), dividing both sides by q^3 and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} D(5n+3)q^n = \frac{f_5^6}{f_1^6}\left(\frac{5}{R^2(q)} - 15qR^3(q)\right)$$
$$= \frac{5}{(q,q^4;q^5)_\infty^3(q^2,q^3;q^5)_\infty^3}\left(\frac{1}{(q,q^4;q^5)_\infty^5} - \frac{3q}{(q^2,q^3;q^5)_\infty^5}\right). \tag{2.74}$$

In order to prove (2.18), that is, $D(5n+3) > 0$ for all nonnegtive integers n, it is enough to show that coefficients in the expansion of

$$\left(\frac{1}{(q,q^4;q^5)_\infty^5} - \frac{3q}{(q^2,q^3;q^5)_\infty^5}\right)$$

are positive.

To that end, define the sequences $(\alpha(n))$ and $(\beta(n))$ by

$$\sum_{n=0}^{\infty} \alpha(n) q^n := \frac{1}{(q, q^4; q^5)_\infty^5}$$

and

$$\sum_{n=0}^{\infty} \beta(n) q^n := \frac{1}{(q^2, q^3; q^5)_\infty^5}.$$

Combinatorially, $\alpha(n)$ counts the number of partitions 5-tuples of n with parts congruent to $\pm 1 \pmod 5$ and $\beta(n)$ counts the number of partitions 5-tuples of n with parts congruent to $\pm 2 \pmod 5$. We first show that

$$\alpha(n) > 3\beta(n-1) \text{ for } n \geq 3. \tag{2.75}$$

Define the following sets:

$R_1(n) := \{\mathcal{P} | \mathcal{P}$ is a partition of n and all parts are $\equiv \pm 1 \pmod 5\}$,

$R_2(n) := \{\mathcal{P} | \mathcal{P}$ is a partition of n and all parts are $\equiv \pm 2 \pmod 5\}$,

$$\mathcal{E}_n := \{\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) | \text{all parts in partition } \pi_i (1 \leq i \leq 5) \text{ are } \equiv \pm 2 \pmod 5 \text{ and } \sum_{i=1}^{5} s(\pi_i) = n\},$$

$$\mathcal{S}_n := \{\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) | \text{all parts in partition } \pi_i (1 \leq i \leq 5) \text{ are } \equiv \pm 1 \pmod 5 \text{ and } \sum_{i=1}^{5} s(\pi_i) = n\},$$

where $s(\pi_i)$ denotes the sum of all parts in partition π_i.

Define the map $\tau : R_2(n) \to R_1(n)$ by $\tau(\pi) = \lambda$, where λ is a partition obtained by subtracting 1 from each of the parts congruent to $2 \pmod 5$ and adding 1 to each of the the parts congruent to $3 \pmod 5$. In case the number of parts congruent to $2 \pmod 5$ and $3 \pmod 5$ are unequal, then we do the following for the extra partitions:

1. If the number of parts congruent to $2 \pmod 5$ are more than the number of parts congruent to $3 \pmod 5$, then we write each remaining part as a part congruent to $1 \pmod 5 + 1$.

2. If the number of parts congruent to $3 \pmod 5$ are more than the number of parts congruent to $2 \pmod 5$, then we write each remaining part as a part

congruent to 1 (mod 5) + 1 + 1.

Clearly, $\lambda \in R_1(n)$. Furthermore, $\pi_1 \neq \pi_2 \iff \tau(\pi_1) \neq \tau(\pi_2)$.

Again, define $\tilde{\tau} : \mathcal{E}_n \to \mathcal{S}_n$ by

$$\tilde{\tau} = (\tau(\pi_1), \tau(\pi_2), \tau(\pi_3), \tau(\pi_4), \tau(\pi_5)) = \lambda'.$$

If $\pi \in \mathcal{E}_{n-1}$, then $\lambda' \in \mathcal{S}_{n-1}$. We now add a part of size one to any one of the components of λ' and let $\tilde{\lambda}'$ be the new partition. Clearly, $\tilde{\lambda}' \in \mathcal{S}_n$. Since, for the partition 5-tuple, there are 5 choices to append this part of size 1, so $\alpha(n) \geq 5\beta(n-1)$. Furthermore, since $\beta(n) \neq 0$ for all $n \geq 2$, we see that $\alpha(n) \geq 5\beta(n-1) > 3\beta(n-1)$ for $n \geq 3$. Thus, (2.75) holds. Using (2.75) and the easily checked facts $D(3) = 5$, $D(8) = 25$, and $D(13) = 155$, in (2.74), we readily arrive at (2.18).

Proof of (2.19). Extracting the terms of the form q^{5n+4} from (2.72), dividing both sides by q^4 and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} D(5n+4)q^n = -5\frac{f_5^6}{f_1^6}\left(\frac{3}{R(q)} + qR^4(q)\right).$$

From the above identity and (2.67), we see that $2D(5n+4) = B(5n+3)$ for all $n \geq 0$. Therefore, (2.9) implies (2.19).

2.7 Proof of Theorem 2.7

Proofs of (2.20) *and* (2.21). From the definitions of $A(n)$ and $B(n)$ given in Theorem 2.2 and Theorem 2.3 respectively, the identity (2.42) may be rewritten as

$$\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} B(n)q^{n+2} = 11q + \frac{f_1^6}{f_5^6}. \tag{2.76}$$

Now, with the aid of the binomial theorem, it can be easily shown that

$$f_1^3 \equiv f_3 \pmod 3. \tag{2.77}$$

Therefore, from (2.76), we have

$$\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} B(n)q^{n+2} \equiv 2q + \frac{f_3^2}{f_{15}^2} \pmod 3. \tag{2.78}$$

Furthermore, by (2.53), we have

$$\begin{aligned}\sum_{n=0}^{\infty} A(n)q^n + \sum_{n=0}^{\infty} B(n)q^{n+2} &= \frac{1}{R^5(q)} + q^2 R^5(q) \\ &= \frac{G^5(q)}{H^5(q)} + q^2 \frac{H^5(q)}{G^5(q)} \\ &= \frac{f_1^5}{f_5^5}\left(G^{10}(q) + q^2 H^{10}(q)\right),\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} A(n)q^n + \sum_{n=0}^{\infty} B(n)q^{n+2} \equiv \frac{f_1^2 f_3}{f_5^2 f_{15}}\left(G(q)G^3(q^3) + q^2 H(q)H^3(q^3)\right) \pmod 3, \tag{2.79}$$

where we also applied the congruences

$$G^3(q) \equiv G(q^3) \pmod 3 \text{ and } H^3(q) \equiv H(q^3) \pmod 3.$$

Now, recall from [83, Theorem 4.1(ii)] that

$$G(q)G^3(q^3) + q^2 H(q)H^3(q^3) = \frac{f_5^3}{f_1 f_3 f_{15}}.$$

Using the above identity in (2.79) and then employing (2.77), we find that

$$\sum_{n=0}^{\infty} A(n)q^n + \sum_{n=0}^{\infty} B(n)q^{n+2} \equiv \frac{f_1 f_5}{f_{15}^2} \pmod 3. \tag{2.80}$$

Adding (2.78) and (2.80), we have

$$2\sum_{n=0}^{\infty} A(n)q^n \equiv 2q + \frac{f_3^2}{f_{15}^2} + \frac{f_1 f_5}{f_{15}^2} \pmod 3. \tag{2.81}$$

Again, from [25, p. 509], we recall that

$$f_1 f_5 = \varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}). \tag{2.82}$$

Employing (2.36) and (2.37) in the above, we have

$$\begin{aligned}f_1 f_5 = &\left(\varphi(q^{45}) + 2q^5 f(q^{15}, q^{75})\right)\left(f(q^6, q^{12}) + q^2\psi(q^{18})\right) - q\left(\varphi(q^9) + 2qf(q^9, q^{15})\right) \\ &\times \left(f(q^{30}, q^{60}) + q^{10}\psi(q^{90})\right).\end{aligned} \tag{2.83}$$

Invoking (2.83) in (2.81), and then extracting the terms involving q^{3n+1}, we find that

$$2\sum_{n=0}^{\infty} A(3n+1)q^n \equiv 2 + \frac{2q^2 f(q^5, q^{25})\psi(q^6) - \varphi(q^3)f(q^{10}, q^{20})}{f_5^2} \pmod 3,$$

which, by (2.28) and (2.29), can be rewritten as

$$2\sum_{n=0}^{\infty} A(3n+1)q^n \equiv 2+2q^2\frac{\psi(q^6)\psi(-q^{15})\chi(q^5)}{f_5^2} - \frac{\varphi(q^3)\varphi(-q^{30})}{f_5^2\chi(-q^{10})} \pmod 3.$$

Employing (1.5), (1.6), (1.7) and (2.30) in the above, we find that

$$\begin{aligned}2\sum_{n=0}^{\infty} A(3n+1)q^n &\equiv 2+2q^2\frac{\psi(q^6)\psi(-q^{15})f_{10}^2}{f_{15}f_{20}} - \frac{\varphi(q^3)\varphi(-q^{30})f_{20}f_5}{f_{15}f_{10}}\\ &\equiv 2+2q^2\frac{\psi(q^6)\psi(-q^{15})}{f_{15}}\varphi(-q^{10}) - \frac{\varphi(q^3)\varphi(-q^{30})}{f_{15}}\psi(-q^5) \pmod 3.\end{aligned}$$

With the aid of (2.36) and (2.37), the above can be written as

$$\begin{aligned}2\sum_{n=0}^{\infty} A(3n+1)q^n &\equiv 2+2q^2\frac{\psi(q^6)\psi(-q^{15})}{f_{15}}\left(\varphi(-q^{90})-2q^{10}f(-q^{30},-q^{150})\right)\\ &\quad - \frac{\phi(q^3)\psi(-q^{30})}{f_{15}}\left(f(-q^{15},q^{30})-q^5\psi(-q^{45})\right) \pmod 3.\end{aligned}$$

Comparing the coefficients of q^{3n+1} for $n\geq 0$, we obtain

$$A(9n+4)\equiv 0 \pmod 3, \tag{2.84}$$

which is (2.20).

Again, extracting the terms involving q^{3n+4} from both sides of (2.78), we find that

$$A(3n+4)\equiv B(3n+2) \pmod 3.$$

Replacing n by $3n$ in the above, and then invoking (2.84), we have

$$A(9n+4)\equiv B(9n+2)\equiv 0 \pmod 3,$$

which is (2.21).

Proof of (2.22). At first, from (2.43) and the definitions of $A(n)$ and $B(n)$ given in Theorem 2.2 and Theorem 2.3, we have

$$\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} B(n)q^{n+2} \equiv \frac{f_2f_5^5}{f_1f_{10}^5} + 4q \pmod 8. \tag{2.85}$$

Next,

$$\begin{aligned}\sum_{n=0}^{\infty} A(n)q^n + \sum_{n=0}^{\infty} B(n)q^{n+2} &= \frac{1}{R^5(q)} + q^2R^5(q)\\ &= -\frac{1}{R(q)R^2(q^2)} - q^2R(q)R^2(q^2) + \left(\frac{R^2(q)}{R(q^2)}+\frac{R(q^2)}{R^2(q)}\right)\end{aligned}$$

$$\times\left(\frac{1}{R^3(q)R(q^2)}+q^2R^3(q)R(q^2)\right).$$

Invoking (2.44)–(2.46) in the above, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}A(n)q^n+\sum_{n=0}^{\infty}B(n)q^{n+2}&=-\frac{f_2^3f_{10}^5}{f_1f_4f_5^3f_{20}^3}-4q^2\frac{f_4f_{20}^3}{f_{10}^4}+\left(2\frac{f_2^2f_{10}^{10}}{f_4f_5^8f_{20}^3}+8q^2\frac{f_1f_4f_{10}f_{20}^3}{f_2f_5^5}\right)\\&\quad\times\left(2q+\frac{f_2f_5^5}{f_1f_{10}^5}+4q^2\frac{f_1f_{10}^5}{f_2f_5^5}\right)\\&\equiv\frac{f_2^3f_{10}^5}{f_1f_4f_5^3f_{20}^3}+4q+4q^2f_4f_{20}\pmod{8}.\end{aligned}\tag{2.86}$$

Without commentary, here and throughout the thesis, we use the fact that for integers $k\geq 1$ and $\ell\geq 1$,

$$f_k^{2^\ell}\equiv f_{2k}^{2^{\ell-1}}\pmod{2^\ell}.\tag{2.87}$$

We now add (2.85) and (2.86), and then use (2.31), (2.33), and (2.35). Accordingly, we find that

$$\begin{aligned}2\sum_{n=0}^{\infty}A(n)q^n&\equiv\frac{f_2^3f_{10}^5}{f_4f_{20}^3}\cdot\frac{1}{f_5^4}\cdot\frac{f_5}{f_1}+\frac{f_2}{f_{10}^5}\cdot f_5^4\cdot\frac{f_5}{f_1}+4q^2f_4f_{20}\\&\equiv\frac{f_2^3f_{10}^5}{f_4f_{20}^3}\left(\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)\left(\frac{f_{20}^{14}}{f_{10}^{14}f_{40}^4}+4q^5\frac{f_{20}^2f_{40}^4}{f_{10}^{10}}\right)+\frac{f_2}{f_{10}^5}\\&\quad\times\left(\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)\left(\frac{f_{20}^{10}}{f_{10}^2f_{40}^4}-4q^5\frac{f_{10}^2f_{40}^4}{f_{20}^2}\right)+4q^2f_4f_{20}\pmod{8}.\end{aligned}$$

Extracting the terms involving odd powers of q, we have

$$\begin{aligned}2\sum_{n=0}^{\infty}A(2n+1)q^n&\equiv\frac{f_2^3f_{10}^9}{f_1^2f_4f_5^6f_{20}^3}+\frac{f_2^2f_{10}^{10}}{f_4f_5^8f_{20}^3}-4q^2\frac{f_4f_{20}^3}{f_1f_5^3}+4q^2\frac{f_1f_4f_{10}f_{20}^3}{f_2f_5^5}\\&\equiv\frac{f_2^3f_{10}^5}{f_4f_{20}^3}\cdot\left(\frac{f_5}{f_1}\right)^2+\frac{f_2^2f_{10}^6}{f_4f_{20}^3}\pmod{8}.\end{aligned}$$

Employing (2.35) in the above and then extracting the terms involving the even powers of q, we obtain

$$2\sum_{n=0}^{\infty}A(4n+1)q^n\equiv\frac{f_{10}}{f_2}\cdot\left(\frac{f_1}{f_5}\right)^2+\frac{f_4^2f_{10}}{f_2f_{20}^2}\cdot\frac{f_5}{f_1}\cdot f_5^4+q\frac{f_2^5f_{20}^2}{f_4^2f_{10}}\cdot\frac{f_1}{f_5}\cdot\frac{1}{f_1^4}\pmod{8}.$$

Using the identities of Lemma 2.9 in the above and then extracting the odd powers of q, we find that

$$2\sum_{n=0}^{\infty}A(8n+5)q^n\equiv\frac{f_2^5f_{20}}{f_4f_{10}}\cdot\frac{1}{f_1^4}-4q^2f_4f_{20}^3\cdot\frac{f_1}{f_5}+\frac{f_4f_{10}^5}{f_2f_{20}}\cdot\frac{1}{f_5^4}-4qf_4^3f_{20}\cdot\frac{f_5}{f_1}$$

$$- 2f_2f_{10} \pmod 8.$$

Once again using the identities of Lemma 2.9 in the above and then extracting the odd powers of q, we have

$$2\sum_{n=0}^{\infty} A(16n+13)q^n \equiv 4\frac{f_2f_4^4f_{10}}{f_1^5f_5} + 4q\frac{f_2^3f_{10}^3f_{20}}{f_4f_5^2} + 4q^2\frac{f_2f_{10}f_{20}^4}{f_1f_5^5} - 4\frac{f_2^3f_4f_{10}^3}{f_1^2f_{20}} \pmod 8$$

which implies that

$$\sum_{n=0}^{\infty} A(16n+13)q^n \equiv 2\frac{f_2f_4^4f_{10}}{f_1^5f_5} + 2q\frac{f_2^3f_{10}^3f_{20}}{f_4f_5^2} + 2q^2\frac{f_2f_{10}f_{20}^4}{f_1f_5^5} - 2\frac{f_2^3f_4f_{10}^3}{f_1^2f_{20}} \pmod 4.$$

Equivalently,

$$\begin{aligned}\sum_{n=0}^{\infty} A(16n+13)q^n &\equiv 2f_1f_5f_4^3 + 2qf_2f_{10}^4 + 2q^2f_1f_5f_{20}^3 - 2f_8f_{10} \\ &\equiv 2f_2f_5(f_2^3 - qf_{10}^3)(f_2^3 + qf_{10}^3 - f_1f_5) \pmod 4. \qquad (2.88)\end{aligned}$$

Now, from (2.82), we have

$$\begin{aligned}f_1f_5 &= \varphi(q^5)\psi(q^2) - q\varphi(q)\psi(q^{10}) \\ &= \frac{f_{10}^5}{f_5^2f_{20}^2}\cdot\frac{f_4^2}{f_2} - q\frac{f_2^5}{f_1^2f_4^2}\cdot\frac{f_{20}^2}{f_{10}} \\ &\equiv f_2^3 + qf_{10}^3 \pmod 2.\end{aligned}$$

Employing the above congruence in (2.88), we find that

$$\sum_{n=0}^{\infty} A(16n+13)q^n \equiv 2f_2f_5(f_2^3 - qf_{10}^3)(f_2^3 + qf_{10}^3 - (f_2^3 + qf_{10}^3)) \equiv 0 \pmod 4,$$

which is (2.22).

Proof of (2.23). From (2.43), we have

$$\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} B(n)q^{n+2} \equiv \frac{f_2f_5}{f_1f_{10}^3} \pmod 4,$$

which, by (2.35), may be written as

$$\sum_{n=0}^{\infty} A(n)q^n - \sum_{n=0}^{\infty} B(n)q^{n+2} \equiv \frac{f_2}{f_{10}^3}\left(\frac{f_8f_{20}^2}{f_2^2f_{40}} + q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right) \pmod 4.$$

Extracting, we have

$$\sum_{n=0}^{\infty} A(2n+1)q^n - \sum_{n=1}^{\infty} B(2n-1)q^n \equiv \frac{f_2^3f_{20}}{f_4f_{10}}\cdot\frac{1}{f_1^2}\cdot\frac{1}{f_5^2} \pmod 4.$$

Employing (2.32) in the above and then extracting, we find that

$$\sum_{n=0}^{\infty} A(4n+1)q^n - \sum_{n=1}^{\infty} B(4n-1)q^n \equiv \frac{f_4^5 f_{20}^5}{f_2 f_8^2 f_{10} f_{40}^2} \cdot \frac{1}{f_1^2} \cdot \frac{1}{f_5^2} \pmod 4.$$

Using (2.32) again in the above and then extracting the odd powers of q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} A(8n+5)q^n - \sum_{n=1}^{\infty} B(8n+3)q^n &\equiv 2\frac{f_2^7 f_8^2 f_{10}^5 f_{20}^3}{f_1^6 f_4^3 f_5^6 f_{40}^2} + 2q^2 \frac{f_2^5 f_4^3 f_{10}^7 f_{40}^2}{f_1^6 f_5^6 f_8^2 f_{20}^3} \\ &\equiv 2\frac{f_2^4 f_8^2 f_{10}^2 f_{20}^3}{f_4^3 f_{40}^2} + 2q^2 \frac{f_2^2 f_4^3 f_{10}^4 f_{40}^2}{f_8^2 f_{20}^3} \pmod 4,\end{aligned}$$

which readily implies that

$$A(16n+13) \equiv B(16n+11) \pmod 4.$$

Using (2.22), we arrive at

$$B(16n+11) \equiv 0 \pmod 4,$$

which is (2.23).

Proofs of (2.24)–(2.27). From (2.58), we have

$$\begin{aligned}\sum_{n=0}^{\infty} A(5n+3)q^n &= \frac{f_5^6}{f_1^6}\left(\frac{5}{R^3(q)} - 15qR^2(q)\right) \\ &\equiv 5\frac{f_5^6}{f_1^6 R^3(q)} \pmod{15} \\ &\equiv 5\frac{f_{15}^2}{f_3^2 R(q^3)} \pmod{15}.\end{aligned}$$

Clearly, the last relation has no terms involving q^{3n+r} for $r \in \{1, 2\}$. Hence, for all $n \geq 0$, we have

$$A(15n+8) \equiv A(15n+13) \equiv 0 \pmod{15}.$$

To prove the remaining two congruences in (2.24), we consider (2.61) and proceed exactly as in the above. Thus, we complete the proof of (2.24).

The proofs of (2.25)–(2.27) are similar to the proof of (2.24). So we only record the required generating functions for the proofs in the following table.

Congruence	Generating functions
(2.25)	(2.63) and (2.65)
(2.26)	(2.71)
(2.27)	(2.73)

2.8 Concluding remarks

In Theorems 2.2, 2.3, and 2.6, we have the sign patterns of the coefficients $A(n)$, $B(n)$, and $D(n)$ of $1/R^5(q)$, $R^5(q)$, and $R(q^5)/R^5(q)$, respectively, except $A(5n)$, $B(5n)$, and $D(5n+1)$. Based on numerical observation, we pose the following conjecture.

Conjecture 2.14. *For all integers $n > 0$,*

$$A(5n) < 0,$$
$$B(5n) < 0,$$
$$D(5n+1) > 0.$$

An affirmative answer to the above conjecture along with Theorems 2.2, 2.3, and 2.6 will prove that the signs of $A(n), B(n)$, and $D(n)$ are periodic with period 5.

There might be more congruences similar to those given in Theorem 2.7. For example, based on numerical calculations, we propose the following conjecture.

Conjecture 2.15. *For all integers $n \geq 0$,*

$$C(27n+18) \equiv 0 \pmod{3},$$
$$D(27n+18) \equiv 0 \pmod{3},$$
$$C(16n+12) \equiv 0 \pmod{4},$$
$$D(16n+12) \equiv 0 \pmod{4},$$
$$C(32n+28) \equiv 0 \pmod{8},$$
$$D(32n+28) \equiv 0 \pmod{8}.$$