Chapter 3

Differences of even and odd numbers of parts of the cubic and some analogous partition functions

3.1 Introduction

In 2021, Merca [110, Definition 1] defined the following functions.

Definition 3.1. *For a positive integer n, let*

1. *$a_e(n)$ be the number of partitions of n into an even number of parts in which the even parts can appear in two colors.*

2. *$a_o(n)$ be the number of partitions of n into an odd number of parts in which the even parts can appear in two colors.*

3. *$\Lambda(n) = a_e(n)$ - $a_o(n)$.*

By considering

$$F(z,q) = \prod_{n=1}^{\infty} \frac{1}{(1-zq^n)(1-zq^{2n})},$$

The contents of this chapter have appeared in *Boletín de la Sociedad Matemática Mexicana* [28].

we can see that the generating function of $\Lambda(n)$ (due to Merca) is given by

$$\sum_{n=0}^{\infty}\Lambda(n)q^n = F(-1,q) = \frac{1}{(-q;q)_\infty(-q^2;q^2)_\infty} = (q;q^2)_\infty(q^2;q^4)_\infty, \tag{3.1}$$

where the last equality arises from Euler's identity

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}.$$

With the aid of Smoot's RaduRK Mathematica package [148] which is based on an algorithm developed by Radu [123], Merca [110, Theorem 6.1] proved that

$$\sum_{n=0}^{\infty}\Lambda(9n+5)q^n = -3q\frac{f_2^3f_3f_{12}^6}{f_1^2f_4^3f_6^5} + 9q^2\frac{f_2^5f_3f_{12}^{10}}{f_1^2f_4^7f_6^7} + 3q^3\frac{f_2^7f_3f_{12}^{14}}{f_1^2f_4^{11}f_6^9} - 9q^4\frac{f_2^9f_3f_{12}^{18}}{f_1^2f_4^{15}f_6^{11}} \tag{3.2}$$

and a similar expression for the generating function of $A(27n+26)$ having 12 terms [110, Theorem 6.1]. From these expressions of the generating functions, Merca readily found the following two Ramanujan-like congruences.

Theorem 3.2. [110, Theorem 1.10] *For all $n \geq 0$,*

$$\Lambda(9n+5) \equiv 0 \pmod 3, \tag{3.3}$$

$$\Lambda(27n+26) \equiv 0 \pmod 3. \tag{3.4}$$

Using classical generating function manipulations and dissections, da Silva and Sellers [68] reproved the above congruences. They also proved an additional congruence and couple of infinite families of congruences modulo 3 as stated in the following theorem.

Theorem 3.3. [68, Theorems 3.1–3.3] *For all $j \geq 0$ and $n \geq 0$,*

$$\Lambda(3n+1) \equiv \Lambda(27n+8) \pmod 3, \tag{3.5}$$

$$\Lambda\left(9^{j+1}n + \frac{39\cdot 9^j+1}{8}\right) \equiv 0 \pmod 3, \tag{3.6}$$

$$\Lambda\left(3\cdot 9^{j+1}n + \frac{23\cdot 9^{j+1}+1}{8}\right) \equiv 0 \pmod 3. \tag{3.7}$$

The first purpose of this chapter is to employ Ramanujan's theta function identities in finding simplified formulas of the generating functions from which proofs of Theorem 3.2 and Theorem 3.3 follow quite naturally.

Now we state our results on $\Lambda(n)$.

Theorem 3.4. *We have*

$$\sum_{n=0}^{\infty}\Lambda(3n)q^n = \frac{\chi(-q)\psi(q^3)\varphi(-q^3)}{\varphi(-q)\varphi(q^2)}, \tag{3.8}$$

$$\sum_{n=0}^{\infty}\Lambda(3n+1)q^n = -\frac{\psi^2(q^3)\chi^2(-q)}{\psi(-q)\psi(q^2)}, \tag{3.9}$$

$$\sum_{n=0}^{\infty}\Lambda(3n+2)q^n = -\frac{\psi(-q^3)\psi(q^6)}{\psi^2(q^2)}, \tag{3.10}$$

$$\sum_{n=1}^{\infty}\Lambda(9n+5)q^n = -3q\frac{f_1f_2^4f_{12}^6}{f_4^{11}}. \tag{3.11}$$

Furthermore, for all $n \geq 0$

$$\Lambda(3n+2) \equiv -\Lambda(27n+17) \pmod 3, \tag{3.12}$$

$$\Lambda(81n+44) \equiv 0 \pmod 3. \tag{3.13}$$

Note that (3.11) is a much simplified form of (3.2) and the congruence (3.3) also readily follows from (3.11).

The second purpose of this chapter is to study the partition function $\Lambda_k(n)$ defined below, where $\Lambda_2(n) = \Lambda(n)$.

Definition 3.5. *For a positive integer* n*, let*

1. $a_e^k(n)$ *be the number of partitions of* n *into an even number of parts in which the parts that are multiples of* k *can appear in two colors.*

2. $a_o^k(n)$ *be the number of partitions of* n *into an odd number of parts in which the parts that are multiples of* k *can appear in two colors.*

3. $\Lambda_k(n)$ = $a_e^k(n)$ - $a_o^k(n)$.

For example, if k= 3 and $n = 4$, then

$a_e^3(4) = 4,$ the relevant partitions being $3_r + 1, 3_b + 1, 2 + 2, 1 + 1 + 1 + 1;$

$a_o^3(4) = 2,$ the relevant partition being 4 and $2 + 1 + 1;$

and hence, $\Lambda_3(4) = 4 - 2 = 2$, where the subscripts r and b depict the two colors of the respective part.

By considering the function

$$G(z,q)=\prod_{n=1}^{\infty}\frac{1}{(1-zq^n)(1-zq^{kn})},$$

we can easily see that the generating function of $\Lambda_k(n)$ is $G(-1,q)$; that is,

$$\sum_{n=0}^{\infty}\Lambda_k(n)q^n=\frac{1}{(-q;q)_\infty(-q^k;q^k)_\infty},$$

which by Euler's identity $(-q;q)_\infty=1/(q;q^2)_\infty$ and (1.7) may be recast as

$$\sum_{n=0}^{\infty}\Lambda_k(n)q^n=(q;q^2)_\infty(q^k;q^{2k})_\infty=\chi(-q)\chi(-q^k). \tag{3.14}$$

In the following theorems we state our results on $\Lambda_k(n)$ for $k\in\{3,5,7,23\}$.

Theorem 3.6. *For all $n\geq 0$,*

$$\Lambda_3(4n+r)\equiv 0 \pmod 2, \quad \textit{where} \quad r\in\{2,3\}. \tag{3.15}$$

Theorem 3.7. *For all $n\geq 0$,*

$$\Lambda_5(10n+r)\equiv 0 \pmod 2, \quad \textit{where} \quad r\in\{2,6\}, \tag{3.16}$$

$$\Lambda_5(25n+r)\equiv 0 \pmod 5, \quad \textit{where} \quad r\in\{14,19,24\}. \tag{3.17}$$

To state the remaining theorems, we require the Legendre symbol, which is defined for a prime $p\geq 3$ by

$$\left(\frac{a}{p}\right)_L:=\begin{cases}1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p\nmid a,\\ 0, & \text{if } p\mid a,\\ -1, & \text{if } a \text{ is a quadratic nonresidue modulo } p.\end{cases}$$

Theorem 3.8. *For all $n\geq 0$ and $j\geq 0$,*

$$\Lambda_7(2n+1)\equiv\Lambda_7(8n+3) \pmod 2, \tag{3.18}$$

$$\Lambda_7\left(2^{2j+3}n+\frac{10\cdot 2^{2j+1}+1}{3}\right)\equiv 0 \pmod 2, \tag{3.19}$$

$$\Lambda_7(16n+r)\equiv 0 \pmod 2, \quad \textit{where} \quad r\in\{9,13\}, \tag{3.20}$$

$$\Lambda_7(50n+r)\equiv 0 \pmod 2, \quad \textit{where} \quad r\in\{7,27,37,47\}, \tag{3.21}$$

if $p > 3$ is a prime such that $\left(\frac{-7}{p}\right)_L = -1$, then for all $\alpha \geq 0$ and $\delta \geq 0$,

$$\Lambda_7\left(2\cdot 7^\alpha\cdot p^{2\delta}(7n+r)+\frac{2\cdot 7^\alpha\cdot p^{2\delta}+1}{3}\right)\equiv 0 \pmod 2, \quad \textit{where} \quad r\in\{3,4,6\}, \tag{3.22}$$

$$\Lambda_7\left(2\cdot 7^\alpha\cdot p^{2\delta+1}(pn+r)+\frac{2\cdot 7^\alpha\cdot p^{2(\delta+1)}+1}{3}\right)\equiv 0 \pmod 2, \tag{3.23}$$

where $r \in \{1, 2, \ldots, p-1\}$.

Theorem 3.9. *For all $n \geq 0$,*

$$\Lambda_{23}(50n+r)\equiv 0 \pmod 2, \quad \textit{where} \quad r\in\{11,21,31,41\}, \tag{3.24}$$

If $p > 3$ is a prime such that $\left(\frac{-23}{p}\right)_L = -1$, then for all $\alpha \geq 0$, $\delta \geq 0$ and $n \geq 0$,

$$\Lambda_{23}\left(2\cdot 23^\alpha\cdot p^{2\delta}(23n+r)+2\cdot 23^\alpha\cdot p^{2\delta}+1\right)\equiv 0 \pmod 2, \tag{3.25}$$

where $r \in \{4, 6, 9, 10, 13, 14, 16, 18, 19, 20, 21\}$,

$$\Lambda_{23}\left(2\cdot 23^\alpha\cdot p^{2\delta+1}(pn+r)+2\cdot 23^\alpha\cdot p^{2(\delta+1)}+1\right)\equiv 0 \pmod 2, \tag{3.26}$$

where $r \in \{1, 2, \ldots, p-1\}$.

Remark 3.10. *Note that congruences equivalent to* (3.15) *and* (3.16) *can be found in the work of Hirschhorn and Sellers [88]. Congruences equivalent to* (3.18) *and* (3.20) *can be found in the work of Radu and Sellers [124]. Further* (3.24) *is equivalent to a special case of [162, Eq. (1.10)]. However, our proofs are different.*

We organize the chapter as follows: In the next section, we state a few well-known dissection formulas. In Section 3.3, we prove Theorems 3.2–3.4. In Sections 3.4–3.7, we prove Theorems 3.6–3.9, respectively.

3.2 Dissection formulas

Some known 2-, 3-, and 5-dissections formulas are stated in the following four lemmas.

Lemma 3.11. [33, p. 40, Entries 25(i) and 25(ii)] *If φ is given by* (1.3), *then*

$$\varphi(q)=\varphi(q^4)+2q\psi(q^8). \tag{3.27}$$

Lemma 3.12. [33, p. 315] *If ψ is given by* (1.4)*, then*

$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}), \tag{3.28}$$

$$\psi(q)\psi(q^7) = \varphi(q^{28})\psi(q^8) + q\psi(q^2)\psi(q^{14}) + q^6\varphi(q^4)\psi(q^{56}). \tag{3.29}$$

Lemma 3.13. [33, p. 49 and p. 51] *We have*

$$\psi(q) = f(q^3, q^6) + q\psi(q^9), \tag{3.30}$$

$$\varphi(q) = \varphi(q^9) + 2q\psi(-q^9)\chi(q^3). \tag{3.31}$$

Lemma 3.14. [33, p. 49] *We have*

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15}, q^{35}) + 2q^4 f(q^5, q^{45}). \tag{3.32}$$

In the following lemma, we recall a p-dissection of f_1.

Lemma 3.15. [64, Theorem 2.2] *For a prime $p > 3$, we have*

$$f_1 = (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2} + \sum_{\substack{k \neq \frac{\pm p-1}{6}, k=-\frac{p-1}{2}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\Big(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}} \Big),$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \dfrac{p-1}{6}, & if\ p \equiv \ 1 \pmod 6, \\ \dfrac{-p-1}{6}, & if\ p \equiv \ -1 \pmod 6. \end{cases}$$

Furthermore, for $\dfrac{-(p-1)}{2} \leq k \leq \dfrac{p-1}{2}$ *and* $k \neq \dfrac{\pm p-1}{6}$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

3.3 Proofs of Theorems 3.2–3.4

Proof. With the aid of (1.7), we rewrite (3.1) in the form

$$\sum_{n=0}^{\infty} \Lambda(n)q^n = \chi(-q)\chi(-q^2). \tag{3.33}$$

From [24, Lemma 3.5] (See also [61, p. 194, Eq. (3.65)]), we recall that

$$\varphi^2(q) - \varphi^2(q^3) = 4q\chi(q)\chi(-q^2)\psi(q^3)\psi(q^6). \tag{3.34}$$

Replacing q by $-q$ in the above and then employing (3.31), we have

$$4q\chi(-q)\chi(-q^2)\psi(-q^3)\psi(q^6) = \varphi^2(-q^3) - \left(\varphi(-q^9) - 2q\psi(q^9)\chi(-q^3)\right)^2. \quad (3.35)$$

It follows from (3.33) and (3.35) that

$$4\sum_{n=0}^{\infty}\Lambda(n)q^{n+1} = \frac{\varphi^2(-q^3)}{\psi(-q^3)\psi(q^6)} - \frac{1}{\psi(-q^3)\psi(q^6)}\left(\varphi(-q^9) - 2q\psi(q^9)\chi(-q^3)\right)^2. \quad (3.36)$$

Extracting the terms involving q^{3n+1} from both sides of the above, we have

$$4\sum_{n=0}^{\infty}\Lambda(3n)q^{3n+1} = 4q\frac{\chi(-q^3)\psi(q^9)\varphi(-q^9)}{\psi(-q^3)\psi(q^6)}.$$

Dividing both sides of the above by $4q$ and then replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty}\Lambda(3n)q^n = \frac{\chi(-q)\psi(q^3)\varphi(-q^3)}{\psi(-q)\psi(q^2)},$$

which is (3.8).

Similarly, extracting the terms involving q^{3n+2} from both sides of (3.36), dividing by $4q^2$ and then replacing q^3 by q, we arrive at (3.9).

Again, extracting the terms involving q^{3n+3} from both sides of (3.36), we have

$$4\sum_{n=0}^{\infty}\Lambda(3n+2)q^{3n+3} = \frac{1}{\psi(-q^3)\psi(q^6)}\left(\varphi^2(-q^3) - \varphi^2(-q^9)\right).$$

Replacing q^3 by q in the above and then employing (3.34), we find that

$$4\sum_{n=0}^{\infty}\Lambda(3n+2)q^{n+1} = -4q\chi(-q)\chi(-q^2)\frac{\psi(-q^3)\psi(q^6)}{\psi(-q)\psi(q^2)},$$

from which it follows that

$$\sum_{n=0}^{\infty}\Lambda(3n+2)q^n = -\frac{\psi(-q^3)\psi(q^6)}{\psi^2(q^2)},$$

which is (3.10).

Now, from [68, Lemma 1.5], we recall that

$$\frac{1}{\psi^2(q^2)} = \left(\frac{f_6^4 f_{18}^6}{f_{12}^{12}} - 2q^6\frac{f_6^7 f_{36}^9}{f_{12}^{16} f_{18}^3}\right) + 3q^4\frac{f_6^6 f_{36}^6}{f_{12}^{14}} - \left(2q^2\frac{f_6^5 f_{18}^3 f_{36}^3}{f_{12}^{13}} - q^8\frac{f_6^8 f_{36}^{12}}{f_{12}^{16} f_{18}^6}\right). \quad (3.37)$$

Employing (3.37) in (3.10), extracting the terms involving q^{3n+1}, dividing both sides by q, and then replacing q^3 by q, we obtain

$$\sum_{n=1}^{\infty}\Lambda(9n+5)q^n = -3q\psi(-q)\psi(q^2)\frac{f_2^6 f_{12}^6}{f_4^{14}} = -3q\frac{f_1 f_4}{f_2}\cdot\frac{f_4^2}{f_2}\cdot\frac{f_2^6 f_{12}^6}{f_4^{14}}$$

$$= -3q\frac{f_1 f_2^4 f_{12}^6}{f_4^{11}},$$

which is (3.11).

Now, it is easy to see that

$$\psi^3(q) \equiv \psi(q^3) \pmod 3.$$

Employing the above in (3.10), we find that

$$\sum_{n=0}^{\infty} \Lambda(3n+2)q^n = -\frac{\psi(-q^3)\psi(q^6)}{\psi^2(q^2)} = -\frac{\psi(-q^3)\psi(q^6)\psi(q^2)}{\psi^3(q^2)}$$
$$\equiv 2\psi(-q^3)\psi(q^2) \pmod 3, \tag{3.38}$$

which by (3.30) can be written as

$$\sum_{n=0}^{\infty} \Lambda(3n+2)q^n \equiv 2\psi(-q^3)\left(f(q^6,q^{12}) + q^2\psi(q^{18})\right) \pmod 3.$$

Extracting the terms involving q^{3n+2} from the above and then employing (3.30) once again, we find that

$$\sum_{n=0}^{\infty} \Lambda(9n+8)q^n \equiv 2\psi(q^6)\psi(-q)$$
$$\equiv 2\psi(q^6)\left(f(-q^3,q^6) - q\psi(-q^9)\right) \pmod 3. \tag{3.39}$$

Equating the coefficients of q^{3n+2} from both sides of the above, we arrive at

$$\Lambda(27n+26) \equiv 0 \pmod 3,$$

which is (3.4).

Next, from (1.4), (2.30) and (3.9), the fact that $f_1^3 \equiv f_3 \pmod 3$, we find that

$$\sum_{n=0}^{\infty} \Lambda(3n+1)q^n = -\frac{\psi^2(q^3)\chi^2(-q)}{\psi(-q)\psi(q^2)} = -\frac{f_1 f_6^4}{f_3^2 f_4^3}$$
$$\equiv 2\frac{f_1 f_6 f_{18}}{f_3^2 f_{12}} \pmod 3. \tag{3.40}$$

Again, extracting the terms involving q^{3n} from both sides of (3.39) and then replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty} \Lambda(27n+8)q^n \equiv 2\psi(q^2)f(-q,q^2) \pmod 3. \tag{3.41}$$

By (1.1), (1.3), and (1.7), we find that

$$f(-q,q^2)=(q;-q^3)_\infty(-q^2;-q^3)_\infty(-q^3;-q^3)_\infty=\frac{(-q^3;q^6)_\infty^2(q^6;q^6)_\infty}{(-q;q^2)_\infty}.$$

Employing the above identity and (1.4) in (3.41), and then simplifying by using the fact $f_1^3\equiv f_3 \pmod 3$ again, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\Lambda(27n+8)q^n &\equiv 2\frac{(q^4;q^4)_\infty(-q^3;q^6)_\infty^2(q^6;q^6)_\infty}{(-q;q^2)_\infty(q^2;q^4)_\infty}\\ &\equiv 2\frac{f_1f_4^3f_6^5}{f_2^3f_3^2f_{12}^2}\\ &\equiv 2\frac{f_1f_6f_{18}}{f_3^2f_{12}} \pmod 3. \end{aligned} \tag{3.42}$$

From (3.40) and (3.42), we conclude that, for all $n\geq 0$,

$$\Lambda(3n+1)\equiv\Lambda(27n+8) \pmod 3,$$

which is (3.5).

Next, extracting the terms involving q^{3n+1} from both sides of (3.39), dividing both sides by q and then replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty}\Lambda(27n+17)q^n\equiv\psi(q^2)\psi(-q^3) \pmod 3. \tag{3.43}$$

From the above congruence and (3.38), we see that, for all $n\geq 0$,

$$\Lambda(3n+2)\equiv-\Lambda(27n+17) \pmod 3,$$

which is (3.12).

With the aid of (3.30), we can rewrite (3.43) as

$$\sum_{n=0}^{\infty}\Lambda(27n+17)q^n\equiv\psi(-q^3)\left(f(q^6,q^{12})+q^2\psi(q^{18})\right) \pmod 3.$$

Equating the coefficients of q^{3n+1} from both sides of the above, we find that

$$\Lambda(81n+44)\equiv 0 \pmod 3,$$

which is (3.13).

Successive iterations of (3.12) give

$$\begin{aligned}\Lambda(3n+2) &\equiv -\Lambda(3(9n+5)+2)\\ &\equiv \Lambda(27(9n+5)+17)\\ &\equiv \Lambda(3^5n+3^3\cdot 5+3\cdot 5+2)\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&\equiv (-1)^j \Lambda(3\cdot 9^j n + 3\cdot 9^{j-1}\cdot 5 + 3.9^{j-2}\cdot 5 + \cdots + 3\cdot 5 + 2) \\
&\equiv (-1)^j \Lambda\left(3\cdot 9^j n + \frac{15\cdot 9^j + 1}{8}\right) \pmod 3. \qquad (3.44)
\end{aligned}
$$

Replacing n by $3n+1$ in the above, we find that

$$\Lambda(9n+5) \equiv (-1)^j \Lambda\left(9^{j+1} n + \frac{39\cdot 9^j + 1}{8}\right) \pmod 3.$$

Employing (3.3), we see that, for all $j \geq 0$ and $n \geq 0$,

$$\Lambda\left(9^{j+1} n + \frac{39\cdot 9^j + 1}{8}\right) \equiv 0 \pmod 3,$$

which is (3.6).

Again, replacing n by $9n+8$ in (3.44), we have

$$\Lambda(27n+26) \equiv (-1)^j \Lambda\left(3\cdot 9^{j+1} n + \frac{23\cdot 9^{j+1} + 1}{8}\right) \pmod 3.$$

Employing (3.4) in the above, we readily arrive at (3.7).

□

3.4 Proof of Theorem 3.6

Setting $k = 3$ in (3.14) and then manipulating the q-products, we have

$$\sum_{n=0}^{\infty} \Lambda_3(n) q^n = \chi(-q)\chi(-q^3) = \frac{f_1 f_3}{f_2 f_6} = \frac{\psi(-q)\psi(-q^3)}{f_4 f_{12}}.$$

Replacing q by $-q$ in (3.28) and then using the resulting identity in the above, we have

$$\sum_{n=0}^{\infty} \Lambda_3(n) q^n = \frac{\varphi(q^6)\psi(q^4) - q\varphi(q^2)\psi(q^{12})}{f_4 f_{12}}.$$

Extracting, in turn, the even and odd terms from both sides of the above, and then using (3.27), we find that

$$\sum_{n=0}^{\infty} \Lambda_3(2n) q^n = \frac{\varphi(q^3)\psi(q^2)}{f_2 f_6} = \frac{\psi(q^2)\left(\varphi(q^{12}) + 2q^3\psi(q^{24})\right)}{f_2 f_6}$$

and

$$\sum_{n=0}^{\infty} \Lambda_3(2n+1) q^n = -\frac{\varphi(q)\psi(q^6)}{f_2 f_6} = -\frac{\psi(q^6)\left(\varphi(q^4) + 2q\psi(q^8)\right)}{f_2 f_6}.$$

Equating the coefficients of q^{2n+1} from both sides of the above two identities, we arrive at

$$\Lambda_3(4n+r) \equiv 0 \pmod 2, \quad \text{where} \quad r \in \{2,3\},$$

which is (3.15).

3.5 Proof of Theorem 3.7

Setting $k = 5$ in (3.14), we have

$$\sum_{n=0}^{\infty} \Lambda_5(n) q^n = \chi(-q)\chi(-q^5). \tag{3.45}$$

Now, recall from [33, p. 258, Entry 9(vii) and p. 262, Entry 10(iv)] that

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7) = 4q\chi(q)f_5 f_{20}. \tag{3.46}$$

Multiplying by $\chi(q^5)$, and then replacing q by $-q$, we find that

$$\begin{aligned} 4q\chi(-q)\chi(-q^5) &= \frac{\chi(-q^5)}{\chi(q^5) f_{10} f_{20}} \left(\varphi^2(-q^5) - \varphi^2(-q)\right) \\ &= \frac{\varphi^2(-q^5) - \varphi^2(-q)}{\psi^2(q^5)}. \end{aligned}$$

With the aid of (3.45) and (3.32), the above may be rewritten as

$$\begin{aligned} &4\sum_{n=0}^{\infty} \Lambda_5(n) q^{n+1} \\ &= \frac{1}{\psi^2(q^5)} \Big(\varphi^2(-q^5) - \big(\varphi(-q^{25}) - 2qf(-q^{15},-q^{35}) + 2q^4 f(-q^5,-q^{45})\big)^2\Big). \end{aligned} \tag{3.47}$$

Extracting, in turn, the terms of the form q^{5n+2} and q^{5n+3} from both sides of the above, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \Lambda_5(5n+1) q^n &= -\frac{f^2(-q^3,-q^7)}{\psi^2(q)} \equiv \frac{f(-q^6,-q^{14})}{\psi(q^2)} \pmod 2, \\ \sum_{n=0}^{\infty} \Lambda_5(5n+2) q^n &= -q\frac{f^2(-q,-q^9)}{\psi^2(q)} \equiv q\frac{f(-q^2,-q^{18})}{\psi(q^2)} \pmod 2, \end{aligned}$$

from which it readily follows that $\Lambda_5(10n+6) \equiv 0 \pmod 2$ and $\Lambda_5(10n+2) \equiv 0 \pmod 2$. This completes the proof of (3.16).

Now, extracting the terms involving q^{5n+5} from both sides of (3.47), replacing

q^5 by q, and then applying (3.46), we find that

$$4\sum_{n=0}^{\infty}\Lambda_5(5n+4)q^{n+1} = \frac{1}{\psi^2(q)}(\varphi^2(-q)-\varphi^2(-q^5)+8qf(-q^3,-q^7)f(-q,-q^9))$$
$$= \frac{4qf(-q^3,-q^7)f(-q,-q^9)}{\psi^2(q)}$$
$$= \frac{4q\chi(-q)(-q^5;q^{10})_\infty f_{10}f_{20}}{\psi^2(q)}.$$

Therefore,

$$\sum_{n=0}^{\infty}\Lambda_5(5n+4)q^n = \frac{f_1^3 f_{10}^3}{f_2^5 f_5} \equiv \frac{f_{10}^2}{f_5}f_1^3 \pmod 5. \tag{3.48}$$

But, well-known Jacobi's identity [34, Eq. (1.3.24)] states that

$$f_1^3 = \sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)/2}. \tag{3.49}$$

Employing this in (3.48), we have

$$\sum_{n=0}^{\infty}\Lambda_5(5n+4)q^n \equiv \frac{f_{10}^2}{f_5}\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)/2} \pmod 5. \tag{3.50}$$

Now, $j(j+1)/2 \equiv 0, 1$ or $3 \pmod 5$. Therefore, equating coefficients of q^{5n+2} and q^{5n+4}, in turn, from both sides of (3.50), we find that

$$\Lambda_5(25n+14) \equiv \Lambda_5(25n+24) \equiv 0 \pmod 5. \tag{3.51}$$

Furthermore, if $j \equiv 2 \pmod 5$, then $j(j+1)/2 \equiv 3 \pmod 5$ and $2j+1 \equiv 0 \pmod 5$. Therefore, equating the coefficients of q^{5n+3} from both sides of (3.50), we find that

$$\Lambda_5(25n+19) \equiv 0 \pmod 5. \tag{3.52}$$

Clearly, (3.51) and (3.52) together give (3.17). This completes the proof.

3.6 Proof of Theorem 3.8

Proofs of (3.18), (3.19), (3.20). Setting $k = 7$ in (3.14), manipulating the q-products, and then employing (3.29), we have

$$\sum_{n=0}^{\infty}\Lambda_7(n)q^n = \chi(-q)\chi(-q^7) = \frac{f_1f_7}{f_2f_{14}} = \frac{\psi(-q)\psi(-q^7)}{f_4f_{28}}$$
$$= \frac{1}{f_4f_{28}}\left(\varphi(q^{28})\psi(q^8) - q\psi(q^2)\psi(q^{14}) + q^6\varphi(q^4)\psi(q^{56})\right). \quad (3.53)$$

Extracting the odd terms from both sides of the above and then employing (3.29) once again, we have

$$\sum_{n=0}^{\infty}\Lambda_7(2n+1)q^n = -\frac{\psi(q)\psi(q^7)}{f_2f_{14}}$$
$$= -\frac{1}{f_2f_{14}}\left(\varphi(q^{28})\psi(q^8) + q\psi(q^2)\psi(q^{14}) + q^6\varphi(q^4)\psi(q^{56})\right). \quad (3.54)$$

Extracting the odd terms, we find that

$$\sum_{n=0}^{\infty}\Lambda_7(4n+3)q^n = -\frac{\psi(q)\psi(q^7)}{f_1f_7} = -\frac{f_2^2f_{14}^2}{f_1^2f_7^2}$$
$$\equiv f_2f_{14} \pmod 2. \quad (3.55)$$

It follows from (3.55) that

$$\sum_{n=0}^{\infty}\Lambda_7(8n+3)q^n \equiv f_1f_7 \pmod 2 \quad (3.56)$$

and

$$\Lambda_7(8n+7) \equiv 0 \pmod 2. \quad (3.57)$$

From (3.54), we also have

$$\sum_{n=0}^{\infty}\Lambda_7(2n+1)q^n \equiv f_1f_7 \pmod 2. \quad (3.58)$$

From the above congruence and (3.56), we readily arrive at (3.18).

Now, iterating (3.18), we find that

$$\Lambda_7(2n+1) \equiv \Lambda_7(2(4n+1)+1)$$
$$\equiv \Lambda_7(2(4^2n+4+1)+1)$$
$$\equiv \Lambda_7(2(4^3n+4^2+4+1)+1)$$

$$\vdots$$
$$\equiv \Lambda_7\left(2\left(4^j n+\frac{4^j-1}{3}\right)+1\right) \pmod 2.$$

Replacing n by $4n+3$ in the above and then employing (3.57), we obtain (3.19).

Now, extracting the even terms on both sides of (3.54), we have

$$\sum_{n=0}^{\infty}\Lambda_7(4n+1)q^n = -\frac{1}{f_1f_7}\left(\varphi(q^{14})\psi(q^4)+q^3\varphi(q^2)\psi(q^{28})\right)$$
$$= \frac{\psi(q)(\psi(q^7)\left(\varphi(q^{14})\psi(q^4)+q^3\varphi(q^2)\psi(q^{28})\right)}{f_2^2f_{14}^2}. \tag{3.59}$$

Now, as $\varphi(q)\equiv 1 \pmod 2$, from (3.29), we have

$$\psi(q)\psi(q^7)\equiv \psi(q^8)+q\psi(q^2)\psi(q^{14})+q^6\psi(q^{56}) \pmod 2.$$

Therefore, from(3.59), we find that

$$\sum_{n=0}^{\infty}\Lambda_7(4n+1)q^n \equiv \frac{\left(\psi(q^8)+q\psi(q^2)\psi(q^{14})+q^6\psi(q^{56})\right)\left(\psi(q^4)+q^3\psi(q^{28})\right)}{f_4f_{28}} \pmod 2. \tag{3.60}$$

Extracting the even terms, we have

$$\sum_{n=0}^{\infty}\Lambda_7(8n+1)q^n$$
$$\equiv \frac{\psi(q^2)\psi(q^4)+q^3\psi(q^2)\psi(q^{28})+q^2\psi(q)\psi(q^7)\psi(q^{14})}{f_2f_{14}}$$
$$\equiv \frac{\psi(q^2)\psi(q^4)+q^3\psi(q^2)\psi(q^{28})+q^2\psi(q^{14})\left(\psi(q^8)+q\psi(q^2)\psi(q^{14})+q^6\psi(q^{56})\right)}{f_2f_{14}} \pmod 2.$$

Extracting the odd terms from both sides of the above, we obtain

$$\sum_{n=0}^{\infty}\Lambda_7(16n+9)q^n \equiv \frac{q\psi(q)\psi(q^{14})+q\psi(q)\psi^2(q^7)}{f_1f_7}$$
$$\equiv 2\frac{q\psi(q)\psi(q^{14})}{f_1f_7}\equiv 0 \pmod 2,$$

from which (3.20) for $r=9$ follows readily.

Next we prove (3.20) for $r=13$. Extracting the odd terms from both sides of (3.60), we find that

$$\sum_{n=0}^{\infty}\Lambda_7(8n+5)q^n$$
$$\equiv \frac{q\psi(q^4)\psi(q^{14})+q^4\psi(q^{14})\psi(q^{28})+\psi(q)\psi(q^2)\psi(q^7)}{f_2f_{14}}$$

$$\equiv \frac{q\psi(q^4)\psi(q^{14}) + q^4\psi(q^{14})\psi(q^{28}) + \psi(q^2)\left(\psi(q^8) + q\psi(q^2)\psi(q^{14}) + q^6\psi(q^{56})\right)}{f_2 f_{14}} \pmod 2.$$

Extracting the odd terms, we obtain

$$\sum_{n=0}^{\infty} \Lambda_7(16n+13)q^n \equiv \frac{\psi(q^2)\psi(q^7) + \psi^2(q)\psi(q^7)}{f_1 f_7}$$
$$\equiv 2\frac{\psi(q^2)\psi(q^7)}{f_1 f_7} \equiv 0 \pmod 2,$$

from which (3.20) for $r = 13$ is apparent. With this, we complete the proof of (3.20).

Proof of (3.21). From (3.58), we have

$$\sum_{n=0}^{\infty} \Lambda_7(2n+1)q^n \equiv f_1 f_7 \pmod 2.$$

Now, using the 5-dissection of f_1 stated in (2.38), we have

$$\sum_{n=0}^{\infty} \Lambda_7(2n+1)q^n \equiv f_{25} f_{7\cdot 25} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5)\right)$$
$$\times \left(\frac{1}{R(q^{35})} - q^7 - q^{14} R(q^{35})\right) \pmod 2.$$

Extracting the terms involving q^{5n+3} from both sides, dividing both sides by q^3 and replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} \Lambda_7(10n+7)q^n \equiv q f_5 f_{35} \pmod 2.$$

Extracting the terms involving q^{5n+r} for $r \in \{0, 2, 3, 4\}$, we obtain

$$\Lambda_7(10(5n+r)+7)q^n \equiv 0 \pmod 2.$$

This completes the proof of (3.21).

Proof of (3.22), (3.23). At first, we show by the mathematical induction that for all $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \Lambda_7\left(2\cdot 7^\alpha n + \frac{2\cdot 7^\alpha + 1}{3}\right) q^n \equiv f_1 f_7 \pmod 2. \tag{3.61}$$

Clearly, by (3.58), the result is true for $\alpha = 0$. Now, suppose that (3.61) holds good for some $\alpha > 0$. Setting $p = 7$ in Lemma 3.15, we have

$$f_1 = q^2 f_{49} + \sum_{k\neq 1, k=-3}^{3} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3\cdot 7^2 + (6k+1)7}{2}}, -q^{\frac{3\cdot 7^2 - (6k+1)7}{2}}\right).$$

Employing the above in (3.61), we have

$$\sum_{n=0}^{\infty} \Lambda_7\left(2\cdot 7^\alpha n + \frac{2\cdot 7^\alpha + 1}{3}\right) q^n$$

$$\equiv q^2 f_7 f_{49} + f_7 \sum_{k\neq 1, k=-3}^{3} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3\cdot 7^2+(6k+1)7}{2}}, -q^{\frac{3\cdot 7^2-(6k+1)7}{2}}\right) \pmod 2.$$

It can be easily verified that $\dfrac{3k^2+k}{2} \not\equiv 2 \pmod 7$ for $k \neq 1$. Therefore, extracting the terms involving q^{7n+2} from both sides of the above, dividing both sides by q^2, and then replacing q^7 by q, we arrive at

$$\sum_{n=0}^{\infty} \Lambda_7 \left(2 \cdot 7^{\alpha+1} n + \frac{2 \cdot 7^{\alpha+1}+1}{3}\right) q^n \equiv f_1 f_7 \pmod 2.$$

Thus, (3.61) holds good for $\alpha+1$ whenever it holds good for some $\alpha > 0$. Hence, by mathematical induction, (3.61) is true for all $\alpha \geq 0$.

Next, we prove by mathematical induction that if p is a prime such that $\left(\dfrac{-7}{p}\right)_L = -1$, then for all $\delta \geq 0$ and $n \geq 0$,

$$\sum_{n=0}^{\infty} \Lambda_7 \left(2 \cdot 7^{\alpha} \cdot p^{2\delta} n + \frac{2 \cdot 7^{\alpha} \cdot p^{2\delta}+1}{3}\right) q^n \equiv f_1 f_7 \pmod 2. \tag{3.62}$$

The case $\delta = 0$ of (3.62) is clearly true by (3.61).

Suppose that (3.62) is true for some $\alpha > 0$. Then, by Lemma 3.15, we have

$$\sum_{n=0}^{\infty} \Lambda_7 \left(2 \cdot 7^{\alpha} \cdot p^{2\delta} n + \frac{2 \cdot 7^{\alpha} \cdot p^{2\delta}+1}{3}\right) q^n$$
$$\equiv \Bigg[\sum_{k\neq \frac{\pm p-1}{6}, k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\Big(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\Big) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}\Bigg]$$
$$\times \Bigg[\sum_{k\neq \frac{\pm p-1}{6}, k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{7\cdot\frac{3k^2+k}{2}} f\Big(-q^{7\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{7\cdot\frac{3p^2-(6k+1)p}{2}}\Big)$$
$$+ (-1)^{\frac{\pm p-1}{6}} q^{7\cdot\frac{p^2-1}{24}} f_{7\cdot p^2}\Bigg] \pmod 2. \tag{3.63}$$

Now we consider the congruence

$$\frac{3k^2+k}{2} + 7\cdot\frac{3m^2+m}{2} \equiv \frac{p^2-1}{3} \pmod p, \tag{3.64}$$

where $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$. Since the above congruence is equivalent to solving the congruence

$$(6k+1)^2 + 7(6m+1)^2 \equiv 0 \pmod p,$$

and $\left(\frac{-7}{p}\right)_L = -1$, it follows that (3.64) has the unique solution $k = m = \frac{\pm p-1}{6}$. Therefore, extracting the terms involving $q^{pn+\frac{p^2-1}{3}}$ from both sides of (3.63), we find that

$$\sum_{n=0}^{\infty} \Lambda_7 \left(2 \cdot 7^{\alpha} \cdot p^{2\delta+1} n + \frac{2 \cdot 7^{\alpha} \cdot p^{2(\delta+1)} + 1}{3} \right) q^n \equiv f_p f_{7p} \pmod{2}. \tag{3.65}$$

Extracting the terms involving q^{pn} from the above and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} \Lambda_7 \left(2 \cdot 7^{\alpha} \cdot p^{2(\delta+1)} n + \frac{2 \cdot 7^{\alpha} \cdot p^{2(\delta+1)} + 1}{3} \right) q^n \equiv f_1 f_7 \pmod{2},$$

which clearly is the $\alpha + 1$ case of (3.62). This completes the proof of (3.62).

Now, it can be seen that $(3k^2+k)/2 \equiv 0, 1, 2, \text{or } 5 \pmod{7}$. Therefore, equating the coefficients of q^{7n+r}, where $r = 3, 4, 6$, from both sides of (3.62), we arrive at

$$\Lambda_7 \left(2 \cdot 7^{\alpha} \cdot p^{2\delta} (7n + r) + \frac{2 \cdot 7^{\alpha} \cdot p^{2\delta} + 1}{3} \right) \equiv 0 \pmod{2},$$

which is (3.22).

Now, equating the coefficients of q^{pn+r} for $r \in \{1, 2, \ldots, p-1\}$ on both sides of (3.65), we readily arrive at (3.23).

Remark 3.16. *It follows from* (3.53) *and* (3.54) *that*

$$\left(\sum_{n=0}^{\infty} \Lambda_7(n) q^n \right) \left(\sum_{n=0}^{\infty} \Lambda_7(2n+1) q^n \right) = -1.$$

3.7 Proof of Theorem 3.9

Proof of (3.24). Setting $k = 23$ in (3.14), we have

$$\sum_{n=0}^{\infty} \Lambda_{23}(n) q^n = \chi(-q)\chi(-q^{23}). \tag{3.66}$$

From [35, Eq. (7.4)], we recall that

$$\chi(-q)\chi(-q^{23}) - \chi(q)\chi(q^{23}) = -2q - 2q^3(-q^2; q^2)_\infty(-q^{46}; q^{46})_\infty,$$

which, by (3.69), may be rewritten as

$$\sum_{n=0}^{\infty} \Lambda_{23}(n) q^n - \sum_{n=0}^{\infty} \Lambda_{23}(n)(-q)^n = -2q - 2q^3(-q^2; q^2)_\infty(-q^{46}; q^{46})_\infty.$$

It follows from the above that

$$\sum_{n=0}^{\infty} \Lambda_{23}(2n+1) q^n = -1 - q(-q; q)_\infty(-q^{23}; q^{23})_\infty$$

$$\equiv 1 + qf_1f_{23} \pmod 2,$$

and hence,

$$\sum_{n=0}^{\infty} \Lambda_{23}(2n+3)q^n \equiv f_1f_{23} \pmod 2, \tag{3.67}$$

From (3.70), we have

$$\sum_{n=0}^{\infty} \Lambda_{23}(2n+3)q^n \equiv f_1f_{23} \pmod 2.$$

Now, using the 5-dissection of f_1 stated in (2.38), we have

$$\sum_{n=0}^{\infty} \Lambda_{23}(2n+3)q^n \equiv f_{25}f_{23\cdot 25}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)$$
$$\times\left(\frac{1}{R(q^{115})} - q^{23} - q^{46}R(q^{115})\right) \pmod 2.$$

Extracting the terms involving q^{5n+4} from both sides, dividing both sides by q^4 and replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} \Lambda_{23}(10n+11)q^n \equiv q^4f_5f_{115} \pmod 2.$$

Extracting the terms involving q^{5n+r} for $r \in \{0,1,2,3\}$, we obtain

$$\Lambda_{23}(10(5n+r)+11)q^n \equiv 0 \pmod 2.$$

This completes the proof of (3.24).

Proof of (3.25), (3.26). At first, we prove by mathematical induction that for all $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \Lambda_{23}\left(2\cdot 23^{\alpha}n + 2\cdot 23^{\alpha} + 1\right)q^n \equiv f_1f_{23} \pmod 2. \tag{3.68}$$

Setting $k = 23$ in (3.14), we have

$$\sum_{n=0}^{\infty} \Lambda_{23}(n)q^n = \chi(-q)\chi(-q^{23}). \tag{3.69}$$

From [35, Eq. (7.4)], we recall that

$$\chi(-q)\chi(-q^{23}) - \chi(q)\chi(q^{23}) = -2q - 2q^3(-q^2;q^2)_\infty(-q^{46};q^{46})_\infty,$$

which, by (3.69), may be rewritten as

$$\sum_{n=0}^{\infty} \Lambda_{23}(n)q^n - \sum_{n=0}^{\infty} \Lambda_{23}(n)(-q)^n = -2q - 2q^3(-q^2;q^2)_\infty(-q^{46};q^{46})_\infty.$$

It follows from the above that

$$\sum_{n=0}^{\infty}\Lambda_{23}(2n+1)q^n = -1-q(-q;q)_\infty(-q^{23};q^{23})_\infty$$
$$\equiv 1+qf_1f_{23} \pmod 2,$$

and hence,

$$\sum_{n=0}^{\infty}\Lambda_{23}(2n+3)q^n \equiv f_1f_{23} \pmod 2, \tag{3.70}$$

which is the case $\alpha=0$ of (3.68).

Now, suppose that (3.68) is true for some $\alpha>0$. We claim that it is then true for $\alpha+1$ as well.

Setting $p=23$ in the p-dissection of f_1 stated in Lemma 3.15, we see that

$$f_1 = q^{22}f_{23^2} + \sum_{k\neq -4, k=-11}^{11} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3\cdot 23^2+23(6k+1)}{2}}, -q^{\frac{3\cdot 23^2-23(6k+1)}{2}}\right).$$

Employing the above in (3.68), we have

$$\sum_{n=0}^{\infty}\Lambda_{23}\left(2\cdot 23^\alpha n+2\cdot 23^\alpha+1\right)q^n$$
$$\equiv q^{22}f_{23}f_{23^2} + \sum_{k\neq -4, k=-11}^{11} (-1)^k q^{\frac{3k^2+k}{2}} f_{23} f\left(-q^{\frac{3\cdot 23^2+23(6k+1)}{2}}, -q^{\frac{3\cdot 23^2-23(6k+1)}{2}}\right) \pmod 2.$$

It is easy to verify that $\dfrac{3k^2+k}{2} \not\equiv 22 \pmod{23}$ for $k\neq -4$. Therefore, extracting the terms involving q^{23n+22} on both sides of the above, dividing by q^{22}, and then replacing q^{23} by q, we find that

$$\sum_{n=0}^{\infty}\Lambda_{23}\left(2\cdot 23^\alpha(23n+22)+2\cdot 23^\alpha+1\right)q^n \equiv f_1f_{23} \pmod 2,$$

which is the $\alpha+1$ case of (3.68). Thus, (3.68) holds good for all $\alpha\geq 0$.

Now, we prove by mathematical induction that if $p>3$ is a prime such that $\left(\dfrac{-23}{p}\right)_L=-1$, then for all $\delta\geq 0$

$$\sum_{n=0}^{\infty}\Lambda_{23}\left(2\cdot 23^\alpha\cdot p^{2\delta}n+2\cdot 23^\alpha\cdot p^{2\delta}+1\right)q^n \equiv f_1f_{23} \pmod 2. \tag{3.71}$$

Clearly, (3.68) is the $\delta=0$ case of (3.71).

Now, suppose that (3.71) is true for some $\alpha > 0$. Then, by Lemma 3.15, we have

$$\sum_{n=0}^{\infty} \Lambda_{23}\left(2\cdot 23^{\alpha}\cdot p^{2\delta}n + 2\cdot 23^{\alpha}\cdot p^{2\delta}+1\right)q^n$$

$$\equiv \Bigg[\sum_{k\neq\frac{\pm p-1}{6},k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\Big(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\Big) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}\Bigg]$$

$$\times \Bigg[\sum_{k\neq\frac{\pm p-1}{6},k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{23\cdot\frac{3k^2+k}{2}} f\Big(-q^{7\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{23\cdot\frac{3p^2-(6k+1)p}{2}}\Big)$$

$$+ (-1)^{\frac{\pm p-1}{6}} q^{23\cdot\frac{p^2-1}{24}} f_{23\cdot p^2}\Bigg] \pmod 2. \tag{3.72}$$

Now, consider the congruence

$$\frac{3k^2+k}{2} + 23\cdot\frac{3m^2+m}{2} \equiv p^2-1 \pmod p,$$

where $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$. As the above congruence is equivalent to solving the congruence

$$(6k+1)^2 + 23(6m+1)^2 \equiv 0 \pmod p,$$

and $\left(\dfrac{-23}{p}\right)_L = -1$, it has a unique solution, namely, $k = m = \frac{\pm p-1}{6}$. Therefore, extracting the terms involving q^{pn+p^2-1} on both sides of the congruence (3.72), dividing by q^{p^2-1}, and then replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} \Lambda_{23}\left(2\cdot 23^{\alpha}\cdot p^{2\delta+1}n + 2\cdot 23^{\alpha}\cdot p^{2(\delta+1)}+1\right)q^n \equiv f_p f_{23p} \pmod 2. \tag{3.73}$$

Extracting the terms involving q^{pn} from both sides of the above and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \Lambda_{23}\left(2\cdot 23^{\alpha}\cdot p^{2(\delta+1)}n + 2\cdot 23^{\alpha}\cdot p^{2(\delta+1)}+1\right)q^n \equiv f_1 f_{23} \pmod 2,$$

which is clearly the $\alpha+1$ case of (3.71). Hence, (3.71) is true for all $\alpha \geq 0$.

It can be easily verified that $\dfrac{3k^2+k}{2} \not\equiv 4, 6, 9, 10, 13, 14, 16, 18, 19, 20, 21 \pmod{23}$. So, equating the coefficients of q^{23n+r} for $r \in \{4, 6, 9, 10, 13, 14, 16, 18, 19, 20, 21\}$ on both sides of (3.71), we find that, for all $\alpha \geq 0$,

$$\Lambda_{23}\left(2\cdot 23^{\alpha}\cdot p^{2\delta}(23n+r) + 2\cdot 23^{\alpha}\cdot p^{2\delta}+1\right) \equiv 0 \pmod 2,$$

which is (3.25).

Equating the coefficients of q^{pn+r} for $r \in \{1, 2, \ldots, p-1\}$ on both sides of (3.73), we readily arrive at (3.26) to complete the proof.

Remark 3.17. *Setting* $\alpha, \delta = 0$ *and* $p = 5$ *in* (3.26) *alongwith the facts that* $\Lambda_{23}(11) \equiv \Lambda_{23}(21) \equiv \Lambda_{23}(31) \equiv \Lambda_{23}(41) \equiv 0 \pmod 2$ *gives* (3.24).