Chapter 4

Arithmetic properties of 5-regular partitions into distinct parts

4.1 Introduction

Let $b'_\ell(n)$ count the number of ℓ-regular partitions into distinct parts of n. For example, $b'_5(10) = 7$ and the relevant 7 partitions of 10 are $9+1$, $8+2$, $7+3$, $7+2+1$, $6+4$, $6+3+1$, and $4+3+2+1$. It is clear that $b'_\ell(n)$ also counts the number of ℓ-regular partitions with odd parts of n. With the convention that $b'_\ell(0) = 1$, the generating function of $b'_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b'_\ell(n) q^n = \frac{(-q;q)_\infty}{(-q^\ell;q^\ell)_\infty}. \tag{4.1}$$

Note that, $b'_2(n)$ counts the number of partitions of n into distinct odd parts, which, in fact, is equal to the number of self-conjugate partitions of n. The function $b'_2(n)$ has been well-studied. There are certain known results on $b'_\ell(n)$ for $\ell \geq 3$. For primes $\ell \geq 3$ and an integer r with $1 \leq r \leq p-1$ such that $24r+1$ is a quadratic nonresidue modulo ℓ, Sellers [141] proved that, for all nonnegative integers n,

$$b'_\ell(\ell n + r) \equiv 0 \pmod 2. \tag{4.2}$$

The contents of this chapter have appeared in *International Journal of Number Theory* [29].

For a given prime $\ell \geq 5$, Cui and Gu [64, p. 523] showed that

$$b'_\ell\left(\ell n + \frac{\ell^2-1}{24}\right) \equiv b_\ell(n) \pmod{2}. \tag{4.3}$$

Therefore, congruences modulo 2 of $b'_\ell(n)$ may be studied from those of $b_\ell(n)$. Thus, many results on congruences modulo 2 for $b'_\ell(n)$ can be derived from the results in papers on $b_\ell(n)$ that we mentioned earlier. Recently, Iwata [91] found some congruences modulo 2 for $b'_\ell(n)$ for $\ell = 9, 25, 41$, and 45 by using modular forms.

In this chapter, we study the arithmetic properties of the function $b'_5(n)$, which counts the number of 5-regular partitions into distinct parts of n. Setting $\ell = 5$ in (4.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b'_5(n)q^n &= \frac{(-q;q)_\infty}{(-q^5;q^5)_\infty}\\ &= 1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + 3q^6 + 4q^7 + 4q^8 + 6q^9 + 7q^{10} + 8q^{11}\\ &\quad + 10q^{12} + 12q^{13} + 14q^{14} + 16q^{15} + 19q^{16} + 22q^{17} + 26q^{18} + \cdots. \end{aligned} \tag{4.4}$$

The sequence $(b'_5(n))$ is A096938 in [147]. Other interpretations of this sequence are also discussed there. The function $b'_5(n)$ is also related to representation theory and studied from that point of view by Andrews *et al.* [13] (see also Andrews *et al.* [12]). Very recently, Ballantine and Feigon [16] gave a new combinatorial interpretation of $b'_5(n)$. There are a few known arithmetic properties of $b'_5(n)$ as well. It follows from (4.2) that

$$b'_5(5n+3) \equiv b'_5(5n+4) \equiv 0 \pmod{2}.$$

Many results on congruences modulo 2 for $b'_5(n)$ can also be derived from (4.3) and the corresponding work on $b_5(n)$. In particular, see [45], [64], and [89] for results on $b_5(n)$ modulo 2.

In this chapter, we prove several new arithmetic results on $b'_5(n)$. We state our results in the following theorems.

The following theorem gives a complete characterization of the parity of $b'_5(2n+1)$.

Theorem 4.1. *For all $n \geq 0$,*

$$b'_5(2n+1) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = 15k^2 - 5k \text{ for } k \in \mathbb{Z}, \\ 0 \pmod 2, & \text{Otherwise.} \end{cases} \tag{4.5}$$

Some congruences modulo 4 for $b'_5(n)$ are given in the next theorem.

Theorem 4.2. *Let p (> 5) be a prime such that $\left(\frac{3}{p}\right)_L \neq \left(\frac{-5}{p}\right)_L$. For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$b'_5(20n+j) \equiv 0 \pmod 4, \text{ where } j \in \{7, 15\}, \tag{4.6}$$

$$b'_5(100n+j) \equiv 0 \pmod 4, \text{ where } j \in \{11, 31\}, \tag{4.7}$$

$$b'_5\left(4 \cdot p^{2\alpha}(5n+j) + \frac{17 \cdot p^{2\alpha}+1}{6}\right) \equiv 0 \pmod 4, \text{ where } j \in \{1, 3\}, \tag{4.8}$$

$$b'_5\left(4 \cdot p^{2\alpha+1}(pn+j) + \frac{17 \cdot p^{2\alpha+2}+1}{6}\right) \equiv 0 \pmod 4, \text{ where } j \in \{1, 2, \ldots, p-1\}. \tag{4.9}$$

In the next theorem, we state the exact generating functions of $b'_5(5n+1)$ and $b'_5(25n+21)$ in terms of q-products, from which an internal congruence modulo 5 is also derived.

Theorem 4.3. *We have*

$$\sum_{n=0}^{\infty} b'_5(5n+1)q^n = \frac{f_2 f_5^3}{f_1^3 f_{10}} \tag{4.10}$$

and

$$\sum_{n=0}^{\infty} b'_5(25n+21)q^n = \frac{f_1 f_{10}^3}{f_2^3 f_5} + 40\frac{f_2^4 f_5^4}{f_1^8} + 500q\frac{f_2^4 f_5^{10}}{f_1^{14}}. \tag{4.11}$$

Furthermore, for all integers $\alpha \geq 0$, we have

$$b'_5(5n+1) \equiv b'_5\left(5^{2\alpha+1}n + \frac{5^{2\alpha+1}+1}{6}\right) \pmod 5. \tag{4.12}$$

In the final result of this chapter, we prove the following theorem which implies that $\sum_{n=0}^{\infty} b'_5(5n+1)q^n$ is lacunary modulo arbitrary positive powers of 5.

Theorem 4.4. *Let k be a positive integer. Then*

$$\lim_{X\to\infty} \frac{\#\{0 \leq n \leq X : b'_5(5n+1) \equiv 0 \pmod{5^k}\}}{X} = 1.$$

In Sections 4.2–4.5, we prove Theorems 4.1–4.4, respectively. We use t-dissections of certain q-products and the theory of modular forms in our proofs. The necessary background material and useful preliminary lemmas are given in the corresponding section.

4.2 Proof of Theorem 4.1

From (4.4), we have

$$\sum_{n=0}^{\infty} b_5'(n)q^n = \frac{(-q;q)_\infty}{(-q^5;q^5)_\infty} = \frac{f_2}{f_{10}} \cdot \frac{f_5}{f_1}. \tag{4.13}$$

Employing (2.35) in (4.13), we have

$$\sum_{n=0}^{\infty} b_5'(n)q^n = \frac{f_8 f_{20}^2}{f_2 f_{10} f_{40}} + q\frac{f_4^3 f_{40}}{f_2^2 f_8 f_{20}}. \tag{4.14}$$

Extracting the terms involving q^{2n+1} from both sides of the above, dividing by q, and then replacing q^2 by q in the resulting identity, we find that

$$\sum_{n=0}^{\infty} b_5'(2n+1)q^n = \frac{f_2^3 f_{20}}{f_1^2 f_4 f_{10}}. \tag{4.15}$$

Employing (2.87) in (4.15), we have

$$\sum_{n=0}^{\infty} b_5'(2n+1)q^n \equiv f_{10} \pmod 2. \tag{4.16}$$

Employing (1.8) with q replaced by q^{10} in (4.16), we see that

$$\sum_{n=0}^{\infty} b_5'(2n+1)q^n \equiv \sum_{k=-\infty}^{\infty} q^{15k^2-5k} \pmod 2,$$

from which we readily arrive at (4.5) to complete the proof.

4.3 Proof of Theorem 4.2

We recall the following 2-dissection is stated in the following lemma, which will be used subsequently.

Lemma 4.5. [26, Lemma 2] *We have*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}. \tag{4.17}$$

The following result which will be helpful in proving the theorem.

Lemma 4.6. ([1, Lemma 2.3]) *If $p \geq 3$ is a prime, then*

$$f_1^3 = \sum_{\substack{k=0,\\ k\neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\cdot\frac{pn+2k+1}{2}} + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} f_{p^2}^3. \tag{4.18}$$

Furthermore, for $0 \leq k \leq p-1$ and $k \neq \frac{p-1}{2}$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

Now we are in a position to prove Theorem 4.2.

Proofs of (4.6) *and* (4.7). By the binomial theorem, for all positive integers j,

$$f_j^4 \equiv f_{2j}^2 \pmod{4}. \tag{4.19}$$

Employing (4.19) in (4.15), and then using (4.17), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_5'(2n+1)q^n &\equiv \frac{f_1^2 f_2 f_{20}}{f_4 f_{10}} \\ &\equiv \frac{f_2^2 f_8 f_{20}}{f_4^3 f_{10}} - 2q\frac{f_2^2 f_{16}^2 f_{20}}{f_4 f_8 f_{10}} \pmod{4}.\end{aligned}$$

Extracting the terms involving q^{2n+1}, we find that

$$\sum_{n=0}^{\infty} b_5'(4n+3)q^n \equiv 2\frac{f_1^2 f_8^2 f_{10}}{f_2 f_4 f_5} \pmod{4}.$$

Applying (2.87) in the above, we have

$$\sum_{n=0}^{\infty} b_5'(4n+3)q^n \equiv 2 f_4^3 f_5 \pmod{4}, \tag{4.20}$$

which, by (2.38), yields

$$\begin{aligned}\sum_{n=0}^{\infty} b_5'(4n+3)q^n \equiv 2 f_5 f_{100}^3 \bigg(&\frac{1}{R(q^{20})^3} + \frac{q^4}{R(q^{20})^2} + q^{12} \\ &+ q^{20}R(q^{20})^2 - \frac{q^{24}}{R(q^{20})^3} \bigg) \pmod{4}.\end{aligned} \tag{4.21}$$

Equating the coefficients of q^{5n+1} and q^{5n+3} from both sides of the above, we arrive at

$$b_5'(20n+7) \equiv b_5'(20n+15) \equiv 0 \pmod{4},$$

which is (4.6).

Next, extracting the coefficients of q^{5n+2} from both sides of (4.21), we find that

$$\sum_{n=0}^{\infty} b_5'(20n+11)q^n \equiv 2q^2 f_1 f_{20}^3 \pmod 4.$$

Employing (2.38) in the above and then equating the coefficients of q^{5n} and q^{5n+1}, we obtain

$$b_5'(100n+11) \equiv b_5'(100n+31) \equiv 0 \pmod 4,$$

which is (4.7).

Proofs of (4.8) *and* (4.9). Employing (3.15) and (4.18), we rewrite (4.20) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_5'(4n+3)q^n \\
&\equiv 2\Bigg[\sum_{k\neq\frac{p-1}{2},k=0}^{p-1} (-1)^k q^{4\cdot\frac{k(k+1)}{2}} \sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{4\cdot pn\cdot\frac{pn+2k+1}{2}} \\
&\quad + p(-1)^{\frac{p-1}{2}} q^{4\cdot\frac{p^2-1}{8}} f_{4p^2}^3\Bigg] \\
&\quad \times \Bigg[\sum_{k\neq\frac{\pm p-1}{6},k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{5\cdot\frac{3k^2+k}{2}} f\left(-q^{5\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{5\cdot\frac{3p^2-(6k+1)p}{2}}\right) \\
&\quad + (-1)^{\frac{\pm p-1}{6}} q^{5\cdot\frac{p^2-1}{24}} f_{5p^2}\Bigg] \pmod 4.
\end{aligned}$$

Consider the congruence

$$4\cdot\frac{k(k+1)}{2} + 5\cdot\frac{3m^2+m}{2} \equiv 17\cdot\frac{p^2-1}{24} \pmod p,$$

where $0 \leq k \leq p-1$ and $-\frac{(p-1)}{2} \leq m \leq \frac{p-1}{2}$. Since the above congruence is equivalent to

$$3(4k+2)^2 + 5(6m+1)^2 \equiv 0 \pmod p$$

and $\left(\frac{3}{p}\right)_L \neq \left(\frac{-5}{p}\right)_L$, it turns out that the only possibilities of satisfying the above congruence are $k = \frac{p-1}{2}$ and $m = \frac{\pm p-1}{6}$. So, extracting the terms involving $q^{pn+17\cdot\frac{p^2-1}{24}}$, dividing both sides by $q^{17\cdot\frac{p^2-1}{24}}$, and then replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} b_5'\left(4pn + \frac{17\cdot p^2+1}{6}\right) q^n \equiv 2pf_{4p}^3 f_{5p} \pmod 4.$$

Extracting the terms involving q^{pn}, we have

$$\sum_{n=0}^{\infty} b'_5\left(4p^2n+\frac{17\cdot p^2+1}{6}\right)q^n \equiv 2pf_4^3f_5 \pmod 4.$$

We now apply (3.15) and (4.18) in the above and repeat the process α times to arrive at

$$\sum_{n=0}^{\infty} b'_5\left(4p^{2\alpha}n+\frac{17\cdot p^{2\alpha}+1}{6}\right)q^n \equiv 2p^\alpha f_4^3f_5 \pmod 4. \tag{4.22}$$

Using (4.18) in the last step and extracting the terms involving q^{5n+1} and q^{5n+3}, we obtain (4.8).

Again, extracting the terms involving $q^{pn+17\cdot\frac{p^2-1}{24}}$ from (4.22), dividing both sides by $q^{17\cdot\frac{p^2-1}{24}}$, and then replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} b'_5\left(4p^{2\alpha+1}n+\frac{17\cdot p^{2\alpha+2}+1}{6}\right)q^n \equiv 2p^{\alpha+1}f_{4p}^3f_{5p} \pmod 4.$$

Comparing the coefficients of q^{pn+r}, where $r\in\{1,2,\ldots,p-1\}$, we obtain (4.9). This completes the proof of Theorem 4.2.

4.4 Proof of Theorem 4.3

First we state some lemmas.

Lemma 4.7. ([22, Eqs (2.6) and (2.7)]) *We have*

$$\frac{f_5}{f_2^2f_{10}} = \frac{f_5^5}{f_1^4f_{10}^3} - 4q\frac{f_{10}^2}{f_1^3f_2}, \tag{4.23}$$

$$\frac{f_2^3f_5^2}{f_1^2f_{10}^2} = \frac{f_5^5}{f_1f_{10}^3} + q\frac{f_{10}^2}{f_2}. \tag{4.24}$$

Lemma 4.8. ([22, Lemma 1.3]) *If $R(q)$ is as defined in* (1.9)*, then*

$$\frac{1}{R(q)R^2(q^2)} - q^2R(q)R^2(q^2) = \frac{f_2f_5^5}{f_1f_{10}^5}, \tag{4.25}$$

$$\frac{R(q^2)}{R^2(q)} - \frac{R^2(q)}{R(q^2)} = 4q\frac{f_1f_{10}^5}{f_2f_5^5}, \tag{4.26}$$

$$\frac{R(q)}{R^3(q^2)} + q^2\frac{R^3(q^2)}{R(q)} = \frac{f_2f_5^5}{f_1f_{10}^5} + 4q^2\frac{f_1f_{10}^5}{f_2f_5^5} - 2q. \tag{4.27}$$

Now we prove Theorem 4.3 by establishing (4.10)–(4.12).

Proof of (4.10). Employing (2.38), with q replaced by q^2 and (2.39) in (4.13), and then extracting the terms involving q^{5n+1}, we find that

$$\sum_{n=0}^{\infty} b_5'(5n+1)q^n = \frac{f_5^5 f_{10}}{f_1^5 f_2}\left(\frac{1}{R^3(q)R(q^2)} + q^2R^3(q)R(q^2) - 2q\left(\frac{R(q^2)}{R^2(q)} - \frac{R^2(q)}{R(q^2)}\right) - 5q\right),$$

which, by (2.44) and (4.26), yields

$$\begin{aligned}\sum_{n=0}^{\infty} b_5'(5n+1)q^n &= \frac{f_5^{10}}{f_1^6 f_{10}^4} - 3q\frac{f_5^5 f_{10}}{f_1^5 f_2} - 4q^2\frac{f_{10}^6}{f_1^4 f_2^2}\\ &= \left(\frac{f_5^{10}}{f_1^6 f_{10}^4} - 4q\frac{f_5^5 f_{10}}{f_1^5 f_2}\right) + q\left(\frac{f_5^5 f_{10}}{f_1^5 f_2} - 4q\frac{f_{10}^6}{f_1^4 f_2^2}\right).\end{aligned}$$

Employing (4.23) and (4.24) in the above, we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_5'(5n+1)q^n &= \frac{f_5^6}{f_1^2 f_2^2 f_{10}^2} + q\frac{f_5 f_{10}^3}{f_1 f_2^3}\\ &= \frac{f_2 f_5^3}{f_1^3 f_{10}}, \qquad (4.28)\end{aligned}$$

which proves (4.10).

Proof of (4.11). With the aid of (4.23), we may rewrite (4.28) as

$$\sum_{n=0}^{\infty} b_5'(5n+1)q^n = \frac{f_1 f_{10}}{f_2 f_5} + 4q\frac{f_{10}^4}{f_1^2 f_5^2}.$$

Employing (2.38) and (2.39) in the above identity, and then extracting the terms involving q^{5n+4}, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} b_5'(25n+21)q^n = \frac{f_5 f_{10}^5}{f_1 f_2^5}&\left(2\left(\frac{1}{R(q)R^2(q^2)} - q^2R(q)R^2(q^2)\right)\right.\\ &\left.- \left(\frac{R(q)}{R^3(q^2)} + q^2\frac{R^3(q^2)}{R(q)}\right) - 5q\right)\\ &+ 20\frac{f_2^4 f_5^{10}}{f_1^{14}}\left(2\left(\frac{1}{R^5(q)} - q^2R^5(q)\right) + 3q\right).\end{aligned}$$

Using (2.42), (4.25), and (4.27) in the above identity, we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_5'(25n+21)q^n = \frac{f_5^6}{f_1^2 f_2^4} - 4q\frac{f_5 f_{10}^5}{f_1 f_2^5} + q\left(\frac{f_5 f_{10}^5}{f_1 f_2^5} - 4q\frac{f_{10}^{10}}{f_2^6 f_5^4}\right) + 40\frac{f_2^4 f_5^4}{f_1^8}\\ + 500q\frac{f_2^4 f_5^{10}}{f_1^{14}}.\end{aligned}$$

With the aid of (4.23) and (4.24), the above identity reduces to

$$\sum_{n=0}^{\infty} b_5'(25n+21)q^n = \frac{f_1 f_{10}^3}{f_2^3 f_5} + 40\frac{f_2^4 f_5^4}{f_1^8} + 500q\frac{f_2^4 f_5^{10}}{f_1^{14}},$$

which is (4.11).

Proof of (4.12). By the binomial theorem, for all positive integers j, we have

$$f_j^5 \equiv f_{5j} \pmod{5}. \tag{4.29}$$

Employing (4.29) in (4.11), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} b_5'(25n+21)q^n &\equiv \frac{f_1 f_{10}^3}{f_2^3 f_5} \\ &\equiv \frac{f_1 f_2^2 f_{10}^2}{f_5} \pmod{5},\end{aligned}$$

which, with the help of (2.38), may be recast as

$$\begin{aligned}\sum_{n=0}^{\infty} b_5'(25n+21)q^n \equiv \frac{f_{10}^2 f_{25} f_{50}^2}{f_5} &\left(\frac{1}{R(q^5)} - q - q^2 R(q^5)\right) \\ \times &\left(\frac{1}{R(q^{10})} - q^2 - q^4 R(q^{10})\right)^2 \pmod{5}.\end{aligned}$$

Extracting the terms involving q^{5n} from both sides of the above, and then replacing q^5 by q, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} b_5'(125n+21)q^n \\ &\equiv \frac{f_2^2 f_5 f_{10}^2}{f_1} \left(\frac{1}{R(q)R^2(q^2)} + q - q^2 R(q) R^2(q^2)\right) \pmod{5}.\end{aligned}$$

Employing (4.25) and (4.24) in the above, and then invoking (4.29), we find that

$$\sum_{n=0}^{\infty} b_5'(125n+21)q^n \equiv \frac{f_2^6 f_5^3}{f_1^3 f_{10}^2} \equiv \frac{f_2 f_5^3}{f_1^3 f_{10}} \pmod{5}.$$

From the above congruence and (4.10), it follows that

$$b_5'(5n+1) \equiv b_5'(125n+21) \pmod{5},$$

which by iteration yields (4.12).

4.5 Proof of Theorem 4.4

Before proving Theorem 4.4, we recall some useful background material on modular forms. Let $\mathbb{H}$ denote the complex upper half-plane. We define the following matrix groups:

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \equiv \ 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv \ d \equiv \ 1 \pmod{N} \right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\},$$

where N is a positive integer.

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ satisfying $\Gamma(N) \subseteq \Gamma$ for some N is called a congruence subgroup and the smallest such N is called the level of Γ. The group

$$\mathrm{GL}_2^+(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}$. We will identify ∞ with $\frac{1}{0}$. We also define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{r}{s} = \frac{ar+bs}{cr+ds}$, where $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This will give an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. If Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then a cusp of Γ is an equivalence class in $\mathbb{Q} \cup \{\infty\}$.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on the functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, then define the slash operator $|_\ell$ by $(f \mid_\ell \gamma)(z) := (det\gamma)^{\ell/2}(cz+d)^{-\ell} f(\gamma z)$.

Definition 4.9. *Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if the following hold:*

1. *We have*

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ *and all* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.

2. *If* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, *then* $(f \mid_\ell \gamma)(z)$ *has a Fourier expansion of the form*

$$(f \mid_\ell \gamma)(z) = \sum_{n=0}^{\infty} a_\gamma(n) q_N^n,$$

where $q_N = e^{\frac{2\pi i z}{N}}$.

For a positive integer ℓ, let $M_\ell(\Gamma)$ denote the complex vector space of modular forms of weight ℓ with respect to Γ.

Definition 4.10. [113, Definition 1.15] *If* χ *is a Dirichlet character modulo* N, *then a modular form* $f \in M_\ell(\Gamma_1(N))$ *has Nebentypus character* χ *if*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ *and all* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. *The space of such modular forms is denoted by* $M_\ell(\Gamma_0(N), \chi)$.

Now recall that the Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty,$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer.

Next, we recall three theorems from [113, p. 18] which will be used to prove Theorem 4.4.

Theorem 4.11. [113, Theorem 1.64] *If* $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ *is an eta-quotient with* $\ell = \frac{1}{2} \sum_{\delta \mid N} r_\delta \in \mathbb{Z}$, *with*

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$*, where*

$$\chi(d) := \left(\frac{(-1)^\ell \prod_{\delta|N} \delta^{r_\delta}}{d}\right).$$

Theorem 4.12. [113, Theorem 1.65] *If c, d, and N are positive integers such that $d|N$ and $\gcd(c,d) = 1$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

Suppose that $f(z)$ is an eta-quotient satisfying the conditions of the last two theorems and the associated weight ℓ is a positive integer. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$. The following result is due to Serre, which we state from [113, p. 43].

Theorem 4.13. *If $f(z) \in M_\ell(\Gamma_0(N), \chi)$ has Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} c(n) q^n \in \mathbb{Z}[[q]],$$

then for each positive integer m there exists a constant $\alpha > 0$ such that

$$|\{n \le X \ : \ c(n) \not\equiv 0 \pmod{m}\}| = \mathcal{O}\left(\frac{X}{(\log X)^\alpha}\right).$$

Now we are in a position to prove Theorem 4.4.

Let

$$A(z) := \prod_{n=1}^{\infty} \frac{(1-q^{12n})^5}{(1-q^{60n})} = \frac{\eta^5(12z)}{\eta(60z)}.$$

Then

$$A^{5^k}(z) = \frac{\eta^{5^{k+1}}(12z)}{\eta^{5^k}(60z)}.$$

Set

$$B_k(z) := \frac{\eta(12z)\eta^3(30z)}{\eta^3(6z)\eta(60z)} A^{5^k}(z) = \frac{\eta^{5^{k+1}+1}(12z)\eta^3(30z)}{\eta^3(6z)\eta^{5^k+1}(60z)}.$$

Working modulo 5^{k+1}, we have

$$B_k(z) \equiv \frac{\eta(12z)\eta^3(30z)}{\eta^3(6z)\eta(60z)} = q\frac{f_{12}f_{30}^3}{f_6^3 f_{60}}. \tag{4.30}$$

From (4.10) and (4.30), we see that

$$B_k(z) \equiv \sum_{n=0}^{\infty} b_5'(5n+1)q^{6n+1} \pmod{5^{k+1}}. \tag{4.31}$$

Clearly, $B_k(z)$ is an eta-quotient with $N = 360$. We now prove that $B_k(z)$ is a modular form for any positive integer k. We know that the cusps of $\Gamma_0(360)$ are given by fractions $\frac{c}{d}$, where $d|360$ and $\gcd(c,d) = 1$. By Theorem 4.12, we find that $B_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$L := (5^{k+2}+5)\frac{\gcd(d,12)^2}{\gcd(d,60)^2} + 6\frac{\gcd(d,30)^2}{\gcd(d,60)^2} - 30\frac{\gcd(d,6)^2}{\gcd(d,60)^2} - 5^k - 1 \geq 0.$$

We verify that the above inequality holds for all the divisors of 360. We illustrate this with the help of the following table.

d	$(5^{k+2}+5)\frac{\gcd(d,12)^2}{\gcd(d,60)^2} + 6\frac{\gcd(d,30)^2}{\gcd(d,60)^2} - 30\frac{\gcd(d,6)^2}{\gcd(d,60)^2} - 5^k - 1$
1,2,3,6,9,18	$24 \cdot 5^k - 20$
4,8,12,24,36,72	$24 \cdot 5^k - 2$
5,10,15,30,45,90	$\frac{1}{25}\left(5^{k+2}+5\right) - 5^k + \frac{19}{5}$
20,40,60,120,180,360	$\frac{1}{25}\left(5^{k+2}+5\right) - 5^k + \frac{1}{5}$

Using Theorem 4.11, we find that the weight of $B_k(z)$ is $2 \cdot 5^k$. Further, the associated character for $B_k(z)$ is $\chi_1(\bullet) = \left(\frac{12^{4\cdot 5^k} \cdot 5^{2-5^k}}{\bullet}\right)$. Thus, $B_k(z) \in M_{2\cdot 5^k}(\Gamma_0(360), \chi_1)$. Also, the Fourier coefficients of $B_k(z)$ are all integers. Hence, by Theorem 4.13, the Fourier coefficients of $B_k(z)$ are almost always divisible by 5^k. By (4.31), the same holds for $b_5'(5n+1)$ and hence Theorem 4.4 follows.