Chapter 5

Arithmetic properties of Andrews' integer partitions with even parts below odd parts

5.1 Introduction

Let $\nu(q)$ denote the following third order mock theta function [153, p. 62]

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}.$$

Let us assume that

$$\nu(-q) = \sum_{n=0}^{\infty} p_\nu(n) q^n.$$

Note that [23, Eq. (1.10)]

$$p_\nu(2n) = f(6n+1),$$

where $f(n)$ counts the number of 1-shell totally symmetric plane partitions of n first introduced by Blecher [37]. Xia [156] proved that

$$\sum_{n=0}^{\infty} f(30n+25)q^n = \sum_{n=0}^{\infty} p_\nu(10n+8)q^n = 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4}. \tag{5.1}$$

Andrews' proved that [8, Corollary 5.2]

$$p_\nu(2n) = \overline{\mathcal{EO}}(2n). \tag{5.2}$$

The contents of this chapter have been submitted for possible publication [140].

Hence, from (5.1) and (5.2), we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n = \sum_{n=0}^{\infty} p_\nu(10n+8)q^n = 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4}. \tag{5.3}$$

Andrews [8] proved several Ramanujan-type congruences. For example, he proved the following result.

Theorem 5.1. [8, Eq. (1.6)] *For all $n \geq 0$, we have*

$$\overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod{5}.$$

Many more congruences involving $\overline{\mathcal{EO}}(n)$ have been proved by several other mathematicians. Goswami and Jha [79] proved a few congruences modulo 2 and 4 for $\overline{\mathcal{EO}}(n)$. They conjectured that $\overline{\mathcal{EO}}(10n+r) \equiv 0 \pmod{2}$ for $r \in \{2, 4\}$. This was recently proved by Baruah, Das, Saikia and Sarma [31]. Goswami and Jha [79] found a few exact generating functions as well which were also found by Pore and Fathima [121]. Pore and Fathima [121] further proved a few congruences modulo 5, 10 and 20. They conjectured the following result which was proved by Ray and Barman [133] by using an algorithm of Radu [122].

Theorem 5.2. [133, Theorem 1.3] *For all $n \geq 0$, we have*

$$\overline{\mathcal{EO}}(50n+r) \equiv 0 \pmod{20} \quad \textit{for} \quad r \in \{18, 28, 38, 48\}. \tag{5.4}$$

The above theorem can also be proved via elementary methods using (5.3). The proof is short, so we complete it here.

Proof of (5.4). From (5.3), we recall that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n &= 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4} \\ &\equiv 5 f_5^2 f_{10} \pmod{20},\end{aligned}$$

where in the last step, we have employed (2.87).

Extracting the terms involving q^{5n+r} for $r \in \{1, 2, 3, 4\}$ in the above, we complete the proof.

Remark 5.3. *Very recently, Guadalupe [81, Theorem 1.1] gave a different proof of* (5.4)*.*

Chern's result [56] reveals that there is an infinite family of congruences modulo arbitrary powers of 5: For all $n \geq 0$ and $\alpha \geq 1$, we have

$$\overline{\mathcal{EO}}\left(2 \cdot 5^{2\alpha-1}n + \frac{5^{2\alpha}-1}{3}\right) \equiv 0 \pmod{5^{\alpha}}. \tag{5.5}$$

Rahman and Saikia [125] found a few infinite families of congruences modulo 2, 4, 5 and 8 for $\overline{\mathcal{EO}}(n)$. Other infinite families for $\overline{\mathcal{EO}}(n)$ were proved in [50, 52, 81, 133]. Passary [116] proved the following result which gives a complete characterisation of the parity of $\overline{\mathcal{EO}}(n)$.

Theorem 5.4. [116, Eq. (2.1.24)] *For all $n \geq 0$,*

$$\overline{\mathcal{EO}}(n) \equiv \begin{cases} 1 \pmod{2}, & \text{if } n = 4k(3k-1) \text{ for } k \in \mathbb{Z}, \\ 0 \pmod{2}, & \text{otherwise.} \end{cases} \tag{5.6}$$

Apart from congruences, a few density results have also been proven for $\overline{\mathcal{EO}}(n)$. One such result was proved by Ray and Barman [133]. Using the theory of modular forms, they proved that for $n \geq 0$, $\overline{\mathcal{EO}}(8n+6)$ is almost always divisible by 8. To be specific, they proved the following theorem.

Theorem 5.5. [133, Theorem 1.5] *For $n \geq 0$, we have*

$$\lim_{X\to\infty} \frac{\#\{0 \leq n \leq X : \overline{\mathcal{EO}}(8n+6) \equiv 0 \pmod{8}\}}{X} = 1.$$

For other related works in this direction, interested readers can look at the following non-exhaustive list of papers and the references therein: Ballantine and Welch [18], Banerjee and Dastidar [19], Bringmann and Jennings-Shaffer [38], Bringmann *et al.* [39], [41], Burson and Eichhorn [42], [43], [44], Y. H. Chen *et al.* [53], Chern [54], [55], Fu and Tang [74].

The goal of this chapter is to extend this study. In this chapter, we prove several new arithmetic properties for the $\overline{\mathcal{EO}}(n)$ partition function. We now state our main results.

We begin our results by presenting some internal congruences modulo 16, namely

Theorem 5.6. *For all $n \geq 0$ and $t \in \{1, 2, 3, 4\}$, we have*

$$\overline{\mathcal{EO}}(1250n + 250t + 208) \equiv 6\overline{\mathcal{EO}}(50n + 10t + 8) \pmod{16}, \tag{5.7}$$

$$\overline{\mathcal{EO}}(10n) \equiv 13\overline{\mathcal{EO}}(250n+8) \equiv 5\overline{\mathcal{EO}}(6250n+208) \pmod{16}, \tag{5.8}$$

$$\overline{\mathcal{EO}}(10n+6) \equiv 13\overline{\mathcal{EO}}(250n+158) \equiv 5\overline{\mathcal{EO}}(6250n+3958) \pmod{16}. \tag{5.9}$$

In the next few theorems, we present infinite families of congruences modulo 10 and 40.

Theorem 5.7. *Let $a(n)$ be defined by*

$$\sum_{n=0}^{\infty} a(n)q^n = f_1^2 f_2.$$

Let $p \geq 5$ be a prime. Define

$$\omega(p) := a\left(\frac{p^2-1}{6}\right) + \left(\frac{-4}{p}\right)_L \left(\frac{\frac{-(p^2-1)}{6}}{p}\right)_L.$$

1. *If $\omega(p) \equiv 0 \pmod{2}$, then for $n, k \geq 0$ and $p \nmid n$, we have*
$$\overline{\mathcal{EO}}\left(50 \cdot p^{4k+3} n + \frac{25 \cdot p^{4k+4} - 1}{3}\right) \equiv 0 \pmod{10}. \tag{5.10}$$
2. *If $\omega(p) \equiv 1 \pmod{2}$, then for $n, k \geq 0$ and $p \nmid n$, we have*
$$\overline{\mathcal{EO}}\left(50 \cdot p^{6k+5} n + \frac{25 \cdot p^{6k+6} - 1}{3}\right) \equiv 0 \pmod{10}. \tag{5.11}$$

Remark 5.8. *For example, if we choose $p = 5$ in the above theorem, we find that $\omega(5) \equiv 0 \pmod{2}$. Then for $5 \nmid n$ and $k = 0$, we have*

$$\overline{\mathcal{EO}}\left(6250n + 5208\right) \equiv 0 \pmod{10}.$$

Theorem 5.9. *Let p be a prime. If $\left(\frac{30r+25}{p}\right)_L = -1$, where $r \in \{1, 2, \ldots, p-1\}$, then for all $n \geq 0$, we have*

$$\overline{\mathcal{EO}}(10(pn+r)+8) \equiv 0 \pmod{10}. \tag{5.12}$$

Remark 5.10. *For example, choosing $p = 7$ and $r = 1$ in the above theorem, we find that $\overline{\mathcal{EO}}(70n+18) \equiv 0 \pmod{10}$.*

Theorem 5.11. *Let p be a prime with $\left(\frac{-3}{p}\right)_L = -1$, then for all $n, \alpha \geq 0$, we have*

$$\overline{\mathcal{EO}}\left(50 \cdot p^{2\alpha+1}(pn+j) + \frac{25 \cdot p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod{10}, \tag{5.13}$$

where $j \in \{1, 2, \ldots, p-1\}$.

Remark 5.12. *For instance, choosing $p = 5, \alpha = 0$ and $j = 1$ in the above theorem, we have $\overline{\mathcal{EO}}(1250n+258) \equiv 0 \pmod{10}$.*

Theorem 5.13. *Let p be a prime with $\left(\frac{-3}{p}\right)_L = -1$, then for all $n, \alpha \geq 0$, we have*

$$\overline{\mathcal{EO}}\,(200n + 40r + 38) \equiv 0 \pmod{40}, \quad \textit{for } r \in \{0, 1, 2, 4\}, \tag{5.14}$$

$$\overline{\mathcal{EO}}\left(200 \cdot p^{2\alpha+1}(pn + j) + \frac{475 \cdot p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod{40}, \quad \textit{for } j \in \{0, 1, \ldots, p - 1\}. \tag{5.15}$$

Remark 5.14. *For example, choosing $p = 5, \alpha = 0$ and $j = 1$ in (5.15), we have $\overline{\mathcal{EO}}(5000n + 4958) \equiv 0 \pmod{40}$.*

With the aid of (5.3), we also study the distribution of $\overline{\mathcal{EO}}(n)$. In the next theorem, we prove that $\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n + 8)q^n$ is lacunary modulo 10, namely

Theorem 5.15. *We have*

$$\lim_{X \to \infty} \frac{\#\{0 \leq n \leq X : \overline{\mathcal{EO}}(10n + 8) \equiv 0 \pmod{10}\}}{X} = 1.$$

The next theorem states that $\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(40n + 38)q^n$ is lacunary modulo 40.

Theorem 5.16. *We have*

$$\lim_{X \to \infty} \frac{\#\{0 \leq n \leq X : \overline{\mathcal{EO}}(40n + 38) \equiv 0 \pmod{40}\}}{X} = 1.$$

The chapter is organised as follows. In Sections 5.2 and 5.3, we prove Theorems 5.6 and 5.7 respectively. In Section 5.4, we prove Theorems 5.9 and 5.11. In Section 5.5, we prove Theorem 5.13. And finally, in Section 5.6, we prove Theorems 5.15 and 5.16. Our proofs make use of elementary techniques. The necessary background material and useful preliminary lemmas are given in the corresponding section.

5.2 Proof of Theorem 5.6

Before proving Theorem 5.6, we recall the following 5-dissection.

Lemma 5.17. [33, p. 49] *We have*

$$\psi(q) = f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}). \tag{5.16}$$

We are now in a position to prove Theorem 5.6.

With the aid of (4.23), (5.3) can be re-written as

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n = 5\left(\frac{f_{10}^3}{f_5^2}+4q\frac{f_2f_{10}^6}{f_1^3f_5^3}\right)$$
$$\equiv 5\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)q^{5n}+20q\frac{f_1f_{10}^8}{f_2f_5^7} \pmod{16}. \tag{5.17}$$

Employing (2.38) and (2.39) in the above and then extracting the terms involving q^{5n}, we find that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(50n+8)q^n \equiv 5\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)q^n+20q\frac{f_{10}^5}{f_5^5}\psi(q) \pmod{16}.$$

Invoking (5.16) in the above, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(50n+8)q^n \equiv 5\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)q^n+20q\frac{f_{10}^5}{f_5^5}\Big(f(q^{10},q^{15})$$
$$+qf(q^5,q^{20})+q^3\psi(q^{25})\Big) \pmod{16}. \tag{5.18}$$

Now, extracting the terms involving q^{5n} and q^{5n+3} from the above, we obtain

$$\overline{\mathcal{EO}}(250n+8) \equiv 5\overline{\mathcal{EO}}(10n) \pmod{16} \tag{5.19}$$

and

$$\overline{\mathcal{EO}}(250n+158) \equiv 5\overline{\mathcal{EO}}(10n+6) \pmod{16} \tag{5.20}$$

Extracting the terms involving q^{5n+4} from both sides of (5.18), we find that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(250n+208)q^n \equiv 5\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n+20\frac{f_2^3f_{10}^4}{f_1f_5^5} \pmod{16}.$$

Using (4.24) in the above, we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(250n+208)q^n \equiv 5\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n+20\frac{f_{10}^3}{f_5^2}+20q\frac{f_1f_{10}^8}{f_2f_5^7} \pmod{16}$$

which on invoking (5.17) can be recast as

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(250n+208)q^n \equiv 5\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n+20\overline{\mathcal{EO}}(2n)q^{5n}+\overline{\mathcal{EO}}(10n+8)q^n$$
$$-5\overline{\mathcal{EO}}(2n)q^{5n}$$
$$\equiv 6\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n+15\overline{\mathcal{EO}}(2n)q^{5n} \pmod{16}. \tag{5.21}$$

Extracting the terms involving q^{5n+r} for $r \in \{1,2,3,4\}$, we arrive at

$$\overline{\mathcal{EO}}(1250n + 250r + 208) \equiv 6\overline{\mathcal{EO}}(50n + 10r + 8) \pmod{16},$$

which readily yields (5.7).

Next, extracting the terms involving q^{25n} from both sides of (5.21), we find that

$$\overline{\mathcal{EO}}(6250n + 208) \equiv 6\overline{\mathcal{EO}}(250n + 8) - \overline{\mathcal{EO}}(10n) \pmod{16}. \tag{5.22}$$

But from (5.19), we have

$$\overline{\mathcal{EO}}(10n) \equiv 13\overline{\mathcal{EO}}(250n + 8) \pmod{16}.$$

So, (5.22) can be written as

$$\begin{aligned}\overline{\mathcal{EO}}(6250n + 208) &\equiv -7\overline{\mathcal{EO}}(250n + 8)\\ &\equiv 9\overline{\mathcal{EO}}(250n + 8) \pmod{16},\end{aligned}$$

which implies that

$$\overline{\mathcal{EO}}(250n + 8) \equiv 9\overline{\mathcal{EO}}(6250n + 208) \pmod{16}, \tag{5.23}$$

Combining (5.19) and (5.23), we readily obtain (5.8).

Again, extracting the terms involving q^{25n+15} from both sides of (5.21), we find that

$$\overline{\mathcal{EO}}(6250n + 3958) \equiv 6\overline{\mathcal{EO}}(250n + 158) - \overline{\mathcal{EO}}(10n + 6) \pmod{16}. \tag{5.24}$$

But from (5.20), we have

$$\overline{\mathcal{EO}}(10n + 6) \equiv 13\overline{\mathcal{EO}}(250n + 158) \pmod{16}.$$

So, (5.24) turns into

$$\begin{aligned}\overline{\mathcal{EO}}(6250n + 3958) &\equiv -7\overline{\mathcal{EO}}(250n + 158)\\ &\equiv 9\overline{\mathcal{EO}}(250n + 158) \pmod{16},\end{aligned}$$

which can be re-written as

$$\overline{\mathcal{EO}}(250n + 158) \equiv 9\overline{\mathcal{EO}}(6250n + 3958) \pmod{16}, \tag{5.25}$$

Combining (5.20) and (5.25), we obtain (5.9). This completes the proof of Theorem 5.6.

5.3 Proof of Theorem 5.7

We recall some necessary background material before going into the proof. The following result of Newman will be useful for the proof. Let p and q denote distinct primes, $r, s \neq 0$, and $r \not\equiv s \pmod 2$. Set

$$\phi(\tau) = \prod_{n=1}^{\infty}(1-x^n)^r(1-x^{nq})^s = \sum_{n=0}^{\infty} a(n)x^n, \tag{5.26}$$

$\epsilon = \frac{1}{2}(r+s), \Delta = \frac{(r+sq)(p^2-1)}{24}$ and $\theta = (-1)^{\frac{1}{2}-\epsilon}2q^s$.

Lemma 5.18. [112, Theorem 3] *With the notations defined above, the coefficients $a(n)$ of $\phi(\tau)$ satisfy*

$$a(np^2+\Delta) - \gamma(n)a(n) + p^{2\epsilon-2}a\left(\frac{n-\Delta}{p^2}\right) = 0,$$

where

$$\gamma(n) = p^{2\epsilon-2}\alpha(p) - \left(\frac{\theta}{p}\right)_L p^{\epsilon-3/2}\left(\frac{n-\Delta}{p}\right)_L,$$

where $\alpha(p)$ is a constant depending on p.

We are now in a position to prove Theorem 5.7.

From (5.1), we have that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n = 5\frac{f_2^2f_5^2f_{10}}{f_1^4} \equiv 5f_5^2f_{10} \pmod{10}.$$

Extracting the terms involving q^{5n} in the above and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(50n+8)q^n \equiv 5f_1^2f_2 \equiv 5\sum_{n=0}^{\infty}a(n)q^n \pmod{10}, \tag{5.27}$$

where $f_1^2f_2 = \sum_{n=0}^{\infty}a(n)q^n$.

Putting $r=2, q=2$ and $s=1$ in (5.26), we have by Lemma 5.18, for any $n \geq 0$

$$a\left(p^2n+\frac{p^2-1}{6}\right) = \gamma(n)a(n) - pa\left(\frac{1}{p^2}\left(n-\frac{p^2-1}{6}\right)\right), \tag{5.28}$$

where

$$\gamma(n) = p\alpha(p) - \left(\frac{-4}{p}\right)_L\left(\frac{n-\frac{p^2-1}{6}}{p}\right)_L. \tag{5.29}$$

Setting $n=0$ in (5.28) and using the fact that $a(0)=1$ and $a\left(\frac{\frac{-(p^2-1)}{6}}{p^2}\right) = 0$, we obtain

$$a\left(\frac{p^2-1}{6}\right)=\gamma(0). \tag{5.30}$$

Setting $n=0$ in (5.29) and then using (5.30), we obtain

$$p\alpha(p)=a\left(\frac{p^2-1}{6}\right)+\left(\frac{-4}{p}\right)_L\left(\frac{\frac{-(p^2-1)}{6}}{p}\right)_L:=\omega(p). \tag{5.31}$$

With the aid of (5.29) and (5.31), (5.28) can be recast as

$$a\left(p^2n+\frac{p^2-1}{6}\right)=\left(\omega(p)-\left(\frac{-4}{p}\right)_L\left(\frac{n-\frac{p^2-1}{6}}{p}\right)_L\right)a(n)$$
$$-pa\left(\frac{1}{p^2}\left(n-\frac{p^2-1}{6}\right)\right). \tag{5.32}$$

Now, replacing n by $pn+\frac{p^2-1}{6}$ in (5.32), we obtain

$$a\left(p^3n+\frac{p^4-1}{6}\right)=\omega(p)a\left(pn+\frac{p^2-1}{6}\right)-pa(n/p). \tag{5.33}$$

Case - 1 : $\omega(p)\equiv 0 \pmod 2$

Since $\omega(p)\equiv 0 \pmod 2$, from (5.33) we have

$$a\left(p^3n+\frac{p^4-1}{6}\right)\equiv pa(n/p) \pmod 2. \tag{5.34}$$

Now, replacing n by pn in (5.34), we obtain

$$a\left(p^4n+\frac{p^4-1}{6}\right)\equiv pa(n)\equiv a(n) \pmod 2.$$

Iterating the above, we obtain that for every integer $k\geq 0$,

$$a\left(p^{4k}n+\frac{p^{4k}-1}{6}\right)\equiv a(n) \pmod 2. \tag{5.35}$$

Now if $p\nmid n$, then (5.34) yields

$$a\left(p^3n+\frac{p^4-1}{6}\right)\equiv 0 \pmod 2.$$

Replacing n by $p^3+\frac{p^4-1}{6}$ in (5.35) and then using the above, we obtain

$$a\left(p^{4k+3}n+\frac{p^{4k+4}-1}{6}\right)\equiv 0 \pmod 2. \tag{5.36}$$

Now, employing (5.36) in (5.27), we readily have (5.10).

Case - 2 : $\omega(p)\equiv 1 \pmod 2$

Next, replacing n by $p^2n+\frac{p(p^2-1)}{6}$ in (5.33), we obtain

$$a\left(p^5n+\frac{p^6-1}{6}\right)=a\left(p^3\left(p^2n+\frac{p(p^2-1)}{6}\right)+\frac{p^4-1}{6}\right)$$

$$= \omega(p) a\left(p^3 n + \frac{p^4-1}{6}\right) - pa\left(pn + \frac{p^2-1}{6}\right)$$

$$= \left(\omega^2(p) - p\right) a\left(pn + \frac{p^2-1}{6}\right) - p\omega(p) a(n/p). \tag{5.37}$$

Now, as $\omega(p) \equiv 1 \pmod 2$ and $p \geq 5$ is an odd prime, we have $\omega^2(p) - p \equiv 0 \pmod 2$. Therefore (5.37) can be written as

$$a\left(p^5 n + \frac{p^6-1}{6}\right) \equiv a(n/p) \pmod 2. \tag{5.38}$$

Replacing n by pn in (5.38), we obtain

$$a\left(p^6 n + \frac{p^6-1}{6}\right) \equiv a(n) \pmod 2.$$

Using the above repeatedly, we see that for every integer $k \geq 0$,

$$a\left(p^{6k} n + \frac{p^{6k}-1}{6}\right) \equiv a(n) \pmod 2. \tag{5.39}$$

Next, if $p \nmid n$, (5.38) yields

$$a\left(p^5 n + \frac{p^6-1}{6}\right) \equiv 0 \pmod 2.$$

Replacing n by $p^5 n + \frac{p^6-1}{6}$ in (5.39) and then using the above, we obtain

$$a\left(p^{6k+5} n + \frac{p^{6k+6}-1}{6}\right) \equiv 0 \pmod 2. \tag{5.40}$$

Invoking (5.40) in (5.27), we readily arrive at (5.11). This completes the proof.

5.4 Proofs of Theorems 5.9 and 5.11

Proof of Theorem 5.9. From (5.1), we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8) q^n \equiv 5 f_{20} \pmod{10}. \tag{5.41}$$

Invoking (1.8) in the above, we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8) q^n \equiv 5 \sum_{k=-\infty}^{\infty} q^{10k(3k-1)} \pmod{10}.$$

If we wish to consider values of the form $\overline{\mathcal{EO}}(10(pn+r)+8)$, then we need to check whether $pn + r = 10j(3j-1)$ is possible for some integer j. Therefore, to achieve our goal it is enough to show that no such representation exists. Note that, if $pn + r = 10j(3j-1)$ is true for some j, then $r \equiv 10j(3j-1) \pmod p$. Equivalently, this can be written as $30r + 25 \equiv (30j-5)^2 \pmod p$, which is not possible as

$\left(\dfrac{30r+25}{p}\right)_L = -1$. Hence, the desired congruence holds and this completes the proof.

Proof of Theorem 5.11. From (5.1), we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n \equiv 5f_5^4 \pmod{10}.$$

Extracting the terms involving q^{5n} and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(50n+8) \equiv 5f_1^4 \equiv 5f_1\cdot f_1^3 \pmod{10}. \tag{5.42}$$

Employing (1.8) and (3.49) in (5.42), we arrive at

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(50n+8)q^n \equiv 5\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} q^{\frac{n(n+1)}{2}+\frac{k(3k-1)}{2}} \pmod{10}.$$

The above can be re-written as

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(50n+8)q^{24n+4} \equiv 5\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} q^{3(2n+1)^2+(6k-1)^2} \pmod{10}.$$

It is not difficult to see that if N is of the form $3x^2+y^2$ and $\left(\frac{-3}{p}\right)_L = -1$, the highest exponent of p dividing N is even. Also, observe that the highest exponent of p dividing $24\left(p^{2\alpha+1}(pn+j)+\frac{p^{2\alpha+2}-1}{6}\right)+4$ for $j\in\{1,2,\ldots,p-1\}$ is $2\alpha+1$, which is odd. Therefore, extracting the terms involving $q^{24\left(p^{2\alpha+1}(pn+j)+\frac{p^{2\alpha+2}-1}{6}\right)+4}$ from both sides with $j\in\{1,2,\ldots,p-1\}$, (5.13) is evident.

Remark 5.19. *As a consequence of* (1.8) *and* (5.41), *we now have the following result.*

$$\overline{\mathcal{EO}}(10n+8) \equiv \begin{cases} 5 \pmod{10}, & \text{if } n = 30k^2-10k \text{ for } k\in\mathbb{Z},\\ 0 \pmod{10}, & \text{Otherwise.}\end{cases}$$

5.5 Proof of Theorem 5.13

From (5.1), we recall that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)q^n = 5\frac{f_2^2f_5^2f_{10}}{f_1^4}.$$

Invoking (4.17) and (2.33) in the above and then extracting the terms involving odd powers of q, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(20n+18)q^n &= 20\frac{f_2^2f_4^4f_{20}^5}{f_{10}^2f_{40}^2}\cdot\frac{f_5^2}{f_1^8} - 10q^2\frac{f_2^{14}f_{40}^2}{f_4^4f_{20}}\cdot\frac{f_5^2}{f_1^{12}}\\ &\equiv 20\frac{f_4^4f_{20}^5}{f_2^2f_{10}f_{40}^2} - 10q^2\frac{f_2^8f_{40}^2}{f_4^4f_{20}}\cdot f_5^2 \pmod{40}.\end{aligned}$$

Employing (4.17) in the above and then extracting the terms involving odd powers of q, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(40n+38)q^n &\equiv 20q^3f_5^{19}\\ &\equiv 20q^3f_{80}f_5^3 \pmod{40}.\end{aligned}\tag{5.43}$$

Comparing the coefficients of the terms of the form q^{5n+r} for $r\in\{0,1,2,4\}$, we readily arrive at (5.14).

Proof of (5.15). From (5.43), we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(40n+38)q^n \equiv 20q^3f_{80}f_5^3 \pmod{40}.\tag{5.44}$$

Extracting the terms of the form q^{5n+3}, and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(200n+158)q^n \equiv 20f_{16}f_1^3 \pmod{40}.$$

Employing (1.8) and (3.49) in the above, we find that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(200n+158)q^n \equiv 20\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}q^{\frac{n(n+1)}{2}+\frac{16\cdot k(3k-1)}{2}} \pmod{40}$$

which can be re-written as

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(200n+158)q^{24n+19} \equiv 20\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}q^{3(2n+1)^2+16(6k-1)^2} \pmod{40}.$$

We know that if N is of the form $3x^2+16y^2$ and $\left(\frac{-3}{p}\right)_L=-1$, the highest exponent of p dividing N is even. Note that the highest power of p dividing $24\left(p^{2\alpha+1}(pn+j)+\frac{19}{24}(p^{2\alpha+2}-1)\right)+19$ for $j\in\{1,2,\ldots,p-1\}$ is $2\alpha+1$, which is odd. Therefore, extracting the terms involving $q^{24\left(p^{2\alpha+1}(pn+j)+\frac{19}{24}(p^{2\alpha+2}-1)\right)+19}$ from both sides with $j\in\{1,2,\ldots,p-1\}$, (5.15) is evident.

5.6 Proof of Theorems 5.15 and 5.16

We present two proofs of Theorem 5.15. But before going into the proofs, we recall the following results. First, we recall a result of D. Chen and R. Chen [50].

Theorem 5.20. [50, Theorem 4.3] *If N is sufficiently large, then*

$$\#\{n \leq N : \overline{\mathcal{EO}}(2n) \equiv 1 \pmod 2\} \leq \sqrt{6N+1}. \tag{5.45}$$

Next, we recall the following result due to Landau [103].

Theorem 5.21. *Let $s(n)$ and $r(n)$ be two quadratic polynomials. Then*

$$\left(\sum_{n\in\mathbb{Z}} q^{r(n)}\right)\left(\sum_{n\in\mathbb{Z}} q^{s(n)}\right)$$

is lacunary modulo 2.

First proof of Theorem 5.15. From (5.1), we have

$$\sum_{n-0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n \equiv 5f_5^2 f_{10} \equiv 5\frac{f_{10}^3}{f_5^2} \equiv 5\sum_{n-0}^{\infty} \overline{\mathcal{EO}}(2n)q^{5n} \pmod{10}. \tag{5.46}$$

Clearly for $n \geq 0$ and $N \geq 1$, we have

$$\#\{0 \leq n \leq N : \overline{\mathcal{EO}}(2n) \equiv 1 \pmod 2\} + \#\{0 \leq n \leq N : \overline{\mathcal{EO}}(2n) \equiv 0 \pmod 2\} = N+1.$$

The above can be re-written as

$$\frac{\#\{0 \leq n \leq N : \overline{\mathcal{EO}}(2n) \equiv 0 \pmod 2\}}{N} = 1 + \frac{1}{N} - \frac{\#\{0 \leq n \leq N : \overline{\mathcal{EO}}(2n) \equiv 1 \pmod 2\}}{N}.$$

With (5.45) in mind and letting $N \to \infty$, we arrive at

$$\lim_{N\to\infty} \frac{\#\{0 \leq n \leq N : \overline{\mathcal{EO}}(2n) \equiv 0 \pmod 2\}}{N} = 1, \tag{5.47}$$

which means that $\overline{\mathcal{EO}}(2n)$ is almost always divisible by 2. Hence with the aid of (5.46), we conclude that $\overline{\mathcal{EO}}(10n+8)$ is almost always divisible by 10, namely

$$\lim_{N\to\infty} \frac{\#\{0 \leq n \leq N : \overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod{10}\}}{N} = 1.$$

This completes the proof.

Remark 5.22. *Note that a different proof of a more general version of* (5.47) *can be found in the work of Burson and Eichhorn [43, Corollary 3.5].*

Second proof of Theorem 5.15. From [86, Eq. (2.3.1)], we recall that

$$f_1^3 = \sum_{k=-\infty}^{\infty} (4k+1)q^{2k^2+k}. \tag{5.48}$$

From (5.1), we have

$$\sum_{n-0}^{\infty} \overline{\mathcal{EO}}(10n+8)q^n \equiv 5f_5^2 f_{10} \equiv 5 \cdot f_5 \cdot f_5^3 \pmod{10}.$$

Invoking (1.8) and (5.48) with $q \to q^5$ in the above and then applying Lemma 5.21, we complete the proof.

Proof of Theorem 5.16. From (5.44), we recall that

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(40n+38)q^n \equiv 20q^3 f_{80} f_5^3 \pmod{40}. \tag{5.49}$$

For $n \geq 0$, let $a(n)$ be defined as

$$\sum_{n=0}^{\infty} a(n)q^n = f_{80} f_5^3.$$

Invoking (1.8) with $q \to q^{80}$ and (5.48) with $q \to q^5$ in the above and then applying Lemma 5.21, we conclude that $a(n)$ is lacunary modulo 2.

Now, re-writing (5.49) as

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(40n+38)q^n \equiv 20 \sum_{n=0}^{\infty} a(n)q^{n+3} \pmod{40}.$$

Due to the fact that $a(n)$ is lacunary modulo 2, clearly $\overline{\mathcal{EO}}(40n+38)$ is lacunary modulo 40. This completes the proof.