Chapter 6

A Restricted Class of F-partitions

6.1 Introduction

In Chapter 1, we have discussed F-partitions whereas in Chapter 4 we discussed a more general class of F-partitions in which each part can be repeated at most h times and comes from $k-$copies of a non-negative integer. In this chapter, we explore a restricted class of F-partitions enumerated by the function $a_{k,\,i}(n)$, which is defined in the next paragraph.

Let $a_{k,\,i}(n)$ denote the k-colored (say $r_1, r_2, \ldots, r_k$) F-partition of a positive integer n in which there are no odd parts of some i colours, say $r_1, r_2, \ldots, r_i$ in the top row and no even parts of the colours $r_1, r_2, \ldots, r_i$ in the bottom row. For example, the F-partitions enumerated by $a_{2,\,1}(2)$ are

$$\begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}.$$

Let $A_{k,\,i}(q)$ denote the generating function for $a_{k,\,i}(n)$.

Padmavathamma [52] obtained the representations for the function $A_{k,\,i}(q)$ for $k = 2, 3$ and $i \leq k$ in terms of $q-$products and proved that $A_{k,\,k}(q) = c\phi_k(q^2)$, where $c\phi_k(q)$ is the generating function for the number of F-partitions with $k-$colors. As an illustration of the work in [52], we have the following:

Theorem 6.1.1. *We have*

$$\sum_{n=0}^{\infty} a_{3,\,2}(n)q^n = \frac{(-q^2;q^2)_\infty\,(-q^4;q^4)_\infty}{(q;q)_\infty}\Big(\left(-q^5;q^8\right)_\infty\left(-q^3;q^8\right)_\infty\left(-q^2;q^4\right)_\infty^2$$

$$+ 2q^2 \left(-q^7;q^8\right)_\infty \left(-q^{-1};q^8\right)_\infty \left(-q^4;q^4\right)_\infty^2 \Big).$$

From Andrews' general principle 1.1.1, the generating function $A_{k,i}(q)$ of $a_{k,i}(n)$ is given by

$$\begin{aligned} A_{k,i}(q) &:= CT_z\left(\prod_{n=0}^{\infty}\left(1+zq^{n+1}\right)^{k-i}\left(1+zq^{2n+1}\right)^{i}\left(1+z^{-1}q^{n}\right)^{k-i}\left(1+z^{-1}q^{2n+1}\right)^{i}\right) \\ &= CT_z\left(\left(-zq;q\right)_\infty^{k-i}\left(-zq;q^2\right)_\infty^{i}\left(-z^{-1};q\right)_\infty^{k-i}\left(-z^{-1}q;q^2\right)_\infty^{i}\right) \\ &= CT_z\left(\frac{1}{f_1^{k-i}f_2^{i}}\, f^{k-i}(z^{-1},zq)f^{i}(z^{-1}q,zq)\right). \end{aligned} \tag{6.1.1}$$

In our work, we obtain the generating functions for $a_{k,i}(n)$ for $k=2,3$ and $i<k$ and find some interesting congruences for these functions. Moreover, we obtain the generating function for $a_{4,2}(n)$ and obtain few congruences. In the last section of this chapter, we derive an identity connecting the generating functions for $a_{2,1}(n)$ and $a_{2p,p}(n)$, where p is a prime.

6.2 Generating function for $a_{2,1}(n)$

Theorem 6.2.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} a_{2,1}(n)q^n = \frac{f_3^2}{f_1^2 f_6}. \tag{6.2.1}$$

Proof. Setting $k=2$ and $i=1$ in (6.1.1), we find that

$$\begin{aligned} A_{2,1}(q) &= CT_z\left(\frac{1}{f_1f_2}\, f(z^{-1},zq)f(z^{-1}q,zq)\right) \\ &= CT_z\left(\frac{1}{f_1f_2}\left(\sum_{m_1=-\infty}^{\infty} z^{m_1}q^{\frac{m_1(m_1+1)}{2}}\right)\left(\sum_{m_2=-\infty}^{\infty} z^{m_2}q^{m_2^2}\right)\right) \\ &= \frac{1}{f_1f_2}\left(\sum_{n=-\infty}^{\infty} q^{3n^2/2+n/2}\right) \\ &= \frac{f(q,q^2)}{f_1f_2}. \end{aligned} \tag{6.2.2}$$

From (1.2.14), we have

$$f(q,q^2)=\frac{f_2f_3^2}{f_1f_6}. \tag{6.2.3}$$

Employing (6.2.3) in (6.2.2), we arrive at (6.2.1). □

Corollary 6.2.2. *We have*

$$\sum_{n=0}^{\infty} a_{2,1}(2n)q^n=\frac{f_2^2f_3f_8^2f_{12}^4}{f_1^4f_4^2f_6^2f_{24}^2}+q\frac{f_3f_4^4f_{24}^2}{f_1^4f_8^2f_{12}^2}, \tag{6.2.4}$$

$$\sum_{n=0}^{\infty} a_{2,1}(2n+1)q^n=2\frac{f_2f_3f_4f_{12}}{f_1^4f_6}. \tag{6.2.5}$$

Proof. Using (1.2.18) in (6.2.2), we have

$$\sum_{n=0}^{\infty} a_{2,1}(n)q^n=\frac{f_4^2f_6f_{16}^2f_{24}^4}{f_2^4f_8^2f_{12}^2f_{48}^2}+2q\frac{f_4f_6f_8f_{24}}{f_1^4f_6}+q^2\frac{f_6f_8^4f_{48}^2}{f_2^4f_{16}^2f_{24}^2}. \tag{6.2.6}$$

Extracting the terms having even powers of q in (6.2.6) and then replacing q^2 by q, we arrive at (6.2.4).

Similarly, extracting the terms having odd powers of q in (6.2.6) followed by dividing the equation by q and then replacing q^2 by q, we arrive at (6.2.5).

□

Corollary 6.2.3. *For $n \geq 0$, we have*

$$a_{2,1}(2n+1)\equiv 0 \pmod 2. \tag{6.2.7}$$

Proof. Congruence (6.2.7) follows directly from (6.2.5). □

Corollary 6.2.4. *If n can not be expressed as a sum of two times a pentagonal number and three times three pentagonal numbers or two times a pentagonal number, three times a pentagonal number, and six times a pentagonal number then*

$$a_{2,1}(2n+1)\equiv 0 \pmod 4. \tag{6.2.8}$$

Proof. From (6.2.5), we have

$$\sum_{n=0}^{\infty} a_{2,1}(2n+1)q^n \equiv 2f_2f_3^3 \tag{6.2.9}$$

$$\equiv 2f_2f_3f_6 \pmod 4, \tag{6.2.10}$$

where we use the fact that $2f_{2k}^m \equiv 2f_k^{2m} \pmod 4$. From (6.2.9) and (6.2.10), we arrive at (6.2.8), in view of Euler's pentagonal number theorem. □

6.3 Generating function for $a_{3,i}(n), i = 1, 2$

In this section we present the generating functions and few congruences for the functions $a_{3,1}(n)$ and $a_{3,2}(n)$.

Theorem 6.3.1. *We have*

$$\sum_{n=0}^{\infty} a_{3,1}(n)q^n = \frac{1}{f_1^2f_2}\left(\varphi(q)f(q^4,q^6) + 2q\psi(q^2)f(q,q^9)\right). \tag{6.3.1}$$

Proof. Setting $k = 3$ and $i = 1$ in (6.1.1), we find that

$$\begin{aligned} A_{3,1}(q) &= CT_z\left(\frac{1}{f_1^2f_2}\,f^2(z^{-1},zq)f(z^{-1}q,zq)\right) \\ &= CT_z\left(\frac{1}{f_1^2f_2}\left(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m(m+1)}{2}}\right)^2\left(\sum_{m=-\infty}^{\infty} z^m q^{m^2}\right)\right) \\ &= CT_z\left(\frac{1}{f_1^2f_2}\sum_{l,m,n=-\infty}^{\infty} z^{l+m+n} q^{\frac{l(l+1)}{2}+\frac{m(m+1)}{2}+n^2}\right). \end{aligned} \tag{6.3.2}$$

Extracting the constant term in (6.3.2), we have

$$\sum_{n=0}^{\infty} a_{3,1}(n)q^n = \frac{1}{f_1^2f_2}\sum_{l,m=-\infty}^{\infty} q^{\frac{1}{2}(3l^2+4lm+3m^2+l+m)}. \tag{6.3.3}$$

Using the integer matrix exact covering system

$$\left\{B\overline{n}, B\overline{n} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

where $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\overline{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$, we can split the right side of (6.3.3) into 2 sums as

$$\begin{aligned}\sum_{n=0}^{\infty} a_{3,1}(n)q^n &= \frac{1}{f_1^2 f_2}\left(\sum_{n_1,n_2=-\infty}^{\infty} q^{5n_1^2+n_2^2+n_1} + \sum_{n_1,n_2=-\infty}^{\infty} q^{5n_1^2+n_2^2+6n_1+n_2+2}\right)\\ &= \frac{1}{f_1^2 f_2}\left(\varphi(q)f(q^4,q^6) + 2q^2\psi(q^2)f(q^{-1},q^{11})\right)\\ &= \frac{1}{f_1^2 f_2}\left(\varphi(q)f(q^4,q^6) + 2q\psi(q^2)f(q,q^9)\right).\end{aligned}$$

This completes the proof of Theorem 6.3.1. □

Theorem 6.3.2. *We have*

$$\sum_{n=0}^{\infty} a_{3,2}(n)q^n = \frac{1}{f_1 f_2^2}\left(\varphi(q^2)f(q^3,q^5) + 2q\psi(q^4)f(q,q^7)\right). \tag{6.3.4}$$

Proof. The proof of (6.3.4) is similar to the proof of (6.3.1). We omit the details. □

6.4 Generating function for $a_{4,2}(n)$

In this section, we present the generating function and few congruences for the function $a_{4,2}(n)$.

Theorem 6.4.1. *We have*

$$\sum_{n=0}^{\infty} a_{4,2}(n)q^n = \frac{f_4^4 f_6^2}{f_1^4 f_8^2 f_{12}} + 4q\frac{f_3 f_8^2 f_{12}}{f_1^3 f_2 f_6}. \tag{6.4.1}$$

Proof. Setting $k=4$ and $i=2$ in (6.1.1) the generating function $A_{4,2}(q)$ of $a_{4,2}(n)$ is given by

$$\begin{aligned}A_{4,2}(q) &= CT_z\left(\frac{1}{f_1^2 f_2^2}\, f(z^{-1},zq)^2 f(z^{-1}q,zq)^2\right)\\ &= CT_z\left(\frac{1}{f_1^2 f_2^2}\left(\sum_{m_1=-\infty}^{\infty} z^{m_1} q^{\frac{m_1(m_1+1)}{2}}\right)^2\left(\sum_{m_2=-\infty}^{\infty} z^{m_2} q^{m_2^2}\right)^2\right)\end{aligned}$$

$$= CT_z\left(\frac{1}{f_1^2 f_2^2}\sum_{m_1,\,m_2,\,m_3,\,m_4=-\infty}^{\infty} z^{m_1+m_2+m_3+m_4} q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+m_3^2+m_4^2}\right)$$
$$= \frac{1}{f_1^2 f_2^2}\sum_{m_1,\,m_2,\,m_3=-\infty}^{\infty} q^{\frac{1}{2}(3m_1^2+3m_2^2+4m_3^2+4m_1m_2+4m_1m_3+4m_2m_3+m_1+m_2)}. \qquad (6.4.2)$$

Using the integer matrix exact covering system

$$\left\{ B\overline{n}, B\overline{n} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

where $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ and $\overline{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, we can split the right side of (6.4.2) into 2 sums as

$$\sum_{n=0}^{\infty} a_{4,\,2}(n)q^n = \frac{1}{f_1^2 f_2^2}\left(\sum_{n_1,\,n_2,\,n_3=-\infty}^{\infty} q^{2n_1^2+3n_2^2+n_3^2+n_2}\right.$$
$$\left. + q^2 \sum_{n_1,\,n_2,\,n_3=-\infty}^{\infty} q^{2n_1^2+3n_2^2+n_3^2+2n_1+4n_2+n_3}\right)$$
$$= \frac{1}{f_1^2 f_2^2}\left(\varphi(q^2)\varphi(q)f(q^2,q^4) + q^2 f(1,q^4)f(q^{-1},q^7)f(1,q^2)\right)$$
$$= \frac{1}{f_1^2 f_2^2}\left(\varphi(q^2)\varphi(q)f(q^2,q^4) + 4q\psi(q^2)\psi(q^4)f(q,q^5)\right). \qquad (6.4.3)$$

Replacing q by q^2 in (1.2.14), we have

$$f(q^2,q^4) = \frac{f_4 f_6^2}{f_2 f_{12}}. \qquad (6.4.4)$$

Using (1.2.6), (1.2.7), (1.2.13), and (6.4.4) in (6.4.3), we find that

$$\sum_{n=0}^{\infty} a_{4,\,2}(n)q^n = \frac{f_4^4 f_6^2}{f_1^4 f_8^2 f_{12}} + 4q\frac{f_3 f_8^2 f_{12}}{f_1^3 f_2 f_6}, \qquad (6.4.5)$$

which is (6.4.1). □

Corollary 6.4.2. *We have*

$$\sum_{n=0}^{\infty} a_{4,\,2}(2n)q^n = \frac{f_2^{18} f_3^2}{f_1^{14} f_4^6 f_6} + 8q\frac{f_2^3 f_8^3 f_{12}^2}{f_1^8 f_{24}} + 4q\frac{f_4^9 f_6 f_{24}}{f_1^8 f_8^3 f_{12}}, \qquad (6.4.6)$$

$$\sum_{n=0}^{\infty} a_{4,2}(2n+1)q^n = 4\frac{f_2^6 f_3^2 f_4^2}{f_1^{10} f_6} + 4\frac{f_2 f_4^6 f_{12}^2}{f_1^8 f_8 f_{24}} + 8q\frac{f_2^2 f_4^3 f_6 f_8 f_{24}}{f_1^8 f_{12}}, \tag{6.4.7}$$

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4,2}(4n)q^n = {} & \frac{f_2^{38} f_3 f_8^2 f_{12}^4}{f_1^{28} f_4^{14} f_6^2 f_{24}^2} + q\frac{f_2^{36} f_3 f_{24}^2}{f_1^{28} f_4^8 f_8^2 f_{12}^2} + 56q\frac{f_2^{25} f_3 f_{12}}{f_1^{24} f_4^3 f_6} \\ & + 64q\frac{f_2^{16} f_4^3 f_6^2}{f_1^{21} f_{12}} + 48q\frac{f_2^{14} f_3 f_4^2 f_8^2 f_{12}^4}{f_1^{20} f_6^2 f_{24}^2} + 48q^2\frac{f_2^{12} f_3 f_4^8 f_{24}^2}{f_1^{20} f_8^2 f_{12}^2} \\ & + 128q^2\frac{f_2 f_3 f_4^{13} f_{12}}{f_1^{16} f_6},\end{aligned} \tag{6.4.8}$$

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4,2}(4n+1)q^n = {} & 4\frac{f_2^{32} f_3 f_8^2 f_{12}^4}{f_1^{26} f_4^{10} f_6^2 f_{24}^2} + 4\frac{f_2^{34} f_6^2}{f_1^{27} f_4^9 f_{12}} + 4q\frac{f_2^{30} f_3 f_{12}^2}{f_1^{26} f_4^4 f_8^2 f_{12}^2} \\ & + 64q\frac{f_2^{10} f_4^7 f_6^2}{f_1^{19} f_{12}} + 64q\frac{f_2^8 f_3 f_4^6 f_8^2 f_{12}^4}{f_1^{18} f_6^2 f_{24}^2} + 128q\frac{f_2^{19} f_3 f_4 f_{12}}{f_1^{22} f_6} \\ & + 64q^2\frac{f_2^6 f_3 f_4^{12} f_{24}^2}{f_1^{18} f_8^2 f_{12}^2},\end{aligned} \tag{6.4.9}$$

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4,2}(4n+2)q^n = {} & 6\frac{f_2^{37} f_3 f_{12}}{f_1^{28} f_4^{11} f_6} + 8\frac{f_2^{28} f_6^2}{f_1^{25} f_4^5 f_{12}} + 12\frac{f_2^{26} f_3 f_8^2 f_{12}^4}{f_1^{24} f_4^6 f_6^2 f_{24}^2} \\ & + 12q\frac{f_2^{24} f_3 f_{24}^2}{f_1^{24} f_8^2 f_{12}^2} + 64q\frac{f_2^2 f_3 f_4^{10} f_8^2 f_{12}^4}{f_1^{16} f_6^2 f_{24}^2} + 128q\frac{f_2^4 f_4^{11} f_6^2}{f_1^{17} f_{12}} \\ & + 160q\frac{f_2^{13} f_3 f_4^5 f_{12}}{f_1^{20} f_6} + 64q^2\frac{f_3 f_4^{16} f_{24}^2}{f_1^{16} f_8^2 f_{12}^2},\end{aligned} \tag{6.4.10}$$

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4,2}(4n+3)q^n = {} & 16\frac{f_2^{31} f_3 f_{12}}{f_1^{26} f_4^7 f_6} + 32\frac{f_2^{22} f_6^2}{f_1^{23} f_4 f_{12}} + 32\frac{f_2^{20} f_3 f_8^2 f_{12}^4}{f_1^{22} f_4^2 f_6^2 f_{24}^2} \\ & + 32q\frac{f_2^{18} f_3 f_4^4 f_{24}^2}{f_1^{22} f_8^2 f_{12}^2} + 256q\frac{f_2^7 f_3 f_4^9 f_{12}}{f_1^{18} f_6}.\end{aligned} \tag{6.4.11}$$

Proof. Using (1.2.16), (1.2.17), and (1.2.18) in (6.4.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4,2}(n)q^n = {} & 8q^3\frac{f_8^3 f_{12} f_{16} f_{48} f_4^2}{f_2^8 f_{24}} + 8q^2\frac{f_{16}^3 f_{24}^2 f_4^3}{f_2^8 f_{48}} + 4q^2\frac{f_8^9 f_{12} f_{48}}{f_2^8 f_{16}^3 f_{24}} \\ & + 4q\frac{f_6^2 f_8^2 f_4^6}{f_2^{10} f_{12}} + 4q\frac{f_8^6 f_{24}^2 f_4}{f_2^8 f_{16} f_{48}} + \frac{f_6^2 f_4^{18}}{f_2^{14} f_8^6 f_{12}}.\end{aligned} \tag{6.4.12}$$

Extracting the terms with even powers of q in (6.4.12) and then replacing q^2 by q, we arrive at (6.4.6).

Extracting the terms with odd powers of q in (6.4.12) followed by dividing the equation by q, and then replacing q^2 by q, we arrive at (6.4.7).

Using (1.2.17) and (1.2.18) in (6.4.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4,2}(2n)q^n &= 64q^5\frac{f_6f_8^{16}f_{48}^2}{f_2^{16}f_{16}^2f_{24}^2} + 48q^4\frac{f_6f_8^8f_{48}^2f_4^{12}}{f_2^{20}f_{16}^2f_{24}^2} + 128q^4\frac{f_6f_8^{13}f_{24}f_4}{f_2^{16}f_{12}} \\ &+ 12q^3\frac{f_6f_{48}^2f_4^{24}}{f_2^{24}f_{16}^2f_{24}^2} + 160q^3\frac{f_6f_8^5f_{24}f_4^{13}}{f_2^{20}f_{12}} + 128q^3\frac{f_8^{11}f_{12}^2f_4^4}{f_2^{17}f_{24}} \\ &+ 64q^3\frac{f_6f_8^{10}f_{16}^2f_{24}^4f_4^2}{f_2^{16}f_{12}^2f_{48}^2} + q^2\frac{f_6f_{48}^2f_4^{36}}{f_2^{28}f_8^8f_{16}^2f_{24}^2} + 56q^2\frac{f_6f_{24}f_4^{25}}{f_2^{24}f_8^3f_{12}} \\ &+ 64q^2\frac{f_8^3f_{12}^2f_4^{16}}{f_2^{21}f_{24}} + 48q^2\frac{f_6f_8^2f_{16}^2f_{24}^4f_4^{14}}{f_2^{20}f_{12}^2f_{48}^2} + 6q\frac{f_6f_{24}f_4^{37}}{f_2^{28}f_8^{11}f_{12}} \\ &+ 8q\frac{f_{12}^2f_4^{28}}{f_2^{25}f_8^5f_{24}} + 12q\frac{f_6f_{16}^2f_{24}^4f_4^{26}}{f_2^{24}f_8^6f_{12}^2f_{48}^2} + \frac{f_6f_{16}^2f_{24}^4f_4^{38}}{f_2^{28}f_8^{14}f_{12}^2f_{48}^2}. \qquad (6.4.13)\end{aligned}$$

Extracting the terms involving even powers of q in (6.4.13) and then replacing q^2 by q, we arrive at (6.4.8).

Extracting the terms with odd powers of q in (6.4.13) followed by dividing the equation by q, and then replacing q^2 by q, we arrive at (6.4.10).

Using (1.2.17) and (1.2.18) again in (6.4.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4,2}(2n+1)q^n &= 64q^4\frac{f_6f_8^{12}f_{48}^2f_4^6}{f_2^{18}f_{16}^2f_{24}^2} + 32q^3\frac{f_6f_8^4f_{48}^2f_4^{18}}{f_2^{22}f_{16}^2f_{24}^2} + 256q^3\frac{f_6f_8^9f_{24}f_4^7}{f_2^{18}f_{12}} \\ &+ 4q^2\frac{f_6f_{48}^2f_4^{30}}{f_2^{26}f_8^4f_{16}^2f_{24}^2} + 128q^2\frac{f_6f_8f_{24}f_4^{19}}{f_2^{22}f_{12}} + 64q^2\frac{f_8^7f_{12}^2f_4^{10}}{f_2^{19}f_{24}} \\ &+ 64q^2\frac{f_6f_8^6f_{16}^2f_{24}^4f_4^8}{f_2^{18}f_{12}^2f_{48}^2} + 16q\frac{f_6f_{24}f_4^{31}}{f_2^{26}f_8^7f_{12}} + 32q\frac{f_{12}^2f_4^{22}}{f_2^{23}f_8f_{24}} \\ &+ 32q\frac{f_6f_{16}^2f_{24}^4f_4^{20}}{f_2^{22}f_8^2f_{12}^2f_{48}^2} + 4\frac{f_{12}^2f_4^{34}}{f_2^{27}f_8^9f_{24}} + 4\frac{f_6f_{16}^2f_{24}^4f_4^{32}}{f_2^{26}f_8^{10}f_{12}^2f_{48}^2}. \qquad (6.4.14)\end{aligned}$$

Extracting the terms involving even powers of q in (6.4.14) and then replacing q^2 by q, we arrive at (6.4.9).

Extracting the terms with odd powers of q in (6.4.14) followed by dividing the equation by q, and then replacing q^2 by q, we arrive at (6.4.11).

$\square$

Corollary 6.4.3. *For $n \geq 0$,*

$$a_{4,2}(4n+2) \equiv 0 \pmod{2}, \tag{6.4.15}$$

$$a_{4,2}(2n+1) \equiv 0 \pmod{8}, \tag{6.4.16}$$

$$a_{4,2}(4n+3) \equiv 0 \pmod{16}. \tag{6.4.17}$$

Proof. Congruences (6.4.15) and (6.4.17) follow directly from (6.4.10) and (6.4.11), respectively.

Also from (6.4.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4,2}(2n+1)q^n &\equiv 4\frac{f_2^6 f_3^2 f_4^2}{f_1^{10} f_6} + 4\frac{f_2 f_4^6 f_{12}^2}{f_1^8 f_8 f_{24}} \\ &\equiv 4f_1^8 + 4f_1^8 \\ &\equiv 8f_1^8 \pmod{8} \\ &\equiv 0 \pmod{8}. \end{aligned} \tag{6.4.18}$$

Congruence (6.4.16) follows from (6.4.18).

□

6.5 Ramanujan-type congruences for $a_{k,i}(n)$

Theorem 6.5.1. *For any prime p and $n \geq 0$, we have*

$$a_{2p,p}(pn) \equiv a_{2,1}(n) \pmod{p}. \tag{6.5.1}$$

Proof. From Andrews' general principle 1.1.1, the generating function $A_{2p,p}(q)$ of $a_{2p,p}(n)$ is given by

$$\begin{aligned}A_{2p,p}(q) &= CT_z\left(\prod_{n=0}^{\infty}\left(1+zq^{n+1}\right)^p\left(1+z^{-1}q^n\right)^p\left(1+zq^{2n+1}\right)^p\left(1+z^{-1}q^{2n+1}\right)^p\right) \\ &\equiv CT_z\left(\prod_{n=0}^{\infty}\left(1+z^pq^{pn+p}\right)\left(1+z^{-p}q^{pn}\right)\left(1+z^pq^{2pn+p}\right)\left(1+z^{-p}q^{2pn+p}\right)\right) \pmod{p}\end{aligned}$$

$$\equiv CT_z\left(\frac{1}{f_p f_{2p}} f(z^{-p}, z^p q^p) f(z^{-p} q^p, z^p q^p)\right) \pmod{p}$$

$$\equiv CT_z\left(\frac{1}{f_p f_{2p}} \sum_{m_1,\, m_2=-\infty}^{\infty} z^{pm_1+pm_2} q^{\frac{pm_1(m_1+1)}{2}+pm_2^2}\right) \pmod{p}. \tag{6.5.2}$$

Extracting the constant term from (6.5.2), we have

$$\sum_{n=0}^{\infty} a_{2p,\,p}(n) q^n \equiv \frac{1}{f_p f_{2p}} \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}pn(3n+1)} \pmod{p}$$

$$\equiv \frac{f(q^p, q^{2p})}{f_p f_{2p}} \pmod{p}. \tag{6.5.3}$$

Replacing q by q^p in (1.2.14), we have

$$f(q^p, q^{2p}) = \frac{f_{2p} f_{3p}^2}{f_p f_{6p}}. \tag{6.5.4}$$

Employing (6.5.4) in (6.5.3), we find that

$$\sum_{n=0}^{\infty} a_{2p,\,p}(n) q^n \equiv \frac{f_{3p}^2}{f_p^2 f_{6p}} \pmod{p}. \tag{6.5.5}$$

Extracting the terms in (6.5.5) where the powers of q are multiple of p and then replacing q^p by q, we have

$$\sum_{n=0}^{\infty} a_{2p,\,p}(pn) q^n \equiv \frac{f_3^2}{f_1^2 f_6} \pmod{p}. \tag{6.5.6}$$

Comparing (6.2.1) and (6.5.6), we arrive at (6.5.1). □

Corollary 6.5.2. *For any prime p and $n \geq 0$,*

$$a_{2p,\,p}(pn + r) \equiv 0 \pmod{p}, \tag{6.5.7}$$

where $r \in \{1, 2, \ldots, p-1\}$.

Proof. Congruence (6.5.7) follows directly from (6.5.5). □