Chapter 7

2^n-dissection of Euler product

7.1 Introduction[1]

This chapter is devoted to an elementary proof of Hirschhorn's conjecture [36] on 2^n-dissection of Euler product $E(q) := (q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$, $|q| < 1$.
We can view $E(q)$ as a particular case of Ramanujan's theta function given by (1.2.1).
In fact,

$$f(-q,-q^2) = (q;q^3)_\infty (q^2;q^3)_\infty (q^3;q^3)_\infty = (q;q)_\infty = E(q).$$

For the 3-dissection of $E(q)$, we have [15, Entry 31, p. 48]

$$E(q) = f(-q^{12},-q^{15}) - qf(-q^6,-q^{21}) - q^2 f(-q^3,-q^{24}).$$

We also have

$$E(q) = E(q^{25})\left[\frac{f(-q^{10},-q^{15})}{f(-q^5,-q^{20})} - q - q^2\frac{f(-q^5,-q^{20})}{f(-q^{10},-q^{15})}\right],$$
$$E(q) = E(q^{49})\left[\frac{f(-q^{14},-q^{35})}{f(-q^7,-q^{42})} - q\frac{f(-q^{21},-q^{28})}{f(-q^{14},-q^{35})} - q^2 + q^3\frac{f(-q^7,-q^{42})}{f(-q^{21},-q^{28})}\right],$$
$$E(q) = E(q^{121})\Bigg[\frac{f(-q^{44},-q^{77})}{f(-q^{22},-q^{99})} - q\frac{f(-q^{22},-q^{99})}{f(-q^{11},-q^{110})} - q^2\frac{f(-q^{55},-q^{66})}{f(-q^{33},-q^{88})} + q^5 + q^7\frac{f(-q^{33},-q^{88})}{f(-q^{44},-q^{77})} - q^{15}\frac{f(-q^{11},-q^{110})}{f(-q^{55},-q^{66})}\Bigg],$$

[1]The contents of this chapter appeared in *Bulletin of the Australian Mathematical Society* [61].

which are respectively, 5-, 7-, and 11- dissections of $E(q)$ and can be found in [15, p. 82, p. 303, p. 363]. A generalized form of these dissections can be found in the following theorem.

Theorem 7.1.1. *Suppose m is a positive integer with $m \equiv 1 \pmod 6$. If $m = 6t+1$ with t positive, then*

$$\frac{E(q^{1/m})}{E(q^m)} = (-1)^t q^{(m^2-1)/(24m)} + \sum_{k=1}^{(m-1)/2} (-1)^{k+t} q^{(k-t)(3k-3t-1)/(2n)} \frac{f(-q^{2k}, -q^{n-2k})}{f(-q^k, -q^{n-k})}.$$

If $m = 6t-1$ with t positive, then

$$\frac{E(q^{1/m})}{E(q^m)} = (-1)^t q^{(m^2-1)/(24m)} + \sum_{k=1}^{(m-1)/2} (-1)^{k+t} q^{(k-t)(3k-3t+1)/(2n)} \frac{f(-q^{2k}, -q^{n-2k})}{f(-q^k, -q^{n-k})}.$$

Theorem 7.1.1 appears as Theorem 12.1 in [15, p. 274] and are due to independent works of Ramanathan [55] and Evans [29]. Recently, this have been reproved by McLaughlin [48] while establishing some other general dissections involving infinite products.

Hirschhorn [36, p. 332] gave the 2- and 4- dissections of $E(q)$ as

$$E(q) = -q(q^4, q^{28}; q^{32})_\infty (q^6, q^{10}, q^{16}; q^{16})_\infty + (q^{12}, q^{20}; q^{32})_\infty (q^2, q^{14}, q^{16}; q^{16})_\infty \tag{7.1.1}$$

$$\begin{aligned} E(q) = {} & q^7 (q^8, q^{120}; q^{128})_\infty (q^{28}, q^{36}, q^{64}; q^{64})_\infty - q^2 (q^{24}, q^{104}; q^{128})_\infty (q^{20}, q^{44}, q^{64}; q^{64})_\infty \\ & + (q^{40}, q^{88}; q^{128})_\infty (q^{12}, q^{52}, q^{64}; q^{64})_\infty - q(q^{56}, q^{72}; q^{128})_\infty (q^4, q^{60}, q^{64}; q^{64})_\infty, \end{aligned} \tag{7.1.2}$$

where $(a_1, a_2, \cdots, a_n; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty$.

Identity (7.1.2) was obtained by 4-dissecting the products $(q^6, q^{10}, q^{16}; q^{16})_\infty$ and $(q^2, q^{14}, q^{16}; q^{16})_\infty$ in (7.1.1). Hirschhorn also remarked that we can continue in the same manner and find the 8-dissection, the 16-dissection and so on. He concluded by noting the following conjecture on 2^n-dissection of $E(q)$.

Conjecture 7.1.2.

$$E(q) = \sum_{k=1}^{2^n} (-1)^{n+k+1} q^{c_k} \left(q^{(2k-1)2^{n+1}}, q^{2^{2n+3}-(2k-1)2^{n+1}}; q^{2^{2n+3}}\right)_\infty$$
$$\times \left(q^{2^{2n+1}+(2k-1)2^n}, q^{2^{2n+1}-(2k-1)2^n}, q^{2^{2n+2}}; q^{2^{2n+2}}\right)_\infty,$$

where

$$\text{if } n \text{ is odd}, \quad c_k = P\left(\frac{2^{n+1}-1}{3} - (k-1)\right), \; k = 1, 2, 3, \ldots, 2^n, \tag{7.1.3}$$
$$\text{if } n \text{ is even}, \quad c_k = P\left(-\frac{2^{n+1}-2}{3} + (k-1)\right), \; k = 1, 2, 3, \ldots, 2^n, \tag{7.1.4}$$

and

$$P(n) = \frac{3n^2 - n}{2}. \tag{7.1.5}$$

Cao [20] discussed product identities for theta functions using integer matrix exact covering system. In particular, he gave the following result [20, Corollary 2.2] involving product of two theta functions.

Theorem 7.1.3. *If $|ab| < 1$ and $cd = (ab)^{k_1k_2}$, where both k_1 and k_2 are positive integers, then*

$$f(a,b)f(c,d) = \sum_{r=0}^{k_1+k_2-1} a^{\frac{r(r+1)}{2}} b^{\frac{r(r-1)}{2}} f(a^{\frac{k_1^2+k_1}{2}+k_1r} b^{\frac{k_1^2-k_1}{2}+k_1r} d, a^{\frac{k_1^2-k_1}{2}-k_1r} b^{\frac{k_1^2+k_1}{2}-k_1r} c)$$
$$\times f(a^{\frac{k_2^2+k_2}{2}+k_2r} b^{\frac{k_2^2-k_2}{2}+k_2r} c, a^{\frac{k_2^2-k_2}{2}-k_2r} b^{\frac{k_2^2+k_2}{2}-k_2r} d). \tag{7.1.6}$$

We also consider the following version of Quintuple Product Identity.

Theorem 7.1.4 (Quintuple Product Identity)**.** *For $a \neq 0$,*

$$(-aq;q)_\infty(-a^{-1};q)_\infty(a^2q;q^2)_\infty(a^{-2}q;q^2)_\infty(q;q)_\infty$$
$$= (a^3q^2;q^3)_\infty(a^{-3}q;q^3)_\infty(q^3;q^3)_\infty + a^{-1}(a^3q;q^3)_\infty(a^{-3}q^2;q^3)_\infty(q^3;q^3)_\infty$$
$$= f(-a^3q^2, -a^{-3}q) + a^{-1}f(-a^3q, -a^{-3}q^2). \tag{7.1.7}$$

We use (7.1.6), (7.1.7) and other properties of Ramanujan's theta function to prove Conjecture 7.1.2. In Section 7.2, we mention few preliminary results involving Ramanujan's theta function and pentagonal numbers $P(n)$. In Section 7.3, we detail out the proof of Conjecture 7.1.2.

7.2 Preliminaries

Lemma 7.2.1. *For any integer n, we have*

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}). \tag{7.2.1}$$

Proof. See [15, Entry 18, p. 34]. □

Lemma 7.2.2. *We have*

$$f(a,b) = af(a^2b, a^{-1}). \tag{7.2.2}$$

Proof. Set $n = 1$ in (7.2.1). □

Lemma 7.2.3. *Let $P(n)$ be as in* (7.1.5)*. For positive integers n and k, we have*

$$P(2^{n+1}+k-1) - P(k-1) = 3\times 2^{2n+1} + 2^n(6k-7). \tag{7.2.3}$$

Proof. We have

$$P(s) - P(t) = \frac{1}{2}(s-t)(3s+3t-1). \tag{7.2.4}$$

Replacing s and t by $2^{n+1}+k-1$ and $k-1$ respectively, in (7.2.4), we find that

$$\begin{aligned} P(2^{n+1}+k-1) - P(k-1) &= \frac{1}{2}\times 2^{n+1}(3\times 2^{n+1}+6k-7) \\ &= 2^n(3\times 2^{n+1}+6k-7) \\ &= 3\times 2^{2n+1} + 2^n(6k-7). \end{aligned}$$

□

Lemma 7.2.4. *If n is even and $c_k = P\left(-\frac{2^{n+1}-2}{3} + (k-1)\right)$, then*

$$P\left(\frac{2^{n+2}+2}{3} - k\right) = c_k + 2^{2n+1} - 2^n(2k-1). \tag{7.2.5}$$

Proof. We have $c_k = P\left(-\frac{2^{n+1}-2}{3} + (k-1)\right) = P\left(\frac{3k-2^{n+1}-1}{3}\right)$.

Hence

$$\begin{aligned}
P\left(\frac{2^{n+2}+2}{3} - k\right) - c_k &= P\left(\frac{2^{n+2}-3k+2}{3}\right) - P\left(\frac{3k-2^{n+1}-1}{3}\right) \\
&= \frac{1}{2}\left(\frac{2^{n+2}-3k+2}{3} - \frac{3k-2^{n+1}-1}{3}\right) \\
&\qquad \times (2^{n+2}-3k+2+3k-2^{n+1}-1-1) \\
&= \frac{1}{2}\left(2^{n+1}-2k+1\right) \times 2^{n+1} \\
&= 2^n\left(2^{n+1}-2k+1\right) \\
&= 2^{2n+1} - 2^n(2k-1). \qquad (7.2.6)
\end{aligned}$$

Identity (7.2.5) follows from (7.2.6). □

Lemma 7.2.5. *If n is even and $c_k = P\left(-\frac{2^{n+1}-2}{3} + (k-1)\right)$, then*

$$P\left(\frac{2^{n+2}+2}{3} + k - 1\right) = c_k + 2^{2n+1} + 3 \times 2^n(2k-1).$$

Proof. Similar to proof of Lemma 7.2.4. □

Lemma 7.2.6. *If n is odd and $c_k = P\left(\frac{2^{n+1}-1}{3} - (k-1)\right)$, then*

$$P\left(\frac{2^{n+1}+2}{3} + k - 1\right) = c_k + 2^{n+1}(2k-1).$$

Proof. Similar to proof of Lemma 7.2.4. □

Lemma 7.2.7. *If n odd is and $c_k = P\left(\frac{2^{n+1}-1}{3} - (k-1)\right)$, then*

$$P\left(\frac{2^{n+3}+2}{3} - k\right) = c_k + 5 \times 2^{2n+1} - 3 \times 2^n(2k-1).$$

Proof. Similar to proof of Lemma 7.2.4. □

7.3 Proof of the main result

Theorem 7.3.1. *Conjecture 7.1.2 holds good.*

Proof. We set $a = -q$, $b = -q^2$, $c = -q^{2^{2n+2}}$ $d = -q^{2^{2n+3}}$, $k_1 = k_2 = 2^{n+1}$, and $r = k-1$ in (7.1.6) to obtain

$$f(-q,-q^2)f(-q^{2^{2n+2}},-q^{2^{2n+3}}) = \sum_{k=1}^{2^{n+2}}\Bigg((-1)^{k+1}q^{\frac{(k-1)(3k-4)}{2}} \times f(-q^{7\times 2^{2n+1}+2^n(6k-7)},-q^{5\times 2^{2n+1}-2^n(6k-7)}) \times f(-q^{5\times 2^{2n+1}+2^n(6k-7)},-q^{7\times 2^{2n+1}-2^n(6k-7)})\Bigg). \tag{7.3.1}$$

Noting that $f(-q,-q^2) = E(q)$, $f(-q^{2^{2n+2}},-q^{2^{2n+3}}) = E(q^{2^{2n+2}})$ and $P(k-1) = \dfrac{(k-1)(3k-4)}{2}$, we rewrite (7.3.1) as

$$\begin{aligned} E(q)E\left(q^{2^{2n+2}}\right) &= \sum_{k=1}^{2^{n+2}}(-1)^{k+1}q^{P(k-1)}f(-q^{7\times 2^{2n+1}+2^n(6k-7)},-q^{5\times 2^{2n+1}-2^n(6k-7)}) \\ &\quad \times f(-q^{5\times 2^{2n+1}+2^n(6k-7)},-q^{7\times 2^{2n+1}-2^n(6k-7)}) \\ &= \sum_{k=1}^{2^{n+2}} T_k, \end{aligned} \tag{7.3.2}$$

where

$$T_k = (-1)^{k+1}q^{P(k-1)}f(-q^{7\times 2^{2n+1}+2^n(6k-7)},-q^{5\times 2^{2n+1}-2^n(6k-7)}) \times f(-q^{5\times 2^{2n+1}+2^n(6k-7)},-q^{7\times 2^{2n+1}-2^n(6k-7)}). \tag{7.3.3}$$

Now

$$\begin{aligned} T_{2^{n+1}+k} &= (-1)^{2^{n+1}+k+1}q^{P(2^{n+1}+k-1)} \\ &\quad \times f(-q^{7\times 2^{2n+1}+2^n(6(2^{n+1}+k)-7)},-q^{5\times 2^{2n+1}-2^n(6(2^{n+1}+k)-7)}) \\ &\quad \times f(-q^{5\times 2^{2n+1}+2^n(6(2^{n+1}+k)-7)},-q^{7\times 2^{2n+1}-2^n(6(2^{n+1}+k)-7)}). \end{aligned} \tag{7.3.4}$$

Employing (7.2.3) and simplifying the exponents of the arguments of the theta functions, we rewrite (7.3.4) as

$$T_{2^{n+1}+k} = (-1)^{k+1}q^{P(k-1)}q^{3\times 2^{2n+1}+2^n(6k-7)}f(-q^{13\times 2^{2n+1}+2^n(6k-7)}, -q^{-2^{2n+1}-2^n(6k-7)}) \times f(-q^{11\times 2^{2n+1}+2^n(6k-7)}, -q^{2^{2n+1}-2^n(6k-7)}). \tag{7.3.5}$$

Adding (7.3.3) and (7.3.5), we find that

$$\begin{aligned} T_k + T_{2^{n+1}+k} = (-1)^{k+1}q^{P(k-1)}\Big(& f(-q^{7\times 2^{2n+1}+2^n(6k-7)}, -q^{5\times 2^{2n+1}-2^n(6k-7)}) \\ & \times f(-q^{5\times 2^{2n+1}+2^n(6k-7)}, -q^{7\times 2^{2n+1}-2^n(6k-7)}) \\ & + q^{3\times 2^{2n+1}+2^n(6k-7)}f(-q^{13\times 2^{2n+1}+2^n(6k-7)}, -q^{-2^{2n+1}-2^n(6k-7)}) \\ & \times f(-q^{11\times 2^{2n+1}+2^n(6k-7)}, -q^{2^{2n+1}-2^n(6k-7)})\Big). \end{aligned} \tag{7.3.6}$$

Setting $a = q^{3\times 2^{2n+1}+2^n(6k-7)}$, $b = q^{3\times 2^{2n+1}-2^n(6k-7)}$, $c = -q^{2^{2n+2}}$, and $d = -q^{2^{2n+3}}$ in (7.3.6) we obtain

$$T_k + T_{2^{n+1}+k} = (-1)^{k+1}q^{P(k-1)}\left(f(ac, bd)f(ad, bc) + af(\frac{b}{c}, \frac{c}{b}abcd)f(\frac{b}{d}, \frac{d}{b}abcd)\right). \tag{7.3.7}$$

Employing (1.2.12) in (7.3.7), we have

$$\begin{aligned} T_k + T_{2^{n+1}+k} &= (-1)^{k+1}q^{P(k-1)}f(a,b)f(c,d) \\ &= (-1)^{k+1}q^{P(k-1)}f(q^{3\times 2^{2n+1}+2^n(6k-7)}, q^{3\times 2^{2n+1}-2^n(6k-7)}) \\ &\quad \times f(-q^{2^{2n+2}}, -q^{2^{2n+3}}) \\ &= (-1)^{k+1}q^{P(k-1)}f(q^{3\times 2^{2n+1}+2^n(6k-7)}, q^{3\times 2^{2n+1}-2^n(6k-7)})E\left(q^{2^{2n+2}}\right). \end{aligned} \tag{7.3.8}$$

From (7.3.2), we have

$$E(q)E\left(q^{2^{2n+2}}\right) = \sum_{k=1}^{2^{n+1}} \left(T_k + T_{2^{n+1}+k}\right). \tag{7.3.9}$$

Using (7.3.8) in (7.3.9), we find that

$$E(q) = \sum_{k=1}^{2^{n+1}} (-1)^{k+1} q^{P(k-1)} f(q^{3\times 2^{2n+1}+2^n(6k-7)}, q^{3\times 2^{2n+1}-2^n(6k-7)})$$
$$= \sum_{k=1}^{2^{n+1}} H_k,$$

where $H_k = (-1)^{k+1} q^{P(k-1)} f(q^{3\times 2^{2n+1}+2^n(6k-7)}, q^{3\times 2^{2n+1}-2^n(6k-7)})$.

We now arrange H_k^{s} in pairs so that each of these pairs can be reduced to quintuple products. For this purpose we consider two separate cases according as n is even or odd.

<u>Case I</u> n is even.

We consider

$$S_1 = \sum_{k=1}^{\frac{2^{n+1}-2}{3}} \left(H_{\frac{2^{n+2}+2}{3}-k+1} + H_{\frac{2^{n+2}+5}{3}+k-1} \right), \tag{7.3.10}$$

$$S_2 = \sum_{k=\frac{2^{n+1}+1}{3}}^{2^n} \left(H_{k+1-\frac{2^{n+1}+1}{3}} + H_{\frac{2^{n+2}+5}{3}-k} \right). \tag{7.3.11}$$

It may be noted that S_2 is the sum of $H_1, H_2, \cdots, H_{\frac{2^{n+1}+4}{3}}$ and S_1 is the sum of $H_{\frac{2^{n+1}+7}{3}}, H_{\frac{2^{n+1}+10}{3}}, \cdots, H_{2^{n+1}}$ and so

$$E(q) = S_1 + S_2. \tag{7.3.12}$$

We also note that c_k is given by (7.1.4).

Now

$$H_{\frac{2^{n+2}+2}{3}-k+1} = (-1)^{\frac{2^{n+2}+2}{3}-k} q^{P\left(\frac{2^{n+2}+2}{3}-k\right)}$$
$$\times f(q^{3\times 2^{2n+1}+2^n\left(6\left(\frac{2^{n+2}+2}{3}-k+1\right)-7\right)}, q^{3\times 2^{2n+1}-2^n\left(6\left(\frac{2^{n+2}+2}{3}-k+1\right)-7\right)}). \tag{7.3.13}$$

Employing Lemma 7.2.4 in (7.3.13) and simplifying the exponents of the arguments of the theta functions, we find that

$$H_{\frac{2^{n+2}+2}{3}-k+1} = (-1)^k q^{c_k+2^{2n+1}-2^n(2k-1)} f\left(q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{-2^{2n+1}+3\times 2^n(2k-1)}\right). \tag{7.3.14}$$

Also

$$H_{\frac{2^{n+2}+5}{3}+k-1} = (-1)^{\frac{2^{n+2}+5}{3}+k} q^{P\left(\frac{2^{n+2}+2}{3}+k-1\right)} \times f\left(q^{3\times 2^{2n+1}+2^n\left(6\left(\frac{2^{n+2}+2}{3}+k\right)-7\right)}, q^{3\times 2^{2n+1}-2^n\left(6\left(\frac{2^{n+2}+2}{3}+k\right)-7\right)}\right). \tag{7.3.15}$$

Employing Lemma 7.2.5 in (7.3.15) and simplifying the exponents, we obtain

$$H_{\frac{2^{n+2}+5}{3}+k-1} = (-1)^{k+1} q^{c_k+2^{2n+1}+3\times 2^n(2k-1)} f\left(q^{7\times 2^{2n+1}+3\times 2^n(2k-1)}, q^{-2^{2n+1}-3\times 2^n(2k-1)}\right). \tag{7.3.16}$$

Using (7.2.2) in (7.3.16), we find that

$$H_{\frac{2^{n+2}+5}{3}+k-1} = (-1)^{k+1} q^{c_k} f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right). \tag{7.3.17}$$

Replacing the expressions for $H_{\frac{2^{n+2}+2}{3}-k+1}$ and $H_{\frac{2^{n+2}+5}{3}+k-1}$ from (7.3.14) and (7.3.17), respectively, in (7.3.10), we find that

$$S_1 = \sum_{k=1}^{\frac{2^{n+1}-2}{3}} (-1)^{k+1} q^{c_k} \left(f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right) - q^{2^{2n+1}-2^n(2k-1)} f\left(q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{-2^{2n+1}+3\times 2^n(2k-1)}\right)\right). \tag{7.3.18}$$

Further,

$$\begin{aligned} H_{k+1-\frac{2^{n+1}+1}{3}} &= (-1)^{k-\frac{2^{n+1}+1}{3}} q^{P\left(k-\frac{2^{n+1}+1}{3}\right)} \\ &\quad \times f\left(q^{3\times 2^{2n+1}+2^n\left(6\left(k+1-\frac{2^{n+1}+1}{3}\right)-7\right)}, q^{3\times 2^{2n+1}-2^n\left(6\left(k+1-\frac{2^{n+1}+1}{3}\right)-7\right)}\right) \\ &= (-1)^{k+1} q^{c_k} f\left(q^{2^{2n+1}+3\times 2^n(2k-1)}, q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}\right). \end{aligned} \tag{7.3.19}$$

Also

$$H_{\frac{2^{n+2}+5}{3}-k} = (-1)^{\frac{2^{n+2}+5}{3}-k+1} q^{P\left(\frac{2^{n+2}+5}{3}-k-1\right)} \times f\left(q^{3\times 2^{2n+1}+2^n\left(6\left(\frac{2^{n+2}+5}{3}-k\right)-7\right)}, q^{3\times 2^{2n+1}-2^n\left(6\left(\frac{2^{n+2}+5}{3}-k\right)-7\right)}\right). \tag{7.3.20}$$

Employing Lemma 7.2.4 in (7.3.20) and simplifying the exponents, we have

$$H_{\frac{2^{n+2}+5}{3}-k} = (-1)^k q^{c_k+2^{2n+1}-(2k-1)2^n} f\left(q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{-2^{2n+1}+3\times 2^n(2k-1)}\right). \tag{7.3.21}$$

Using (7.3.19) and (7.3.21) in (7.3.11), we find that

$$S_2 = \sum_{k=\frac{2^{n+1}+1}{3}}^{2^n} (-1)^{k+1} q^{c_k}\Bigg(f\left(q^{2^{2n+1}+3\times 2^n(2k-1)}, q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}\right) - q^{2^{2n+1}-(2k-1)2^n} f\left(q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{-2^{2n+1}+3\times 2^n(2k-1)}\right)\Bigg). \tag{7.3.22}$$

From (7.3.12), (7.3.18), and (7.3.22), we find that

$$\begin{aligned} E(q) &= \left(\sum_{k=1}^{\frac{2^{n+1}-2}{3}} + \sum_{k=\frac{2^{n+1}+1}{3}}^{2^n}\right)(-1)^{k+1} q^{c_k}\Bigg(f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right) \\ &\qquad - q^{2^{2n+1}-2^n(2k-1)} f\left(q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{-2^{2n+1}+3\times 2^n(2k-1)}\right)\Bigg) \\ &= \sum_{k=1}^{2^n} (-1)^{n+k+1} q^{c_k}\Bigg(f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right) \\ &\qquad - q^{2^{2n+1}-2^n(2k-1)} f\left(q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{-2^{2n+1}+3\times 2^n(2k-1)}\right)\Bigg). \end{aligned} \tag{7.3.23}$$

Case II n is odd.

In this case we consider the sums

$$S_3 = \sum_{k=1}^{\frac{2^{n+1}+2}{3}} \left(H_{\frac{2^{n+1}+2}{3}-k+1} + H_{\frac{2^{n+1}+5}{3}+k-1}\right), \tag{7.3.24}$$

$$S_4 = \sum_{k=\frac{2^{n+1}+5}{3}}^{2^n} \left(H_{\frac{2^{n+1}+2}{3}+k} + H_{\frac{2^{n+3}+5}{3}-k} \right). \tag{7.3.25}$$

It may be noted that S_3 is the sum of $H_1, H_2, \cdots, H_{\frac{2^{n+2}+4}{3}}$ and S_4 is the sum of $H_{\frac{2^{n+2}+7}{3}}, H_{\frac{2^{n+2}+10}{3}}, \cdots, H_{2^{n+1}}$ and so

$$E(q) = S_3 + S_4. \tag{7.3.26}$$

It may also be noted that c_k is given by (7.1.3).

We have

$$\begin{aligned} H_{\frac{2^{n+1}+2}{3}-k+1} &= (-1)^{\frac{2^{n+1}+2}{3}-k} q^{P\left(\frac{2^{n+1}+2}{3}-k\right)} \\ &\quad \times f\left(q^{3\times 2^{2n+1}+2^n\left(6\left(\frac{2^{n+1}+2}{3}-k+1\right)-7\right)}, q^{3\times 2^{2n+1}-2^n\left(6\left(\frac{2^{n+1}+2}{3}-k+1\right)-7\right)}\right) \\ &= (-1)^k q^{c_k} f\left(q^{2^{2n+1}+3\times 2^n(2k-1)}, q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}\right). \end{aligned} \tag{7.3.27}$$

Similarly

$$\begin{aligned} H_{\frac{2^{n+1}+5}{3}+k-1} &= (-1)^{\frac{2^{n+1}+5}{3}+k} q^{P\left(\frac{2^{n+1}+5}{3}+k-2\right)} \\ &\quad \times f\left(q^{3\times 2^{2n+1}+2^n\left(6\left(\frac{2^{n+1}+5}{3}+k-1\right)-7\right)}, q^{3\times 2^{2n+1}-2^n\left(6\left(\frac{2^{n+1}+5}{3}+k-1\right)-7\right)}\right). \end{aligned} \tag{7.3.28}$$

Using Lemma 7.2.6 in (7.3.28) and simplifying the exponents, we have

$$H_{\frac{2^{n+1}+5}{3}+k-1} = (-1)^{k+1} q^{c_k+(2k-1)2^{n+1}} f\left(q^{2^{2n+1}-3\times 2^n(2k-1)}, q^{5\times 2^{2n+1}+3\times 2^n(2k-1)}\right). \tag{7.3.29}$$

Now we employ (7.2.2) in (7.3.29) to obtain

$$\begin{aligned} H_{\frac{2^{n+1}+5}{3}+k-1} &= (-1)^{k+1} q^{c_k+(2k-1)2^{n+1}} q^{2^{2n+1}-3\times 2^n(2k-1)} \\ &\quad \times f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right) \\ &= (-1)^{k+1} q^{c_k+2^{2n+1}-2^n(2k-1)} f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right). \end{aligned} \tag{7.3.30}$$

Using (7.3.27) and (7.3.30) in (7.3.24), we find that

$$S_3 = \sum_{k=1}^{\frac{2^{n+1}+2}{3}} (-1)^k q^{c_k} \Bigg(f\left(q^{2^{2n+1}+3\times 2^n(2k-1)}, q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}\right) - q^{2^{2n+1}-2^n(2k-1)} f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right) \Bigg). \tag{7.3.31}$$

Also

$$H_{\frac{2^{n+1}+2}{3}+k} = (-1)^{\frac{2^{n+1}+2}{3}+k+1} q^{P\left(\frac{2^{n+1}+2}{3}+k-1\right)} \times f\left(q^{3\times 2^{2n+1}+2^n\left(6\left(\frac{2^{n+1}+2}{3}+k\right)-7\right)}, q^{3\times 2^{2n+1}-2^n\left(6\left(\frac{2^{n+1}+2}{3}+k\right)-7\right)}\right). \tag{7.3.32}$$

Using Lemma 7.2.6 in (7.3.32) and simplifying the exponents, we obtain

$$H_{\frac{2^{n+1}+2}{3}+k} = (-1)^{k+1} q^{c_k+2^{n+1}(2k-1)} f\left(q^{2^{2n+1}-3\times 2^n(2k-1)}, q^{5\times 2^{2n+1}+3\times 2^n(2k-1)}\right). \tag{7.3.33}$$

Employing (7.2.1) in (7.3.33), we find that

$$\begin{aligned} H_{\frac{2^{n+1}+2}{3}+k} &= (-1)^{k+1} q^{c_k+2^{n+1}(2k-1)} q^{2^{2n+1}-3\times 2^n(2k-1)} \\ &\quad \times f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right) \\ &= (-1)^{k+1} q^{c_k} q^{2^{2n+1}-2^n(2k-1)} f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right). \end{aligned} \tag{7.3.34}$$

Further,

$$H_{\frac{2^{n+3}+5}{3}-k} = (-1)^{\frac{2^{n+3}+5}{3}-k+1} q^{P\left(\frac{2^{n+3}+5}{3}-k-1\right)} \times f\left(q^{3\times 2^{2n+1}+2^n\left(6\left(\frac{2^{n+3}+5}{3}-k\right)-7\right)}, q^{3\times 2^{2n+1}-2^n\left(6\left(\frac{2^{n+3}+5}{3}-k\right)-7\right)}\right). \tag{7.3.35}$$

Using Lemma 7.2.7 in (7.3.35), we find that

$$H_{\frac{2^{n+3}+5}{3}-k} = (-1)^k q^{c_k+5\times 2^{2n+1}-3\times 2^n(2k-1)} f\left(q^{-5\times 2^{2n+1}+3\times 2^n(2k-1)}, q^{11\times 2^{2n+1}-3\times 2^n(2k-1)}\right). \tag{7.3.36}$$

Employing (7.2.2) in (7.3.36), we have

$$\begin{aligned}H_{\frac{2^{n+3}+5}{3}-k} &= (-1)^k q^{c_k+5\times 2^{2n+1}-3\times 2^n(2k-1)} q^{-5\times 2^{2n+1}+3\times 2^n(2k-1)} \\ &\quad \times f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right) \\ &= (-1)^k q^{c_k} f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right). \qquad (7.3.37)\end{aligned}$$

From (7.3.34), (7.3.37), and(7.3.25), we find that

$$\begin{aligned}S_4 = \sum_{k=\frac{2^{n+1}+5}{3}}^{2^n} (-1)^k q^{c_k} \Bigg(& f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right) \\ & - q^{2^{2n+1}-2^n(2k-1)} f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right)\Bigg). \qquad (7.3.38)\end{aligned}$$

Replacing the expressions for S_3 and S_4 from (7.3.31) and (7.3.38), respectively, in (7.3.26), we have

$$\begin{aligned}E(q) &= \left(\sum_{k=1}^{\frac{2^{n+1}+2}{3}} + \sum_{k=\frac{2^{n+1}+5}{3}}^{2^n}\right)(-1)^k q^{c_k}\Bigg(f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right) \\ &\qquad - q^{2^{2n+1}-2^n(2k-1)} f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right)\Bigg) \\ &= \sum_{k=1}^{2^n}(-1)^{n+k+1} q^{c_k}\Bigg(f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right) \\ &\qquad - q^{2^{2n+1}-2^n(2k-1)} f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right)\Bigg).\end{aligned}$$

(7.3.39)

From (7.3.23) and (7.3.39) we find that, for any positive integer n

$$\begin{aligned}E(q) = \sum_{k=1}^{2^n}(-1)^{n+k+1} q^{c_k}\Bigg(& f\left(q^{5\times 2^{2n+1}-3\times 2^n(2k-1)}, q^{2^{2n+1}+3\times 2^n(2k-1)}\right) \\ & - q^{2^{2n+1}-2^n(2k-1)} f\left(q^{-2^{2n+1}+3\times 2^n(2k-1)}, q^{7\times 2^{2n+1}-3\times 2^n(2k-1)}\right)\Bigg),\end{aligned}$$

(7.3.40)

where appropriate expressions for c_k are chosen according as n is even or odd.

Setting $A=-q^{-2^{2n+1}+2^n(2k-1)}$ and $Q=q^{2^{2n+2}}$ in (7.3.40) we obtain

$$E(q)=\sum_{k=1}^{2^n}(-1)^{n+k+1}q^{c_k}\left(f(-A^3Q^2,-A^{-3}Q)+A^{-1}f(-A^3Q,-A^{-3}Q^2)\right). \quad (7.3.41)$$

Employing quintuple product identity in (7.3.41), we find that

$$\begin{aligned}E(q)&=\sum_{k=1}^{2^n}(-1)^{n+k+1}q^{c_k}(-AQ;Q)_\infty(-A^{-1};Q)_\infty(A^2Q;Q^2)_\infty(A^{-2}Q;Q^2)_\infty(Q;Q)_\infty\\&=\sum_{k=1}^{2^n}(-1)^{n+k+1}q^{c_k}(A^2Q,A^{-2}Q;Q^2)_\infty(-AQ,-A^{-1},Q;Q)_\infty\\&=\sum_{k=1}^{2^n}(-1)^{n+k+1}q^{c_k}(q^{2^{n+1}(2k-1)},q^{2^{2n+3}-2^{n+1}(2k-1)};q^{2^{2n+3}})_\infty\\&\qquad\times(q^{2^{2n+1}+2^n(2k-1)},q^{2^{2n+1}-2^n(2k-1)},q^{2^{2n+2}};q^{2^{2n+2}})_\infty.\end{aligned}$$

Thus we have completed the proof of Conjecture 7.1.2. □