CONTRIBUTIONS TO CONGRUENCES FOR GENERALIZED FROBENIUS PARTITIONS AND DISSECTION OF EULER PRODUCT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
MATHEMATICAL SCIENCES

By
Satyajit Gayan
Registration No. TZ155749 of 2015

Department of Mathematical Sciences
School of Sciences
Tezpur University, Napaam–784028
Assam, India

SEPTEMBER 2025

Chapter 8

Conclusion and Future work

In this thesis, we studied generating functions and related congruences for various types of generalized Frobenius partitions, also referred to as F-partitions. Specifically, we discuss F-partitions with h repetitions, F-partitions with k colors and h repetitions, $(k,a)-$colored F-partitions, and a restricted class of F-partitions enumerated by $a_{k,i}(n)$. We also explored derivation of modular equations from theta function identities that arise in the context of F-partitions. Another contribution of this work is the proof of 2^n-dissection formula for the Euler product $E(q)$, originally conjectured by M.D. Hirschhorn. Existence of congruences for $\phi_k(n)$ is rare compared to $c\phi_k(n)$. However computational evidences support possibility of the following congruences for $\phi_8(n)$.

Conjecture 8.0.1. *For $n \geq 0$, we have*

$$\phi_8(200n+179) \equiv 0 \pmod 2,$$
$$\phi_8(400n+242) \equiv 0 \pmod 2,$$
$$\phi_8(400n+342) \equiv 0 \pmod 2.$$

In our future work we will try to find the generating functions for various values of k and to explore more such congruences.

The q-series expansions of $c\Phi_{2,2}(q)$ and $c\Phi_{2,3}(q)$ up to q^{20} are

$$c\Phi_{2,2}(q) = 1 + 4q + 17q^2 + 40q^3 + 99q^4 + 216q^5 + 453q^6 + 888q^7 + 1705q^8 + 3124q^9$$

$$+ 5614q^{10} + 9800q^{11} + 16792q^{12} + 28164q^{13} + 46547q^{14} + 75600q^{15}$$
$$+ 121239q^{16} + 191796q^{17} + 300017q^{18} + 463976q^{19} + 710648q^{20} + \cdots$$

and

$$\begin{aligned} c\Phi_{2,3}(q) =& 1 + 4q + 17q^2 + 52q^3 + 131q^4 + 308q^5 + 682q^6 + 1424q^7 + 2847q^8 \\ & + 5496q^9 + 10286q^{10} + 18748q^{11} + 33375q^{12} + 58184q^{13} + 99589q^{14} \\ & + 167620q^{15} + 277822q^{16} + 454124q^{17} + 732883q^{18} + 1168820q^{19} \\ & + 1843728q^{20} + \cdots . \end{aligned}$$

Using elementary techniques we are able to establish a few congruences modulo powers of 2 and 3 for $c\phi_{2,2}(n)$ and $c\phi_{2,3}(n)$. However, computational evidences suggest that these results are not exhaustive. In particular, the q-series expansions of the above functions support the following congruences.

Conjecture 8.0.2. *For $n \geq 0$, we have*

$$\begin{aligned} c\phi_{2,2}(20n + 11) &\equiv 0 \pmod{5}, \\ c\phi_{2,3}(45n + 19) &\equiv 0 \pmod{5}, \\ c\phi_{2,3}(50n + 39) &\equiv 0 \pmod{16}, \\ c\phi_{2,3}(50n + 49) &\equiv 0 \pmod{16}. \end{aligned}$$

A closer look at the respective generating functions of the associated (k, a)-colored Frobenius partition functions reveals the following relations.

Theorem 8.0.3. *We have*

$$\begin{aligned} c\psi_{4,3}(n) &= c\psi_{4,1}(n-1) \quad n \geq 1, \\ c\psi_{6,4}(n) &= c\psi_{6,2}(n-2) \quad n \geq 2, \\ c\psi_{6,5}(n) &= c\psi_{6,1}(n-3) \quad n \geq 3. \end{aligned}$$

From the above identities, it will be interesting to explore possibility of existence of other such results.

We have established few congruences for $c\psi_{6,0}(n)$, $c\psi_{6,1}(n)$ and $c\psi_{6,2}(n)$ modulo powers of 2. However, computational evidence supports the existence of more such congruences. For instance, we have the following conjecture.

Conjecture 8.0.4. *For $n \geq 0$, we have*

$$c\psi_{6,0}(64n+k) \equiv 0 \pmod{16}, \quad k \in \{5, 25, 33, 41, 45\}.$$

We can explore more such congruences in future and if possible prove the above congruences. In our work on modular equations we compare the generating functions for F-partitions with 5 repetitions. In future we can explore more such functions, comparing which more such new modular equations may be formulated.

The main tool that is being used in our work are integer matrix exact covering systems and properties of Ramanujan's general theta functions. In our future work we are looking forward to use these tools extensively to establish more such results related to theta function.