Chapter 1

Introduction

1.1 Fundamental Concepts

A *partition* of a positive integer n is a way of expressing n as a sum of positive integers, known as *parts*, where different orderings with the same set of parts are regarded as identical. Alternatively, a partition λ of a positive integer n can be represented as a finite non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ satisfying $n = \lambda_1 + \lambda_2 + \ldots + \lambda_k$. For example the five partitions of 4 are:

$$(4), (3, 1), (2, 2), (2, 1, 1), \text{and} \, (1, 1, 1, 1).$$

The total number of such distinct representations for a given positive integer n is denoted by $p(n)$.

Beyond the classical definition of partition, there exist several other types of partitions including overpartitions, plane partitions, regular partitions, tagged-part partitions, smallest-part partitions, to name a few. By imposing additional conditions on the parts, one can derive numerous other specialized forms of partitions. These partitions and their associated partition functions are studied from various perspectives like arithmetic behaviour, combinatorial interpretations, recurrence relations, identification of generating functions etc.

The generating function for $p(n)$, as given by Euler, is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, for $|q| < 1$, $(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$ and $p\,(0) = 1$.

Ramanujan [56]–[58] found elegant congruence properties for $p(n)$ modulo 5, 7, and 11, namely, for any non-negative integer n,

$$p(5n + 4) \equiv 0 \pmod 5,$$

$$p(7n + 5) \equiv 0 \pmod 7,$$

$$p(11n + 6) \equiv 0 \pmod{11}.$$

Moreover, Ramanujan offered a more general conjecture which states that if $\delta = 5^a 7^b 11^c$ and λ is an integer such that $24\lambda \equiv 1 \pmod{\delta}$, then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}.$$

Ramanujan's work on partition congruences inspired many researchers to explore the existence of similar congruences, not only for the partition function $p(n)$ but also a variety of other partition functions, which are collectively known as *Ramanujan-type congruences.*

In this thesis, we primarily investigate arithmetic properties of several classes of generalized Frobenius partition functions [3]. Our methods are largely elementary and rely significantly on the properties of Ramanujan's general theta functions, which we introduce in Section 1.2.

This thesis is organized into seven chapters including the introductory chapter. In the next few paragraphs, we present a brief introduction to the fundamental concepts and terminology that will be used throughout the subsequent chapters, along with an overview of the thesis structure.

Frobenius pioneered the study of representation of a partition in terms of a 2− rowed array which was further developed by Andrews [3] in his study of generalized Frobenius partitions. The basic object used in such studies is the Ferrers diagram (or Ferrers graph), attributed to N. M. Ferrers by Sylvester [70], which is explained in detail in the following paragraph.

In a *Ferrers diagram*, a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is represented with the help of left justified rows of evenly-spaced dots, where the i^{th} row contains λ_i dots. For example, the Ferrers diagram for the partition (5, 4, 4, 3, 2, 1) of 19 is given by

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \\ \bullet & & & & \end{array}.$$

The *conjugate partition* λ' of a partition λ is obtained by interchanging the rows and columns in the Ferrers diagram. Frobenius introduced a method of representing a partition such that its conjugate can be readily determined. In this method, the dots above and below the main diagonal in the Ferrers diagram of a partition are enumerated by rows and columns, respectively, to obtain two strictly decreasing sequences $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$ of non-negative integers. Subsequently, these sequences are presented as rows in a $2 \times r$ matrix also known as *Frobenius symbol* which is given by

$$\begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}. \tag{1.1.1}$$

For example, the Frobenius symbol representation for the partition $\lambda = (5, 4, 4, 3, 2, 1)$ is $\begin{pmatrix} 4 & 2 & 1 \\ 5 & 3 & 1 \end{pmatrix}$. We can also verify that the Frobenius symbol representation for the conjugate (6, 5, 4, 3, 1) of λ is $\begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, which is a straightforward swap of the rows of Frobenius notation for λ.

The mention of such objects can be found in the study of group representation theory by Frobenius [30]. Littlewood [46] and Robinson [59] employed the Frobenius notation in their study of representation theory of symmetric groups. For additional information on the use of Frobenius symbol, one can see [72, 73, 74].

In an AMS Memoir [3], Andrews introduced the idea of generalized Frobenius partitions, or simply F-partitions, which arise naturally as a combinatorial object associated to elliptic theta functions. A *generalized Frobenius partition* is an array as in (1.1.1) but the entries in the rows are allowed to be non-increasing. Suppose $\phi_k(n)$ denotes the number of F-partitions of n that allow up to k repetitions of an integer in any row. For example, the eleven partitions enumerated by $\phi_3(4)$ are

$$\begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}20\\00\end{pmatrix}, \begin{pmatrix}00\\20\end{pmatrix}, \begin{pmatrix}11\\00\end{pmatrix}, \begin{pmatrix}00\\11\end{pmatrix}, \begin{pmatrix}10\\10\end{pmatrix}, \begin{pmatrix}100\\000\end{pmatrix}, \text{ and } \begin{pmatrix}000\\100\end{pmatrix}.$$

Andrews gave the generating function of $\phi_k(n)$ and obtained elegant q-product representations for the generating functions for $\phi_1(n)$, $\phi_2(n)$, and $\phi_3(n)$.
In [3], it was also proved that, for all $n \geq 0$,

$$\phi_2(5n+3) \equiv 0 \pmod 5.$$

Andrews also considered another general class of F-partitions, enumerated by $c\phi_k(n)$, using k-copies $j_1, j_2, \cdots, j_k$ of each nonnegative integer j with strict decrease in each row. An order relation between two copies j_i and l_h is defined by "$j_i < l_h$ if and only if $j < l$ or $j = l$ and $i < h$". Also j_i is said to be distinct from l_h unless $j = l$ and $i = h$.

Since the publication of the Memoir [3], generalized Frobenius partitions emerged as a rich area of study, with several researchers discovering congruences for $\phi_k(n)$ and $c\phi_k(n)$. For example, Sellers [66] established that, for all $n \geq 0$,

$$\phi_3(3n+2) \equiv 0 \pmod 3.$$

Recently, Andrews et.al. [5] proved that, for all $n \geq 0$,

$$\phi_{pl-1}(pn+r) \equiv 0 \pmod 2,$$

where $p \geq 5$ is a prime, l is a positive integer and $0 < r < p$ such that $24r+1$ is quadratic nonresidue modulo p.

Chapter 2 of this thesis is mainly devoted to representations of the generating functions for $\phi_k(n)$, $k = 4, 7, 8, 11, 15$. We also prove a few congruences modulo small powers of 2 and 5 for $\phi_4(n)$ in Section 2.4. Moreover, we establish two infinite families of congruences modulo 2 satisfied by $\phi_8(n)$ in Section 2.7.

We now define a modular equation as introduced by Ramanujan. For $0 < k < 1$, the complete elliptic integral of the first kind associated with the modulus k, is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The number $k' := \sqrt{1-k^2}$ is called the complementary modulus. One of the classical results in the theory of elliptic functions asserts that [15, p. 101]

$$z := \varphi^2(q) = \frac{2}{\pi} K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \tag{1.1.2}$$

where ${}_2F_1(a, b; c; z)$, $|z| < 1$, denotes the ordinary hypergeometric series, $\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$,

$$q := \exp\left(-\pi \frac{K'}{K}\right) \quad \text{with} \quad K' = K(k'), \tag{1.1.3}$$

and

$$k = \sqrt{1 - \frac{\varphi^4(-q)}{\varphi^4(q)}}. \tag{1.1.4}$$

Let $K,\ K',\ L$, and L' denote complete elliptic integrals of the first kind associated with the moduli $k,\ k',\ l$, and l', respectively. Suppose that

$$n \frac{K'}{K} = \frac{L'}{L} \tag{1.1.5}$$

for some positive integer n. A relation between k and l induced by (1.1.5) is called a modular equation of degree n. Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. We often say that β has degree n over α. If we set $z_n := \varphi^2(q^n)$, then the multiplier m of degree n is defined by $m := \dfrac{z_1}{z_n}$. In his notebooks [56], Ramanujan recorded more than 100 modular equations and in his lost notebook [57], Ramanujan recorded additional modular equations.

In Chapter 3 of this thesis, we find two representations for the generating function for $\phi_5(n)$ in terms of q-products. Equating these two representations, we derive two mixed modular equations for the quadruple of degrees 1, 3, 5, and 15.

Let $c\phi_{k,\,h}(n)$ represent the number of F-partitions of a positive integer n where each part may appear at most h times, is taken from k copies of the non-negative integers, and a strict order relation between the colored parts is maintained. For example, the four partitions enumerated by $c\phi_{2,\,2}(1)$ are

$$\begin{pmatrix}0_1\\0_1\end{pmatrix},\ \begin{pmatrix}0_2\\0_1\end{pmatrix},\ \begin{pmatrix}0_1\\0_2\end{pmatrix},\ \text{and}\ \begin{pmatrix}0_2\\0_2\end{pmatrix}.$$

It is easy to see that the function $c\phi_{k,\,h}(n)$ indeed is a generalisation of the functions $\phi_k(n)$ and $c\phi_k(n)$, since $\phi_k(n) = c\phi_{1,\,k}(n)$ and $c\phi_k(n) = c\phi_{k,\,1}(n)$.

Padmavathamma [53] outlined a method for obtaining representations of the generating functions for $c\phi_{k,\,h}(n)$ for arbitrary positive integers k and h in terms of infinite products, more precisely for the functions $c\phi_{2,\,2}(n)$ and $c\phi_{2,\,3}(n)$. As an illustration of this work, we include the following result.

$$\sum_{n=0}^{\infty} c\phi_{2,2}(n)q^n = A_0(q)^2\left(q^4;q^4\right)_\infty\left(-q^2;q^4\right)_\infty^2 + 2q^{-1}(qB_0(q))^2\left(q^4;q^4\right)_\infty\left(-q^4;q^4\right)_\infty^2,$$

where $A_0(q) = \Phi_2(q)$, the generating function for F-partitions with 2 repetitions and $qB_0(q)$ is the generating function for symbols of the form

$$\begin{pmatrix}\alpha_1 & \alpha_2 & . & . & . & \alpha_r & \alpha_{r+1}\\ \beta_1 & \beta_2 & . & . & . & \beta_r & \end{pmatrix},$$

which is very much similar to that of $\Phi_2(q)$ with the additional condition that there is an extra element in the top row.

In Sections 4.3 and 4.5, we derive representations for the generating functions of $c\phi_{2,\,2}(n)$ and $c\phi_{2,\,3}(n)$ followed by congruences modulo small powers of 2 and 3 for these functions in Sections 4.4 and 4.6.

Chapter 5 is dedicated to a more generalized class of F-partitions called (k,a)-*colored F-partitions*, enumerated by $c\psi_{k,\,a}(n)$, which is a two rowed array of the form

$$\begin{pmatrix}a_1 & a_2 & \dots & a_r\\ b_1 & b_2 & \dots & b_s\end{pmatrix}$$

such that

$$n = r + \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j,$$

where the parts a_i, b_j are taken from k copies of nonnegative integers, each row is decreasing with respect to the lexicographical ordering and $r - s = a - \frac{k}{2}$, $(r, s) \neq (0, 0)$. It is observed that there is a difference in the number of parts in the top and bottom row which is given by $a - \frac{k}{2}$. In the case, when this difference is zero, that is, $a = \frac{k}{2}$, we have $c\psi_{k,\, k/2}(n) = c\phi_k(n)$.
The notion of $(k, a)-$colored F-partitions was introduced by Jiang, Rolen, and Woodbury in [37], where they established a bijection between (k, a)-colored F-partitions and equivalence classes of (k, a)-Motzkin paths. Recently, Sellers and Eichhorn [28] found some interesting results for the function $c\psi_{2,\,0}(n)$ which is equivalent to the Drake's [26] function $\psi_{2l}(n)$. In our work, we derive q-product representations for the generating functions of $c\psi_{2,0}(n)$, $c\psi_{3,\,\frac{1}{2}}(n)$, $c\psi_{4,\,0}(n)$, $c\psi_{4,\,1}(n)$, $c\psi_{6,\,0}(n)$, $c\psi_{6,\,1}(n)$, $c\psi_{6,\,2}(n)$, dissect these generating functions, and obtain a few congruences satisfied by these functions.

In Chapter 6, we discuss a restricted family of F-partitions, enumerated by $a_{k,\,i}(n)$, which counts the number of F-partitions with $k-$colors for any positive integer n with the restriction that there are no odd parts of some i colors in the top row and no even parts of these colors in the bottom row. For example, the F-partitions enumerated by $a_{3,\,1}(1)$ are

$$\binom{0_1}{0_2}, \binom{0_1}{0_3}, \binom{0_2}{0_2}, \binom{0_3}{0_3}, \binom{0_3}{0_2}, \binom{0_2}{0_3}.$$

The idea of such restricted F-partitions originated in chapter 7 of [4], where a representation for the generating function of $a_{2,\,1}(n)$ was given with an alternate notation $a(n)$. Padmavathamma [52] continued the work and obtained representations for the generating functions of $a_{k,\,i}(n)$ for $k = 2, 3$ and $i < k$.

In Sections 6.2, 6.3, and 6.4 of this chapter we find representations for the generating functions of $a_{2,\,1}(n)$, $a_{3,\,i}(n), i < 3$, and $a_{4,\,2}(n)$, respectively. Additionally, we obtain few congruences modulo small powers of 2 for these functions in Section 6.5.

A t-dissection of any power series $A(q)$ is given by $A(q) = \sum_{k=0}^{t-1} q^k A_k(q^t)$, where $A_k(q^t)$ are power series in q^t.

In the final chapter of the thesis, that is, Chapter 7 we present an elementary proof of 2^n-dissection formula for $E(q) := (q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$ as conjectured by Hirschhorn [36, p. 332]. Our proof relies upon sum to product identities for Ramanujan's general theta functions as explained by Cao in [20].

Andrews [3] devised the following general principle for determining the generating functions for F-partitions.

Andrews' General Principle 1.1.1. *If $f_A(z) := f_A(z,q) = \sum P_A(m,n)z^m q^n$ denotes the generating function for $P_A(m,n)$, the number of ordinary partitions of n into m parts subject to the set of restriction A, then $f_A(zq)f_B(z^{-1})$ has as its constant term (coefficient of z^0) the generating function*

$$\Phi_{A,B}(q) := \textstyle\sum_{n\geq 0}\phi_{A,B}(n)q^n,$$

where $\phi_{A,B}(n)$ is the number of F-partitions of n of the form (1.1.1) *in which the top row is subject to the set of restrictions A and the bottom row is subject to the set of restrictions B.*

In Chapters 2–6, we use Andrews' general principle 1.1.1 extensively to obtain the generating functions for various F-partitions.

We conclude this section with a brief discussion of an integer matrix exact covering system as described by Cao [20].

An *exact covering system* is a partition of the set of integers into a finite set of arithmetic sequences. An *integer matrix exact covering system* is a partition of $\mathbb{Z}^n$ (The set of all n-tuples with entries from $\mathbb{Z}$) into a lattice and a finite number of its translates without overlap.

Let

$$S = \sum_{x_1,\, x_2,\, \cdots,\, x_n=-\infty}^{\infty} f(x_1,\, x_2,\, \cdots,\, x_n).$$

We change the variables from x_i to y_i by the transformation $y = Ax$, where A is an integer matrix with $\det A \neq 0$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$. Then as given in [8], S can be written as a linear combination of k parts using the integer matrix exact covering system $\left\{By + \frac{1}{d}Bc_r\right\}_{r=0}^{k-1}$ for $\mathbb{Z}^n$, where $B = \text{sgn}(s_n(A))\dfrac{A^*}{d_{n-1}(A)}$ with $A^* =$ adjoint of A, $d_k(A) = k^{th}$ determinantal divisor of A, $s_n(A) = \frac{d_n(A)}{d_{n-1}(A)}$, $d = |s_n(A)|$ and $y \equiv c_r \pmod{d}$, $r = 0, 1, \cdots, k-1$ is the solution set of $By \equiv 0 \pmod{d}$.

Several integer matrix exact covering systems were developed and used by Baruah and Sarmah [8, 10] to obtain the generating functions for F-partitions with 4 and 6 colors respectively. In our work, we use those covering systems together with few new covering systems for $\mathbb{Z}^n$ with small values of n using the procedure for obtaining series-product identities given by Cao [20, 21]. In the next section, we introduce Ramanujan's general theta functions and record few useful identities related to these functions.

1.2 Ramanujan's general theta functions

Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \tag{1.2.1}$$

which is equivalent to Jacobi's classical theta function

$$\vartheta_3(z,q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}, \quad \text{with} \quad |q| < 1.$$

In fact

$$f(a,b) = \vartheta_3(z,q), \quad \text{where} \quad a = qe^{2iz},\ b = qe^{-2iz}.$$

For $z \neq 0$ and $|q| < 1$, Jacobi's triple product identity is given by

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \left(-zq;\, q^2\right)_\infty \left(-q/z;\, q^2\right)_\infty \left(q^2;\, q^2\right)_\infty, \tag{1.2.2}$$

which can be recasted in the form

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\frac{n(n+1)}{2}} = (zq;\, q)_\infty \left(z^{-1};\, q\right)_\infty (q;\, q)_\infty . \tag{1.2.3}$$

Using Ramanujan's general theta function, we can rewrite (1.2.2) in the form

$$f(a,b) = (-a;\, ab)_\infty(-b;\, ab)_\infty(ab;\, ab)_\infty. \tag{1.2.4}$$

It is easy to verify that

$$\begin{aligned} f(a,b) &= af(a^{-1},a^2b), \\ f(a,b) &= f(b,a), \\ f(1,a) &= 2f(a,a^3), \\ f(-1,a) &= 0. \end{aligned} \tag{1.2.5}$$

We also use the following two special cases of $f(a,\, b)$:

$$\varphi(q) := f(q,\, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.2.6}$$

$$\psi(q) := f(q,\, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}, \tag{1.2.7}$$

where, here and throughout the thesis, $f_k := (q^k;\, q^k)_\infty$. The product notation used in the above two identities appear from Jacobi's triple product identity (1.2.4). Further, it is easy to note that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}, \tag{1.2.8}$$

$$\varphi(-q) = f(-q,-q) = (q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_1^2}{f_2}. \tag{1.2.9}$$

After Ramanujan, we also define

$$\chi(q) := (-q;\, q^2)_\infty = \frac{f_2^2}{f_1 f_4}. \tag{1.2.10}$$

We also use the following identities involving Ramanujan's theta functions.

Lemma 1.2.1. *For $|ab| < 1$, we have*

$$f(a,b) = f(a^3b,ab^3) + af(\frac{b}{a},a^5b^3). \tag{1.2.11}$$

Proof. For proof see [20, p. 3, Entry 1.1]. □

Lemma 1.2.2. *For $ab = cd$,*

$$f(a,\, b)f(c,\, d) = f(ad,\, bc)f(ac,\, bd) + af(c/a,\, a^2bd)f(d/a,\, a^2bc). \tag{1.2.12}$$

Proof. For proof see [15, p. 45, Entry 29]. □

Lemma 1.2.3. *We have*

$$f(q, q^5) = \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6}, \tag{1.2.13}$$

$$f(q, q^2) = \frac{f_2 f_3^2}{f_1 f_6}. \tag{1.2.14}$$

Proof. The proofs follows from direct applications of Jacobi's triple product identity (1.2.4). □

Lemma 1.2.4. *The following 2-dissections hold.*

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \tag{1.2.15}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{1.2.16}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{1.2.17}$$

Proof. Identity (1.2.15) is Equation (1.10.1) in [36], whereas identity (1.2.16) is simply another variant of (1.2.15). Identity (1.2.17) is equivalent to Equation (1.9.4) in [36]. □

Lemma 1.2.5. *The following 2-dissections hold.*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{1.2.18}$$

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}, \tag{1.2.19}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \tag{1.2.20}$$

Proof. Identities (1.2.18) and (1.2.19) have been proved by Xia and Yao [79] whereas identity (1.2.20) is Equation (30.13.1) in [36]. □

Lemma 1.2.6. *We have*

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} = 3\frac{f_3^3}{f_1}. \tag{1.2.21}$$

Proof. Identity (1.2.21) is obtained by combining (22.3.1) and (22.1.7) in [36]. □

Lemma 1.2.7. *The following 5-dissection holds.*

$$\varphi(q) = a + 2qb + 2q^4c, \tag{1.2.22}$$

where $a = \varphi(q^{25}),\ b = f(q^{15}, q^{35}),\ c = f(q^5, q^{45})$.

Proof. Identity (1.2.22) is Equation (36.3.2) in [36]. □

Lemma 1.2.8. *The following 5-dissection hold.*

$$f_1 = f_{25}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right), \tag{1.2.23}$$

where

$$R(q) := \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

Proof. Identity (1.2.23) is given by Equation (8.1.1) in [36]. □

Lemma 1.2.9. *We have*

$$f_1^3 \equiv J_0 + J_1 \pmod 5, \tag{1.2.24}$$

where $J_0 = (q^{10},\, q^{15},\, q^{25};\, q^{25})_\infty$, $J_1 = -3q(q^5,\, q^{20},\, q^{25};\, q^{25})_\infty$, *and*

$$(a_1,\, a_2,\, \ldots a_n;\, q)_\infty = (a_1;\, q)_\infty(\, a_2;\, q)_\infty \ldots (a_n;\, q)_\infty. \tag{1.2.25}$$

Proof. Identity (1.2.24) is given by Equation (3.2.6) in [36]. □

Lemma 1.2.10. *The following identities hold for* $\varphi(q)$.

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \tag{1.2.26}$$

$$\varphi(q) + \varphi(q^2) = 2\frac{(q^3,\, q^5,\, q^8;\, q^8)_\infty}{(q,\, q^4,\, q^7;\, q^8)_\infty}, \tag{1.2.27}$$

$$\varphi(q) + \varphi(q^5) = 2\frac{(q^2,\, q^8,\, q^{10},\, q^{12},\, q^{18},\, q^{20};\, q^{20})_\infty}{(q,\, q^4,\, q^9,\, q^{11},\, q^{16},\, q^{19};\, q^{20})_\infty}, \tag{1.2.28}$$

$$\varphi^2(q) - \varphi^2(q^5) = 4q(-q,\, -q^3,\, -q^7,\, -q^9,\, q^{10},\, q^{10};\, q^{10})_\infty. \tag{1.2.29}$$

Proof. Equations (1.2.26), (1.2.27), (1.2.28), and (1.2.29) are Equations (1.10.1), (34.1.1), (34.1.7), and (34.1.20) in [36] respectively. □