Chapter 2

Generating Functions and Congruences for F-partitions

2.1 Introduction[1]

In Section 1.1 we discussed F-partitions and two particular cases of it enumerated by $\phi_k(n)$ and $c\phi_k(n)$. Andrews [3] gave the following general form for the generating function $\Phi_k(q)$ of $\phi_k(n)$.

$$\Phi_k(q) = \frac{\displaystyle\sum_{m_1,\, m_2,\, \ldots,\, m_{k-1}=-\infty}^{\infty} \zeta^{(k-1)m_1+(k-2)m_2+\ldots+m_{k-1}} q^{Q(m_1,\, m_2,\, \ldots,\, m_{k-1})}}{(q;\, q)_\infty^k}, \tag{2.1.1}$$

where

$$Q(m_1,\, m_2,\, \ldots,\, m_{k-1}) = m_1^2 + m_2^2 + \ldots + m_{k-1}^2 + \sum_{1\le i<j\le k-1} m_i m_j, \tag{2.1.2}$$

and $\zeta = e^{2\pi i/(k+1)}$.

In particular, Andrews found the following elegant infinite product representations for $\Phi_1(q)$, $\Phi_2(q)$, and $\Phi_3(q)$.

$$\Phi_1(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$
$$\Phi_2(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{12n-2})(1-q^{12n-3})(1-q^{12n-9})(1-q^{12n-10})},$$

[1]Some contents of Sections 2.3 and 2.4 appeared in *Integers* [60].

$$\Phi_3(q) = \prod_{n=1}^{\infty} \frac{(1-q^{12n-6})}{(1-q^{6n-1})(1-q^{6n-2})^2(1-q^{6n-3})^3(1-q^{6n-4})^2(1-q^{6n-5})(1-q^{12n})}.$$

For the second general class of F-partitions, k-copies $j_1, j_2, \cdots, j_k$ of each non-negative integer j are considered and an order relation between two copies j_i and l_h is defined by "$j_i < l_h$ if and only if $j < l$ or $j = l$ and $i < h$". Also j_i is said to be distinct from l_h unless $j = l$ and $i = h$. Further, $c\phi_k(n)$ represents the number of F-partitions of n using these k-copies of integers with strict decrease in each row.

Moreover, the generating function for $c\phi_k(n)$ is given by [3],

$$\sum_{n=0}^{\infty} c\phi_k(n) q^n = \frac{\displaystyle\sum_{m_1, m_2, \ldots, m_{k-1}=-\infty}^{\infty} q^{Q(m_1, m_2, \ldots, m_{k-1})}}{(q;\, q)_\infty^k},$$

where $Q(m_1, m_2, \ldots, m_{k-1})$ is defined in (2.1.2).

In [3], it was also proved that

$$\phi_2(5n+3) \equiv c\phi_2(5n+3) \equiv 0 \pmod 5,$$

$$c\phi_k(n) \equiv 0 \pmod{k^2} \text{ if } k \text{ is a prime and does not divide } n.$$

Since the publication of the Memoir [3] a number of authors worked on these partition functions and uncovered a host of congruences mostly for $c\phi_k(n)$. For example, Sellers [68] established that

$$\phi_3(3n+2) \equiv 0 \pmod 3.$$

Lovejoy [47] established modular forms whose Fourier coefficients are related to $c\phi_3(n)$ and proved the following congruences modulo 5, 7, 11, and 19 for $c\phi_3(n)$:

$$c\phi_3(45n+23) \equiv 0 \pmod 5,$$

$$c\phi_3(45n+41) \equiv 0 \pmod 5,$$

$$c\phi_3(63n+50) \equiv 0 \pmod 7,$$

$$c\phi_3(99n+95) \equiv 0 \pmod{11},$$

$$c\phi_3(171n+50) \equiv 0 \pmod{19}.$$

Baruah and Sarmah [8] represented the generating function for $c\phi_4(n)$ in terms of q-products and established the following congruences modulo powers of 4 for $c\phi_4(n)$:

$$c\phi_4(2n+1) \equiv 0 \pmod{4^2},$$

$$c\phi_4(4n+3) \equiv 0 \pmod{4^4},$$

$$c\phi_4(4n+2) \equiv 0 \pmod{4}.$$

Xia [77] proved the following congruences modulo 5 for $c\phi_4(n)$:

$$c\phi_4(20n+11) \equiv 0 \pmod{5}.$$

Hirschhorn and Sellers [35] proved the following characterization of $c\phi_4(10n+1)$ modulo 5:

$$c\phi_4(10n+1) \equiv \begin{cases} k+1 & \pmod{5} \text{ if } n = k(3k+1) \text{ for some integer } k, \\ 0 & \pmod{5} \text{ otherwise.} \end{cases}$$

From the above characterization they found the following infinite set of Ramanujan-type congruences modulo 5 satisfied by $c\phi_4(n)$: Let $p \geq 5$ be prime and let r be an integer, $1 \leq r \leq p-1$, such that $12r+1$ is a quadratic non-residue modulo p. Then, for all $n \geq 0$,

$$c\phi_4(10pn+10r+1) \equiv 0 \pmod{5}.$$

Garvan and Sellers [31] proved several infinite families of congruences for $c\phi_k(n)$, where k is allowed to grow arbitrarily large. In particular, they proved that, if p is a prime, r is an integer such that $0 < r < p$ and if

$$c\phi_k(pn+r) \equiv 0 \pmod{p}$$

for all $n \geq 0$, then

$$c\phi_{pN+k}(pn+r) \equiv 0 \pmod{p}$$

for all $N \geq 0$ and $n \geq 0$.

As a corollary, they proved a number of congruences modulo 3, 5, 7, and 11 for

$c\phi_{pN+k}(pn+r)$ for $p = 3,\ 5,\ 7,$ and 11 and particular values of k.

For some other congruences and families of congruences involving generalized Frobenius partition we refer to [1, 10, 22, 24, 25, 28, 34, 40, 41, 42, 45, 51, 54, 69, 71, 76, 78].

Kolitsch [38, 39] introduced the function $\overline{c\phi}_k(n)$, which denotes the number of F-partitions of n with k colors whose order is k under cyclic permutation of the k-colors. For example, the F-partitions enumerated by $\overline{c\phi}_2(2)$ are $\begin{pmatrix}1_r\\0_r\end{pmatrix}$, $\begin{pmatrix}1_g\\0_r\end{pmatrix}$, $\begin{pmatrix}1_r\\0_g\end{pmatrix}$, $\begin{pmatrix}1_g\\0_g\end{pmatrix}$, $\begin{pmatrix}0_r\\1_r\end{pmatrix}$, $\begin{pmatrix}0_r\\1_g\end{pmatrix}$, $\begin{pmatrix}0_g\\1_r\end{pmatrix}$ and $\begin{pmatrix}0_g\\1_g\end{pmatrix}$, where the subscripts represent the two colors viz. red and green of the non-negative integers. The generating function for $\overline{c\phi}_k(n)$ is given by [39],

$$\sum_{n=0}^{\infty} \overline{c\phi}_k(n)q^n = \frac{k\sum q^{Q(\mathbf{m})}}{(q;\,q)_\infty^k},$$

where the sum of the right extends over all vectors $\mathbf{m} = (m_1,\, m_2,\, \ldots,\, m_k)$ with $\mathbf{m}.\overline{1} = 1$ and $Q(\mathbf{m}) = \frac{1}{2}\sum_{i=1}^{k}(m_i - m_{i+1})^2$ wherein $\overline{1}$=$(1, 1, \ldots, 1)$ and $m_{k+1} = m_1$. Kolitsch [38] found that, for all integers $k \geq 2$,

$$\overline{c\phi}_k(n) \equiv 0 \pmod{k^2}.$$

Sellers [66, 67] established that

$$\overline{c\phi}_k(kn) \equiv 0 \pmod{k^3} \text{ for } k = 2,\ 3,\ 5,\ 7, \text{ and } 11.$$

The above results are then generalised by Kolitsch [44].

Baruah and Sarmah [8] established the following congruences modulo powers of 4 for $\overline{c\phi}_4(n)$:

$$\overline{c\phi}_4(2n) \equiv 0 \pmod{4^3},$$
$$\overline{c\phi}_4(4n+3) \equiv 0 \pmod{4^4},$$
$$\overline{c\phi}_4(4n) \equiv 0 \pmod{4^4}.$$

Existence of such wide variety of results for $c\phi_k(n)$ and $\overline{c\phi}_k(n)$ for various values of k motivates us to investigate the functions $\phi_k(n)$, and to search for new results. A key feature of this chapter is the representation of the generating function for

$\phi_k(n)$ for various values of k in terms of q-products and dissection of some of them to obtain congruences.

A key tool employed in this study is integer matrix exact covering system, as described in Section 1.1. We also use identities involving Ramanujan's general theta functions and Jacobi's triple product identity, the details of which are presented in Section 1.2.

2.2 Preliminaries

In this section, we list few lemmas that play important roles in the proofs of our main results.

Lemma 2.2.1. *The following 2-dissection holds.*

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}. \tag{2.2.1}$$

Proof. Identity (2.2.1) is Equation (3.41) in [75]. □

Lemma 2.2.2. *The following 2-dissection holds.*

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{2.2.2}$$

Proof. For proof, see [33, 50]. □

Lemma 2.2.3. *The following 2-dissection holds.*

$$\frac{1}{f_1^3 f_5} = \frac{f_4^4}{f_2^7 f_{10}} - 2q\frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q\frac{f_4^3 f_{20}}{f_2^8} + 2q^2\frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}. \tag{2.2.3}$$

Proof. For proof, see [49]. □

Lemma 2.2.4. *The following identities hold.*

$$\frac{f_5^5}{f_1^4 f_{10}^3} = \frac{f_5}{f_2^2 f_{10}} + 4q\frac{f_{10}^2}{f_1^3 f_2}, \tag{2.2.4}$$

$$\frac{f_2^3 f_5^2}{f_1^5 f_{10}^2} = \frac{f_5}{f_2^2 f_{10}} + 5q\frac{f_{10}^2}{f_1^3 f_2}. \tag{2.2.5}$$

Proof. For proofs, see [13]. □

2.3 Generating function for $\phi_4(n)$

Theorem 2.3.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \phi_4(2n)q^n = \frac{f_2^{10} f_5}{f_1^9 f_4^2 f_{10}} + 2q\frac{f_2^{15} f_{20}^2}{f_1^{12} f_4^4 f_{10}^2} - 8q\frac{f_4^6 f_{10}}{f_1^7 f_5}, \tag{2.3.1}$$

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n = \frac{f_2^7 f_{10}^2}{f_1^6 f_4^2 f_5^2}. \tag{2.3.2}$$

Proof. From (2.1.1), we have

$$\Phi_4(q) = \frac{S_4}{(q;\, q)_\infty^4}, \tag{2.3.3}$$

where $S_4 = \sum_{m_1,\, m_2,\, m_3=-\infty}^{\infty} \zeta^{3m_1+2m_2+m_3} q^{m_1^2+m_2^2+m_3^2+m_1m_2+m_2m_3+m_1m_3}$ with $\zeta = e^{2\pi i/5}$.

Using the integer matrix exact covering system

$$\left\{ B\overline{n}, B\overline{n} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, B\overline{n} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, B\overline{n} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \tag{2.3.4}$$

where

$B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ and $\overline{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, obtained in [8], we can write S_4 as a linear combination of four parts as

$$\begin{aligned} S_4 = &\sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} \zeta^{2n_2+4n_3}\, q^{2n_1^2+2n_2^2+2n_3^2} \\ &+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} \zeta^{3+2n_2+4n_3}\, q^{2n_1^2+2n_2^2+2n_3^2+2n_2+2n_3+1} \\ &+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} \zeta^{2+2n_2+4n_3}\, q^{2n_1^2+2n_2^2+2n_3^2+2n_1+2n_3+1} \\ &+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} \zeta^{1+2n_2+4n_3}\, q^{2n_1^2+2n_2^2+2n_3^2+2n_1+2n_2+1} \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2}\right)\left(\sum_{n_2=-\infty}^{\infty} \zeta^{2n_2} q^{2n_2^2}\right)\left(\sum_{n_3=-\infty}^{\infty} \zeta^{4n_3} q^{2n_3^2}\right) \\
&\quad + q\zeta^3\left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2}\right)\left(\sum_{n_2=-\infty}^{\infty} \zeta^{2n_2} q^{2n_2^2+2n_2}\right)\left(\sum_{n_3=-\infty}^{\infty} \zeta^{4n_3} q^{2n_3^2+2n_3}\right) \\
&\quad + q\zeta^2\left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2+2n_1}\right)\left(\sum_{n_2=-\infty}^{\infty} \zeta^{2n_2} q^{2n_2^2}\right)\left(\sum_{n_3=-\infty}^{\infty} \zeta^{4n_3} q^{2n_3^2+2n_3}\right) \\
&\quad + q\zeta\left(\sum_{n_1=-\infty}^{\infty} q^{2n_1^2+2n_1}\right)\left(\sum_{n_2=-\infty}^{\infty} \zeta^{2n_2} q^{2n_2^2+2n_2}\right)\left(\sum_{n_3=-\infty}^{\infty} \zeta^{4n_3} q^{2n_3^2}\right). \qquad (2.3.5)
\end{aligned}
$$

Now using Jacobi's triple product identity (1.2.2), we have

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \zeta^k q^{k^2} &= \prod_{k\geq 1}\left(1+\zeta^{-1}q^{2k-1}\right)\left(1+\zeta q^{2k-1}\right)\left(1-q^{2k}\right) \\
&= \left(-\zeta^{-1}q;\, q^2\right)_\infty \left(-\zeta q;\, q^2\right)_\infty \left(q^2;\, q^2\right)_\infty \\
&= f(\zeta q,\, \zeta^4 q). \qquad (2.3.6)
\end{aligned}
$$

From the second version of Jacobi's triple product identity (1.2.3), we obtain

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} (-1)^k \zeta^k q^{\frac{k^2+k}{2}} &= \left(\zeta^{-1};\, q\right)_\infty (\zeta q;\, q)_\infty (q;\, q)_\infty \\
&= f(-\zeta^{-1},\, -\zeta q).
\end{aligned}
$$

Therefore, we have

$$
\sum_{k=-\infty}^{\infty} \zeta^k q^{\frac{k^2+k}{2}} = f(\zeta^{-1},\, \zeta q) = f(\zeta^4,\, \zeta q). \qquad (2.3.7)
$$

Using (2.3.6) and (2.3.7), we have

$$
\sum_{n=-\infty}^{\infty} \zeta^{2n} q^{2n^2} = f(\zeta^2 q^2,\, \zeta^8 q^2) = f(\zeta^2 q^2,\, \zeta^3 q^2), \qquad (2.3.8)
$$

$$
\sum_{n=-\infty}^{\infty} \zeta^{4n} q^{2n^2} = f(\zeta^4 q^2,\, \zeta^{16} q^2) = f(\zeta^4 q^2,\, \zeta q^2), \qquad (2.3.9)
$$

$$
\sum_{n=-\infty}^{\infty} \zeta^{2n} q^{2n^2+2n} = f(\zeta^8,\, \zeta^2 q^4) = f(\zeta^3,\, \zeta^2 q^4), \qquad (2.3.10)
$$

$$
\sum_{n=-\infty}^{\infty} \zeta^{4n} q^{2n^2+2n} = f(\zeta^{16},\, \zeta^4 q^4) = f(\zeta,\, \zeta^4 q^4). \qquad (2.3.11)
$$

We recall (1.2.6) and (1.2.7) to note that

$$\varphi(q) := f(q,\, q) = \sum_{k=-\infty}^{\infty} q^{k^2}, \tag{2.3.12}$$

$$\psi(q) := f(q,\, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{1}{2}\sum_{k=-\infty}^{\infty} q^{k(k+1)/2}. \tag{2.3.13}$$

Using (2.3.8), (2.3.9), (2.3.10), (2.3.11), (2.3.12), and (2.3.13) in (2.3.5), we have

$$\begin{aligned} S_4 &= \varphi(q^2) f(\zeta^2 q^2,\, \zeta^3 q^2) f(\zeta^4 q^2,\, \zeta q^2) + q\zeta^3 \varphi(q^2) f(\zeta^3,\, \zeta^2 q^4) f(\zeta,\, \zeta^4 q^4) \\ &\quad + 2q\zeta^2 \psi(q^4) f(\zeta^2 q^2,\, \zeta^3 q^2) f(\zeta,\, \zeta^4 q^4) + 2q\zeta \psi(q^4) f(\zeta^4 q^2,\, \zeta q^2) f(\zeta^3,\, \zeta^2 q^4). \end{aligned} \tag{2.3.14}$$

Now

$$\begin{aligned} & f(\zeta^2 q^2,\, \zeta^3 q^2) f(\zeta^4 q^2,\, \zeta q^2) \\ &= \left(-\zeta^2 q^2;\, q^4\right)_\infty \left(-\zeta^3 q^2;\, q^4\right)_\infty \left(-\zeta^4 q^2;\, q^4\right)_\infty \left(-\zeta q^2;\, q^4\right)_\infty \left(q^4;\, q^4\right)_\infty^2 \\ &= \left(-\zeta q^2;\, q^4\right)_\infty \left(-\zeta^2 q^2;\, q^4\right)_\infty \left(-\zeta^3 q^2;\, q^4\right)_\infty \left(-\zeta q^4 q^2;\, q^4\right)_\infty \left(-q^2;\, q^4\right)_\infty \\ &\qquad \times \frac{\left(q^4;\, q^4\right)_\infty^2}{\left(-q^2;\, q^4\right)_\infty} \\ &= \prod_{n=0}^{\infty} \left(1+\zeta q^{4n+2}\right)\left(1+\zeta^2 q^{4n+2}\right)\left(1+\zeta^3 q^{4n+2}\right)\left(1+\zeta^4 q^{4n+2}\right)\left(1+q^{4n+2}\right) \\ &\qquad \times \frac{\left(q^4;\, q^4\right)_\infty^2}{\left(-q^2;\, q^4\right)_\infty} \\ &= \prod_{n=0}^{\infty} \left(1+q^{20n+10}\right) \frac{\left(q^4;\, q^4\right)_\infty^2}{\left(-q^2;\, q^4\right)_\infty} \\ &= \left(-q^{10};\, q^{20}\right)_\infty \left(q^2;\, q^2\right)_\infty \left(q^8;\, q^8\right)_\infty \\ &= \frac{\left(q^{20};\, q^{20}\right)_\infty^2}{\left(q^{40};\, q^{40}\right)_\infty \left(q^{10};\, q^{10}\right)_\infty} \left(q^2;\, q^2\right)_\infty \left(q^8;\, q^8\right)_\infty \\ &= \frac{f_2 f_8 f_{20}^2}{f_{10} f_{40}}. \end{aligned} \tag{2.3.15}$$

Similarly, we find that

$$\begin{aligned} & f(\zeta^3,\, \zeta^2 q^4) f(\zeta,\, \zeta^4 q^4) \\ &= \left(-\zeta^3;\, q^4\right)_\infty \left(-\zeta^2 q^4;\, q^4\right)_\infty \left(-\zeta;\, q^4\right)_\infty \left(-\zeta^4 q^4;\, q^4\right)_\infty \left(q^4;\, q^4\right)_\infty^2 \end{aligned}$$

$$= \prod_{n=0}^{\infty}\left(1+\zeta^3q^{4n}\right)\left(1+\zeta^2q^{4n+4}\right)\left(1+\zeta q^{4n}\right)\left(1+\zeta^4q^{4n+4}\right)\left(q^4;\,q^4\right)_\infty^2$$

$$= (1+\zeta)\left(1+\zeta^3\right)\prod_{n=0}^{\infty}\left(1+\zeta^3q^{4n+4}\right)\left(1+\zeta^2q^{4n+4}\right)\left(1+\zeta q^{4n+4}\right)\left(1+\zeta^4q^{4n+4}\right)$$

$$\times\left(q^4;\,q^4\right)_\infty^2$$

$$= -\frac{1}{\zeta^3}\prod_{n=0}^{\infty}\left(1+q^{20n+20}\right)\frac{\left(q^4;\,q^4\right)_\infty^2}{\left(-q^4;\,q^4\right)_\infty}$$

$$= -\frac{1}{\zeta^3}\left(-q^{20};\,q^{20}\right)_\infty\frac{\left(q^4;\,q^4\right)_\infty^2}{\left(-q^4;\,q^4\right)_\infty}$$

$$= -\frac{1}{\zeta^3}\frac{f_{40}}{f_{20}}\frac{f_4^2f_4}{f_8}$$

$$= -\frac{1}{\zeta^3}\frac{f_4^3f_{40}}{f_8f_{20}}. \qquad (2.3.16)$$

Next, we take

$$A(q) = f(\zeta^3,\,\zeta^2q^4)f(\zeta^4q^2,\,\zeta q^2) + \zeta f(\zeta,\,\zeta^4q^4)f(\zeta^2q^2,\,\zeta^3q^2).$$

Setting $a=\zeta$, $b=\zeta^4q^2$, $c=\zeta^3q^2$, and $d=\zeta^2$ in (1.2.12), we have

$$A(q) = f(\zeta,\,\zeta^4q^2)f(\zeta^2,\,\zeta^3q^2). \qquad (2.3.17)$$

Now

$$f(\zeta,\,\zeta^4q^2)f(\zeta^2,\,\zeta^3q^2)$$

$$= \left(-\zeta;\,q^2\right)_\infty\left(-\zeta^4q^2;\,q^2\right)_\infty\left(-\zeta^2;\,q^2\right)_\infty\left(-\zeta^3q^2;\,q^2\right)_\infty\left(q^2;\,q^2\right)_\infty^2$$

$$= \prod_{n=0}^{\infty}\left(1+\zeta q^{2n}\right)\left(1+\zeta^4q^{2n+2}\right)\left(1+\zeta^2q^{2n}\right)\left(1+\zeta^3q^{2n+2}\right)\left(q^2;\,q^2\right)_\infty^2$$

$$= (1+\zeta)\left(1+\zeta^2\right)\prod_{n=0}^{\infty}\left(1+\zeta q^{2n+2}\right)\left(1+\zeta^4q^{2n+2}\right)\left(1+\zeta^2q^{2n+2}\right)\left(1+\zeta^3q^{2n+2}\right)$$

$$\times\left(q^2;\,q^2\right)_\infty^2$$

$$= -\frac{1}{\zeta}\prod_{n=0}^{\infty}\left(1+q^{10n+10}\right)\frac{\left(q^2;\,q^2\right)_\infty^2}{\left(-q^2;\,q^2\right)_\infty}$$

$$= -\frac{1}{\zeta}\left(-q^{10};\,q^{10}\right)_\infty\frac{\left(q^2;\,q^2\right)_\infty^2}{\left(-q^2;\,q^2\right)_\infty}$$

$$= -\frac{1}{\zeta}\frac{f_{20}}{f_{10}}\frac{f_2^3}{f_4}. \tag{2.3.18}$$

From (2.3.17) and (2.3.18), we have

$$A(q) = f(\zeta^3, \zeta^2 q^4) f(\zeta^4 q^2, \zeta q^2) + \zeta f(\zeta, \zeta^4 q^4) f(\zeta^2 q^2, \zeta^3 q^2) = -\frac{1}{\zeta}\frac{f_{20}}{f_{10}}\frac{f_2^3}{f_4}. \tag{2.3.19}$$

Employing (2.3.15), (2.3.16), and (2.3.19) in (2.3.14), we have

$$\begin{aligned} S_4 &= \varphi(q^2)\frac{f_2 f_8 f_{20}^2}{f_{10} f_{40}} + q\zeta^3\varphi(q^2)\left(-\frac{1}{\zeta^3}\frac{f_4^3 f_{40}}{f_8 f_{20}}\right) + 2q\zeta\psi(q^4)\left(-\frac{1}{\zeta}\frac{f_2^3 f_{20}}{f_4 f_{10}}\right) \\ &= \frac{f_4^5}{f_2^2 f_8^2} \times \frac{f_2 f_8 f_{20}^2}{f_{10} f_{40}} - q\frac{f_4^5}{f_2^2 f_8^2} \times \frac{f_4^3 f_{40}}{f_8 f_{20}} - 2q\frac{f_8^2}{f_4} \times \frac{f_2^3 f_{20}}{f_4 f_{10}} \\ &= \frac{f_4^5 f_{20}^2}{f_2 f_8 f_{10} f_{40}} - q\frac{f_4^8 f_{40}}{f_2^2 f_8^3 f_{20}} - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}. \end{aligned} \tag{2.3.20}$$

Using (2.3.20) in (2.3.3), we find that

$$\begin{aligned} \Phi_4(q) &= \frac{1}{f_1^4}\left(\frac{f_4^5 f_{20}^2}{f_2 f_8 f_{10} f_{40}} - q\frac{f_4^8 f_{40}}{f_2^2 f_8^3 f_{20}} - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}\right) \\ &= \frac{1}{f_1^4}\left\{\frac{f_4^6 f_{10}^2}{f_2^2 f_8^2 f_{20}}\left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}\right) - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}\right\}. \end{aligned} \tag{2.3.21}$$

Using (2.2.2) in (2.3.21), we have

$$\begin{aligned} \Phi_4(q) &= \frac{1}{f_1^4}\left(\frac{f_4^6 f_{10}^2}{f_2^2 f_8^2 f_{20}} \times \frac{f_1}{f_5} - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}\right) \\ &= \frac{f_4^6 f_{10}^2}{f_1^3 f_2^2 f_5 f_8^2 f_{20}} - 2q\frac{f_2^3 f_8^2 f_{20}}{f_1^4 f_4^2 f_{10}}. \end{aligned} \tag{2.3.22}$$

Using (1.2.17) and (2.2.3) in (2.3.22), we have

$$\begin{aligned} \Phi_4(q) &= \frac{f_4^6 f_{10}^2}{f_2^2 f_8^2 f_{20}}\left(\frac{f_4^4}{f_2^7 f_{10}} - 2q\frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q\frac{f_4^3 f_{20}}{f_2^8} + 2q^2\frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}\right) \\ &\quad - 2q\frac{f_2^3 f_8^2 f_{20}}{f_4^2 f_{10}}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right) \\ &= \left(\frac{f_4^{10} f_{10}}{f_2^9 f_8^2 f_{20}} + 2q^2\frac{f_4^{15} f_{40}^2}{f_2^{12} f_8^4 f_{20}^2} - 8q^2\frac{f_8^6 f_{20}}{f_2^7 f_{10}}\right) \\ &\quad + \left(5q\frac{f_4^9 f_{10}^2}{f_2^{10} f_8^2} - 2q\frac{f_4^{12} f_{20}}{f_2^{11} f_8^2 f_{10}} - 2q\frac{f_4^{12} f_{20}}{f_2^{11} f_8^2 f_{10}}\right). \end{aligned} \tag{2.3.23}$$

Extracting the terms involving q^{2n} in (2.3.23) and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}\phi_4(2n)q^n = \frac{f_2^{10}f_5}{f_1^9f_4^2f_{10}} + 2q\frac{f_2^{15}f_{20}^2}{f_1^{12}f_4^4f_{10}^2} - 8q\frac{f_4^6f_{10}}{f_1^7f_5},$$

which is (2.3.1).

Similarly, extracting the terms involving q^{2n+1} in (2.3.23), dividing both sides of the resulting identity by q, and replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}\phi_4(2n+1)q^n = 5\frac{f_2^9f_5^2}{f_1^{10}f_4^2} - 4\frac{f_2^{12}f_{10}}{f_1^{11}f_4^2f_5}. \tag{2.3.24}$$

Now multiplying (2.2.4) by 5 and (2.2.5) by 4, then subtracting the resulting equations, we have

$$5\frac{f_5^5}{f_1^4f_{10}^3} - 4\frac{f_2^3f_5^2}{f_1^5f_{10}^2} = \frac{f_5}{f_2^2f_{10}}. \tag{2.3.25}$$

Multiplying (2.3.25) by $\dfrac{f_2^9f_{10}^3}{f_1^6f_4^2f_5^3}$, we obtain

$$5\frac{f_2^9f_5^2}{f_1^{10}f_4^2} - 4\frac{f_2^{12}f_{10}}{f_1^{11}f_4^2f_5} = \frac{f_2^7f_{10}^2}{f_1^6f_4^2f_5^2}. \tag{2.3.26}$$

Using (2.3.26) in (2.3.24), we find that

$$\sum_{n=0}^{\infty}\phi_4(2n+1)q^n = \frac{f_2^7f_{10}^2}{f_1^6f_4^2f_5^2},$$

which is (2.3.2). □

2.4 Congruences for $\phi_4(n)$

Corollary 2.4.1. *For $n \geq 0$, we have*

$$\phi_4(4n+3) \equiv 0 \pmod{2}. \tag{2.4.1}$$

Proof. Using (1.2.6) and (1.2.9), we rewrite (2.3.2) as

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n = \frac{f_2^7 f_{10}^2}{f_1^6 f_4^2 f_5^2} = \frac{\varphi(q)}{\varphi^2(-q)} \frac{f_{10}^2}{f_5^2}. \tag{2.4.2}$$

From (1.2.27), we find that

$$\varphi(q) = -\varphi(q^2) + 2\frac{(q^3,\, q^5,\, q^8;\, q^8)_\infty}{(q,\, q^4,\, q^7;\, q^8)_\infty} \equiv \varphi(q^2) \pmod 2. \tag{2.4.3}$$

Similarly, (1.2.26), we have

$$\varphi^2(-q) \equiv \varphi^2(q^2) \pmod 2. \tag{2.4.4}$$

Using (2.4.3) and (2.4.4) in (2.4.2), we find that

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n \equiv \frac{f_{10}}{\varphi(q^2)} \pmod 2,$$

from which (2.4.1) follows. □

Corollary 2.4.2. *If N is a positive integer such that N is not a multiple of 5, then we have*

$$\phi_4(2N+1) \equiv 0 \pmod 2. \tag{2.4.5}$$

Proof. From (2.4.2), we have

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n = \frac{\varphi(q)}{\varphi^2(-q)} \frac{f_{10}^2}{f_5^2}. \tag{2.4.6}$$

From (1.2.28) and (1.2.29), we have

$$\varphi(q) \equiv \varphi(q^5) \pmod 2, \tag{2.4.7}$$

$$\varphi^2(-q) \equiv \varphi^2(-q^5) \pmod 2. \tag{2.4.8}$$

Using (2.4.7) and (2.4.8) in (2.4.6), we arrive at (2.4.5). □

Corollary 2.4.3. *For $n \geq 0$, we have*

$$\phi_4(4n+1) \equiv \begin{cases} 1 & \pmod 2 \ \text{if } n = \frac{5k(3k\pm1)}{2} \text{ for some integer } k, \\ 0 & \pmod 2 \ \text{otherwise.} \end{cases} \tag{2.4.9}$$

Proof. Employing (2.2.3) in (2.3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_4(2n+1)q^n &= \frac{f_4^6}{f_2^7} - 4q\frac{f_4^8 f_{20}^2}{f_2^9 f_{10}^2} + 10q\frac{f_4^5 f_{20} f_{10}}{f_2^8} + 4q^2\frac{f_4^{11} f_{40}^2}{f_2^{10} f_8^2 f_{10} f_{20}} \\ &+ 4q^2\frac{f_4^{10} f_{20}^4}{f_2^{11} f_{10}^4} - 20q^2\frac{f_4^7 f_{20}^3}{f_2^{10} f_{10}} - 8q^3\frac{f_4^{13} f_{20} f_{40}^2}{f_2^{12} f_8^2 f_{10}^3} \\ &+ 25q^2\frac{f_4^4 f_{10}^2 f_{20}^2}{f_2^9} + 20q^3\frac{f_4^{10} f_{40}^2}{f_2^{11} f_8^2} + 4q^4\frac{f_4^{16} f_{40}^4}{f_2^{13} f_8^4 f_{10}^2 f_{20}^2}.\end{aligned} \tag{2.4.10}$$

Extracting the terms involving q^{2n} in (2.4.10) and then replacing q^2 by q, we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} \phi_4(4n+1)q^n \\ &= \frac{f_2^6}{f_1^7} + 4q\frac{f_2^{11} f_{20}^2}{f_1^{10} f_4^2 f_5 f_{10}} + 4q\frac{f_2^{10} f_{10}^4}{f_1^{11} f_5^4} - 20q\frac{f_2^7 f_{10}^3}{f_1^{10} f_5} + 25q\frac{f_2^4 f_5^2 f_{10}^2}{f_1^9} + 4q^2\frac{f_2^{16} f_{20}^4}{f_1^{13} f_4^4 f_5^2 f_{10}^2} \\ &\equiv \frac{f_2^3}{f_1} + q\frac{f_{10}^3}{f_1} \pmod 2.\end{aligned} \tag{2.4.11}$$

Now, from (2.2.3), we have

$$\frac{f_2^3}{f_1} + q\frac{f_{10}^3}{f_1} \equiv f_5 \pmod 2. \tag{2.4.12}$$

From (2.4.11) and (2.4.12), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_4(4n+1)q^n &\equiv f_5 \pmod 2, \\ &\equiv 1 + \sum_{n=1}^{\infty} q^{\frac{5n(3n\pm1)}{2}} \pmod 2.\end{aligned} \tag{2.4.13}$$

Now (2.4.9) follows from (2.4.13). □

Corollary 2.4.4. *For $n \geq 0$, we have*

$$\phi_4(10n+6) \equiv 0 \pmod 5. \tag{2.4.14}$$

Proof. We have

$$\sum_{n=0}^{\infty}\phi_4(2n)q^n = \frac{f_2^{10}f_5}{f_1^9f_4^2f_{10}} + 2q\frac{f_2^{15}f_{20}^2}{f_1^{12}f_4^4f_{10}^2} - 8q\frac{f_4^6f_{10}}{f_1^7f_5}$$
$$\equiv \frac{f_1^3}{f_5}\varphi(q) + 2q\frac{f_{20}}{f_5^2}\varphi(q)f_4^3 - 8q\frac{f_{10}f_{20}}{f_5^3}f_1^3f_4 \pmod 5. \tag{2.4.15}$$

From (1.2.22), we have

$$\varphi(q) = L_0 + L_1 + L_4, \tag{2.4.16}$$

where $L_0 = \varphi(q^{25})$, $L_1 = 2qf(q^{15}, q^{35})$ and $L_4 = 2q^4f(q^5, q^{45})$.

From (1.2.23), we have

$$f_1 = E_0 + E_1 + E_2, \tag{2.4.17}$$

where $E_0 = \dfrac{f_{25}}{R(q^5)}$, $E_1 = -qf_{25}$, and $E_2 = -q^2f_{25}R(q^5)$ and from (1.2.24), we have

$$f_1^3 \equiv J_0 + J_1 \pmod 5, \tag{2.4.18}$$

where $J_0 = (q^{10}, q^{15}, q^{25}; q^{25})_\infty$ and $J_1 = -3q(q^5, q^{20}, q^{25}; q^{25})_\infty$.

Using (2.4.16), (2.4.17), and (2.4.18) in (2.4.15), we find that there are no terms of the form q^{5n+3}, $n \geq 0$ in the resulting congruence, from which (2.4.14) follows. □

Corollary 2.4.5. *For $n \geq 0$, we have*

$$\phi_4(10n+1) \equiv \begin{cases} k+1 \pmod 5 & \text{if } n = k(3k+1) \text{ for some integer } k, \\ 0 \pmod 5 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\sum_{n=0}^{\infty}\phi_4(2n+1)q^n = \frac{f_2^7f_{10}^2}{f_1^6f_4^2f_5^2}$$
$$\equiv \frac{f_2^{17}}{f_1^{16}f_4^2} \pmod 5.$$

For the remaining part of the proof see [35, Theorem 1.3]. □

Corollary 2.4.6. *For $n \geq 0$, we have*

$$\phi_4(10n+5) \equiv 0 \pmod{4}, \tag{2.4.19}$$

$$\phi_4(10n+7) \equiv 0 \pmod{4}. \tag{2.4.20}$$

Proof. From (2.3.2), we have

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n \equiv \frac{f_2^5 f_5^2}{f_1^2 f_4^2} \pmod{4}$$
$$= \varphi(q) f_5^2. \tag{2.4.21}$$

Using (1.2.22) in (2.4.21), we have

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n \equiv \left(\varphi(q^{25}) - 2qf(q^{15}, q^{35}) - 2q^4 f(q^5, q^{45})\right) f_5^2 \pmod{4}. \tag{2.4.22}$$

Extracting the coefficients of q^{5n+2} and q^{5n+3} in (2.4.22), we arrive at (2.4.19) and (2.4.20) respectively. □

Corollary 2.4.7. *For $n \geq 0$, we have*

$$\phi_4(20n+7) \equiv 0 \pmod{8}, \tag{2.4.23}$$

$$\phi_4(20n+15) \equiv 0 \pmod{8}. \tag{2.4.24}$$

Proof. From (2.3.2), we have

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n \equiv \left(\frac{f_1}{f_5}\right)^2 \times \frac{f_2^3 f_{10}^2}{f_4^2} \pmod{8}. \tag{2.4.25}$$

Using (2.2.2) in (2.4.25), and then extracting the terms involving q^{2n+1}, we have

$$\sum_{n=0}^{\infty} \phi_4(4n+3)q^n \equiv 6\frac{f_1^4 f_{10}^3}{f_2 f_5^3}$$
$$\equiv 6 f_2 \times \frac{f_{10}^3}{f_5^3} \pmod{8}. \tag{2.4.26}$$

Using (1.2.23) in (2.4.26), we have

$$\sum_{n=0}^{\infty} \phi_4(4n+3)q^n \equiv \left(\frac{1}{R(q^{10})} - q^2 - q^4 R(q^{10})\right) \frac{f_{10}^3 f_{50}}{f_5^3} \pmod{8}. \tag{2.4.27}$$

Extracting the coefficients of q^{5n+1} and q^{5n+3} in (2.4.27), we arrive at (2.4.23) and (2.4.24) respectively. $\square$

Corollary 2.4.8. *For $n \geq 0$, we have*

$$\phi_4(10n+4) \equiv 0 \pmod 4, \tag{2.4.28}$$

$$\phi_4(10n+8) \equiv 0 \pmod 4. \tag{2.4.29}$$

Proof. From (2.3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_4(2n)q^n &\equiv \frac{f_2^{10} f_5}{f_1^9 f_4^2 f_{10}} + 2q \frac{f_2^{15} f_{20}^2}{f_1^{12} f_4^4 f_{10}^2} \\ &\equiv f_1^3 \times \frac{f_5}{f_{10}} + 2q f_4 f_{10}^2 \pmod 4. \end{aligned} \tag{2.4.30}$$

Now, from (1.2.23), we have

$$f_2 = \frac{1}{R(q^{10})} - q^2 - q^4 R(q^{10}), \tag{2.4.31}$$

$$f_1^3 = \frac{1}{R^3(q^5)} - \frac{3q}{R^2(q^5)} - 3q^5 R^2(q^5) - q^6 R^3(q^5) + 5q^3. \tag{2.4.32}$$

Using (2.4.31) and (2.4.32) in (2.4.30), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_4(2n)q^n \equiv{}& \frac{f_5}{f_{10} R^3(q^5)} + 2q \frac{f_{10}^2}{R(q^{10})} - 3q \frac{f_5}{f_{10} R^2(q^5)} - 2q^3 f_{10}^2 + 5q^3 \frac{f_5}{f_{10}} \\ &- 3q^5 \frac{f_5 R^2(q^5)}{f_{10}} - 2q^5 f_{10}^2 R(q^{10}) - q^6 \frac{f_5 R^3(q^5)}{f_{10}} \pmod 4. \end{aligned} \tag{2.4.33}$$

Congruences (2.4.28) and (2.4.29) follow from (2.4.33). $\square$

2.5 Generating function for $\phi_7(n)$

Theorem 2.5.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \phi_7(n)q^n = \frac{f_{12}^5 f_{42}^5}{f_1 f_2 f_4 f_6^2 f_{21}^2 f_{24}^2 f_{84}^2} + 2q^3 \frac{f_4 f_6 f_{14}^2 f_{21} f_{24} f_{84}}{f_1 f_2^2 f_7 f_8 f_{12} f_{28} f_{42}}. \tag{2.5.1}$$

Proof. From Andrews' general principle 1.1.1, the generating function $\Phi_7(q)$ of $\phi_7(n)$ is given by

$$\phi_7(q) = CT_z\left(\prod_{n=0}^{\infty}\left(\sum_{j=0}^{7} z^j q^{nj+j}\right)\left(\sum_{j=0}^{7} z^{-j} q^{nj}\right)\right), \tag{2.5.2}$$

where $CT_z(S(z,q))$ is the coefficient of z^0 in the sum $S(z,q)$. We also refer $CT_z(S(z,q))$ as the *constant term* in $S(z,q)$.

Factoring the sums on the right side of (2.5.2), we find that

$$\begin{aligned}\Phi_7(q) &= CT_z\Bigg(\prod_{n=0}^{\infty}\left(1+zq^{n+1}\right)\left(1+z^2q^{2n+2}\right)\left(1+z^4q^{4n+4}\right)\\ &\qquad \times\left(1+z^{-1}q^n\right)\left(1+z^{-2}q^{2n}\right)\left(1+z^{-4}q^{4n}\right)\Bigg)\\ &= CT_z\left(\frac{1}{f_1f_2f_4}f(z^{-1},zq)f(z^{-2},z^2q^2)f(z^{-4},z^4q^4)\right)\\ &= CT_z\left(\frac{1}{f_1f_2f_4}\sum_{m_1,\,m_2,\,m_3=-\infty}^{\infty} z^{m_1+2m_2+4m_3}q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}}\right).\end{aligned} \tag{2.5.3}$$

For the constant term, we require

$$m_1+2m_2+4m_3=0. \tag{2.5.4}$$

Using (2.5.4) in (2.5.3), we find that

$$\phi_7(q) = \frac{1}{f_1f_2f_4}\sum_{l_1,\,l_2=-\infty}^{\infty} q^{3l_1^2+8l_1l_2+10l_2^2}. \tag{2.5.5}$$

Now, the set $S = \left\{B\overline{n}, B\overline{n}+\begin{pmatrix}1\\0\end{pmatrix}, B\overline{n}+\begin{pmatrix}2\\0\end{pmatrix}\right\}$, where $B = \begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix}$ and $\overline{n} = \begin{pmatrix}n_1\\ n_2\end{pmatrix}$, forms an integer matrix exact covering system for $\mathbb{Z}^2$. Using this integer matrix exact covering system in (2.5.5), we have

$$\sum_{n=0}^{\infty}\phi_7(n)q^n = \frac{1}{f_1f_2f_4}\Bigg(\sum_{n_1,\,n_2=-\infty}^{\infty} q^{6n_1^2+21n_2^2} + q^3\sum_{n_1,\,n_2=-\infty}^{\infty} q^{6n_1^2+21n_2^2+4n_1+14n_2}$$

$$+ q^{12} \sum_{n_1, n_2=-\infty}^{\infty} q^{6n_1^2+21n_2^2+8n_1+28n_2} \Bigg)$$

$$= \frac{1}{f_1 f_2 f_4} \big(\varphi(q^6)\varphi(q^{21}) + q^3 f(q^2, q^{10}) f(q^7, q^{35})$$

$$+ q^{12} f(q^{-2}, q^{14}) f(q^{-7}, q^{49})\big)$$

$$= \frac{1}{f_1 f_2 f_4} \left(\varphi(q^6)\varphi(q^{21}) + 2q^3 f(q^2, q^{10}) f(q^7, q^{35})\right). \tag{2.5.6}$$

From lemma 1.2.13, we have

$$f(q, q^5) = \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6}. \tag{2.5.7}$$

Replacing q by q^2 and q^7 respectively in (2.5.7), we find that

$$f(q^2, q^{10}) = \frac{f_4^2 f_6 f_{24}}{f_2 f_8 f_{12}}, \tag{2.5.8}$$

$$f(q^7, q^{35}) = \frac{f_{14}^2 f_{21} f_{84}}{f_7 f_{28} f_{42}}. \tag{2.5.9}$$

Employing (1.2.6), (2.5.8), and (2.5.9) in (2.5.6) we derive (2.5.1). □

2.6 Generating function for $\phi_8(n)$

In this section, we present a q-product representation for the generating function of $\phi_8(n)$.

Theorem 2.6.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \phi_8(n) q^n = \frac{f_4 f_6 f_{18}^2 f_{24}^5}{f_1 f_2 f_3^2 f_9 f_{12}^2 f_{36} f_{48}^2} + 2q^4 \frac{f_2^2 f_6 f_{36} f_{48}^2}{f_1^2 f_3^2 f_4 f_{18} f_{24}}. \tag{2.6.1}$$

Proof. From Andrews' general principle 1.1.1, the generating function $\Phi_8(q)$ of $\phi_8(n)$ is given by

$$\Phi_8(q) = CT_z \left(\prod_{n=0}^{\infty} \left(\sum_{j=0}^{8} z^j q^{nj+j} \right) \left(\sum_{j=0}^{8} z^{-j} q^{nj} \right) \right). \tag{2.6.2}$$

Factoring the sums on the right side of (2.6.2), we find that

$$\begin{aligned}\Phi_8(q) &= CT_z\Bigg(\prod_{n=0}^{\infty}\left(1+zq^{n+1}+z^2q^{2n+2}\right)\left(1+z^3q^{3n+3}+z^6q^{6n+6}\right)\\ &\qquad\times\left(1+z^{-1}q^{n}+z^{-2}q^{2n}\right)\left(1+z^{-3}q^{3n}+z^{-6}q^{6n}\right)\Bigg)\\ &= CT_z\Big((\omega zq;q)_\infty(\omega^2z^{-1};q)_\infty(\omega^2zq;q)_\infty(\omega z^{-1};q)_\infty\\ &\qquad\times(\omega z^3q^3;q^3)_\infty(\omega^2z^{-3};q^3)_\infty(\omega^2z^3q^3;q^3)_\infty(\omega z^{-3};q^3)_\infty\Big), \qquad (2.6.3)\end{aligned}$$

where ω is a cube root of unity other than 1.

Employing (1.2.4) in (2.6.3), we have

$$\begin{aligned}\Phi_8(q) = CT_z\Bigg(&\frac{1}{f_1^2f_3^2}f(-\omega zq,-\omega^2z^{-1})f(-\omega^2zq,-\omega z^{-1})f(-\omega z^3q^3,-\omega^2z^{-3})\\ &\times f(-\omega^2z^3q^3,-\omega z^{-3})\Bigg). \qquad (2.6.4)\end{aligned}$$

Using (1.2.12) in (2.6.4), we have

$$\begin{aligned}\Phi_8(q) &= CT_z\Bigg(\frac{1}{f_1^2f_3^2}\left(f(z^2q^2,z^{-2})f(\omega^2q,\omega q)-\omega z^{-1}f(\omega,\omega^2q^2)f(z^2q,z^{-2}q)\right)\\ &\qquad\times\left(f(z^6q^6,z^{-6})f(\omega^2q^3,\omega q^3)-\omega z^{-3}f(\omega,\omega^2q^6)f(z^6q^3,z^{-6}q^3)\right)\Bigg)\\ &= CT_z\Bigg(\frac{1}{f_1^2f_3^2}\Bigg(f(\omega^2q,\omega q)f(\omega^2q^3,\omega q^3)f(z^2q^2,z^{-2})f(z^6q^6,z^{-6})\\ &\qquad-\omega z^{-3}f(\omega^2q,\omega q)f(\omega,\omega^2q^6)f(z^2q^2,z^{-2})f(z^6q^3,z^{-6}q^3)\\ &\qquad-\omega z^{-1}f(\omega,\omega^2q^2)f(\omega^2q^3,\omega q^3)f(z^2q,z^{-2}q)f(z^6q^6,z^{-6})\\ &\qquad+\omega^2z^{-4}f(\omega,\omega^2q^2)f(\omega,\omega^2q^6)f(z^2q,z^{-2}q)f(z^6q^3,z^{-6}q^3)\Bigg)\Bigg). \qquad (2.6.5)\end{aligned}$$

We observe that $f(z^2q^2,z^{-2})f(z^6q^3,z^{-6}q^3)$ is even in z and hence

$$CT_z\left(\omega z^{-3}f(\omega^2q,\omega q)f(\omega,\omega^2q^6)f(z^2q^2,z^{-2})f(z^6q^3,z^{-6}q^3)\right)=0. \qquad (2.6.6)$$

Similarly

$$CT_z\left(\omega z^{-1}f(\omega,\omega^2q^2)f(\omega^2q^3,\omega q^3)f(z^2q,z^{-2}q)f(z^6q^6,z^{-6})\right)=0. \qquad (2.6.7)$$

Using (2.6.6) and (2.6.7) in (2.6.5), we find that

$$\begin{aligned}\Phi_8(q) = CT_z\Bigg(&\frac{1}{f_1^2f_3^2}\big(f(\omega^2q,\omega q)f(\omega^2q^3,\omega q^3)f(z^2q^2,z^{-2})f(z^6q^6,z^{-6})\\&+\omega^2z^{-4}f(\omega,\omega^2q^2)f(\omega,\omega^2q^6)f(z^2q,z^{-2}q)f(z^6q^3,z^{-6}q^3)\big)\Bigg)\\=CT_z\Bigg(&A(q)\sum_{m,n=-\infty}^{\infty}z^{2m+6n}q^{m(m+1)+3n(n+1)}+B(q)\sum_{m,n=-\infty}^{\infty}z^{2m+6n-4}q^{m^2+3n^2}\Bigg),\end{aligned} \tag{2.6.8}$$

where

$$A(q)=\frac{1}{f_1^2f_3^2}f(\omega^2q,\omega q)f(\omega^2q^3,\omega q^3), \tag{2.6.9}$$

$$B(q)=\frac{\omega^2}{f_1^2f_3^2}f(\omega,\omega^2q^2)f(\omega,\omega^2q^6). \tag{2.6.10}$$

Extracting the constant term in (2.6.8), we obtain

$$\begin{aligned}\Phi_8(q)&=A(q)\sum_{n=-\infty}^{\infty}q^{12n^2}+B(q)\sum_{n=-\infty}^{\infty}q^{12n^2-12n+4}\\&=A(q)\varphi(q^{12})+2q^4B(q)\psi(q^{24}).\end{aligned} \tag{2.6.11}$$

Now,

$$\begin{aligned}f(\omega q,\omega^2q)&=\left(-\omega q;q^2\right)_\infty\left(-\omega^2q;q^2\right)_\infty\left(q^2;q^2\right)_\infty\\&=f_2\prod_{n=0}^{\infty}\left(1+\omega q^{2n+1}\right)\left(1+\omega^2q^{2n+1}\right)\\&=f_2\prod_{n=0}^{\infty}\left(\frac{1+q^{6n+3}}{1+q^{2n+1}}\right)\\&=f_2\frac{(-q^3;q^6)_\infty}{(-q;q^2)_\infty}\\&=\frac{f_1f_4f_6^2}{f_2f_3f_{12}}.\end{aligned} \tag{2.6.12}$$

Replacing q by q^3 in (2.6.12), we have

$$f(\omega q^3,\omega^2q^3)=\frac{f_3f_{12}f_{18}^2}{f_6f_9f_{36}}. \tag{2.6.13}$$

Also,

$$\begin{aligned}
f(\omega,\omega^2 q) &= (-\omega;q)_\infty \left(-\omega^2 q;q\right)_\infty (q;q)_\infty \\
&= f_1 \prod_{n=0}^{\infty} (1+\omega q^n)\left(1+\omega^2 q^{n+1}\right) \\
&= f_1 (1+\omega) \prod_{n=0}^{\infty} \left(1+\omega q^{n+1}\right)\left(1+\omega^2 q^{n+1}\right) \\
&= f_1 (1+\omega) \prod_{n=0}^{\infty} \left(\frac{1+q^{3n+3}}{1+q^{n+1}}\right) \\
&= f_1 (1+\omega) \frac{(-q^3;q^3)_\infty}{(-q;q)_\infty} \\
&= (1+\omega) \frac{f_1^2 f_6}{f_2 f_3}. \qquad (2.6.14)
\end{aligned}$$

Replacing q with q^2 and q^6 in (2.6.14), we obtain

$$f(\omega,\omega^2 q^2) = (1+\omega) \frac{f_2^2 f_{12}}{f_4 f_6} \qquad (2.6.15)$$

and

$$f(\omega,\omega^2 q^6) = (1+\omega) \frac{f_6^2 f_{36}}{f_{12} f_{18}}, \qquad (2.6.16)$$

respectively. Using (2.6.12) and (2.6.13) in (2.6.9), we have

$$A(q) = \frac{f_4 f_6 f_{18}^2}{f_1 f_2 f_3^2 f_9 f_{36}}. \qquad (2.6.17)$$

Also, using (2.6.15) and (2.6.16) in (2.6.10), we have

$$B(q) = \frac{f_2^2 f_6 f_{36}}{f_1^2 f_3^2 f_4 f_{18}}. \qquad (2.6.18)$$

Employing (1.2.6), (1.2.7), (2.6.17), and (2.6.18) in (2.6.11), we arrive at (2.6.1). □

2.7 Congruences for $\phi_8(n)$

In this section, we establish two congruences satisfied by $\phi_8(n)$ and extend these congruences into infinite families.

Theorem 2.7.1. *For $n \geq 0$, we have*

$$\phi_8(16n+3) \equiv 0 \pmod 2, \tag{2.7.1}$$

$$\phi_8(128n+107) \equiv 0 \pmod 2. \tag{2.7.2}$$

Proof. From (2.6.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_8(n) q^n &\equiv \frac{f_4 f_6 f_{18}^2 f_{24}^5}{f_1 f_2 f_3^2 f_9 f_{12}^2 f_{36} f_{48}^2} \\ &\equiv \frac{f_9}{f_1} \times \frac{f_2}{f_{18}} \pmod 2. \end{aligned} \tag{2.7.3}$$

Using (2.2.1) in (2.7.3), extracting the terms having odd powers of q, then dividing the resulting expression by q and replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_8(2n+1) q^n &\equiv \frac{f_2^2 f_3 f_{18}}{f_1^2 f_6 f_9} \\ &\equiv \frac{f_2 f_9}{f_3} \pmod 2. \end{aligned} \tag{2.7.4}$$

Replacing q by q^3 in (1.2.18) and using it in (2.7.4), isolating the terms with odd powers of q, then dividing the expression by q and replacing q^2 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_8(4n+3) q^n &\equiv q \frac{f_1 f_9 f_{12}^2 f_{72}}{f_3^2 f_{24} f_{36}} \\ &\equiv q \frac{f_9}{f_1} \times \frac{f_2 f_{36}}{f_6} \pmod 2. \end{aligned} \tag{2.7.5}$$

Using (2.2.1) in (2.7.5), extracting the terms with even powers of q, and then replacing q^2 by q, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_8(8n+3) q^n &\equiv q \frac{f_2^2 f_3 f_{18}}{f_1^3 f_6} \times \frac{f_1 f_{18}}{f_3} \\ &\equiv q \frac{f_2 f_{18}^2}{f_6} \pmod 2. \end{aligned} \tag{2.7.6}$$

Congruence (2.7.1) follows from (2.7.6).

Extracting the terms with odd powers of q in (2.7.6), then dividing the expression by q and replacing q^2 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_8(16n+11)q^n &\equiv \frac{f_1f_9^2}{f_3}\\ &\equiv \frac{f_1f_{18}}{f_3} \pmod 2. \qquad (2.7.7)\end{aligned}$$

Using (1.2.19) in (2.7.7), extracting the terms with even powers of q, and replacing q^2 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_8(32n+11)q^n &\equiv \frac{f_1f_8f_{12}^2}{f_3^2f_4f_{24}}\times f_9\\ &\equiv \frac{f_9}{f_1}\times\frac{f_2f_4}{f_6} \pmod 2. \qquad (2.7.8)\end{aligned}$$

Using (2.2.1) in (2.7.8), extracting the terms with odd powers of q, then dividing the equation by q and replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_8(64n+43)q^n &\equiv \frac{f_1f_2}{f_3}\times\frac{f_2^2f_3f_{18}}{f_1^3f_6}\\ &\equiv \frac{f_4f_{18}}{f_6} \pmod 2. \qquad (2.7.9)\end{aligned}$$

Congruence (2.7.2) follows from (2.7.9). □

Corollary 2.7.2. *For $k\geq 0$, and $n\geq 0$, we have*

$$\phi_8\left(2^{6k+4}n+\frac{2^{6k+3}+1}{3}\right)\equiv 0 \pmod 2, \qquad (2.7.10)$$

$$\phi_8\left(2^{6k+7}n+\frac{5\times 2^{6k+6}+1}{3}\right)\equiv 0 \pmod 2. \qquad (2.7.11)$$

Proof. Extracting the terms with even powers of q in (2.7.9) and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty}\phi_8(128n+43)q^n \equiv \frac{f_2f_9}{f_3} \pmod 2. \qquad (2.7.12)$$

From (2.7.4) and (2.7.12), we find that

$$\phi_8(2n+1)\equiv\phi_8(128n+43) \pmod 2. \qquad (2.7.13)$$

Iterating (2.7.13), we obtain for $k \geq 0$,

$$\phi_8(2n+1) \equiv \phi_8\left(2^{6k+1}n + \frac{2^{6k+1}+1}{3}\right) \pmod 2. \tag{2.7.14}$$

Replacing n by $8n+1$ and $64n+53$ respectively in (2.7.14), we have

$$\phi_8\left(2^{6k+4}n + \frac{2^{6k+3}+1}{3}\right) \equiv \phi_8(16n+3) \pmod 2, \tag{2.7.15}$$

$$\phi_8\left(2^{6k+7}n + \frac{5\times 2^{6k+6}+1}{3}\right) \equiv \phi_8(128n+107) \pmod 2. \tag{2.7.16}$$

Using (2.7.1) and (2.7.2) in (2.7.15) and (2.7.16), respectively, we obtain (2.7.10) and (2.7.11). □

2.8 Generating function for $\phi_{11}(n)$

Theorem 2.8.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \phi_{11}(n)q^n = \frac{f_4 f_6 f_{18}^5 f_{198}^5}{f_1 f_2 f_3^2 f_9^2 f_{12} f_{36}^2 f_{99}^2 f_{396}^2} + 2q^4 \frac{f_2^2 f_9 f_{36} f_{132} f_{198}^2}{f_1^2 f_3^2 f_4 f_{18} f_{66} f_{396}} + 2q^{12} \frac{f_4 f_6^3 f_9 f_{36} f_{66}^2 f_{99} f_{396}}{f_1 f_2 f_3^3 f_{12}^2 f_{18} f_{33} f_{132} f_{198}} + 2q^{25} \frac{f_2^2 f_{12} f_{18}^5 f_{396}^2}{f_1^2 f_3 f_4 f_6^2 f_9^2 f_{36}^2 f_{198}}. \tag{2.8.1}$$

Proof. From Andrews' general principle 1.1.1, the generating function $\Phi_{11}(q)$ of $\phi_{11}(n)$ is given by

$$\begin{aligned}
\Phi_{11}(q) &= CT_z\left(\prod_{n=0}^{\infty}\left(\sum_{j=0}^{11} z^j q^{nj+j}\right)\left(\sum_{j=0}^{11} z^{-j} q^{nj}\right)\right). \\
&= CT_z\Bigg(\prod_{n=0}^{\infty}\left(1+z^3q^{3n+3}\right)\left(1+z^6q^{6n+6}\right)\left(1+zq^{n+1}+z^2q^{2n+2}\right)\left(1+z^{-3}q^{3n}\right) \\
&\qquad \left(1+z^{-6}q^{6n}\right)\left(1+z^{-1}q^n+z^{-2}q^{2n}\right)\Bigg) \\
&= CT_z\Bigg(\frac{1}{f_1^2 f_3 f_6} f(z^3q^3, z^{-3}) f(z^6q^6, z^{-6}) f(-\omega zq, -\omega^{-1}z^{-1}) \\
&\qquad \times f(-\omega^{-1}zq, -\omega z^{-1})\Bigg),
\end{aligned} \tag{2.8.2}$$

where ω is a cube root of unity other that one.

Using (1.2.12) in (2.8.2), we have

$$\begin{aligned}\Phi_{11}(q) &= CT_z\Bigg(\frac{1}{f_1^2 f_3 f_6} f(z^6q^6, z^{-6}) \times \big(f(z^{-6}q^3, z^6q^9) + z^{-3}f(z^6q^3, z^{-6}q^9)\big)\\ &\quad \times \big(f(z^{-2}, z^2q^2)f(\omega q, \omega^{-1}q) - z^{-1}\omega^{-1}f(z^2q, z^{-2}q)f(\omega^{-1}, \omega q^2)\big)\Bigg)\\ &= \frac{1}{f_1^2 f_3 f_6} CT_z\Bigg(f(z^{-6}, z^6q^6)f(z^{-6}q^3, z^6q^9)f(z^{-2}, z^2q^2)f(\omega q, \omega^{-1}q)\\ &\quad - \omega^{-1}z^{-1}f(z^{-6}, z^6q^6)f(z^{-6}q^3, z^6q^9)f(z^2q, z^{-2}q)f(\omega^{-1}, \omega q^2)\\ &\quad + z^{-3}f(z^{-6}, z^6q^6)f(z^6q^3, z^{-6}q^9)f(z^{-2}, z^2q^2)f(\omega q, \omega^{-1}q)\\ &\quad - \omega^{-1}z^{-4}f(z^{-6}, z^6q^6)f(z^6q^3, z^{-6}q^9)f(z^2q, z^{-2}q)f(\omega^{-1}, \omega q^2)\Bigg). \qquad (2.8.3)\end{aligned}$$

Now,

$$CT_z\Big(\omega^{-1}z^{-1}f(z^{-6}, z^6q^6)f(z^{-6}q^3, z^6q^9)f(z^2q, z^{-2}q)f(\omega^{-1}, \omega q^2)\Big) = 0, \text{ and}$$
$$CT_z\Big(z^{-3}f(z^{-6}, z^6q^6)f(z^6q^3, z^{-6}q^9)f(z^{-2}, z^2q^2)f(\omega q, \omega^{-1}q)\Big) = 0.$$

Hence, (2.8.3) reduces to

$$\begin{aligned}\Phi_{11}(q) &= \frac{1}{f_1^2 f_3 f_6} CT_z\Bigg(f(z^{-6}, z^6q^6)f(z^{-6}q^3, z^6q^9)f(z^{-2}, z^2q^2)f(\omega q, \omega^{-1}q)\\ &\quad - \omega^{-1}z^{-4}f(z^{-6}, z^6q^6)f(z^6q^3, z^{-6}q^9)f(z^2q, z^{-2}q)f(\omega^{-1}, \omega q^2)\Bigg)\\ &= \frac{1}{f_1^2 f_3 f_6} CT_z\Bigg(f(\omega q, \omega^{-1}q)\sum_{m_1, m_2, m_3=-\infty}^{\infty} z^{2m_1+6m_2+6m_3}q^{m_1^2+3m_2^2+6m_3^2+m_1+3m_2+3m_3}\\ &\quad - \omega^{-1}f(\omega^{-1}, \omega q^2)\sum_{m_1, m_2, m_3=-\infty}^{\infty} z^{2m_1+6m_2+6m_3-4}q^{m_1^2+3m_2^2+6m_3^2+3m_2-3m_3}\Bigg).\end{aligned}$$

(2.8.4)

Extracting the constant term in (2.8.4), we have

$$\begin{aligned}\Phi_{11}(q) &= \frac{1}{f_1^2 f_3 f_6}\Bigg(f(\omega q, \omega^{-1}q)\sum_{m_1, m_2=-\infty}^{\infty} q^{12m_1^2+18m_1m_2+15m_2^2}\\ &\quad - \omega^{-1}f(\omega^{-1}, \omega q^2)\sum_{m_1, m_2=-\infty}^{\infty} q^{12m_1^2+18m_1m_2+15m_2^2-m_1-15m_2+4}\Bigg). \qquad (2.8.5)\end{aligned}$$

Using the integer matrix exact covering system

$$\left\{ B\overline{n}, B\overline{n} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, B\overline{n} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\},$$

where

$B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ and $\overline{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$, we can split the sum on the right side of (2.8.4) into 3 sums as

$$\begin{aligned}
\Phi_{11}(q) = \frac{1}{f_1^2 f_3 f_6} CT_z \Bigg(& f(\omega q, \omega^{-1} q) \Bigg(\sum_{n_1, n_2 = -\infty}^{\infty} q^{9n_1^2 + 99n_2^2} \\
& + 2 \sum_{n_1, n_2 = -\infty}^{\infty} q^{9n_1^2 + 99n_2^2 - 6n_1 - 66n_2 + 12} \Bigg) \\
& - \omega^2 f(\omega^{-1}, \omega q^2) \Bigg(\sum_{n_1, n_2 = -\infty}^{\infty} q^{9n_1^2 + 99n_2^2 - 99n_2 + 25} \\
& + 2 \sum_{n_1, n_2 = -\infty}^{\infty} q^{9n_1^2 + 99n_2^2 + 12n_1 + 33n_2 + 7} \Bigg) \Bigg) \\
= \frac{1}{f_1^2 f_3 f_6} \Bigg(& f(\omega q, \omega^{-1} q) \Big(\varphi(q^9)\varphi(q^{99}) + 2q^{12} f(q^3, q^{15}) f(q^{33}, q^{165}) \Big) \\
& - \omega^2 f(\omega^{-1}, \omega q^2) \Big(2q^{25} \varphi(q^9) \psi(q^{198}) + 2q^7 f(q^{-3}, q^{21}) f(q^{66}, q^{132}) \Big) \Bigg).
\end{aligned} \tag{2.8.6}$$

Using (1.2.6), (1.2.7), (1.2.13), (1.2.14), (2.6.12), and (2.6.14) in (2.8.6), we arrive at (2.8.1). □

2.9 Generating function for $\phi_{15}(n)$

Theorem 2.9.1. *For $n \geq 0$, we have*

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi_{15}(n) q^n = \frac{1}{f_1 f_2 f_2 f_8} \Big(& \varphi(q^{10})\varphi(q^{15})\varphi(q^{20}) + 2q^6 f(q^3, q^{27}) f(q^6, q^{14}) f(q^4, q^{36}) \\
& + 2q^3 f(q^9, q^{21}) f(q^2, q^{18}) f(q^{12}, q^{28}) \Big).
\end{aligned} \tag{2.9.1}$$

Proof. From Andrews' general principle 1.1.1, the generating function $\Phi_{15}(q)$ of $\phi_{15}(n)$ is given by

$$\begin{aligned}
\Phi_{15}(q) &= CT_z\left(\prod_{n=0}^{\infty}\left(\sum_{j=0}^{15} z^j q^{nj+j}\right)\left(\sum_{j=0}^{15} z^{-j}q^{nj}\right)\right)\\
&= CT_z\Bigg(\prod_{n=0}^{\infty}\left(1+zq^{n+1}\right)\left(1+z^2q^{2n+2}\right)\left(1+z^4q^{4n+4}\right)\left(1+z^8q^{8n+8}\right)\\
&\qquad \times\left(1+z^{-1}q^{n}\right)\left(1+z^{-2}q^{2n}\right)\left(1+z^{-4}q^{4n}\right)\left(1+z^{-8}q^{8n}\right)\Bigg)\\
&= CT_z\left(\frac{1}{f_1f_2f_4f_8}f(zq,z^{-1})f(z^2q^2,z^{-2})f(z^4q^4,z^{-4})f(z^8q^8,z^{-8})\right)\\
&= CT_z\left(\frac{1}{f_1f_2f_4f_8}\sum_{m_1,m_2,m_3,m_4=-\infty}^{\infty} z^{m_1+2m_2+4m_3+8m_4}X(q)\right), \qquad (2.9.2)
\end{aligned}$$

where

$$X(q) = q^{\frac{m_1(m_1+1)}{2}+\frac{2m_2(m_2+1)}{2}+\frac{4m_3(m_3+1)}{2}+\frac{8m_4(m_4+1)}{2}}.$$

Extracting the constant term in (2.9.2), we have

$$\Phi_{15}(q) = \frac{1}{f_1f_2f_4f_8}\sum_{m_1,m_2,m_3=-\infty}^{\infty} q^{3m_1^2+10m_2^2+36m_3^2+8m_1m_2+16m_1m_3+32m_2m_3}. \qquad (2.9.3)$$

Using the integer matrix exact covering system

$$\left\{B\overline{n}, B\overline{n}+\begin{pmatrix}1\\0\\0\end{pmatrix}, B\overline{n}+\begin{pmatrix}-1\\0\\0\end{pmatrix}, B\overline{n}+\begin{pmatrix}2\\0\\0\end{pmatrix}, B\overline{n}+\begin{pmatrix}-2\\0\\0\end{pmatrix}\right\},$$

where

$B = \begin{pmatrix}-1 & 0 & 4\\ 2 & -1 & 0\\ -1 & 0 & 1\end{pmatrix}$ and $\overline{n} = \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$, we can split the right side of (2.9.3) into 5 sums as

$$\Phi_{15}(q) = \frac{1}{f_1f_2f_4f_8}\left(\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{15n_1^2+10n_2^2+20n_3^2}\right.$$

$$
\begin{aligned}
&+\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{15n_1^2+10n_2^2+20n_3^2-12n_1-16n_2-16n_3+12}\\
&+\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{15n_1^2+10n_2^2+20n_3^2-6n_1-8n_2-8n_3+3}\\
&+\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{15n_1^2+10n_2^2+20n_3^2+6n_1+8n_2+8n_3+3}\\
&+\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{15n_1^2+10n_2^2+20n_3^2+12n_1+16n_2+16n_3+12}\Bigg)\\
&=\frac{1}{f_1f_2f_4f_8}\Bigg(\varphi(q^{10})\varphi(q^{15})\varphi(q^{20})+2q^{12}\left(q^3,q^{27}\right)\left(q^{-6},q^{26}\right)\left(q^4,q^{36}\right)\\
&+2q^3\left(q^9,q^{21}\right)\left(q^2,q^{18}\right)\left(q^{12},q^{28}\right)\Bigg).
\end{aligned}
$$

This completes the proof of Theorem 2.9.1. □

2.10 Concluding remarks

Existence of congruences for $\phi_k(n)$ is rare compared to $c\phi_k(n)$. However computational evidences support possibility of the following congruences for $\phi_8(n)$.

Conjecture 2.10.1. *For $n \geq 0$, we have*

$$
\begin{aligned}
\phi_8(200n+179) &\equiv 0 \pmod{2},\\
\phi_8(400n+242) &\equiv 0 \pmod{2},\\
\phi_8(400n+342) &\equiv 0 \pmod{2}.
\end{aligned}
$$