Chapter 3

Generalized Frobenius Partitions and Modular Equations

3.1 Introduction

In this chapter, we establish a connection between F-partitions and modular equations. For a brief description of F-partitions and modular equations, one can refer to Section 1.1 of Chapter 1. We also recall a method [15, Entry 24(v), p .216] of obtaining a new modular equation from a known modular equation.

Theorem 3.1.1 (Method of Reciprocation). *If we replace α by $1-\beta$, β by $1-\alpha$, and m by n/m in a modular equation of degree n, then we obtain a new modular equation of the same degree.*

Ramanujan defined a mixed modular equation or a modular equation of composite degree as follows. Let $K,\ K',\ L_1,\ L_1',\ L_2,\ L_2',\ L_3$, and L_3' denote the complete elliptic integrals of the first kind corresponding to the moduli $\sqrt{\alpha}$, $\sqrt{1-\alpha}$, $\sqrt{\beta}$, $\sqrt{1-\beta}$, $\sqrt{\gamma}$, $\sqrt{1-\gamma}$, $\sqrt{\delta}$, and $\sqrt{1-\delta}$, respectively. Let $n_1,\ n_2,\ n_3$ are three positive integers such that $n_1 n_2 = n_3$. Suppose that the relations

$$n_1\frac{K'}{K}=\frac{L_1'}{L_1},\quad n_2\frac{K'}{K}=\frac{L_2'}{L_2},\quad n_3\frac{K'}{K}=\frac{L_3'}{L_3} \tag{3.1.1}$$

hold. Then a mixed modular equation is a relation among the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$ induced by the relations (3.1.1). In such a case, we say that β, γ, and δ are of degrees n_1, n_2, and n_3, respectively, over α, or α, β, γ, and δ have degrees 1, n_1, n_2, and n_3, respectively. The adaptation of the method of reciprocation as described in Theorem 3.1.1 in case of mixed modular equations is given in the following theorem.

Theorem 3.1.2 (Method of Reciprocation for mixed modular equations)**.** *Suppose* n_1*,* n_2*,* n_3 *are three positive integers such that* $n_1 n_2 = n_3$ *and* β*,* γ*, and* δ *are of degrees* n_1*,* n_2*, and* n_3*, respectively, over* α*. Let* m *and* m' *denote the multipliers connecting the pairs* α*,* β *and* γ*,* δ*, respectively.*

If we replace α *by* $1-\beta$*,* β *by* $1-\alpha$*,* γ *by* $1-\delta$*,* δ *by* $1-\gamma$*, and* mm' *by* $\dfrac{n_1^2}{mm'}$ *in a modular equation for the quadruple of degrees* 1*,* n_1*,* n_2*, and* n_3*, then we obtain a new modular equation of the same quadruple of degrees.*

Partition-theoretic interpretations of Ramanujan's modular equations has been initiated by Berndt with the publication of [16]. There are several subsequent works on the subject in the recent past. For example, see [7, 11, 12, 14, 17, 19, 80, 81, 82]. In majority of these studies, modular equations are transformed into theta function identities and q-products leading to interpretations in terms of partition functions.

In this work, we obtain two different q-product representations of the generating function for $\phi_5(n)$. Equating the representations we derive a new theta function identity. Then we transcribe the theta function identity to a mixed modular equation of the quadruple of degrees 1, 3, 5, 15, and then obtain the reciprocal modular equation as well. To the best of our knowledge, the modular equations are new. It is interesting to note that a partition function leads to the discovery of new modular equations. For the known mixed modular equations of the quadruple of degrees 1, 3, 5, 15 and their proofs, we refer to [6, 15].

We use properties of Ramanujan's general theta functions and integer matrix exact covering systems in our proofs. A brief introduction to these tools has been provided in Section 1.1 and Section 1.2 of Chapter 1.

3.2 Preliminaries

Lemma 3.2.1. *With q, z, and α(or k^2) defined by* (1.1.3), (1.1.2), *and* (1.1.4), *respectively, we have*

$$f_1 = f(-q) = \sqrt{z}2^{-1/6}(1-\alpha)^{1/6}\left(\frac{\alpha}{q}\right)^{1/24}, \tag{3.2.1}$$

$$f_2 = f(-q^2) = \sqrt{z}2^{-1/3}\left(\frac{\alpha(1-\alpha)}{q}\right)^{1/12}, \tag{3.2.2}$$

$$f_4 = f(-q^4) = \sqrt{z}2^{-2/3}(1-\alpha)^{1/24}\left(\frac{\alpha}{q}\right)^{1/6}, \tag{3.2.3}$$

$$f_8 = f(-q^8) = \sqrt{z}2^{-13/12}\alpha^{1/12}(1-\alpha)^{1/48}(1-\sqrt{1-\alpha})^{1/4}q^{-1/3}. \tag{3.2.4}$$

Suppose that β has degree n over α. If we replace q by q^n above, then the same evaluations hold with α replaced by β and z replaced by $z_n := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

Proof. Formulae (3.2.1), (3.2.2), and (3.2.3) are given by [15, Entry 12(ii)–(iv), p. 124]. Further, it is easy to verify that

$$f_8 = \sqrt{f(-q^4)\psi(q^4)}. \tag{3.2.5}$$

Employing [15, Entry 11(iv) p. 123] and [15, Entry 12(iv), p. 124] in (3.2.5), we arrive at (3.2.4). □

3.3 Two representations for $\Phi_5(q)$

In this section, we derive two different representations for $\Phi_5(q)$.

Theorem 3.3.1. *We have*

$$\Phi_5(q) = \frac{f_3f_4^5f_{10}f_{40}f_{60}^2}{f_1^4f_2f_6f_8^2f_{20}f_{30}f_{120}} - 3q\frac{f_2f_6f_{15}}{f_1^4} + 2q^3\frac{f_2f_3f_8^2f_{20}^2f_{120}}{f_1^4f_4f_6f_{40}f_{60}}. \tag{3.3.1}$$

Proof. From (2.1.1), we have

$$\Phi_5(q) = \frac{S_5}{(q;q)_\infty^5}, \tag{3.3.2}$$

where

$$S_5 = \sum_{m_1, m_2, m_3, m_4 = -\infty}^{\infty} \zeta^{4m_1+3m_2+2m_3+m_4} q^{Q(m_1,m_2,m_3,m_4)},$$

$$Q(m_1, m_2, m_3, m_4) = m_1^2 + m_2^2 + m_3^2 + m_4^2 + \sum_{1 \le i < j \le 4} m_i m_j,$$

and $\zeta = e^{\frac{\pi i}{3}}$. We choose the matrix $A = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & -4 & -4 \\ 4 & -4 & -4 & 4 \\ 4 & -4 & 4 & -4 \end{pmatrix}$ to set up the integer matrix exact covering system

$$\left\{ B\mathbf{n} + \frac{1}{16} B c_r \right\}_{r=0}^{15}, \tag{3.3.3}$$

where $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$, and $c_0, c_1, \ldots, c_{15}$ are the solutions of the congruences $B\mathbf{n} \equiv 0 \pmod{16}$. An exclusive representation of (3.3.3) is as follows:

$$B\mathbf{n},\ B\mathbf{n} + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},\ B\mathbf{n} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},\ B\mathbf{n} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},\ B\mathbf{n} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ B\mathbf{n} + \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix},$$

$$B\mathbf{n} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix}, B\mathbf{n} + \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix},\ B\mathbf{n} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ B\mathbf{n} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ B\mathbf{n} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ B\mathbf{n} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$B\mathbf{n}+\begin{pmatrix}3\\0\\0\\0\end{pmatrix},\ B\mathbf{n}+\begin{pmatrix}2\\0\\1\\0\end{pmatrix},\ B\mathbf{n}+\begin{pmatrix}2\\0\\0\\1\end{pmatrix},\ B\mathbf{n}+\begin{pmatrix}2\\1\\0\\0\end{pmatrix}.$$

Corresponding to this integer matrix exact covering system, we can write S_5 as a sum of sixteen parts as

$$\begin{aligned}
S_5 = &\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+2}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+2n_2+2n_3+1}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+3}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+2n_2+2n_4+1}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+1}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+2n_3+2n_4+1}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+4}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+5n_1+n_2+n_3+n_4+1}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+6}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+5n_1+3n_2+3n_3+n_4+3}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+7}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+5n_1+3n_2+n_3+3n_4+3}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+5}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+5n_1+n_2+3n_3+3n_4+3}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+8}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+10n_1+2n_2+2n_3+2n_4+4}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+6}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+10n_1+2n_4+3}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+5}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+10n_1+2n_3+3}\\
&+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} \zeta^{10n_1+4n_2+2n_4+7}\, q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+10n_1+2n_2+3}
\end{aligned}$$

$$+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty}\zeta^{10n_1+4n_2+2n_4+12}\,q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+15n_1+3n_2+3n_3+3n_4+9}$$
$$+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty}\zeta^{10n_1+4n_2+2n_4+10}\,q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+15n_1+n_2+n_3+3n_4+7}$$
$$+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty}\zeta^{10n_1+4n_2+2n_4+9}\,q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+15n_1+n_2+3n_3+n_4+7}$$
$$+\sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty}\zeta^{10n_1+4n_2+2n_4+11}\,q^{10n_1^2+2n_2^2+2n_3^2+2n_4^2+15n_1+3n_2+n_3+n_4+7}. \quad (3.3.4)$$

We divide the sums on the right side of (3.3.4) into four groups T_1, T_2, T_3, and T_4, where T_1 consists of the first four sums, T_2 consists of the next four sums, and so on. We have

$$\begin{aligned}
T_1 &= \left(\sum_{n=-\infty}^{\infty}\zeta^{10n}q^{10n^2}\right)\left(\sum_{n=-\infty}^{\infty}\zeta^{4n}q^{2n^2}\right)\left(\sum_{n=-\infty}^{\infty}q^{2n^2}\right)\left(\sum_{n=-\infty}^{\infty}\zeta^{2n}q^{2n^2}\right)\\
&\quad + q\zeta^2\left(\sum_{n=-\infty}^{\infty}\zeta^{10n}q^{10n^2}\right)\left(\sum_{n=-\infty}^{\infty}\zeta^{4n}q^{2n^2+2n}\right)\left(\sum_{n=-\infty}^{\infty}q^{2n^2+2n}\right)\left(\sum_{n=-\infty}^{\infty}\zeta^{2n}q^{2n^2}\right)\\
&\quad + q\zeta^3\left(\sum_{n=-\infty}^{\infty}\zeta^{10n}q^{10n^2}\right)\left(\sum_{n=-\infty}^{\infty}\zeta^{4n}q^{2n^2+2n}\right)\left(\sum_{n=-\infty}^{\infty}q^{2n^2}\right)\left(\sum_{n=-\infty}^{\infty}\zeta^{2n}q^{2n^2+2n}\right)\\
&\quad + q\zeta\left(\sum_{n=-\infty}^{\infty}\zeta^{10n}q^{10n^2}\right)\left(\sum_{n=-\infty}^{\infty}\zeta^{4n}q^{2n^2}\right)\left(\sum_{n=-\infty}^{\infty}q^{2n^2+2n}\right)\left(\sum_{n=-\infty}^{\infty}\zeta^{2n}q^{2n^2+2n}\right)\\
&= f(\zeta^{-10}q^{10},\,\zeta^{10}q^{10})f(\zeta^{-4}q^2,\,\zeta^4q^2)\varphi(q^2)f(\zeta^{-2}q^2,\,\zeta^2q^2)\\
&\quad + 2q\zeta^2 f(\zeta^{-10}q^{10},\,\zeta^{10}q^{10})f(\zeta^{-4},\,\zeta^4q^4)\psi(q^4)f(\zeta^{-2}q^2,\,\zeta^2q^2)\\
&\quad + q\zeta^3 f(\zeta^{-10}q^{10},\,\zeta^{10}q^{10})f(\zeta^{-4},\,\zeta^4q^4)\varphi(q^2)f(\zeta^{-2},\,\zeta^2q^4)\\
&\quad + 2q\zeta f(\zeta^{-10}q^{10},\,\zeta^{10}q^{10})f(\zeta^{-4}q^2,\,\zeta^4q^2)\psi(q^4)f(\zeta^{-2},\,\zeta^2q^4)\\
&= f(\zeta^2q^{10},\,\zeta^{-2}q^{10})(\varphi(q^2)(f(\zeta^2q^2,\,\zeta^{-2}q^2)f(\zeta^{-2}q^2,\,\zeta^2q^2) - qf(\zeta^2,\,\zeta^{-2}q^4)f(\zeta^{-2},\,\zeta^2q^4))\\
&\quad + 2\zeta q\psi(q^4)(f(\zeta^2q^2,\,\zeta^{-2}q^2)f(\zeta^{-2},\,\zeta^2q^4) + \zeta f(\zeta^2,\,\zeta^{-2}q^4)f(\zeta^{-2}q^2,\,\zeta^2q^2))).
\end{aligned}\quad (3.3.5)$$

Setting $a = -q,\ b = -q,\ c = -\zeta^2 q,\ d = -\zeta^{-2} q$ in (1.2.12), we have

$$f(\zeta^2 q^2,\, \zeta^{-2} q^2) f(\zeta^{-2} q^2,\, \zeta^2 q^2) - q f(\zeta^2,\, \zeta^{-2} q^4) f(\zeta^{-2},\, \zeta^2 q^4) = \varphi(-q) f(-\zeta^2 q,\, -\zeta^{-2} q). \tag{3.3.6}$$

Also setting $a = \zeta$, $b = \zeta^5 q^2$, $c = \zeta^{-3},\ d = \zeta^{-3} q^2$ in (1.2.12), we have

$$\begin{aligned} f(\zeta^2 q^2,\, \zeta^{-2} q^2) f(\zeta^{-2},\, \zeta^2 q^4) + \zeta f(\zeta^2,\, \zeta^{-2} q^4) f(\zeta^{-2} q^2,\, \zeta^2 q^2) &= f(\zeta,\, \zeta^5 q^2) f(\zeta^{-3} q^2,\, \zeta^{-3}) \\ &= f(\zeta,\, \zeta^{-1} q^2) f(-q^2,\, -1) \\ &= 0, \end{aligned} \tag{3.3.7}$$

where we use (1.2.5) in the last equality.

Using (3.3.6) and (3.3.7) in (3.3.5), we have

$$T_1 = \varphi(-q)\varphi(q^2) f(\zeta^2 q^{10},\, \zeta^{-2} q^{10}) f(-\zeta^2 q,\, -\zeta^{-2} q). \tag{3.3.8}$$

In a similar way, we have

$$T_2 = -q\psi(q)\varphi(-q) f(\zeta^{-2} q^{15},\, \zeta^2 q^5) f(\zeta^{-1},\, \zeta q^2), \tag{3.3.9}$$

$$T_3 = -2q^3\zeta^2\psi(q^4)\varphi(-q) f(\zeta^2,\, \zeta^{-2} q^{20}) f(-\zeta^2 q,\, -\zeta^{-2} q), \tag{3.3.10}$$

$$T_4 = -q\zeta\psi(q)\varphi(-q) f(\zeta^{-2} q^5,\, \zeta^2 q^{15}) f(\zeta^{-1}, \zeta q^2). \tag{3.3.11}$$

Using (3.3.8), (3.3.9), (3.3.10), and (3.3.11) in (3.3.4), we have

$$\begin{aligned} S_5 &= \varphi(-q)\varphi(q^2) f(\zeta^2 q^{10},\, \zeta^{-2} q^{10}) f(-\zeta^2 q,\, -\zeta^{-2} q) \\ &\quad - q\psi(q)\varphi(-q) f(\zeta^{-2} q^{15},\, \zeta^2 q^5) f(\zeta^{-1},\, \zeta q^2) \\ &\quad - 2q^3\zeta^2\psi(q^4)\varphi(-q) f(\zeta^2,\, \zeta^{-2} q^{20}) f(-\zeta^2 q,\, -\zeta^{-2} q) \\ &\quad - q\zeta\psi(q)\varphi(-q) f(\zeta^{-2} q^5, \zeta^2 q^{15}) f(\zeta^{-1}, \zeta q^2) \\ &= \varphi(-q)\big(f(-\zeta^{-2} q,\, -\zeta^2 q)(\varphi(q^2) f(\zeta^2 q^{10},\, \zeta^{-2} q^{10}) - 2q^3\zeta^2\psi(q^4) f(\zeta^2,\, \zeta^{-2} q^{20})) \\ &\quad - q\psi(q) f(\zeta^{-1}, \zeta q^2)(f(\zeta^{-2} q^{15},\, \zeta^2 q^5) + \zeta f(\zeta^{-2} q^5,\, \zeta^2 q^{15}))\big). \end{aligned} \tag{3.3.12}$$

Using (1.2.4), we have

$$\begin{aligned}
f(-\zeta^{-2}q,-\zeta^{2}q) &= (\zeta^{-2}q;q^2)_\infty(\zeta^2q;q^2)_\infty(q^2;q^2)_\infty \\
&= (q^2;q^2)_\infty\prod_{k\geq1}(1-\zeta^{-2}q^{2k-1})(1-\zeta^2q^{2k-1}) \\
&= (q^2;q^2)_\infty\prod_{k\geq1}(1-(\zeta^{-2}+\zeta^2)q^{2k-1}+q^{4k-2}) \\
&= (q^2;q^2)_\infty\prod_{k\geq1}(1+q^{2k-1}+q^{4k-2}) \\
&= \frac{(q^2;q^2)_\infty(q^3;q^6)_\infty}{(q;q^2)_\infty} \\
&= \frac{f_2^2f_3}{f_1f_6}. \qquad (3.3.13)
\end{aligned}$$

Replacing q by $-q^{10}$ in (3.3.13), we have

$$f(\zeta^2q^{10},\,\zeta^{-2}q^{10}) = \frac{(q^{20};q^{20})_\infty(-q^{30};q^{60})_\infty}{(-q^{10};q^{20})_\infty} = \frac{f_{10}f_{40}f_{60}^2}{f_{20}f_{30}f_{120}}. \qquad (3.3.14)$$

Using (1.2.4), we have

$$\begin{aligned}
\zeta^2f(\zeta^2,\,\zeta^{-2}q^{20}) &= \zeta^2(-\zeta^2;q^{20})_\infty(-\zeta^{-2}q^{20};q^{20})_\infty(q^{20};q^{20})_\infty \\
&= \zeta^2(q^{20};q^{20})_\infty\prod_{k\geq1}(1+\zeta^2q^{20k-20})(1+\zeta^{-2}q^{20k}) \\
&= \zeta^2(1+\zeta^2)(q^{20};q^{20})_\infty\prod_{k\geq1}(1+\zeta^2q^{20k})(1+\zeta^{-2}q^{20k}) \\
&= -(q^{20};q^{20})_\infty\prod_{k\geq1}(1-q^{20k}+q^{40k}) \\
&= -\frac{(q^{20};q^{20})_\infty(-q^{60};q^{60})_\infty}{(-q^{20};q^{20})_\infty} \\
&= -\frac{f_{20}^2f_{120}}{f_{40}f_{60}}. \qquad (3.3.15)
\end{aligned}$$

Using (1.2.4) once again, we find that

$$\begin{aligned}
f(\zeta^{-1},\,\zeta q^2) &= (-\zeta^{-1};q^2)_\infty(-\zeta q^2;q^2)_\infty(q^2;q^2)_\infty \\
&= (q^2;q^2)_\infty\prod_{k\geq1}(1+\zeta^{-1}q^{2k-2})(1+\zeta q^{2k})
\end{aligned}$$

$$
\begin{aligned}
&= (1+\zeta^{-1})(q^2;q^2)_\infty \prod_{k\geq 1}(1+\zeta^{-1}q^{2k})(1+\zeta q^{2k}) \\
&= (1+\zeta^{-1})(q^2;q^2)_\infty \prod_{k\geq 1}(1+q^{2k}+q^{4k}) \\
&= (1+\zeta^{-1})(q^6;q^6)_\infty \\
&= (1+\zeta^{-1})f_6. \qquad (3.3.16)
\end{aligned}
$$

Replacing q^2 by q^5 in (3.3.16), and then multiplying by ζ, we find that

$$
f(\zeta,\zeta^{-1}q^5) = (1+\zeta)(q^{15};q^{15})_\infty = (1+\zeta)f_{15}. \qquad (3.3.17)
$$

Setting $a=\zeta$ and $b=\zeta^{-1}q^5$ in (1.2.11), we find that

$$
f(\zeta,\zeta^{-1}q^5) = f(\zeta^2q^5,\zeta^{-2}q^{15}) + \zeta f(\zeta^2q^{15},\zeta^{-2}q^5). \qquad (3.3.18)
$$

From (3.3.17) and (3.3.18), we have

$$
f(\zeta^2q^5,\zeta^{-2}q^{15}) + \zeta f(\zeta^2q^{15},\zeta^{-2}q^5) = (1+\zeta)f_{15}. \qquad (3.3.19)
$$

Using (3.3.13), (3.3.14), (3.3.15), (3.3.16) and (3.3.19) in (3.3.12), we obtain

$$
\begin{aligned}
S_5 = \varphi(-q)\Bigg(&\frac{f_2^2f_3}{f_1f_6}\left(\varphi(q^2)\frac{f_{10}f_{40}f_{60}^2}{f_{20}f_{30}f_{120}} + 2q^3\psi(q^4)\frac{f_{20}^2f_{120}}{f_{40}f_{60}}\right) \\
&- q(1+\zeta)(1+\zeta^{-1})\psi(q)f_6f_{15}\Bigg). \qquad (3.3.20)
\end{aligned}
$$

Using (1.2.6), (1.2.7), (1.2.9) in (3.3.20) and noting the fact that $(1+\zeta)(1+\zeta^{-1})=3$, we have

$$
S_5 = \frac{f_1f_3f_4^5f_{10}f_{40}f_{60}^2}{f_2f_6f_8^2f_{20}f_{30}f_{120}} - 3qf_1f_2f_6f_{15} + 2q^3\frac{f_1f_2f_3f_8^2f_{20}^2f_{120}}{f_4f_6f_{40}f_{60}}. \qquad (3.3.21)
$$

Using the expression for S_5 from (3.3.21) in (3.3.2), we arrive at (3.3.1). □

We present another representation of $\Phi_5(q)$ in terms of q-products by a similar but shorter method.

Theorem 3.3.2. *We have*

$$\Phi_5(q) = \frac{f_8 f_{12}^2 f_{20}^5}{f_1 f_2 f_4 f_6 f_{10}^2 f_{24} f_{40}^2} + 2q^3 \frac{f_4^2 f_{24} f_{40}^2}{f_1 f_2^2 f_8 f_{12} f_{20}}. \tag{3.3.22}$$

Proof. From Andrews' general principle 1.1.1, we have

$$\begin{aligned}\Phi_5(q) = CT_z\Bigg(&\prod_{n=0}^{\infty}(1 + zq^{n+1} + z^2q^{2n+2} + z^3q^{3n+3} + z^4q^{4n+4} + z^5q^{5n+5})\\ &\times (1 + z^{-1}q^n + z^{-2}q^{2n} + z^{-3}q^{3n} + z^{-4}q^{4n} + z^{-5}q^{5n})\Bigg),\end{aligned} \tag{3.3.23}$$

where $CT_z(S(z,q))$ is the co-efficient of z^0 in the expansion of $S(z,q)$.

Factoring the products on the right side of (3.3.23), we find that

$$\Phi_5(q) = CT_z\big((-zq;q)_\infty(\omega z^2q^2;q^2)_\infty(\omega^2z^2q^2;q^2)_\infty(-z^{-1};q)_\infty(\omega z^{-2};q^2)_\infty(\omega^2z^{-2};q^2)_\infty\big), \tag{3.3.24}$$

where ω is a cube root of unity other than 1. Employing (1.2.4) in (3.3.24), we have

$$\begin{aligned}\Phi_5(q) &= CT_z\left(\frac{1}{(q;q)_\infty(q^2;q^2)_\infty^2} f(z^{-1}, zq) f(-\omega z^{-2}, -\omega^2 z^2q^2) f(-\omega^2 z^{-2}, -\omega z^2 q^2)\right)\\ &= CT_z\Bigg(\frac{1}{(q;q)_\infty(q^2;q^2)_\infty^2}\\ &\qquad \times \sum_{m_1,\, m_2,\, m_3=-\infty}^{\infty} (-1)^{m_1+m_2}\omega^{\frac{1}{2}(3m_1^2+3m_2^2+m_1-m_2)} z^{2m_1+2m_2+m_3}\\ &\qquad \times q^{m_1(m_1+1)+m_2(m_2+1)+\frac{m_3(m_3+1)}{2}}\Bigg)\\ &= \frac{1}{f_1f_2^2} \sum_{m_1,\, m_2=-\infty}^{\infty} (-1)^{m_1+m_2}\omega^{\frac{1}{2}(3m_1^2+3m_2^2+m_1-m_2)} q^{3m_1^2+4m_1m_2+3m_2^2}.\end{aligned} \tag{3.3.25}$$

Using the integer matrix exact covering system

$$\left\{\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}n_1\\ n_2\end{pmatrix},\ \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}n_1\\ n_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\right\}$$

in (3.3.25), we find that

$$\Phi_5(q) = \frac{1}{f_1f_2^2}\left(\sum_{n_1,\, n_2=-\infty}^{\infty} \omega^{-n_2} q^{10n_1^2+2n_2^2} - \sum_{n_1,\, n_2=-\infty}^{\infty} \omega^{-n_2+1} q^{10n_1^2+2n_2^2+10n_1+2n_2+3}\right)$$

$$= \frac{1}{f_1 f_2^2}\left(\varphi(q^{10})f(\omega q^2,\omega^2 q^2) - 2\omega q^3\psi(q^{20})f(\omega,\omega^2 q^4)\right). \tag{3.3.26}$$

We note that

$$f(\omega q^2,\omega^2 q^2) =(-\omega q^2;q^4)_\infty(-\omega^2 q^2;q^4)_\infty(q^4;q^4)_\infty = \frac{f_2 f_8 f_{12}^2}{f_4 f_6 f_{24}} \tag{3.3.27}$$

and

$$f(\omega,\omega^2 q^4) =(-\omega;q^4)_\infty(-\omega^2 q^4;q^4)_\infty(q^4;q^4)_\infty = (1+\omega)\frac{f_4^2 f_{24}}{f_8 f_{12}}. \tag{3.3.28}$$

Using (1.2.6), (1.2.7), (3.3.27), and (3.3.28) in (3.3.26), we arrive at (3.3.22). □

3.4 Modular equations for the quadruple of degrees 1, 3, 5, and 15

In this section, we derive two modular equations for the quadruple of degrees 1, 3, 5, and 15 using Theorem 3.3.1 and Theorem 3.3.2.

Theorem 3.4.1. *Let α, β, γ, and δ be of the first, third, fifth, and fifteenth degrees, respectively. Let m denote the multiplier connecting α and β, and let m' be the multiplier relating γ and δ. If*

$$P = \left(\frac{\gamma(1-\sqrt{1-\alpha})^4(1-\sqrt{1-\delta})^2}{\alpha^2\delta(1-\sqrt{1-\gamma})^2}\right)^{1/8} \tag{3.4.1}$$

and

$$Q = \left(\frac{\alpha(1-\sqrt{1-\gamma})^4(1-\sqrt{1-\beta})^2}{\gamma^2\beta(1-\sqrt{1-\alpha})^2}\right)^{1/8}, \tag{3.4.2}$$

then

$$\left(\frac{\alpha^4\gamma^2(1-\gamma)}{\beta^2(1-\delta)^3}\right)^{1/16}\left(P+\frac{1}{P}\right) - 2^{1/3}\left(\frac{\gamma^{12}(1-\alpha)^{15}}{\beta^2\delta^2(1-\beta)^5(1-\delta)^8}\right)^{1/48}\left(Q+\frac{1}{Q}\right) = 3\sqrt{\frac{2}{mm'}}. \tag{3.4.3}$$

Proof. From (3.3.1) and (3.3.22), we have

$$\frac{f_3 f_4^5 f_{10} f_{40} f_{60}^2}{f_1^3 f_2 f_6 f_8^2 f_{20} f_{30} f_{120}} - 3q\frac{f_2 f_6 f_{15}}{f_1^3} + 2q^3 \frac{f_2 f_3 f_8^2 f_{20}^2 f_{120}}{f_1^3 f_4 f_6 f_{40} f_{60}} = \frac{f_8 f_{12}^2 f_{20}^5}{f_2 f_4 f_6 f_{10}^2 f_{24} f_{40}^2} + 2q^3 \frac{f_4^2 f_{24} f_{40}^2}{f_2^2 f_8 f_{12} f_{20}}. \tag{3.4.4}$$

Rearranging the q-products in (3.4.4), we have

$$\left(\frac{f_4^5}{f_1^3 f_2 f_8^2}\right)\left(\frac{f_3}{f_6}\right)\left(\frac{f_{10} f_{40}}{f_{20}}\right)\left(\frac{f_{60}^2}{f_{30} f_{120}}\right) - 3q\left(\frac{f_2}{f_1^3}\right) f_6 f_{15} + 2q^3\left(\frac{f_2 f_8^2}{f_1^3 f_4}\right)\left(\frac{f_3}{f_6}\right)\left(\frac{f_{20}^2}{f_{40}}\right)\left(\frac{f_{120}}{f_{60}}\right)$$
$$= \left(\frac{f_8}{f_2 f_4}\right)\left(\frac{f_{12}^2}{f_6 f_{24}}\right)\left(\frac{f_{20}^5}{f_{10}^2 f_{40}^2}\right) + 2q^3\left(\frac{f_4^2}{f_2^2 f_8}\right)\left(\frac{f_{24}}{f_{12}}\right)\left(\frac{f_{40}^2}{f_{20}}\right). \tag{3.4.5}$$

Employing (3.2.1), (3.2.2), (3.2.3), and (3.2.4) in (3.4.5) while noting that β, γ, and δ are of degrees 3, 5, and 15, respectively, over α, we obtain

$$\begin{aligned}
&2^{-5/6}\sqrt{\frac{z_5}{z_1}}\frac{\alpha^{11/24}}{(1-\alpha)^{5/12}(1-\sqrt{1-\alpha})^{1/2}} \cdot \frac{(1-\beta)^{1/12}}{\beta^{1/24}} \\
&\quad \cdot (1-\gamma)^{1/16}(1-\sqrt{1-\gamma})^{1/4} \cdot \frac{\delta^{1/6}}{(1-\delta)^{1/48}(1-\sqrt{1-\delta})^{1/4}} \\
&\quad - 3\cdot 2^{-1/3}\frac{\sqrt{z_3 z_{15}}}{z_1}\frac{1}{\alpha^{1/24}(1-\alpha)^{5/12}} \cdot (\beta(1-\beta))^{1/12} \cdot \delta^{1/24}(1-\delta)^{1/6} \\
&\quad + 2^{-5/6}\sqrt{\frac{z_5}{z_1}}\frac{(1-\sqrt{1-\alpha})^{1/2}}{\alpha^{1/24}(1-\alpha)^{5/12}} \cdot \frac{(1-\beta)^{1/12}}{\beta^{1/24}} \cdot \frac{\gamma^{1/4}(1-\gamma)^{1/16}}{(1-\sqrt{1-\gamma})^{1/4}} \cdot \frac{(1-\sqrt{1-\delta})^{1/4}}{\delta^{1/12}(1-\delta)^{1/48}} \\
&= 2^{-1/2}\sqrt{\frac{z_5}{z_1}}\frac{(1-\sqrt{1-\alpha})^{1/4}}{\alpha^{1/6}(1-\alpha)^{5/48}} \cdot \frac{\beta^{1/6}}{(1-\beta)^{1/48}(1-\sqrt{1-\beta})^{1/4}} \cdot \frac{\gamma^{1/2}}{(1-\sqrt{1-\gamma})^{1/2}} \\
&\quad + 2^{-1/2}\sqrt{\frac{z_5}{z_1}}\frac{\alpha^{1/12}}{(1-\alpha)^{5/48}(1-\sqrt{1-\alpha})^{1/4}} \cdot \frac{(1-\sqrt{1-\beta})^{1/4}}{\beta^{1/12}(1-\beta)^{1/48}} \cdot (1-\sqrt{1-\gamma})^{1/2}.
\end{aligned} \tag{3.4.6}$$

Multiplying (3.4.6) by $2^{5/6}\sqrt{\dfrac{z_1}{z_5}}$ and noting that $m = \dfrac{z_1}{z_3}$ and $m' = \dfrac{z_5}{z_{15}}$, we arrive at (3.4.3) after using the expressions for P and Q from (3.4.1) and (3.4.2), respectively.

□

Theorem 3.4.2. *We have*

$$\left(\frac{(1-\beta)^4(1-\delta)^2\delta}{(1-\alpha)^2\gamma^3}\right)^{1/16}\left(P' + \frac{1}{P'}\right)$$

$$-2^{1/3}\left(\frac{(1-\delta)^{12}\beta^{15}}{(1-\alpha)^2(1-\gamma)^2\alpha^5\gamma^8}\right)^{1/48}\left(Q'+\frac{1}{Q'}\right)=\sqrt{2mm'}, \tag{3.4.7}$$

where

$$P'=\left(\frac{(1+\sqrt{\delta})(1-\sqrt{\gamma})(1-\sqrt{\beta})^2}{(1-\sqrt{\delta})(1+\sqrt{\gamma})(1+\sqrt{\beta})^2}\right)^{1/8}$$

and

$$Q'=\left(\frac{(1+\sqrt{\beta})(1-\sqrt{\alpha})(1-\sqrt{\delta})^2}{(1-\sqrt{\beta})(1+\sqrt{\alpha})(1+\sqrt{\delta})^2}\right)^{1/8}.$$

Proof. We employ the method of reciprocation 3.1.2 ($\alpha \to 1-\beta$, $\beta \to 1-\alpha$, $\gamma \to 1-\delta$, $\delta \to 1-\gamma$, and $mm' \to 9/mm'$) to the modular equation (3.4.3) to obtain (3.4.7). □

The theta function identity (3.4.4) underlying the modular equations (3.4.3) is also quite intriguing. We can easily recast (3.4.4) in the following form

Theorem 3.4.3. *We have*

$$\begin{aligned}&f_2f_3f_4^6f_{10}^3f_{12}f_{24}f_{40}^3f_{60}^3+2q^3f_2^3f_3f_8^4f_{10}^2f_{12}f_{20}^3f_{24}f_{30}f_{40}f_{120}^2\\&\quad=f_1^3f_2f_8^3f_{12}^3f_{20}^6f_{30}f_{60}f_{120}+3qf_2^3f_4f_6^2f_8^2f_{10}^2f_{12}f_{15}f_{20}f_{24}f_{30}f_{40}^2f_{60}f_{120}\\&\qquad+2q^3f_1^3f_4^3f_6f_8f_{10}^2f_{24}^2f_{30}f_{40}^4f_{60}f_{120}.\end{aligned} \tag{3.4.8}$$

Remark 3.4.1. *Few curious observations on the identity* (3.4.8) *are as follows:*

1. *The identity* (3.4.8) *is of level 120 in the sense that in each term* $\prod_{i,j} f_i^j$, *we have i's are divisors of* 120. *Further, all divisors of 120 appears in* (3.4.8), *except the divisor* 5.

2. *The identity* (3.4.8) *is of degree 19 in the sense that in each term* $\prod_{i,j} f_i^j$, *we have* $\sum j = 19$. *Further, each exponent j is a divisor of* 12.

3. *For each term* $\prod_{i,j} f_i^j$ *in* (3.4.8), *we have* $\sum ij \equiv 11 \pmod{24}$. *As such, we can easily recast* (3.4.8) *in terms of Dedekind's eta-function* $\eta(z) := q^{1/24}(q;q)_\infty$, $q = \exp(2\pi i z)$.