Chapter 4

F-partitions with k colors and h repetitions

4.1 Introduction[1]

In Chapter 1, we discussed F-partitions which are two rowed array representations of the form (1.1.1), with additional conditions on the parts. We emphasised mainly on two F-partition functions $\phi_k(n)$ and $c\phi_k(n)$. The function $\phi_k(n)$ counts the number of F-partitions where a part can be repeated at most k times while $c\phi_k(n)$ counts the number of F-partitions where each part comes from k copies of non-negative integers. In this chapter, we discuss a more generalized class of F-partitions, which can be interpreted as a generalization of both $\phi_k(n)$ and $c\phi_k(n)$.

Let $c\phi_{k,h}(n)$ denote the number of F-partitions of n in which each part is repeated at most h times and is taken from k copies of the nonnegative integers. The order relation between two colored parts j_i and l_h is defined by '$j_i < l_h$ if and only if $j < l$ or $j = h$ and $i < h$'. For example, the F-partitions enumerated by $c\phi_{2,2}(2)$ are

$$\begin{pmatrix}1_1\\0_1\end{pmatrix}, \begin{pmatrix}1_2\\0_1\end{pmatrix}, \begin{pmatrix}1_1\\0_2\end{pmatrix}, \begin{pmatrix}1_2\\0_2\end{pmatrix}, \begin{pmatrix}0_1\\1_1\end{pmatrix}, \begin{pmatrix}0_2\\1_1\end{pmatrix}, \begin{pmatrix}0_1\\1_2\end{pmatrix}, \begin{pmatrix}0_2\\1_2\end{pmatrix}$$

$$\begin{pmatrix}0_2 & 0_1\\0_2 & 0_1\end{pmatrix}, \begin{pmatrix}0_2 & 0_2\\0_2 & 0_2\end{pmatrix}, \begin{pmatrix}0_2 & 0_2\\0_2 & 0_1\end{pmatrix}, \begin{pmatrix}0_2 & 0_2\\0_1 & 0_1\end{pmatrix}, \begin{pmatrix}0_2 & 0_1\\0_2 & 0_2\end{pmatrix},$$
$$\begin{pmatrix}0_2 & 0_1\\0_1 & 0_1\end{pmatrix}, \begin{pmatrix}0_1 & 0_1\\0_2 & 0_1\end{pmatrix}, \begin{pmatrix}0_1 & 0_1\\0_1 & 0_1\end{pmatrix}, \begin{pmatrix}0_1 & 0_1\\0_2 & 0_2\end{pmatrix}.$$

[1]Some of contents of this chapter appeared in *The Ramanujan Journal* [62].

It is easy to see that $\phi_k(n) := c\phi_{1,k}(n)$ and $c\phi_k(n) := c\phi_{k,1}(n)$. Andrews [3] also indicated that further study of $c\phi_{k,h}(n)$ is possible beyond the cases with $h = 1$ or $k = 1$.

Padmavathamma [53] outlined a method for obtaining representations of the generating functions for $c\phi_{k,h}(n)$ for arbitrary positive integers k and h in terms of infinite products. By employing this method, an infinite product representation for $c\Phi_{2,2}(q)$, the generating function for $c\phi_{2,2}(n)$, was found [53, Corollary 1]. However, there is a misprint in the result and the correct version is as follows:

Theorem 4.1.1. *We have,*

$$\begin{aligned} c\Phi_{2,2}(q) = &\frac{(-q^2;q^2)_\infty^2(-q^3;q^6)_\infty^2(-q^2;q^4)_\infty^2(q^4;q^4)_\infty}{(q;q)_\infty^2} \\ &+ 2q\frac{(-q;q^2)_\infty^2(-q^6;q^6)_\infty^2(-q^4;q^4)_\infty^2(q^4;q^4)_\infty}{(q;q)_\infty^2}. \end{aligned}$$

Kolitsch [43] derived the following congruences for $c\phi_{k,h}(n)$:

$$\sum_{d|(k,n)} \mu(d) c\phi_{k/d,h}\left(\frac{n}{d}\right) \equiv 0 \pmod{k}$$

and

$$\sum_{d|(k,n)} \mu(d) c\phi_{k/d,h}\left(\frac{n}{d}\right) \equiv 0 \pmod{kK},$$

where K is the product of all prime power factors of k which are relatively prime to $h+1$.

In this chapter, we present expressions for the generating functions of $c\phi_{2,2}(n)$ and $c\phi_{2,3}(n)$, dissect these generating functions, and obtain a number of congruences satisfied by these functions.

A key tool employed in this study is integer matrix exact covering system, as described in Section 1.1 of Chapter 1. We also use identities involving Ramanujan's general theta functions and Jacobi's triple product identity, the details of which have been presented in Section 1.2 of Chapter 1.

4.2 Preliminaries

Lemma 4.2.1. *The following 2-dissection holds.*

$$\frac{1}{f_1f_3}=\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}. \tag{4.2.1}$$

Proof. Identity (4.2.1) was proved by Baruah and Ojah [9]. □

Lemma 4.2.2. *We have*

$$a(q):=\sum_{m,\,n=-\infty}^{\infty}q^{m^2+mn+n^2}=\varphi(q)\varphi(q^3)+4q\psi(q^2)\psi(q^4), \tag{4.2.2}$$

$$a(q)\equiv 1 \pmod 6. \tag{4.2.3}$$

Proof. Identity (4.2.2) is given by Equation (3.7.18) in [18] and Identity (4.2.3) follows from Equation (21.1.1) in [36]. □

Lemma 4.2.3. *We have*

$$\psi(q)=\frac{f_6f_9^2}{f_3f_{18}}+q\psi(q^9), \tag{4.2.4}$$

$$\varphi(q)=\varphi(q^9)+2q\Omega(q^3), \tag{4.2.5}$$

where

$$\Omega(q):=\sum_{n=-\infty}^{\infty}q^{3n^2+2n}=\frac{f_2^2f_3f_{12}}{f_1f_4f_6}.$$

Proof. Identities (4.2.4) and (4.2.5) are Eequations (22.6.13) and (26.1.1), respectively, in [36]. □

Lemma 4.2.4. *We have*

$$f_1^3=f_3a(q^3)-3qf_9^3, \tag{4.2.6}$$

$$\frac{1}{f_1^3}=a^2(q^3)\frac{f_9^3}{f_3^{10}}+3qa(q^3)\frac{f_9^6}{f_3^{11}}+9q^2\frac{f_9^9}{f_3^{12}}, \tag{4.2.7}$$

$$\frac{1}{f_1f_2}=a(q^6)\frac{f_9^3}{f_3^4f_6^3}+qa(q^3)\frac{f_{18}^3}{f_3^3f_6^4}+3q^2\frac{f_9^3f_{18}^3}{f_3^4f_6^4}, \tag{4.2.8}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{4.2.9}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{4.2.10}$$

Proof. Identities (4.2.6) and (4.2.7) follow from Equations (21.3.1) and (39.2.8), respectively, in [36]. Identity (4.2.8) is Equation (22.9.4) in [36]. Identity (4.2.9) is equivalent to Equation (14.3.2) in [36].

To prove (4.2.10), we first note that

$$(\omega q^k; \omega q^k)_\infty (\omega^2 q^k; \omega^2 q^k)_\infty = \frac{f_{3k}^4}{f_k f_{9k}}, \tag{4.2.11}$$

where ω denotes a primitive cube root of unity. Replacing q with ωq and $\omega^2 q$ in (4.2.9), multiplying the two resulting equations, and then employing (4.2.11), we obtain (4.2.10). □

4.3 Generating function for $c\phi_{2,2}(n)$

In this section, we present a representation of the generating function for $c\phi_{2,2}(n)$.

Theorem 4.3.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\phi_{2,2}(n) q^n = \frac{f_2^{11} f_8 f_{12}^2}{f_1^8 f_4^5 f_6 f_{24}} - 4q\frac{f_4^6 f_{24}}{f_1^4 f_2^2 f_8 f_{12}}.$$

Proof. From Andrews' general principle 1.1.1, the generating function $c\Phi_{2,2}(q)$ of $c\phi_{2,2}(n)$ is given by

$$\begin{aligned}
c\Phi_{2,2}(q) &= CT_z\left(\prod_{n=0}^{\infty}\left(1 + zq^{n+1} + z^2 q^{2n+2}\right)^2\left(1 + z^{-1}q^n + z^{-2}q^{2n}\right)^2\right)\\
&= CT_z\left(\prod_{n=0}^{\infty}\left(1 - \omega z q^{n+1}\right)^2\left(1 - \omega^2 z q^{n+1}\right)^2\left(1 - \omega z^{-1} q^n\right)^2\left(1 - \omega^2 z^{-1} q^n\right)^2\right)\\
&= CT_z\left(\left(\omega z q; q\right)_\infty^2\left(\omega^2 z q; q\right)_\infty^2\left(\omega z^{-1}; q\right)_\infty^2\left(\omega^2 z^{-1}; q\right)_\infty^2\right)\\
&= CT_z\left(\frac{1}{f_1^4} \times f^2\left(-\omega z q, -\omega^2 z^{-1}\right) f^2\left(-\omega^2 z q, -\omega z^{-1}\right)\right)
\end{aligned}$$

$$= CT_z\Bigg(\frac{1}{f_1^4}\sum_{m_1,\,m_2,\,m_3,\,m_4=-\infty}^{\infty}(-1)^{m_1+m_2+m_3+m_4}\,q^{\sum_{k=1}^{4}\frac{m_k(m_k+1)}{2}} \times z^{m_1+m_2+m_3+m_4}\omega^{m_1+m_2-m_3-m_4}\Bigg), \tag{4.3.1}$$

where ω is a cube root of unity other than 1. Extracting the constant term in (4.3.1), we find that

$$c\Phi_{2,\,2}(q) = \frac{1}{f_1^4}\sum_{m_1,\,m_2,\,m_3=-\infty}^{\infty}\omega^{2m_1+2m_2}q^{m_1^2+m_2^2+m_3^3+m_1m_2+m_2m_3+m_3m_1}. \tag{4.3.2}$$

Using the integer matrix exact covering system (2.3.4), we can write (4.3.2) as a linear combination of four parts as

$$\begin{aligned}
c\Phi_{2,\,2}(q) &= \frac{1}{f_1^4}\Bigg(\sum_{n_1,\,n_2,\,n_3=-\infty}^{\infty}\omega^{4n_3}q^{2n_1^2+2n_2^2+2n_3^2} + 2\sum_{n_1,\,n_2,\,n_3=-\infty}^{\infty}\omega^{4n_3+2}q^{2n_1^2+2n_2^2+2n_3^2+2n_2+2n_3+1} \\
&\quad + \sum_{n_1,\,n_2,\,n_3=-\infty}^{\infty}\omega^{4n_3}q^{2n_1^2+2n_2^2+2n_3^2+2n_1+2n_2+1}\Bigg) \\
&= \frac{1}{f_1^4}\Bigg(\Bigg(\sum_{n=-\infty}^{\infty}q^{2n^2}\Bigg)^2\Bigg(\sum_{n=-\infty}^{\infty}\omega^n q^{2n^2}\Bigg) + q\Bigg(\sum_{n=-\infty}^{\infty}q^{2n^2+2n}\Bigg)^2\Bigg(\sum_{n=-\infty}^{\infty}\omega^n q^{2n^2}\Bigg) \\
&\quad + 2q\Bigg(\sum_{n=-\infty}^{\infty}q^{2n^2}\Bigg)\Bigg(\sum_{n=-\infty}^{\infty}q^{2n^2+2n}\Bigg)\Bigg(\sum_{n=-\infty}^{\infty}\omega^{n+2}q^{2n^2+2n}\Bigg)\Bigg) \\
&= \frac{1}{f_1^4}\big(\varphi^2(q^2)f(\omega q^2,\omega^{-1}q^2) + 4q\psi^2(q^4)f(\omega q^2,\omega^{-1}q^2) \\
&\quad + 4q\omega^2\varphi(q^2)\psi(q^4)f(\omega^{-1},\omega q^4)\big) \\
&= \frac{1}{f_1^4}\big(f(\omega q^2,\omega^{-1}q^2)\{\varphi^2(q^2)+4q\psi^2(q^4)\} + 4q\omega^2\psi(q^2)\psi(q^4)f(\omega^{-1},\omega q^4)\big).
\end{aligned} \tag{4.3.3}$$

Using (1.2.15) in (4.3.3), we have

$$c\Phi_{2,\,2}(q) = \frac{1}{f_1^4}\left(f(\omega q^2,\omega^{-1}q^2)\varphi^2(q) + 4q\omega^2\psi(q^2)\psi(q^4)f(\omega^{-1},\omega q^4)\right). \tag{4.3.4}$$

Now, from (2.6.12), we have

$$f(\omega q,\omega^{-1}q) = \frac{f_1f_4f_6^2}{f_2f_3f_{12}}. \tag{4.3.5}$$

Also, from (2.6.14), we have

$$f(\omega^{-1},\omega q)=\left(1+\omega^{-1}\right)\frac{f_1^2f_6}{f_2f_3}. \tag{4.3.6}$$

Replacing q by q^2 and q^4 in (4.3.5) and (4.3.6) respectively, we have

$$f(\omega q^2,\omega^{-1}q^2)=\frac{f_2f_8f_{12}^2}{f_4f_6f_{24}}, \tag{4.3.7}$$

$$f(\omega^{-1},\omega q^4)=\left(1+\omega^{-1}\right)\frac{f_4^2f_{24}}{f_8f_{12}}. \tag{4.3.8}$$

Using (1.2.6), (1.2.7), (4.3.7), and (4.3.8) in (4.3.4), we have

$$\begin{aligned}c\Phi_{2,2}(q)&=\frac{1}{f_1^4}\left(\frac{f_2f_8f_{12}^2}{f_4f_6f_{24}}\times\frac{f_2^{10}}{f_1^4f_4^4}+4q\omega^2\frac{f_4^5}{f_2^2f_8^2}\times\frac{f_8^2}{f_4}\times\left(1+\omega^{-1}\right)\times\frac{f_4^2f_{24}}{f_8f_{12}}\right)\\&=\frac{1}{f_1^4}\left(\frac{f_2^{11}f_8f_{12}^2}{f_1^4f_4^5f_6f_{24}}-4q\frac{f_4^6f_{24}}{f_2^2f_8f_{12}}\right).\end{aligned} \tag{4.3.9}$$

This completes the proof of Theorem 4.3.1. □

4.4 Dissections and congruences for $c\phi_{2,2}(n)$

In this section, we present 2- and 4-dissections for $c\Phi_{2,2}(q)$, and some congruences satisfied by $c\phi_{2,2}(n)$.

Theorem 4.4.1. *For $n\geq 0$, we have*

$$\sum_{n=0}^{\infty}c\phi_{2,2}(2n+1)q^n=8\frac{f_2^{11}f_4f_6^2}{f_1^{13}f_3f_{12}}-4\frac{f_2^{20}f_{12}}{f_1^{16}f_4^5f_6}, \tag{4.4.1}$$

$$\sum_{n=0}^{\infty}c\phi_{2,2}(2n)q^n=\frac{f_2^{23}f_6^2}{f_1^{17}f_3f_4^7f_{12}}-16q\frac{f_2^8f_4^3f_{12}}{f_1^{12}f_6}+16q\frac{f_4^9f_6^2}{f_1^9f_2f_3f_{12}}, \tag{4.4.2}$$

$$\begin{aligned}\sum_{n=0}^{\infty}c\phi_{2,2}(4n+3)q^n&=8\frac{f_2^{48}f_{12}^2}{f_1^{35}f_4^{14}f_6^2}-64\frac{f_2^{39}f_6}{f_1^{32}f_3f_4^8}+96\frac{f_2^{30}f_6^4}{f_1^{29}f_3^2f_4^2f_{12}^2}+384q\frac{f_2^{24}f_4^2f_{12}^2}{f_1^{27}f_6^2}\\&\quad-1024q\frac{f_2^{15}f_4^8f_6}{f_1^{24}f_3}+512q\frac{f_2^6f_4^{14}f_6^4}{f_1^{21}f_3^2f_{12}^2},\end{aligned} \tag{4.4.3}$$

$$\sum_{n=0}^{\infty}c\phi_{2,2}(4n+1)q^n=8\frac{f_2^{42}f_6^4}{f_1^{33}f_3^2f_4^{10}f_{12}^2}-4\frac{f_2^{51}f_6}{f_1^{36}f_3f_4^{16}}+96q\frac{f_2^{36}f_{12}^2}{f_1^{31}f_4^6f_6^2}-384q\frac{f_2^{27}f_6}{f_1^{28}f_3}$$

$$+ 384q\frac{f_2^{18}f_4^6f_6^4}{f_1^{25}f_3^2f_{12}^2} + 512q^2\frac{f_2^{12}f_4^{10}f_{12}^2}{f_1^{23}f_6^2} - 1024q^2\frac{f_2^3f_4^{16}f_6}{f_1^{20}f_3}. \tag{4.4.4}$$

Proof. From (4.3.9), we have

$$\sum_{n=0}^{\infty} c\phi_{2,2}(n)q^n = \frac{f_2^{11}f_8f_{12}^2}{f_1^8f_4^5f_6f_{24}} - 4q\frac{f_4^6f_{24}}{f_1^4f_2^2f_8f_{12}}. \tag{4.4.5}$$

Using (1.2.17) in (4.4.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,2}(n)q^n &= \frac{f_2^{11}f_8f_{12}^2}{f_4^5f_6f_{24}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2 - 4q\frac{f_4^6f_{24}}{f_2^2f_8f_{12}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\\ &= \frac{f_4^{23}f_{12}^2}{f_2^{17}f_6f_8^7f_{24}} + 8q\frac{f_4^{11}f_8f_{12}^2}{f_2^{13}f_6f_{24}} + 16q^2\frac{f_8^9f_{12}^2}{f_2^9f_4f_6f_{24}} - 4q\frac{f_4^{20}f_{24}}{f_2^{16}f_8^5f_{12}}\\ &\quad - 16q^2\frac{f_4^8f_8^3f_{24}}{f_2^{12}f_{12}}.\end{aligned} \tag{4.4.6}$$

Extracting the terms involving q^{2n+1} and q^{2n} in (4.4.6) we obtain, respectively,

$$\sum_{n=0}^{\infty} c\phi_{2,2}(2n+1)q^n = 8\frac{f_2^{11}f_4f_6^2}{f_1^{13}f_3f_{12}} - 4\frac{f_2^{20}f_{12}}{f_1^{16}f_4^5f_6} \tag{4.4.7}$$

and

$$\sum_{n=0}^{\infty} c\phi_{2,2}(2n)q^n = \frac{f_2^{23}f_6^2}{f_1^{17}f_3f_4^7f_{12}} - 16q\frac{f_2^8f_4^3f_{12}}{f_1^{12}f_6} + 16q\frac{f_4^9f_6^2}{f_1^9f_2f_3f_{12}},$$

which are (4.4.1) and (4.4.2).

Using (1.2.17) and (4.2.1) in (4.4.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,2}(2n+1)q^n &= 8\frac{f_2^{11}f_4f_6^2}{f_{12}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^3\left(\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2} + q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right)\\ &\quad - 4\frac{f_2^{20}f_{12}}{f_4^5f_6}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^4.\end{aligned} \tag{4.4.8}$$

Extracting the terms involving q^{2n+1} and q^{2n} in (4.4.8), we obtain, respectively,

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,2}(4n+3)q^n &= 8\frac{f_2^{48}f_{12}^2}{f_1^{35}f_4^{14}f_6^2} - 64\frac{f_2^{39}f_6}{f_1^{32}f_3f_4^8} + 96\frac{f_2^{30}f_6^4}{f_1^{29}f_3^2f_4^2f_{12}^2} + 384q\frac{f_2^{24}f_4^2f_{12}^2}{f_1^{27}f_6^2}\\ &\quad - 1024q\frac{f_2^{15}f_4^8f_6}{f_{24}f_3} + 512q\frac{f_2^6f_4^{14}f_6^4}{f_1^{21}f_3^2f_{12}^2}\end{aligned}$$

and

$$\sum_{n=0}^{\infty} c\phi_{2,2}(4n+1)q^n = 8\frac{f_2^{42} f_6^4}{f_1^{33} f_3^2 f_4^{10} f_{12}^2} - 4\frac{f_2^{51} f_6}{f_1^{36} f_3 f_4^{16}} + 96q\frac{f_2^{36} f_{12}^2}{f_1^{31} f_4^6 f_6^2} - 384q\frac{f_2^{27} f_6}{f_1^{28} f_3}$$
$$+ 384q\frac{f_2^{18} f_4^6 f_6^4}{f_1^{25} f_3^2 f_{12}^2} + 512q^2\frac{f_2^{12} f_4^{10} f_{12}^2}{f_1^{23} f_6^2} - 1024q^2\frac{f_2^3 f_4^{16} f_6}{f_1^{20} f_3},$$

which are (4.4.3) and (4.4.4). □

Corollary 4.4.2. *For $n \geq 0$, we have*

$$c\phi_{2,2}(2n+1) \equiv 0 \pmod{4}. \tag{4.4.9}$$

Proof. Congruence (4.4.9) immediately follows from (4.4.1). □

Corollary 4.4.3. *For $n \geq 0$, we have*

$$c\phi_{2,2}(4n+3) \equiv 0 \pmod{8}. \tag{4.4.10}$$

Proof. Congruence (4.4.10) immediately follows from (4.4.3). □

Corollary 4.4.4. *If n can not be expressed as a sum of a pentagonal number, four times a pentagonal number, and twelve times a pentagonal number then*

$$c\phi_{2,2}(4n+3) \equiv 0 \pmod{16}. \tag{4.4.11}$$

Proof. From (4.4.3), we have

$$\sum_{n=0}^{\infty} c\phi_{2,2}(4n+3)q^n \equiv 8\frac{f_2^{48} f_{12}^2}{f_1^{35} f_4^{14} f_6^2} \pmod{16}. \tag{4.4.12}$$

Using the elementary facts that $f_m^{16k} \equiv f_{2m}^{8k} \pmod{16}$ and $8f_m^{2k} \equiv 8f_{2m}^k \pmod{16}$ in (4.4.12), we find that

$$\sum_{n=0}^{\infty} c\phi_{2,2}(4n+3)q^n \equiv 8\frac{f_4^2 f_{12}^2}{f_1^3 f_6^2}$$
$$\equiv 8f_1 f_4 f_{12} \pmod{16},$$

which yields (4.4.11). □

Corollary 4.4.5. *If n can not be expressed as a sum of two times a pentagonal number and three times a pentagonal number then*

$$c\phi_{2,2}(4n+1) \equiv 0 \pmod 8. \tag{4.4.13}$$

Proof. From (4.4.4), we have

$$\sum_{n=0}^{\infty} c\phi_{2,2}(4n+1)q^n \equiv 4\frac{f_2^{51}f_6}{f_1^{36}f_3f_4^{16}} \pmod 8. \tag{4.4.14}$$

Now using the elementary facts that $f_m^{8k} \equiv f_{2m}^{4k} \pmod 8$ and $4f_m^{2k} \equiv 4f_{2m}^{k} \pmod 8$ in (4.4.14), we find that

$$\sum_{n=0}^{\infty} c\phi_{2,2}(4n+1)q^n \equiv 4f_2f_3 \pmod 8, \tag{4.4.15}$$

from which (4.4.13) follows. □

Corollary 4.4.6. *If n can not be expressed as a sum of two times a pentagonal number and three times a triangular number then*

$$c\phi_{2,2}(4n+1) \equiv 0 \pmod 8. \tag{4.4.16}$$

Proof. Replacing q by $-q$ in (4.4.15), we have

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n c\phi_{2,2}(4n+1)q^n &\equiv 4f_2\left(-q^3;-q^3\right)_\infty \\ &\equiv 4\frac{f_2f_6^4}{f_3f_{12}} \pmod 8,\end{aligned} \tag{4.4.17}$$

where we have also used (1.2.8).

Employing the elementary fact $4f_6^2 \equiv 4f_{12} \pmod 8$ in (4.4.17), we find that

$$\sum_{n=0}^{\infty}(-1)^n c\phi_{2,2}(4n+1)q^n \equiv 4f_2\psi(q^3) \pmod 8,$$

from which (4.4.16) follows. □

4.5 Generating function for $c\phi_{2,3}(n)$

In this section, we present a representation of the generating function for $c\phi_{2,3}(n)$.

Theorem 4.5.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\phi_{2,3}(n) q^n = \frac{\varphi(q) a(q^2)}{f_1^2 f_2^2}. \tag{4.5.1}$$

Proof. From Andrews' general principle 1.1.1, the generating function $c\Phi_{2,3}(q)$ of $c\phi_{2,3}(n)$ is given by

$$\begin{aligned} c\Phi_{2,3}(q) &= CT_z\left(\prod_{n=0}^{\infty} \left(1 + zq^{n+1} + z^2 q^{2n+2} + z^3 q^{3n+3}\right)^2 \left(1 + z^{-1} q^n + z^{-2} q^{2n} + z^{-3} q^{3n}\right)^2\right) \\ &= CT_z\left(\prod_{n=0}^{\infty} \left(1 + zq^{n+1}\right)^2 \left(1 + z^2 q^{2n+2}\right)^2 \left(1 + z^{-1} q^n\right)^2 \left(1 + z^{-2} q^{2n}\right)^2\right) \\ &= CT_z\left((-zq; q)_\infty^2 \left(-z^2 q^2; q^2\right)_\infty^2 \left(-z^{-1}; q\right)_\infty^2 \left(-z^{-2}; q^2\right)_\infty^2\right). \end{aligned} \tag{4.5.2}$$

Using Jacobi's triple product identity (1.2.3) in (4.5.2), we obtain

$$\begin{aligned} c\Phi_{2,3}(q) &= CT_z\left(\frac{1}{f_1^2 f_2^2} \left(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m(m+1)}{2}}\right)^2 \left(\sum_{n=-\infty}^{\infty} z^{2n} q^{n(n+1)}\right)^2\right) \\ &= CT_z\left(\frac{1}{f_1^2 f_2^2} \sum_{m_1, m_2, m_3, m_4=-\infty}^{\infty} z^{m_1+m_2+2m_3+2m_4} q^{\frac{m_1(m_1+1)}{2} + \frac{m_2(m_2+1)}{2} + m_3(m_3+1) + m_4(m_4+1)}\right). \end{aligned} \tag{4.5.3}$$

Extracting the constant term in (4.5.3), we find that

$$c\Phi_{2,3}(q) = \frac{1}{f_1^2 f_2^2} \sum_{m_1, m_2, m_3=-\infty}^{\infty} q^{m_1^2 + 3m_2^2 + 3m_3^2 + 2m_1 m_2 + 2m_1 m_3 + 4m_2 m_3}. \tag{4.5.4}$$

Now we consider the matrix

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 2 & 2 \end{bmatrix}.$$

Clearly $det(B) = 16 = 4^{3-1}$, and

$$4B^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$

which is an integer matrix. Therefore by the procedure for obtaining series-product identities obtained in [20], the matrix B can be used to obtain integer matrix exact covering system for $\mathbb{Z}^3$ which is given by

$$\left\{ \begin{pmatrix} n_1 \\ n_2 - n_3 \\ n_2 + n_3 \end{pmatrix}, \begin{pmatrix} n_1 \\ n_2 - n_3 \\ n_2 + n_3 + 1 \end{pmatrix} \right\}.$$

Employing this integer matrix exact covering system in (4.5.4), we have

$$\begin{aligned} c\Phi_{2,3}(q) &= \frac{1}{f_1^2 f_2^2} \Bigg(\sum_{n_1, n_2, n_3 = -\infty}^{\infty} q^{n_1^2 + 10n_2^2 + 2n_3^2 + 4n_1 n_2} \\ &\qquad + \sum_{n_1, n_2, n_3 = -\infty}^{\infty} q^{n_1^2 + 10n_2^2 + 2n_3^2 + 4n_1 n_2 + 2n_1 + 10n_2 + 2n_3 + 3} \Bigg) \\ &= \frac{1}{f_1^2 f_2^2} \Bigg(\left(\sum_{n_3=-\infty}^{\infty} q^{2n_3^2} \right) \left(\sum_{n_1, n_2 = -\infty}^{\infty} q^{(n_1 + 2n_2)^2 + 6n_2^2} \right) \\ &\qquad + q^3 \left(\sum_{n_3=-\infty}^{\infty} q^{2n_3^2 + 2n_3} \right) \left(\sum_{n_1, n_2 = -\infty}^{\infty} q^{(n_1+2n_2)^2 + 6n_2^2 + 2(n_1 + 2n_2) + 6n_2} \right) \Bigg) \\ &= \frac{1}{f_1^2 f_2^2} \left(\varphi(q^2)\varphi(q)\varphi(q^6) + 2q^3 \psi(q^4) \left(\sum_{l=-\infty}^{\infty} q^{l^2 + 2l} \right) \left(\sum_{k=-\infty}^{\infty} q^{6k^2 + 6k} \right) \right) \\ &= \frac{1}{f_1^2 f_2^2} \varphi(q) \left(\varphi(q^2)\varphi(q^6) + 4q^2 \psi(q^4)\psi(q^{12}) \right) \\ &= \frac{\varphi(q) a(q^2)}{f_1^2 f_2^2}, \end{aligned} \tag{4.5.5}$$

which is (4.5.1). □

4.6 Dissections and congruences for $c\phi_{2,3}(n)$

In this section, we give a 2-dissection for $c\Phi_{2,3}(q)$ and deduce some congruences satisfied by $c\phi_{2,3}(n)$.

Theorem 4.6.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\phi_{2,3}(2n)q^n = \frac{f_2^{17}f_6^5}{f_1^{13}f_3^2f_4^6f_{12}^2} + 4q\frac{f_2^{11}f_{12}^2}{f_1^{11}f_4^2f_6} \tag{4.6.1}$$

and

$$\sum_{n=0}^{\infty} c\phi_{2,3}(2n+1)q^n = 4\frac{f_2^5f_4^2f_6^5}{f_1^9f_3^2f_{12}^2} + 16q\frac{f_4^6f_{12}^2}{f_1^7f_2f_6}. \tag{4.6.2}$$

Proof. Using (4.2.2), (1.2.6), and (1.2.7) in (4.5.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,3}(n)q^n &= \frac{f_2^3}{f_1^4f_4^2}\left(\varphi(q^2)\varphi(q^6) + 4q^2\psi(q^4)\psi(q^{12})\right)\\ &= \frac{f_2^3}{f_1^4f_4^2}\left(\frac{f_4^5}{f_2^2f_8^2} \times \frac{f_{12}^5}{f_6^2f_{24}^2} + 4q^2\frac{f_8^2}{f_4} \times \frac{f_{24}^2}{f_{12}}\right)\\ &= \frac{f_2f_4^3f_{12}^5}{f_1^4f_6^2f_8^2f_{24}^2} + 4q^2\frac{f_2^3f_8^2f_{24}^2}{f_1^4f_4^3f_{12}}. \end{aligned} \tag{4.6.3}$$

Using (1.2.17) in (4.6.3) and extracting the terms involving q^{2n} and q^{2n+1} respectively, we have

$$\sum_{n=0}^{\infty} c\phi_{2,3}(2n)q^n = \frac{f_2^{17}f_6^5}{f_1^{13}f_3^2f_4^6f_{12}^2} + 4q\frac{f_2^{11}f_{12}^2}{f_1^{11}f_4^2f_6}$$

and

$$\sum_{n=0}^{\infty} c\phi_{2,3}(2n+1)q^n = 4\frac{f_2^5f_4^2f_6^5}{f_1^9f_3^2f_{12}^2} + 16q\frac{f_4^6f_{12}^2}{f_1^7f_2f_6},$$

which are (4.6.1) and (4.6.2). □

Corollary 4.6.2. *For $n \geq 0$, we have*

$$c\phi_{2,3}(2n+1) \equiv 0 \pmod 4. \tag{4.6.4}$$

Proof. The congruence (4.6.4) follows directly from (4.6.2). □

Corollary 4.6.3. *For $n \geq 0$, we have*

$$c\phi_{2,3}(81n+34) \equiv 0 \pmod{3}. \tag{4.6.5}$$

Proof. Using (4.2.3), (4.2.5), and (4.2.8) in (4.5.5), we have

$$\sum_{n=0}^{\infty} c\phi_{2,3}(n)q^n \equiv \left(\frac{f_9^3}{f_3^4 f_6^3} + q\frac{f_{18}^3}{f_3^3 f_6^4}\right)^2 (\varphi(q^9) + 2q\Omega(q^3)) \pmod{3}. \tag{4.6.6}$$

Extracting the terms involving q^{3n+1} in (4.6.6) and using the fact that $f_k^3 \equiv f_{3k}$ (mod 3), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,3}(3n+1)q^n &\equiv 2\Omega(q)\frac{f_1 f_3^3}{f_6^2} + 2\varphi(q^3)\frac{f_3 f_6}{f_1 f_2} \\ &\equiv 2\frac{f_3^4 f_{12}}{f_2 f_4 f_6^2} + 2\varphi(q^3)\frac{f_3 f_6}{f_1 f_2} \pmod{3}. \end{aligned} \tag{4.6.7}$$

Using (4.2.8) in (4.6.7) and then extracting the terms involving q^{3n+2}, we find that

$$\sum_{n=0}^{\infty} c\phi_{2,3}(9n+7)q^n \equiv 2\frac{f_3^2 f_{12}^2}{f_1^2 f_2^2 f_6} \pmod{3}. \tag{4.6.8}$$

Employing (4.2.8) in (4.6.8) and extracting the terms involving q^{3n}, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,3}(27n+7)q^n &\equiv 2\frac{f_3^6 f_4^2}{f_1^6 f_2^7} \\ &\equiv 2\frac{f_3^4 f_{12}}{f_2 f_4 f_6^2} \pmod{3}. \end{aligned} \tag{4.6.9}$$

Using (4.2.8) again in (4.6.9) and extracting the coefficients of q^{3n+1}, we arrive at (4.6.5). □

Corollary 4.6.4. *For $n \geq 0$, we have*

$$c\phi_{2,3}(18n+13) \equiv 0 \pmod{8}. \tag{4.6.10}$$

Proof. From (4.6.2), we have

$$\sum_{n=0}^{\infty} c\phi_{2,3}(2n+1)q^n \equiv 4\frac{f_2^5 f_4^2 f_6^5}{f_1^9 f_3^2 f_{12}^2} \pmod{8}. \tag{4.6.11}$$

Using the elementary fact that $4f_k^2 \equiv 4f_{2k} \pmod 8$ in (4.6.11), we find that

$$\sum_{n=0}^{\infty} c\phi_{2,3}(2n+1)q^n \equiv 4f_1^9 \equiv 4\frac{f_4^3}{f_1 f_2} \pmod 8. \quad (4.6.12)$$

Using (4.2.8) and (4.2.6) in (4.6.12) and extracting the terms involving q^{3n}, we obtain

$$\sum_{n=0}^{\infty} c\phi_{2,3}(6n+1)q^n \equiv 4a(q^4)a(q^2)\frac{f_3^3 f_4}{f_1^4 f_2^3} - 4q^2\frac{f_{12}^3 f_3^3 f_6^3}{f_1^4 f_2^4} \equiv 4\frac{f_3^3}{f_2^3} - 4q^2\frac{f_3^{21}}{f_4^3} \pmod 8, \quad (4.6.13)$$

where we also use the fact that $4a(q) \equiv 4 \pmod 8$.

Using (4.2.7) in (4.6.13) and extracting the terms involving q^{3n+2}, we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,3}(18n+13)q^n &\equiv 4a(q^2)\frac{f_1^3 f_{12}^3}{f_2^3 f_4^4} - 4a(q^8)\frac{f_1^{21} f_{12}^3}{f_4^4 f_8^3} \\ &\equiv 4\frac{f_1^3 f_{12}^3}{f_2^3 f_4^4} - 4\frac{f_1^{21} f_{12}^3}{f_4^4 f_8^3} \\ &\equiv \frac{f_1^3 f_{12}^3}{f_2^3 f_4^4} - \frac{f_1^3 f_{12}^3}{f_2^3 f_4^4} \\ &\equiv 0 \pmod 8. \quad (4.6.14)\end{aligned}$$

Congruence (4.6.10) follows from (4.6.14). □

Corollary 4.6.5. *For $n \geq 0$, we have*

$$c\phi_{2,3}(162n+7) \equiv 0 \pmod{16}. \quad (4.6.15)$$

Proof. Using the elementary facts that $f_m^{16k} \equiv f_{2m}^{8k} \pmod{16}$ and $4f_m^{4k} \equiv 4f_{2m}^{2k} \pmod{16}$ in (4.6.2), we obtain

$$\sum_{n=0}^{\infty} c\phi_{2,3}(2n+1)q^n \equiv 4\frac{f_2^5 f_4^2 f_6^5}{f_1^9 f_3^2 f_{12}^2} \equiv 4\psi(q)f_2^3 \times \frac{f_6}{f_3^2} \pmod{16}. \quad (4.6.16)$$

Employing (4.2.4) and (4.2.6) in (4.6.16) and extracting the terms involving q^{3n}, we find that

$$\sum_{n=0}^{\infty} c\phi_{2,3}(6n+1)q^n \equiv 4\psi(-q)\varphi(q^6)\frac{f_3^2}{f_6} + 4q\psi(q^3)\frac{f_2 f_6^3}{f_1^2} \pmod{16}. \tag{4.6.17}$$

Using (4.2.4) and (4.2.10) in (4.6.17) and extracting the terms involving q^{3n+1}, we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,3}(18n+7)q^n &\equiv -4\psi(-q^3)\varphi(q^2)\frac{f_1^2}{f_2} + 4\psi(q)\frac{f_2^3 f_3^6}{f_6^3}\\ &\equiv -4\psi(-q^3)\frac{f_1^2 f_4}{f_2^3} + 4\psi(q)\frac{f_2^3 f_3^6}{f_6^3} \pmod{16}. \end{aligned}\tag{4.6.18}$$

Employing (4.2.4), (4.2.6), and (4.2.9) in (4.6.18) and extracting the terms involving q^{3n}, we have

$$\sum_{n=0}^{\infty} c\phi_{2,3}(54n+7)q^n \equiv 4q\psi(q^3)\frac{f_1^2 f_6^3}{f_2} \pmod{16}. \tag{4.6.19}$$

Using (4.2.9) in (4.6.19) and extracting the coefficients of q^{3n}, we obtain (4.6.15). □

Corollary 4.6.6. *For $n \geq 0$ and $k \in \{9, 29, 39, 49\}$, we have*

$$c\phi_{2,3}(50n+k) \equiv 0 \pmod{8}. \tag{4.6.20}$$

Proof. Using the elementary facts that $f_m^{8k} \equiv f_{2m}^{4k} \pmod{8}$ and $4f_m^{2k} \equiv 4f_{2m}^{k} \pmod{8}$ in (4.6.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,3}(2n+1)q^n &= 4\frac{f_2^5 f_4^2 f_6^5}{f_1^9 f_3^2 f_{12}^2}\\ &\equiv 4f_1^9\\ &\equiv 4f_1 f_8 \pmod{8}. \end{aligned}\tag{4.6.21}$$

Now, using (1.2.23) in (4.6.21), we have

$$\sum_{n=0}^{\infty} c\phi_{2,3}(2n+1)q^n \equiv 4f_{25}f_{200}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right)$$

$$\times \left(\frac{1}{R(q^{40})} - q^8 - q^{16}R(q^{40}) \right) \pmod 8. \tag{4.6.22}$$

Extracting the terms involving q^{5n+4} in (4.6.22), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_{2,3}(10n+9)q^n &\equiv 4qf_5f_{40} \\ &\equiv 4qf_5^9 \pmod 8.\end{aligned} \tag{4.6.23}$$

The congruences (4.6.20) follow from (4.6.23). □

Corollary 4.6.7. *For $n \geq 0$ and $k \geq 0$, we have*

$$c\phi_{2,3}\left(2 \times 5^{2k+2}n + 9 \times \frac{5^{2k+2}-1}{12} + 1\right) \equiv c\phi_{2,3}(2n+1) \pmod 8. \tag{4.6.24}$$

Proof. Extracting the terms involving q^{5n+1} in (4.6.23), we have

$$\sum_{n=0}^{\infty} c\phi_{2,3}(50n+19)q^n \equiv 4f_1^9 \pmod 8. \tag{4.6.25}$$

Comparing (4.6.21) and (4.6.25), we have

$$c\phi_{2,3}(2n+1) \equiv c\phi_{2,3}(50n+19) \pmod 8. \tag{4.6.26}$$

Now (4.6.24) follows from (4.6.26) by induction on n. □

4.7 Ramanujan-type congruences for $c\phi_{k,h}(n)$

Theorem 4.7.1. *Suppose that p is a prime and $k \in \mathbb{Z}^+$. If n is not a multiple of p, then*

$$c\phi_{pk,\,p-1}(n) \equiv 0 \pmod p. \tag{4.7.1}$$

Proof. From Andrews' general principle 1.1.1, the generating function $C\Phi_{pk,\,p-1}(q)$ of $c\phi_{pk,\,p-1}(n)$ is given by

$$C\Phi_{pk,\,p-1}(q) = CT_z\left(\prod_{n=0}^{\infty}\left(\sum_{k=0}^{p-1} z^k q^{k(n+1)}\right)^{pk}\left(\sum_{k=0}^{p-1} z^{-k}q^{kn}\right)^{pk}\right)$$

$$= CT_z\left(\prod_{n=0}^{\infty}\left(\prod_{j=1}^{p-1}\left(1-\xi^j zq^{n+1}\right)^{pk}\left(1-\xi^j z^{-1}q^n\right)^{pk}\right)\right), \tag{4.7.2}$$

where ξ is a p^{th} root of unity other than 1.

From (4.7.2), we have

$$\begin{aligned} C\Phi_{pk,\,p-1}(q) &\equiv CT_z\left(\prod_{n=0}^{\infty}\left(\prod_{j=1}^{p-1}\left(1-\xi^{pj}z^p q^{pn+p}\right)^k\left(1-\xi^{pj}z^{-p}q^{pn}\right)^k\right)\right)\\ &\equiv CT_z\left(\prod_{n=0}^{\infty}\left(\left(1-z^p q^{pn+p}\right)^{pk-k}\left(1-z^{-p}q^{pn}\right)^{pk-k}\right)\right)\\ &\equiv CT_z\left(\frac{1}{f_p^{pk-k}}\sum_{\overline{m}=\infty}^{\infty}(-z)^{X(\overline{m})}q^{Y(\overline{m})}\right) \pmod p, \end{aligned} \tag{4.7.3}$$

where

$$\begin{aligned} \overline{m} &= \left(m_1, m_2, \ldots, m_{pk-k}\right),\\ X(\overline{m}) &= \sum_{i=1}^{pk-k} pm_i,\\ Y(\overline{m}) &= \sum_{i=1}^{pk-k}\frac{pm_i(m_i+1)}{2}. \end{aligned}$$

Extracting the constant term in (4.7.3), we have

$$\sum_{n=0}^{\infty} c\phi_{pk,\,p-1}(n)q^n \equiv \frac{1}{f_p^{pk-k}}\sum_{\overline{m'}=\infty}^{\infty} q^{Z(\overline{m'})} \pmod p, \tag{4.7.4}$$

where

$$\overline{m'} = \left(m_1, m_2, \ldots, m_{pk-k-1}\right), \tag{4.7.5}$$

$$Z(\overline{m'}) = \sum_{1\le s,\,t\le pk-k-1} pm_s m_t. \tag{4.7.6}$$

Combining (4.7.5), (4.7.6) with (4.7.4), we arrive at (4.7.1). □

4.8 Concluding remarks

The q-series expansions of $c\Phi_{2,\,2}(q)$ and $c\Phi_{2,\,3}(q)$ up to q^{20} are

$$c\Phi_{2,\,2}(q) = 1 + 4q + 17q^2 + 40q^3 + 99q^4 + 216q^5 + 453q^6 + 888q^7 + 1705q^8 + 3124q^9$$

$$+ 5614q^{10} + 9800q^{11} + 16792q^{12} + 28164q^{13} + 46547q^{14} + 75600q^{15}$$
$$+ 121239q^{16} + 191796q^{17} + 300017q^{18} + 463976q^{19} + 710648q^{20} + \cdots$$

and

$$\begin{aligned} c\Phi_{2,3}(q) =& 1 + 4q + 17q^2 + 52q^3 + 131q^4 + 308q^5 + 682q^6 + 1424q^7 + 2847q^8 \\ &+ 5496q^9 + 10286q^{10} + 18748q^{11} + 33375q^{12} + 58184q^{13} + 99589q^{14} \\ &+ 167620q^{15} + 277822q^{16} + 454124q^{17} + 732883q^{18} + 1168820q^{19} \\ &+ 1843728q^{20} + \cdots . \end{aligned}$$

Using elementary techniques we are able to establish a few congruences modulo powers of 2 and 3 for $c\phi_{2,2}(n)$ and $c\phi_{2,3}(n)$. However, computational evidences suggest that these results are not exhaustive. In particular, the q-series expansions of the above functions support the following congruences.

Conjecture 4.8.1. *For $n \geq 0$, we have*

$$\begin{aligned} c\phi_{2,2}(20n + 11) &\equiv 0 \pmod{5}, \\ c\phi_{2,3}(45n + 19) &\equiv 0 \pmod{5}, \\ c\phi_{2,3}(50n + 39) &\equiv 0 \pmod{16}, \\ c\phi_{2,3}(50n + 49) &\equiv 0 \pmod{16}. \end{aligned}$$