Chapter 5

$(k,\, a)-$colored F-partitions

5.1 Introduction

In the introductory chapter, we discussed in details the generalized Frobenius partitions or, more simply, F-partitions which is a two-rowed array of the form (1.1.1). We also saw that the number of parts in each row of (1.1.1) were equal. However, one can consider F-partitions with unequal row lengths in the Frobenius symbols. Jiang, Rolen, and Woodbury [37] pioneered the study of such F-partitions while working on Motzkin paths and introduced the notion of $(k,\, a)$-colored F-partition which is defined as follows:

Given $k \in \mathbb{Z}^+$ and $a \in \mathbb{Z} + \frac{k}{2}$ $(a \geq 0)$, a $(k,\, a)$-colored F-partition of a positive integer n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{pmatrix}$$

such that

$$n = r + \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j$$

satisfying the following conditions:

1. Each entry a_i, b_j belongs to one of k copies of nonnegative integers.

2. Each row is decreasing with respect to the lexicographical ordering. (Meaning if l_i and m_j are i^{th} and j^{th} copies of the integers l and m respectively, then we have $l_i < m_j$ if and only if $l < m$ or $l = m$ and $i < j$.)

3. The pair $(r,\, s) \neq (0, 0)$ of non-negative integers satisfies

$$r - s = a - \frac{k}{2}.$$

They considered the function

$$F_k(z,\, \tau) := \left(\frac{-\vartheta(z + \frac{1}{2},\, \tau)}{q^{\frac{1}{12}} \eta(\tau)} \right)^k,$$

where $z,\, \tau \in \mathbb{C}$, $Re(\tau) > 0$, $k \in \mathbb{Z}^+$, $q := e^{2\pi i \tau}$, and $\vartheta(z,\, \tau)$ is the Jacobi theta function defined as

$$\vartheta(z,\, \tau) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n (\frac{1}{2} + z)}.$$

Using $\zeta := e^{2\pi i z}$, they established the following:

Theorem 5.1.1. *The ζ^a coefficient of $F_k(z, \tau)$ is*

$$C\Psi_{k,\,a}(q) := \sum_{n=0}^{\infty} c\psi_{k,\,a}(n) q^n,$$

where $c\psi_{k,\,a}(n)$ is the number of $(k,\, a)-$colored F-partitions of n.

In particular, $C\Psi_{k,\, \frac{k}{2}}(q) = C\Phi_k(q)$, where $C\Phi_k(q) := \sum_{n=0}^{\infty} c\phi_k(n) q^n$ is the generating function for $k-$colored F-partitions introduced by Andrews [3].

Using Jacobi's triple product identity (1.2.3), Jiang, Rolen, and Woodbury [37] established the following key result.

Theorem 5.1.2. *The function $C\Psi_{k,\,a}(q)$ is the constant term (with respect to ζ) of*

$$\zeta^{\frac{k}{2} - a} \prod_{n=0}^{\infty} \left(1 + \zeta q^{n+1}\right)^k \left(1 + \zeta^{-1} q^n\right)^k$$

with $a \in \mathbb{Z} + \frac{k}{2}$.

Recently, Eckland and Sellers [27] considered the function $c\psi_{2,\,0}(n)$ and derived a number of congruences satisfied by this function. In [32], Garvan, Sellers, and Smoot obtained the following infinite family of congruences for $c\psi_{2,\,0}(n)$.

Theorem 5.1.3. *If n and α are positive integers with $6n \equiv -1 \pmod{5^\alpha}$, then $c\psi_{2,0}(n) \equiv 0 \pmod{5^\alpha}$.*

In this chapter, we consider specific cases of $c\psi_{k,a}(n)$ and establish some interesting Ramanujan-type congruences satisfied by these functions. We also present expressions for the generating functions of $c\psi_{3,\frac{1}{2}}(n)$, $c\psi_{4,0}(n)$, $c\psi_{4,1}(n)$, $c\psi_{6,0}(n)$, $c\psi_{6,1}(n)$, and $c\psi_{6,2}(n)$ in terms of q-products, dissect these generating functions, and obtain a number of congruences satisfied by these functions.

A key tool employed in this study is integer matrix exact covering system, as described in Section 1.1 of Chapter 1. We also use identities involving Ramanujan's general theta functions and Jacobi's triple product identity, the details of which have been presented in Section 1.2 of Chapter 1.

5.2 Ramanujan-type congruences for $c\psi_{k,a}(n)$

In this section, we consider two particular classes of (k, a)-colored F-partitions and obtain congruences satisfied by these functions.

Theorem 5.2.1. *For any prime p and $l \in \mathbb{Z}^+$,*

$$c\psi_{pl,a}(n) \equiv 0 \pmod{p} \tag{5.2.1}$$

for all $a \in \mathbb{Z}^* + \frac{pl}{2}$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{kp | k \in \mathbb{Z}\}$.

Proof. From Theorem 5.1.2, the generating function $C\Psi_{pl,a}(q)$ of $c\psi_{pl,a}(n)$ is the constant term in

$$G_z(q) := z^{\frac{pl}{2}-a} \prod_{n=0}^{\infty} \left(1 + zq^{n+1}\right)^{pl} \left(1 + z^{-1}q^n\right)^{pl}. \tag{5.2.2}$$

From (5.2.2), we have

$$G_z(q) \equiv z^{\frac{pl}{2}-a} \prod_{n=0}^{\infty} \left(1 + z^p q^{pn+p}\right)^{l} \left(1 + z^{-p}q^{pn}\right)^{l} \pmod{p}. \tag{5.2.3}$$

Using Jacobi's triple product identity (1.2.3) in (5.2.3), we have

$$G_z(q) \equiv \frac{1}{f_p^l} z^{\frac{pl}{2}-a} \sum_{\overline{m}=-\infty}^{\infty} z^{A(\overline{m})} q^{B(\overline{m})} \pmod{p}, \tag{5.2.4}$$

where $\overline{m} = (m_1, m_2, \ldots, m_l)$, $A(\overline{m}) = \sum_{i=1}^{l} pm_i$, and $B(\overline{m}) = \sum_{i=1}^{l} \frac{pm_i(m_i+1)}{2}$.
For the constant term in (5.2.4), we must have

$$A(\overline{m}) + \frac{pl}{2} - a = 0,$$

which is interchangeable with

$$pm_1 + pm_2 + \ldots + pm_l = a - \frac{pl}{2}. \tag{5.2.5}$$

The linear Diophantine equation (5.2.5) has a solution if and only if p divides $a - \frac{pl}{2}$. It means that the constant term in (5.2.2) vanishes modulo p if p does not divide $a - \frac{pl}{2}$, from which the result follows. □

Corollary 5.2.2. *If $l \in \mathbb{Z}^+$ is odd, then for any prime $p \geq 3$*

$$c\psi_{pl,\,\frac{1}{2}}(n) \equiv 0 \pmod{p}. \tag{5.2.6}$$

Proof. If l and p both odd, then $\frac{1}{2} \in \mathbb{Z} + \frac{lp}{2}$.
Also, for $a = \frac{1}{2}$, the congruence (5.2.1) is not true if

$$1 = (2k + l)\, p,$$

which does not have a solution k. Hence (5.2.6) follows from (5.2.1). □

Corollary 5.2.3. *If $l \in \mathbb{Z}^+$ is odd, then for all $m \in \mathbb{Z}^+ \bigcup \{0\}$*

$$c\psi_{2l,\,2m}(n) \equiv 0 \pmod{2}. \tag{5.2.7}$$

Proof. For $p = 2$, the congruence (5.2.1) is not true if

$$a = 2k + l,$$

which does not have a solution k, if l is odd and a is even. Thus (5.2.7) follows from (5.2.1). □

Theorem 5.2.4. *For any prime p and $a \in \mathbb{Z} + \dfrac{p+1}{2}$, we have*

$$\sum_{n=0}^{\infty} c\psi_{p+1,\,a}(n) q^n \equiv \frac{q^{\frac{b(b+1)}{2}}}{f_1 f_p} \sum_{m=-\infty}^{\infty} q^{\frac{p(p+1)}{2}m^2 - pbm} \pmod{p}, \tag{5.2.8}$$

where, $b = a - \dfrac{p+1}{2}$.

Proof. From Theorem 5.1.2, the generating function $C\Psi_{p+1,\,a}(q)$ of $c\psi_{p+1,\,a}(n)$ is the constant term in

$$G_z(q) := z^{\frac{p+1}{2} - a} \prod_{n=0}^{\infty} \left(1 + zq^{n+1}\right)^{p+1} \left(1 + z^{-1} q^n\right)^{p+1}. \tag{5.2.9}$$

From (5.2.9), we find that

$$G_z(q) \equiv z^{-b} \prod_{n=0}^{\infty} \left(1 + z^p q^{pn+p}\right)\left(1 + z^{-p} q^{pn}\right)\left(1 + zq^{n+1}\right)\left(1 + z^{-1} q^n\right) \pmod{p}. \tag{5.2.10}$$

Using Jacobi's triple product identity (1.2.3) in (5.2.10), we have

$$G_z(q) \equiv \frac{1}{f_1 f_p} \sum_{m,\, n=-\infty}^{\infty} z^{pm+n-b} q^{\frac{pm(m+1)}{2} + \frac{n(n+1)}{2}} \pmod{p}. \tag{5.2.11}$$

For constant term in (5.2.11), we must have $pm + n - b = 0$, which yields

$$n = -pm + b. \tag{5.2.12}$$

Using the value of n from (5.2.12) in (5.2.11), we find that

$$\sum_{n=0}^{\infty} c\psi_{p+1,\,a}(n) q^n \equiv \frac{q^{\frac{b(b+1)}{2}}}{f_1 f_p} \sum_{m=-\infty}^{\infty} q^{\frac{p(p+1)}{2}m^2 - pbm} \pmod{p},$$

which is (5.2.8). □

We recall Ramanujan's famous partition congruences given by

$$p(5n+4) \equiv 0 \pmod{5}, \tag{5.2.13}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{5.2.14}$$

$$p(11+6) \equiv 0 \pmod{11}, \tag{5.2.15}$$

where $p(n)$ denotes the number of unrestricted partitions of n. We also note that $\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1}$, $|q| < 1$. We use (5.2.8) and Ramanujan's partition congruences (5.2.13)–(5.2.15) to establish few Ramanujan-type congruences for the functions $c\psi_{k,\,a}(n)$.

Corollary 5.2.5. *For $n \geq 0$, we have*

$$c\psi_{6,\,0}(5n+2) \equiv 0 \pmod{5}, \tag{5.2.16}$$

$$c\psi_{6,\,1}(5n) \equiv 0 \pmod{5}, \tag{5.2.17}$$

$$c\psi_{6,\,2}(5n+4) \equiv 0 \pmod{5}. \tag{5.2.18}$$

Proof. Setting $p = 5$ and $a = 2$ in (5.2.8), we find that

$$\sum_{n=0}^{\infty} c\psi_{6,\,2}(n)q^n \equiv \frac{1}{f_1 f_5} \sum_{m=-\infty}^{\infty} q^{15m^2+5m}$$
$$\equiv \frac{f_{20} f_{30}^2}{f_1 f_5 f_{10} f_{60}} \pmod{5}. \tag{5.2.19}$$

In view of (5.2.13), we observe that the coefficients of q^{5n+4} on the right side of (5.2.19) are multiples of 5. This is enough to prove (5.2.18). The proofs of (5.2.16) and (5.2.17) are similar to the proof of (5.2.18). □

Corollary 5.2.6. *For $n \geq 0$, we have*

$$c\psi_{8,\,0}(7n+4) \equiv 0 \pmod{7}, \tag{5.2.20}$$

$$c\psi_{8,\,1}(7n+1) \equiv 0 \pmod{7}, \tag{5.2.21}$$

$$c\psi_{8,\,2}(7n+6) \equiv 0 \pmod{7}, \tag{5.2.22}$$

$$c\psi_{8,\,3}(7n+5) \equiv 0 \pmod{7}. \tag{5.2.23}$$

Proof. For $p = 7$, setting $a = 0, 1, 2,$ and 3 in (5.2.8) and using (5.2.14) we obtain (5.2.20), (5.2.21), (5.2.22), and (5.2.23) respectively. □

Corollary 5.2.7. *For $n \geq 0$, we have*

$$c\psi_{12,0}(11n+10) \equiv 0 \pmod{11}, \tag{5.2.24}$$

$$c\psi_{12,1}(11n+5) \equiv 0 \pmod{11}, \tag{5.2.25}$$

$$c\psi_{12,2}(11n+1) \equiv 0 \pmod{11}, \tag{5.2.26}$$

$$c\psi_{12,3}(11n+9) \equiv 0 \pmod{11}, \tag{5.2.27}$$

$$c\psi_{12,4}(11n+7) \equiv 0 \pmod{11}, \tag{5.2.28}$$

$$c\psi_{12,5}(11n+6) \equiv 0 \pmod{11}. \tag{5.2.29}$$

Proof. For $p = 11$, setting $a = 0, 1, 2, 3, 4$, and 5 in (5.2.8) and using (5.2.15), we obtain (5.2.24), (5.2.25), (5.2.26), (5.2.27), (5.2.28), and (5.2.29) respectively. □

Remarks 5.2.8. *The listing of congruences in the above corollaries is by no means exhaustive. One can choose other values of a to derive more such congruences.*

We now emphasize on the exact generating functions of few particular instances of $c\psi_{k,a}(n)$.

5.3 Generating function for $c\psi_{3,\frac{1}{2}}(n)$

In this section, we present a representation of the generating function for $c\psi_{3,\frac{1}{2}}(n)$.

Theorem 5.3.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\psi_{3,\frac{1}{2}}(n)q^n = 3\frac{f_3^3}{f_1^4}. \tag{5.3.1}$$

Proof. From Theorem 5.1.2, the generating function $C\Psi_{3,\frac{1}{2}}(q)$ of $c\psi_{3,\frac{1}{2}}(n)$ is the constant term in

$$\begin{aligned} G_q(z) &= z\prod_{n=0}^{\infty}\left(1+zq^{n+1}\right)^3\left(1+z^{-1}q^n\right)^3 \\ &= z \times \frac{1}{f_1^3}\left(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m(m+1)}{2}}\right)^3 \end{aligned}$$

$$= \frac{1}{f_1^3} \sum_{m_1, m_2, m_3=-\infty}^{\infty} z^{m_1+m_2+m_3+1} q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}}. \tag{5.3.2}$$

Extracting the constant term in (5.3.2), we have

$$\sum_{n=0}^{\infty} c\psi_{3,\frac{1}{2}}(n) q^n = \frac{1}{f_1^3} \sum_{n_1, n_2=-\infty}^{\infty} q^{n_1^2+n_2^2+n_1n_2+n_1+n_2}. \tag{5.3.3}$$

Using (1.2.21) in (5.3.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{3,\frac{1}{2}}(n) q^n &= \frac{c(q)}{f_1^3} \\ &= 3\frac{f_3^3}{f_1^4},\end{aligned}$$

which is (5.3.1). □

Corollary 5.3.2. *For $n \geq 0$, we have*

$$c\psi_{3,\frac{1}{2}}(n) \equiv 0 \pmod 3. \tag{5.3.4}$$

Proof. (5.3.4) follows directly from (5.3.1). □

Remarks 5.3.3. *The corollary* (5.3.4) *also follows from corollary* (5.2.1).

5.4 Generating function and congruences for $c\psi_{4,0}(n)$

In this section, we present a representation of the generating function for $c\psi_{4,0}(n)$.

Theorem 5.4.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\psi_{4,0}(n) q^n = 4\frac{f_4^9}{f_1^4 f_2^4 f_8^2} + 2\frac{f_2^{10} f_8^2}{f_1^8 f_4^5}. \tag{5.4.1}$$

Proof. From Theorem 5.1.2, the generating function $C\Psi_{4,0}(q)$ of $c\psi_{4,0}(n)$ is the constant term in

$$G_z(q) := z^2 \prod_{n=0}^{\infty} \left(1 + zq^{n+1}\right)^4 \left(1 + z^{-1}q^n\right)^4$$

$$= z^2 \times \frac{1}{f_1^4}\left(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m(m+1)}{2}}\right)^4$$

$$= \frac{1}{f_1^4}\sum_{m_1,\, m_2,\, m_3,\, m_4=-\infty}^{\infty} z^{m_1+m_2+m_3+m_4+2} q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}+\frac{m_4(m_4+1)}{2}}. \tag{5.4.2}$$

Extracting the constant term in (5.4.2), we find that

$$\sum_{n=0}^{\infty} c\psi_{4,\,0}(n)q^n = \frac{1}{f_1^4}\sum_{m_1,\, m_2,\, m_3=-\infty}^{\infty} q^{m_1^2+m_2^2+m_3^3+m_1m_2+m_1m_3+m_2m_3+2m_1+2m_2+2m_3+1}. \tag{5.4.3}$$

Using the integer matrix exact covering system (2.3.4), we can split the right side of (5.4.3) into 4 sums as

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{4,\,0}(n)q^n = \frac{1}{f_1^4}\Bigg(&\sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+2n_1+2n_2+2n_3+1} \\ &+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+4n_1+4n_2+2n_3+4} \\ &+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+4n_1+2n_2+4n_3+4} \\ &+ \sum_{n_1,\, n_2,\, n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+2n_1+4n_2+4n_3+4}\Bigg) \\ = \frac{1}{f_1^4}&\left(8q\psi^3(q^4) + 6q^4 f^2(q^{-2}, q^6)\psi(q^4)\right).\end{aligned} \tag{5.4.4}$$

Using (1.2.15) in (5.4.4), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{4,\,0}(n)q^n &= \frac{\psi(q^4)}{f_1^4}\left\{4\varphi^2(q^2) + 2\varphi^2(q)\right\} \\ &= 4\frac{f_4^9}{f_1^4 f_2^4 f_8^2} + 2\frac{f_2^{10} f_8^2}{f_1^8 f_4^5}.\end{aligned}$$

This completes the proof of (5.4.1). □

Theorem 5.4.2. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\psi_{4,0}(2n+1)q^n = 32\frac{f_2^{11}f_4^2}{f_1^{14}}, \tag{5.4.5}$$

$$\sum_{n=0}^{\infty} c\psi_{4,0}(2n)q^n = 6\frac{f_2^{23}}{f_1^{18}f_4^6} + 32q\frac{f_4^{10}}{f_1^{10}f_2}, \tag{5.4.6}$$

$$\sum_{n=0}^{\infty} c\psi_{4,0}(4n+1)q^n = 32\frac{f_2^{44}}{f_1^{36}f_4^7f_8^2} + 1536q\frac{f_2^{20}f_4^9}{f_1^{28}f_8^2} + 768q\frac{f_2^{34}f_8^2}{f_1^{32}f_4^5} + 4096q^2\frac{f_2^{10}f_4^{11}f_8^2}{f_1^{24}}, \tag{5.4.7}$$

$$\sum_{n=0}^{\infty} c\psi_{4,0}(4n+3)q^n = 384\frac{f_2^{32}f_4}{f_1^{32}f_8^2} + 2048q\frac{f_2^8f_4^{17}}{f_1^{24}f_8^2} + 64\frac{f_2^{46}f_8^2}{f_1^{36}f_4^{13}} + 3072q\frac{f_2^{22}f_4^3f_8^2}{f_1^{28}}. \tag{5.4.8}$$

Proof. Using (1.2.17) in (5.4.1), we find that

$$\sum_{n=0}^{\infty} c\psi_{4,0}(n)q^n = 6\frac{f_4^{23}}{f_2^{18}f_8^6} + 32q\frac{f_4^{11}f_8^2}{f_2^{14}} + 32q^2\frac{f_8^{10}}{f_2^{10}f_4}. \tag{5.4.9}$$

Equating the terms involving q^{2n+1} in (5.4.9), dividing the resulting identity by q, and replacing q^2 by q we arrive at (5.4.5). Similarly, equating the terms involving q^{2n} in (5.4.9) and replacing q^2 by q we derive (5.4.6).

Using (1.2.16) and (1.2.17) in (5.4.5) and extracting the terms involving q^{2n} and q^{2n+1}, we arrive at (5.4.7) and (5.4.8), respectively. □

Corollary 5.4.3. *For $n \geq 0$, we have*

$$c\psi_{4,0}(n) \equiv 0 \pmod{2}, \tag{5.4.10}$$

$$c\psi_{4,0}(2n+1) \equiv 0 \pmod{32}, \tag{5.4.11}$$

$$c\psi_{4,0}(4n+3) \equiv 0 \pmod{64}, \tag{5.4.12}$$

$$c\psi_{4,0}(8n+5) \equiv 0 \pmod{128}, \tag{5.4.13}$$

$$c\psi_{4,0}(8n+7) \equiv 0 \pmod{256}, \tag{5.4.14}$$

$$c\psi_{4,0}(16n+11) \equiv 0 \pmod{1024}, \tag{5.4.15}$$

$$c\psi_{4,0}(16n+15) \equiv 0 \pmod{1024}. \tag{5.4.16}$$

Proof. The congruences (5.4.10), (5.4.11), and (5.4.12) follow directly from (5.4.1), (5.4.5), and (5.4.8), respectively.

From (5.4.7), we find that

$$\sum_{n=0}^{\infty} c\psi_{4,0}(4n+1)q^n \equiv 32\frac{f_2^{44}}{f_1^{36}f_4^7f_8^2} \equiv 32\frac{f_2^{26}}{f_4^7f_8^2} \pmod{128}. \tag{5.4.17}$$

Extracting the odd parts in (5.4.17), we obtain (5.4.13).

From (5.4.8), we have

$$\sum_{n=0}^{\infty} c\psi_{4,0}(4n+3)q^n \equiv 384\frac{f_2^{32}f_4}{f_1^{32}f_8^2} + 64\frac{f_2^{46}f_8^2}{f_1^{36}f_4^{13}} \equiv 384\frac{f_2^{16}f_4}{f_8^2} + 64\frac{f_2^{28}f_8^2}{f_4^{13}} \pmod{256}. \tag{5.4.18}$$

Extracting the odd parts in (5.4.18), we obtain (5.4.14).

Further, from (5.4.8), we have

$$\sum_{n=0}^{\infty} c\psi_{4,0}(4n+3)q^n \equiv 384\frac{f_2^{32}f_4}{f_1^{32}f_8^2} + 64\frac{f_2^{46}f_8^2}{f_1^{36}f_4^{13}} \tag{5.4.19}$$

$$\equiv 384\frac{f_2^{16}f_4}{f_8^2} + 64\frac{f_2^{30}f_8^2}{f_1^4f_4^{13}} \pmod{1024}. \tag{5.4.20}$$

Using (1.2.17) in (5.4.20) and extracting the terms involving q^{2n}, we have

$$\sum_{n=0}^{\infty} c\psi_{4,0}(8n+3)q^n \equiv 384\frac{f_1^{16}f_2}{f_4^2} + 64\frac{f_1^{16}f_2}{f_4^2} \equiv 448\frac{f_2^9}{f_4^2} \pmod{1024},$$

from which (5.4.15) follows.

From (5.4.19), we have

$$\sum_{n=0}^{\infty} c\psi_{4,0}(4n+3)q^n \equiv 384\frac{f_2^{32}f_4}{f_1^{32}f_8^2} + 64\frac{f_2^{46}f_8^2}{f_1^{36}f_4^{13}} \equiv 384\frac{f_2^{16}f_4}{f_8^2} + 64\frac{f_2^{30}f_8^2}{f_1^4f_4^{13}} \pmod{1024}. \tag{5.4.21}$$

Using (1.2.17) in (5.4.21) and extracting terms involving q^{2n+1}, we find that

$$\sum_{n=0}^{\infty} c\psi_{4,0}(8n+7)q^n \equiv 256\frac{f_1^{20}f_4^6}{f_2^{11}} \equiv 256\frac{f_4^6}{f_2} \pmod{1024}. \tag{5.4.22}$$

Congruence (5.4.16) follows from (5.4.22). □

Remarks 5.4.4. *Congruences* (5.4.11) *and* (5.4.12) *have been proved in [37].*

5.5 Generating function and congruences for $c\psi_{4,1}(n)$

In this section, we present a representation of the generating function for $c\psi_{4,1}(n)$.

Theorem 5.5.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\psi_{4,1}(n)q^n = 4\frac{f_2^6}{f_1^7}. \tag{5.5.1}$$

Proof. From Theorem 5.1.2, the generating function $C\Psi_{4,1}(q)$ of $c\psi_{4,1}(n)$ is the constant term in

$$\begin{aligned} G_z(q) &:= z\prod_{n=0}^{\infty}\left(1+zq^{n+1}\right)^4\left(1+z^{-1}q^n\right)^4 \\ &= z \times \frac{1}{f_1^2}\left(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m(m+1)}{2}}\right)^4 \\ &= \frac{1}{f_1^4}\sum_{m_1,m_2,m_3,m_4=-\infty}^{\infty} z^{m_1+m_2+m_3+m_4+1} q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}+\frac{m_4(m_4+1)}{2}}. \end{aligned} \tag{5.5.2}$$

Extracting the constant term in (5.5.2), we find that

$$\sum_{n=0}^{\infty} c\psi_{4,1}(n)q^n = \frac{1}{f_1^4}\sum_{m_1,m_2,m_3=-\infty}^{\infty} q^{m_1^2+m_2^2+m_3^2+m_1m_2+m_1m_3+m_2m_3+m_1+m_2+m_3}. \tag{5.5.3}$$

Using the integer matrix exact covering system (2.3.4), we write the right side of (5.5.3) as

$$\sum_{n=0}^{\infty} c\psi_{4,1}(n)q^n = \frac{1}{f_1^4}\left(\sum_{n_1,n_2,n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+n_1+n_2+n_3}\right.$$

$$
\begin{aligned}
&+ q^2 \sum_{n_1, n_2, n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+3n_1+3n_2+n_3} \\
&+ q^2 \sum_{n_1, n_2, n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+3n_1+n_2+3n_3} \\
&+ q^2 \sum_{n_1, n_2, n_3=-\infty}^{\infty} q^{2n_1^2+2n_2^2+2n_3^2+n_1+3n_2+3n_3} \Bigg) \\
= &\frac{1}{f_1^4}\left(\psi^3(q) + 3\psi^3(q)\right) \\
= &4\frac{f_2^6}{f_1^7}.
\end{aligned}
$$

□

Corollary 5.5.2. *For $n \geq 0$, we have*

$$c\psi_{4,1}(n) \equiv 0 \pmod{4}, \tag{5.5.4}$$

$$c\psi_{4,1}(7n+3) \equiv 0 \pmod{7}, \tag{5.5.5}$$

$$c\psi_{4,1}(25n+k) \equiv 0 \pmod{5}, \quad k \in \{5, 15, 20\}. \tag{5.5.6}$$

Proof. Congruence (5.5.4) follows directly from (5.5.1).

From (5.5.1), we have

$$\sum_{n=0}^{\infty} c\psi_{4,1}(n)q^n \equiv 4\frac{(f_2^3)^2}{f_7} \pmod{7}. \tag{5.5.7}$$

From [36, Eqn. 3.7.1], we have

$$f_2^3 \equiv H(q^{14}) - 3q^2 J(q^{14}) + 5q^6 K(q^{14}) \pmod{7}, \tag{5.5.8}$$

where

$$
\begin{aligned}
H(q) &= \left(q^3, q^4, q^7; q^7\right)_\infty, \\
J(q) &= \left(q^2, q^5, q^7; q^7\right)_\infty, \\
K(q) &= \left(q, q^6, q^7; q^7\right)_\infty,
\end{aligned}
$$

and $(a_1, a_2, \ldots, a_k; q)_\infty$ is given by (1.2.25).

Using (5.5.8) in (5.5.7), we find that

$$\sum_{n=0}^{\infty} c\psi_{4,1}(n)q^n \equiv \frac{4}{f_7}\big(H^2(q^{14}) - 6q^2H(q^{14})J(q^{14}) + 9q^4J^2(q^{14}) + 10q^6H(q^{14})K(q^{14}) - 30q^8J(q^{14})K(q^{14}) + 25q^{12}K^2(q^{14})\big) \pmod 7. \tag{5.5.9}$$

Congruence (5.5.5) follows from (5.5.9).

From (5.5.1), we observe that, modulo 5,

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{4,1}(n)q^n &\equiv 4 \times \frac{f_{10}}{f_5} \times \frac{1}{\varphi(-q)}\\ &\equiv 4 \times \frac{f_{10}}{f_5} \times \frac{\varphi^4(-q)}{\varphi(-q^5)}.\end{aligned} \tag{5.5.10}$$

Employing 1.2.22, extracting the terms involving q^{5n}, and using (1.2.29) in (5.5.10), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{4,1}(5n)q^n &\equiv 4 \times \frac{f_2}{f_1} \times \frac{\varphi^4(-q)}{\varphi(-q)}\\ &\equiv 4 \times \frac{f_5}{f_{10}} \times f_2^3 \pmod 5.\end{aligned} \tag{5.5.11}$$

Replacing q by q^2 in (1.2.24), we have

$$f_2^3 \equiv J_0 + J_2 \pmod 5, \tag{5.5.12}$$

where J_i consists of terms in which the power of q is congruent to i modulo 5. Using (5.5.12) in (5.5.11), we arrive at (5.5.6). □

Remarks 5.5.3. *Congruence* (5.5.5) *appeared as a conjecture in [37].*

5.6 Generating functions and congruences for $c\psi_{6,a}(n)$, $a = 0, 1, 2$

In this section, we present representations of the generating functions for $c\psi_{6,0}(n)$, $c\psi_{6,1}(n)$, and $c\psi_{6,2}(n)$.

Theorem 5.6.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\psi_{6,0}(n)q^n = 4\frac{f_2^9 f_8^2 f_{12}^5}{f_1^{10} f_4^3 f_6^2 f_{24}^2} + 8\frac{f_2^{11} f_6^2}{f_1^{11} f_3 f_4^2} + 8\frac{f_4^{11} f_{12}^5}{f_1^6 f_2^5 f_6^2 f_8^2 f_{24}^2} + 4q\frac{f_2^7 f_4^3 f_{24}^2}{f_1^{10} f_8^2 f_{12}} + 32q^2\frac{f_4^5 f_8^2 f_{24}^2}{f_1^6 f_2^3 f_{12}}. \tag{5.6.1}$$

Proof. From Theorem 5.1.2, the generating function $C\Psi_{6,0}(q)$ of $c\psi_{6,0}(n)$ is the constant term in

$$\begin{aligned} G_z(q) &:= z^3 \prod_{n=0}^{\infty} \left(1 + zq^{n+1}\right)^6 \left(1 + z^{-1}q^n\right)^6 \\ &= z^3 \times \frac{1}{f_1^6} \left(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m(m+1)}{2}} \right)^6 \\ &= \frac{1}{f_1^6} \sum_{m_1, m_2, m_3, m_4, m_5, m_6 = -\infty}^{\infty} \Bigg(z^{m_1+m_2+m_3+m_4+m_5+m_6+3} \\ &\qquad \times q^{\frac{m_1(m_1+1)}{2} + \frac{m_2(m_2+1)}{2} + \frac{m_3(m_3+1)}{2} + \frac{m_4(m_4+1)}{2} + \frac{m_5(m_5+1)}{2} + \frac{m_6(m_6+1)}{2}} \Bigg). \end{aligned} \tag{5.6.2}$$

Extracting the constant term in (5.6.2), we have

$$\sum_{n=0}^{\infty} c\psi_{6,0}(n)q^n = \frac{1}{f_1^6} \sum_{m_1, m_2, m_3, m_4, m_5 = -\infty}^{\infty} q^{Q(\overline{m})}, \tag{5.6.3}$$

where

$$Q(\overline{m}) = \sum_{1 \leq i, j \leq 5} m_i m_j + 3\sum_{i=1}^{5} m_i + 3.$$

Using the integer matrix exact covering system

$$\left\{ B\overline{n}, B\overline{n} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, B\overline{n} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, B\overline{n} + \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}, \tag{5.6.4}$$

where $B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ and $\overline{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{pmatrix}$, obtained in [10], we can split the right side of (5.6.3) into 4 parts as

$$\sum_{n=0}^{\infty} c\psi_{6,0}(n)q^n = \frac{1}{f_1^6}\Bigg[q^3\left(\sum_{n_1,\,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+3n_1+3n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2\left(\sum_{n=-\infty}^{\infty} q^{n^2+3n}\right)$$
$$+ 2q^7\left(\sum_{n_1,\,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+5n_1+4n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)$$
$$\times\left(\sum_{n=-\infty}^{\infty} q^{n^2+n}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2+4n}\right)$$
$$+ q^8\left(\sum_{n_1,\,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+6n_1+6n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2+n}\right)^2\left(\sum_{n=-\infty}^{\infty} q^{n^2+3n}\right)\Bigg]. \tag{5.6.5}$$

Using the integer matrix exact covering system

$$\left\{B\overline{n}, B\overline{n} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \tag{5.6.6}$$

where $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\overline{n} = \begin{pmatrix} l \\ m \end{pmatrix}$, we find that

$$\sum_{n_1,\,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+3n_1+3n_4} = \left(\sum_{n=-\infty}^{\infty} q^{2n^2}\right)\left(\sum_{n=-\infty}^{\infty} q^{6n^2+6n}\right)$$
$$+ q^5\left(\sum_{n=-\infty}^{\infty} q^{2n^2+2n}\right)\left(\sum_{n=-\infty}^{\infty} q^{6n^2+12n}\right)$$
$$= \varphi(q^2)f(1,q^{12}) + 2q^5\psi(q^4)f(q^{-6},q^{18})$$
$$= 2\varphi(q^2)\psi(q^{12}) + 2q^{-1}\psi(q^4)\varphi(q^6). \tag{5.6.7}$$

Similarly, we deduce that

$$\sum_{n_1,\, n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+5n_1+4n_4} = 2q^{-3}\psi(q)\psi(q^3), \tag{5.6.8}$$

and

$$\sum_{n_1,\, n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+6n_1+6n_4} = q^{-6}\varphi(q^2)\varphi(q^6) + 4q^{-4}\psi(q^4)\psi(q^{12}). \tag{5.6.9}$$

Using (5.6.7), (5.6.8), and (5.6.9) in (5.6.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{6,0}(n)q^n = \frac{1}{f_1^6}\Big(&4\varphi^2(q)\varphi(q^6)\psi(q^2)\psi(q^4) + 8\varphi^2(q)\psi(q)\psi(q^2)\psi(q^3) \\ &+ 8\varphi(q^2)\varphi(q^6)\psi^3(q^2) + 4q\varphi^2(q)\varphi(q^2)\psi(q^2)\psi(q^{12}) \\ &+ 32q^2\psi^3(q^2)\psi(q^4)\psi(q^{12})\Big).\end{aligned} \tag{5.6.10}$$

Using (1.2.6) and (1.2.7) in (5.6.10), we arrive at (5.6.1). □

Corollary 5.6.2. *For $n \geq 0$, we have*

$$c\psi_{6,0}(n) \equiv 0 \pmod 4, \tag{5.6.11}$$

$$c\psi_{6,0}(4n+2) \equiv 0 \pmod 8, \tag{5.6.12}$$

$$c\psi_{6,0}(4n+3) \equiv 0 \pmod 8. \tag{5.6.13}$$

Proof. Congruence (5.6.11) follows directly from (5.6.1).

From (5.6.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{6,0}(n)q^n &\equiv 4\frac{f_2^9 f_8^2 f_{12}^5}{f_1^{10} f_4^3 f_6^2 f_{24}^2} + 4q\frac{f_2^7 f_4^3 f_{24}^2}{f_1^{10} f_8^2 f_{12}} \\ &\equiv 4f_2^6 + 4qf_6^6 \pmod 8.\end{aligned} \tag{5.6.14}$$

Extracting the terms involving q^{2n} and q^{2n+1} in (5.6.14), we obtain

$$\sum_{n=0}^{\infty} c\psi_{6,0}(2n)q^n \equiv 4f_2^3 \pmod 8, \tag{5.6.15}$$

and

$$\sum_{n=0}^{\infty} c\psi_{6,0}(2n+1)q^n \equiv 4f_6^3 \pmod{8}, \tag{5.6.16}$$

respectively.

Congruences (5.6.12) and (5.6.13) follow from (5.6.15) and (5.6.16), respectively. □

Theorem 5.6.3. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\psi_{6,1}(n)q^n = 3\frac{f_2^{14}f_6^3}{f_1^{12}f_4^6} + 4\frac{f_2f_4^6f_{12}^2}{f_1^8f_8f_{24}} + 8\frac{f_2^6f_3^2f_4^2}{f_1^{10}f_6} + 8q\frac{f_2^2f_4^3f_6f_8f_{24}}{f_1^8f_{12}}. \tag{5.6.17}$$

Proof. From Theorem 5.1.2, the generating function $C\Psi_{6,1}(q)$ of $c\psi_{6,1}(n)$ is the constant term in

$$\begin{aligned}
G_z(q) &= z^2\prod_{n=0}^{\infty}\left(1+zq^{n+1}\right)^6\left(1+z^{-1}q^n\right)^6 \\
&= z^2 \times \frac{1}{f_1^6}\left(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m(m+1)}{2}}\right)^6 \\
&= \frac{1}{f_1^6}\sum_{m_1,m_2,m_3,m_4,m_5,m_6=-\infty}^{\infty}\Bigg(z^{m_1+m_2+m_3+m_4+m_5+m_6+2} \\
&\times q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}+\frac{m_4(m_4+1)}{2}+\frac{m_5(m_5+1)}{2}+\frac{m_6(m_6+1)}{2}}\Bigg).
\end{aligned} \tag{5.6.18}$$

Extracting the constant term in (5.6.18), we have

$$\sum_{n=0}^{\infty} c\psi_{6,1}(n)q^n = \frac{1}{f_1^6}\sum_{m_1,m_2,m_3,m_4,m_5=-\infty}^{\infty} q^{Q(\overline{m})}, \tag{5.6.19}$$

where

$$Q(\overline{m}) = \sum_{1\leq i,j\leq 5} m_im_j + 2\sum_{i=1}^{5} m_i.$$

Using Integer matrix exact covering system (5.6.4) in (5.6.19), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} c\psi_{6,1}(n)q^n = \frac{1}{f_1^6}\Bigg[&q\left(\sum_{n_1,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+2n_1+2n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2\left(\sum_{n=-\infty}^{\infty} q^{n^2+2n}\right) \\
&+2q^4\left(\sum_{n_1,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+4n_1+3n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)
\end{aligned}$$

$$\times \left(\sum_{n=-\infty}^{\infty} q^{n^2+3n}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2+n}\right)$$

$$+ q^5 \left(\sum_{n_1,\, n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+5n_1+5n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2+n}\right)^2 \left(\sum_{n=-\infty}^{\infty} q^{n^2+2n}\right)\Bigg]. \tag{5.6.20}$$

Employing Lemma 1.2.21, we have

$$\sum_{n_1,\, n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+2n_1+2n_4} = c(q^2)$$

$$= 3\frac{f_6^3}{f_2}. \tag{5.6.21}$$

Using the integer matrix exact covering system (5.6.6), we find that

$$\sum_{n_1,\, n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+4n_1+3n_4} = q^{-1}\psi(q)\left(f(q,q^{11}) + q^{-1}f(q^5,q^7)\right), \tag{5.6.22}$$

and

$$\sum_{n_1,\, n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+5n_1+5n_4} = q^{-4}\phi(q^2)f(q^4,q^8) + 2q^{-3}\psi(q^4)f(q^2,q^{10}). \tag{5.6.23}$$

Now setting $a = q^{-1}, b = q^4$ in (1.2.11), we have

$$f(q,q^{11}) + q^{-1}f(q^5,q^7) = f(q^{-1},q^4). \tag{5.6.24}$$

Using (5.6.24) in (5.6.22), we have

$$\sum_{n_1,\, n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+4n_1+3n_4} = q^{-1}\psi(q)f(q^{-1},q^4) = q^{-2}\psi(q)f(q,q^2). \tag{5.6.25}$$

Employing (5.6.21), (5.6.23), and (5.6.25) in (5.6.20), we have

$$\sum_{n=0}^{\infty} c\psi_{6,1}(n)q^n = \frac{1}{f_1^6}\Bigg[\frac{3f_6^3}{f_2}\varphi^3(q) + 8\varphi(q)\psi(q)\psi^2(q^2)f(q,q^2) + 4\varphi(q)\varphi(q^2)\psi^2(q^2)f(q^4,q^8)$$

$$+ 8q\varphi(q)\psi(q^4)\psi^2(q^2)f(q^2,q^{10})\Bigg] \tag{5.6.26}$$

Using (1.2.6), (1.2.7), (1.2.13), and (1.2.14) in (5.6.26), we arrive at (5.6.17).

□

Corollary 5.6.4. *For $n \geq 0$, we have*

$$c\psi_{6,1}(2n+1) \equiv 0 \pmod{4}. \tag{5.6.27}$$

Proof. From (5.6.17), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{6,1}(n)q^n &\equiv 3\frac{f_2^{14}f_6^3}{f_1^{12}f_4^6} \\ &\equiv 3\frac{f_2^8 f_6^3}{f_4^6} \pmod{4}. \end{aligned} \tag{5.6.28}$$

Congruence (5.6.27) follows from (5.6.28). □

Theorem 5.6.5. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} c\psi_{6,2}(n)q^n = 2\frac{f_2^7 f_4^2 f_{12}^2}{f_1^{10} f_8 f_{24}} + 4\frac{f_2^{12} f_3^2}{f_1^{12} f_4^2 f_6} + 4q\frac{f_2^8 f_6 f_8 f_{24}}{f_1^{10} f_4 f_{12}} + 24q\frac{f_4^6 f_6^3}{f_1^6 f_2^4}. \tag{5.6.29}$$

Proof. From Theorem 5.1.2, the generating function $C\Psi_{6,2}(q)$ of $c\psi_{6,2}(n)$ is the constant term in

$$\begin{aligned} G_z(q) &= z\prod_{n=0}^{\infty}\left(1+zq^{n+1}\right)^6\left(1+z^{-1}q^n\right)^6 \\ &= z \times \frac{1}{f_1^6}\left(\sum_{m=-\infty}^{\infty} z^m q^{\frac{m(m+1)}{2}}\right)^6 \\ &= \frac{1}{f_1^6}\sum_{m_1,m_2,m_3,m_4,m_5,m_6=-\infty}^{\infty}\Bigg(z^{m_1+m_2+m_3+m_4+m_5+m_6+1} \\ &\times q^{\frac{m_1(m_1+1)}{2}+\frac{m_2(m_2+1)}{2}+\frac{m_3(m_3+1)}{2}+\frac{m_4(m_4+1)}{2}+\frac{m_5(m_5+1)}{2}+\frac{m_6(m_6+1)}{2}}\Bigg). \end{aligned} \tag{5.6.30}$$

Extracting the constant term in (5.6.30), we have

$$\sum_{n=0}^{\infty} c\psi_{6,2}(n)q^n = \frac{1}{f_1^6}\sum_{m_1,m_2,m_3,m_4,m_5=-\infty}^{\infty} q^{Q(\overline{m})}, \tag{5.6.31}$$

where

$$Q(\overline{m}) = \sum_{1\leq i,j\leq 5} m_i m_j + \sum_{i=1}^{5} m_i$$

Using Integer matrix exact covering system (5.6.4), we have

$$\sum_{n=0}^{\infty} c\psi_{6,2}(n)q^n = \frac{1}{f_1^6}\Bigg[\left(\sum_{n_1,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+n_1+n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2\left(\sum_{n=-\infty}^{\infty} q^{n^2+n}\right)$$
$$+ 2q^2\left(\sum_{n_1,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+3n_1+2n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)$$
$$\times\left(\sum_{n=-\infty}^{\infty} q^{n^2+2n}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2+n}\right)$$
$$+ q^3\left(\sum_{n_1,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+4n_1+4n_4}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2+n}\right)^3\Bigg]. \quad (5.6.32)$$

Using the integer matrix exact covering system (5.6.6), we find that

$$\sum_{n_1,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+n_1+n_4} = \left(\sum_{n=-\infty}^{\infty} q^{2n^2}\right)\left(\sum_{n=-\infty}^{\infty} q^{6n^2+2n}\right)$$
$$+ q^3\left(\sum_{n=-\infty}^{\infty} q^{2n^2+2n}\right)\left(\sum_{n=-\infty}^{\infty} q^{6n^2+8n}\right)$$
$$= \varphi(q^2)f(q^4,q^8) + 2q^3\psi(q^4)f(q^{-2},q^{14})$$
$$= \varphi(q^2)f(q^4,q^8) + 2q\psi(q^4)f(q^2,q^{10}). \quad (5.6.33)$$

In a similar way, we find that

$$\sum_{n_1,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+3n_1+2n_4} = q^{-1}\psi(q)f(q,q^2), \quad (5.6.34)$$

and

$$\sum_{n_1,n_4=-\infty}^{\infty} q^{2n_1^2+2n_4^2+2n_1n_4+4n_1+4n_4} = 3q^{-2}\frac{f_6^3}{f_2}. \quad (5.6.35)$$

Using (5.6.33), (5.6.34), and (5.6.35) in (5.6.32), we find that

$$\sum_{n=0}^{\infty} c\psi_{6,2}(n)q^n = \frac{1}{f_1^6}\Big(2\varphi(q^2)\varphi^2(q)\psi(q^2)f(q^4,q^8) + 4\psi(q)\psi(q^2)\varphi^2(q)f(q,q^2)$$
$$+ 4q\varphi^2(q)\psi(q^2)\psi(q^4)f(q^2,q^{10}) + 24q\frac{f_6^3}{f_2}\psi^3(q^2)\Big). \quad (5.6.36)$$

Using (1.2.6), (1.2.7), (1.2.13), and (1.2.13) in (5.6.36), we arrive at (5.6.29). □

Corollary 5.6.6. *For $n \geq 0$, we have*

$$c\psi_{6,2}(2n+1) \equiv 0 \pmod{4}, \tag{5.6.37}$$

$$c\psi_{6,2}(4n+2) \equiv 0 \pmod{4}, \tag{5.6.38}$$

$$c\psi_{6,2}(4n+1) \equiv 0 \pmod{8}. \tag{5.6.39}$$

Proof. From (5.6.29), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{6,2}(n)q^n &\equiv 2\frac{f_2^7 f_4^2 f_{12}^2}{f_1^{10} f_8 f_{24}} \\ &\equiv 2f_2^2 \pmod{4}, \end{aligned} \tag{5.6.40}$$

from which (5.6.37) follows.

Extracting the terms involving q^{2n} in (5.6.40) we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{6,2}(2n)q^n &\equiv 2f_1^2 \\ &\equiv 2f_2 \pmod{4}. \end{aligned} \tag{5.6.41}$$

Congruence (5.6.38) follows from (5.6.41).

From (5.6.29), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{6,2}(n)q^n &\equiv 2\frac{f_2^7 f_4^2 f_{12}^2}{f_1^{10} f_8 f_{24}} + 4\frac{f_2^{12} f_3^2}{f_1^{12} f_4^2 f_6} + 4q\frac{f_2^8 f_6 f_8 f_{24}}{f_1^{10} f_4 f_{12}} \pmod{8} \\ &\equiv 2\frac{f_2^7 f_4^2 f_{12}^2}{f_1^2 f_4^2 f_8 f_{24}} + 4\frac{f_2^{12} f_6}{f_4^3 f_4^2 f_6} + 4q\frac{f_2^8 f_6 f_8 f_{24}}{f_2^5 f_4 f_{12}} \pmod{8} \\ &= 2\frac{f_2^7 f_{12}^2}{f_1^2 f_8 f_{24}} + 4\frac{f_2^{12}}{f_4^5} + 4q\frac{f_2^3 f_6 f_8 f_{24}}{f_4 f_{12}}. \end{aligned} \tag{5.6.42}$$

Employing (1.2.16) in (5.6.42) and extracting the terms involving odd powers of q, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c\psi_{6,2}(2n+1)q^{2n+1} &\equiv 2\frac{f_2^7 f_{12}^2}{f_8 f_{24}}\left(2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) + 4q\frac{f_2^3 f_6 f_8 f_{24}}{f_4 f_{12}} \pmod{8} \\ &= 4q\frac{f_2^2 f_4^2 f_{12}^2 f_{16}^2}{f_8^2 f_{24}} + 4q\frac{f_2^3 f_6 f_8 f_{24}}{f_4 f_{12}}. \end{aligned} \tag{5.6.43}$$

Dividing (5.6.43) by q and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty} c\psi_{6,2}(2n+1)q^n \equiv 4\frac{f_1^2f_2^2f_6^2f_8^2}{f_4^2f_{12}} + 4\frac{f_1^3f_3f_4f_{12}}{f_2f_6} \pmod 8$$

$$\equiv 4\frac{f_2^3f_6^2f_8^2}{f_4^2f_{12}} + 4\frac{f_1f_3f_4f_{12}}{f_6} \pmod 8. \tag{5.6.44}$$

Employing (1.2.20) in (5.6.44), extracting the terms having even powers of q and then replacing q^2 by q we obtain

$$\sum_{n=0}^{\infty} c\psi_{6,2}(4n+1)q^n \equiv 4\frac{f_1^3f_3^2f_4^2}{f_2^2f_6} + 4\frac{f_1f_4^2f_6^5}{f_2f_3^2f_{12}^2} \pmod 8$$

$$\equiv 4\frac{f_1^3f_3^2f_1^8}{f_1^4f_3^2} + 4\frac{f_1f_1^8f_3^{10}}{f_1^2f_3^{10}} \pmod 8$$

$$= 8f_1^7. \tag{5.6.45}$$

Congruence (5.6.39) follows from (5.6.45). □

5.7 Concluding remarks

A closer look at the respective generating functions of the associated (k, a)-colored Frobenius partition functions reveals the following relations.

Theorem 5.7.1. *We have*

$$c\psi_{4,3}(n) = c\psi_{4,1}(n-1) \quad n \geq 1,$$

$$c\psi_{6,4}(n) = c\psi_{6,2}(n-2) \quad n \geq 2,$$

$$c\psi_{6,5}(n) = c\psi_{6,1}(n-3) \quad n \geq 3.$$

From the above identities, it will be interesting to explore possibility of existence of other such results.

We have established few congruences for $c\psi_{6,0}(n)$, $c\psi_{6,1}(n)$ and $c\psi_{6,2}(n)$ modulo powers of 2. However, computational evidence supports the existence of more such congruences. For instance, we have the following conjecture.

Conjecture 5.7.2. *For* $n \geq 0$*, we have*

$$c\psi_{6,0}(64n+k) \equiv 0 \pmod{16}, \quad k \in \{5, 25, 33, 41, 45\}.$$