Chapter 6

Distance Laplacian spectrum and energy of Commuting Conjugacy Class graphs

In this chapter, we compute distance Laplacian spectrum and distance Laplacian energy of CCC-graphs of the groups $D_{2n} = \langle x, y : x^n = y^2 = 1,\ yxy^{-1} = x^{-1} \rangle$, $Q_{4n} = \langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1} \rangle$, $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1,\ x^{-1}yx = y^{-1} \rangle$, $V_{8n} = \langle x, y : x^{2n} = y^4 = 1,\ yx = x^{-1}y^{-1},\ y^{-1}x = x^{-1}y \rangle$, $SD_{8n} = \langle x, y : x^{4n} = y^2 = 1, yxy = x^{2n-1} \rangle$ along with finite groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2n}. As a consequence of our results, it follows that CCC-graphs of the above mentioned groups are DL-integral. The computation of DL-energy of CCC-graphs will also lead towards examining the equality in Problem 1.1.13 of for CCC-graphs. This chapter is based on our paper [71] published in the proceeding of the *International Conference on Algebra and its Applications-2023*.

6.1 DL-spectrum and DL-energy of CCC-graphs

We begin this section by considering CCC-graph of the dihedral group.

Theorem 6.1.1. *Let D_{2n} $(n \geq 3)$ be the dihedral group.*

(a) *If n is odd then*

$$\text{DL-spec}(\Gamma_{\text{ccc}}(D_{2n})) = \left\{[0]^1, [n+2]^1, \left[\frac{n+5}{2}\right]^{\frac{n-3}{2}}, \left[\frac{n+3}{2}\right]^1\right\}$$

$$\textit{and} \quad \text{E}_{\text{DL}}(\Gamma_{\text{ccc}}(D_{2n})) = \begin{cases} \frac{16}{3}, & \textit{when } n = 3 \\ \frac{(n-1)(n+11)}{n+3}, & \textit{when } n \geq 5. \end{cases}$$

(b) *If n and $\frac{n}{2}$ are even then*

$$\text{DL-spec}(\Gamma_{\text{ccc}}(D_{2n})) = \left\{[0]^1, [n+4]^2, \left[\frac{n+10}{2}\right]^{\frac{n-4}{2}}, \left[\frac{n+6}{2}\right]^2\right\}$$

$$\textit{and} \quad \text{E}_{\text{DL}}(\Gamma_{\text{ccc}}(D_{2n})) = \begin{cases} \frac{n^2+30n-24}{n+6}, & \textit{when } 4 \leq n \leq 20 \\ \frac{2n(n+2)+64}{n+6}, & \textit{when } n \geq 24. \end{cases}$$

(c) *If n is even and $\frac{n}{2}$ is odd then*

$$\text{DL-spec}(\Gamma_{\text{ccc}}(D_{2n})) = \left\{[0]^1, [n+2]^1, [n+4]^1, \left[\frac{n+10}{2}\right]^{\frac{n-4}{2}}, \left[\frac{n+6}{2}\right]^2\right\}$$

$$\textit{and} \quad \text{E}_{\text{DL}}(\Gamma_{\text{ccc}}(D_{2n})) = \begin{cases} \frac{n^2+30n-48}{n+6}, & \textit{when } 6 \leq n \leq 22 \\ \frac{2(n^2+28)}{n+6}, & \textit{when } n \geq 26. \end{cases}$$

Proof. We know that

$$|Z(D_{2n})| = \begin{cases} 1, & \text{when } n \text{ is odd} \\ 2, & \text{when } n \text{ is even.} \end{cases}$$

By Result 1.2.20 and (1.2.a), we have

$$\Gamma_{\text{ccc}}(D_{2n}) = \begin{cases} K_1 + (K_{\frac{n-1}{2}} \cup K_1), & \text{when } n \text{ is odd} \\ K_2 + (K_{\frac{n}{2}-1} \cup 2K_1), & \text{when } n \text{ and } \frac{n}{2} \text{ are even} \\ K_2 + (K_{\frac{n}{2}-1} \cup K_2), & \text{when } n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

(a) If n is odd then by Result 1.1.15, we get

$$\mathrm{Ch}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(D_{2n}),x) = x\left(x+\frac{1-n}{2}-2\right)(x-n-2)\left(x+\frac{n-1}{2}-n-2\right)^{\frac{n-1}{2}-1}.$$

Therefore, DL-spec$(\Gamma_{\mathrm{ccc}}(D_{2n})) = \left\{[0]^1, [n+2]^1, [\frac{n+5}{2}]^{\frac{n-3}{2}}, [\frac{n+3}{2}]^1\right\}$.

Here $|v(\Gamma_{\mathrm{ccc}}(D_{2n}))| = \frac{n+3}{2}$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\mathrm{ccc}}(D_{2n})) = \frac{1}{4}(n(n+8)-1)$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n})) = \frac{n(n+8)-1}{2(n+3)}$. We have

$$\alpha_1 := \left|0-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n}))\right| = \left|-\frac{n(n+8)-1}{2(n+3)}\right| = \frac{n(n+8)-1}{2(n+3)},$$

$$\alpha_2 := \left|(n+2)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n}))\right| = \left|\frac{n^2+2n+13}{2n+6}\right| = \frac{n^2+2n+13}{2n+6},$$

$$\alpha_3 := \left|\frac{n+5}{2}-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n}))\right| = \left|\frac{8}{n+3}\right| = \frac{8}{n+3}$$

and

$$\alpha_4 := \left|\frac{n+3}{2}-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n}))\right| = \left|\frac{5-n}{n+3}\right| = \begin{cases} \frac{1}{3}, & \text{when } n=3 \\ \frac{n-5}{n+3}, & \text{when } n\geq 5. \end{cases}$$

Hence, by (1.1.*f*), we get

$$\begin{aligned} \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(D_{2n})) &= 1\times\alpha_1 + 1\times\alpha_2 + \frac{n-3}{2}\times\alpha_3 + 1\times\alpha_4 \\ &= \begin{cases} \frac{16}{3}, & \text{when } n=3 \\ \frac{(n-1)(n+11)}{n+3}, & \text{when } n\geq 5. \end{cases} \end{aligned}$$

(b) If n and $\frac{n}{2}$ are even then by Result 1.1.15, we get

$$\mathrm{Ch}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(D_{2n}),x) = x\left(x-2\left(\frac{n}{2}-1\right)-6\right)^2\left(x-\frac{n}{2}-5\right)^{\frac{n}{2}-2}\left(x-\frac{n}{2}-3\right)^2.$$

Therefore, DL-spec$(\Gamma_{\mathrm{ccc}}(D_{2n})) = \left\{[0]^1, [n+4]^2, [\frac{n+10}{2}]^{\frac{n-4}{2}}, [\frac{n+6}{2}]^2\right\}$.

Here $|v(\Gamma_{\rm ccc}(D_{2n}))| = \frac{n+6}{2}$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\rm ccc}(D_{2n})) = \frac{1}{4}(n^2+18n+16)$. Therefore, $\Delta_{\rm D}(\Gamma_{\rm ccc}(D_{2n})) = \frac{n^2+18n+16}{2(n+6)}$. We have

$$\beta_1 := \left|0 - \Delta_{\rm D}(\Gamma_{\rm ccc}(D_{2n}))\right| = \left| -\frac{n^2+18n+16}{2(n+6)}\right| = \frac{n^2+18n+16}{2(n+6)},$$

$$\beta_2 := \left|(n+4) - \Delta_{\rm D}(\Gamma_{\rm ccc}(D_{2n}))\right| = \left|\frac{n^2+2n+32}{2(n+6)}\right| = \frac{n^2+2n+32}{2(n+6)},$$

$$\beta_3 := \left|\frac{n+10}{2} - \Delta_{\rm D}(\Gamma_{\rm ccc}(D_{2n}))\right| = \left|\frac{22-n}{n+6}\right| = \begin{cases} \frac{22-n}{n+6}, & \text{when } n \le 21 \\ \frac{n-22}{n+6}, & \text{when } n \ge 22 \end{cases}$$

and

$$\beta_4 := \left|\frac{n+6}{2} - \Delta_{\rm D}(\Gamma_{\rm ccc}(D_{2n}))\right| = \left|\frac{10-3n}{n+6}\right| = \frac{3n-10}{n+6}, \qquad \text{as } n \ge 4$$

Hence, by (1.1.*f*), we get

$$\begin{aligned} E_{\rm DL}(\Gamma_{\rm ccc}(D_{2n})) &= 1\times\beta_1 + 2\times\beta_2 + \frac{n-4}{2}\times\beta_3 + 2\times\beta_4 \\ &= \begin{cases} \frac{n^2+30n-24}{n+6}, & \text{when } 4 \le n \le 20 \\ \frac{2n(n+2)+64}{n+6}, & \text{when } n \ge 24. \end{cases} \end{aligned}$$

(c) If n is even and $\frac{n}{2}$ is odd then, by Result 1.1.15, we get

$$\begin{aligned} \mathrm{Ch}_{\rm DL}(\Gamma_{\rm ccc}(D_{2n}), x) = x\left(x - 2\left(\frac{n}{2}-1\right) - 6\right)\left(x - 2\left(\frac{n}{2}-1\right) - 4\right)\left(x - \frac{n}{2} - 3\right)^2 \\ \times \left(x - 2\left(\frac{n}{2}-1\right) + \frac{n}{2} - 7\right)^{\frac{n}{2}-2}. \end{aligned}$$

Therefore, $\text{DL-spec}(\Gamma_{\rm ccc}(D_{2n})) = \left\{[0]^1, [n+2]^1, [n+4]^1, [\frac{n+10}{2}]^{\frac{n-4}{2}}, [\frac{n+6}{2}]^2\right\}$.

Here $|v(\Gamma_{\rm ccc}(D_{2n}))| = \frac{n+6}{2}$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\rm ccc}(D_{2n})) = \frac{1}{4}(n^2+18n+8)$. Therefore, $\Delta_{\rm D}(\Gamma_{\rm ccc}(D_{2n}))$ $= \frac{n^2+18n+8}{2(n+6)}$. We have

$$\gamma_1 := \left|0 - \Delta_{\rm D}(\Gamma_{\rm ccc}(D_{2n}))\right| = \left| -\frac{n^2+18n+8}{2(n+6)}\right| = \frac{n^2+18n+8}{2(n+6)},$$

$$\gamma_2 := \left|(n+2) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n}))\right| = \left|\frac{n^2-2n+16}{2(n+6)}\right| = \frac{n^2-2n+16}{2(n+6)},$$

$$\gamma_3 := \left|(n+4) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n}))\right| = \left|\frac{n^2+2n+40}{2(n+6)}\right| = \frac{n^2+2n+40}{2(n+6)},$$

$$\gamma_4 := \left|\frac{n+10}{2} - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n}))\right| = \left|\frac{26-n}{n+6}\right| = \begin{cases} \frac{26-n}{n+6}, & \text{when } n \leq 25 \\ \frac{n-26}{n+6}, & \text{when } n \geq 26 \end{cases}$$

and

$$\gamma_5 := \left|\frac{n+6}{2} - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(D_{2n}))\right| = \left|\frac{14-3n}{n+6}\right| = \frac{3n-14}{n+6}, \qquad \text{as } n \geq 6$$

Therefore, by (1.1.*f*), we get

$$\begin{aligned} \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(D_{2n})) &= 1 \times \gamma_1 + 1 \times \gamma_2 + 1 \times \gamma_3 + \frac{n-4}{2} \times \gamma_4 + 2 \times \gamma_5 \\ &= \begin{cases} \frac{n^2+30n-48}{n+6}, & \text{when } 6 \leq n \leq 22 \\ \frac{2\left(n^2+28\right)}{n+6}, & \text{when } n \geq 26. \end{cases} \end{aligned}$$

Hence the result follows. □

Theorem 6.1.2. *Let Q_{4n} $(n \geq 2)$ be the dicyclic group.*

(a) *If n is even then*

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(Q_{4n})) = \left\{[0]^1, [2(n+2)]^2, [n+5]^{(n-2)}, [n+3]^2\right\}$$

$$\textit{and } \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(Q_{4n})) = \begin{cases} \frac{2(n^2+15n-6)}{n+3}, & \textit{when } n \leq 10 \\ \frac{4(n^2+n+8)}{n+3}, & \textit{when } n \geq 12. \end{cases}$$

(b) *If n is odd then*

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(Q_{4n})) = \left\{[0]^1, [2(n+2)]^1, [2(n+1)]^1, [n+3]^2, [n+5]^{n-2}\right\}$$

$$\textit{and } \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(Q_{4n})) = \begin{cases} \frac{2\left(n^2+15n-12\right)}{n+3}, & \textit{when } n \leq 11 \\ \frac{4\left(n^2+7\right)}{n+3}, & \textit{when } n \geq 13. \end{cases}$$

Proof. We know that $|Z(Q_{4n})| = 2$. Now, by Result 1.2.21 and (1.2.a), we have

$$\Gamma_{\text{ccc}}(Q_{4n}) = \begin{cases} K_2 + (2K_1 \cup K_{n-1}), & \text{when } n \text{ is even} \\ K_2 + (K_{n-1} \cup K_2), & \text{when } n \text{ is odd.} \end{cases}$$

(a) If n is even then by Result 1.1.15, we get

$$\text{Ch}_{\text{DL}}(\Gamma_{\text{ccc}}(Q_{4n}), x) = x(x - 2(n-1) - 6)^2(x - n - 5)^{n-2}(x - n - 3)^2.$$

Therefore, DL-spec$(\Gamma_{\text{ccc}}(Q_{4n})) = \{[0]^1, [2(n+2)]^2, [n+5]^{n-2}, [n+3]^2\}$.

Here $|v(\Gamma_{\text{ccc}}(Q_{4n}))| = n + 3$ and $\text{tr}(\text{DL}(\Gamma_{\text{ccc}}(Q_{4n})) = n^2 + 9n + 4$. Therefore, $\Delta_{\text{D}}(\Gamma_{\text{ccc}}(Q_{4n})) = \frac{n^2+9n+4}{n+3}$. We have

$$\alpha_1 := \left|0 - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(Q_{4n}))\right| = \left|-\frac{n^2+9n+4}{n+3}\right| = \frac{n^2+9n+4}{n+3},$$

$$\alpha_2 := \left|2(n+2) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(Q_{4n}))\right| = \left|\frac{n^2+n+8}{n+3}\right| = \frac{n^2+n+8}{n+3},$$

$$\alpha_3 := \left|(n+5) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(Q_{4n}))\right| = \left|\frac{11-n}{n+3}\right| = \begin{cases} \frac{11-n}{n+3}, & \text{when } n \le 10 \\ \frac{n-11}{n+3}, & \text{when } n \ge 11 \end{cases}$$

and

$$\alpha_4 := \left|(n+3) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(Q_{4n}))\right| = \left|\frac{5-3n}{n+3}\right| = \frac{3n-5}{n+3}, \quad \text{as } n \ge 2.$$

Hence, by (1.1.f), we get

$$\begin{aligned} \text{E}_{\text{DL}}(\Gamma_{\text{ccc}}(Q_{4n})) &= 1 \times \alpha_1 + 2 \times \alpha_2 + (n-2) \times \alpha_3 + 2 \times \alpha_4 \\ &= \begin{cases} \frac{2(n^2+15n-6)}{n+3}, & \text{when } n \le 10 \\ \frac{4(n^2+n+8)}{n+3}, & \text{otherwise.} \end{cases} \end{aligned}$$

(b) If n is odd then by Result 1.1.15, we get

$$\mathrm{Ch}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(Q_{4n}), x) = x(x-2(n-1)-6)(x-2(n-1)-4)(x-n-3)^2 \times (x-2(n-1)+n-7)^{n-2}.$$

Therefore, DL-spec$(\Gamma_{\mathrm{ccc}}(Q_{4n})) = \{[0]^1, [2(n+2)]^1, [2(n+1)]^1, [n+3]^2, [n+5]^{n-2}\}$.

Here $|v(\Gamma_{\mathrm{ccc}}(Q_{4n}))| = n+3$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\mathrm{ccc}}(Q_{4n}))) = n^2+9n+2$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(Q_{4n})) = \frac{n^2+9n+2}{n+3}$. We have

$$\beta_1 := \left|0 - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(Q_{4n}))\right| = \left|-\frac{n^2+9n+2}{n+3}\right| = \frac{n^2+9n+2}{n+3},$$

$$\beta_2 = \left|2(n+2) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(Q_{4n}))\right| = \left|\frac{n^2+n+10}{n+3}\right| = \frac{n^2+n+10}{n+3},$$

$$\beta_3 := \left|2(n+1) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(Q_{4n}))\right| = \left|\frac{n^2-n+4}{n+3}\right| = \frac{n^2-n+4}{n+3},$$

$$\beta_4 := \left|(n+3) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(Q_{4n}))\right| = \left|\frac{7-3n}{n+3}\right| = \frac{3n-7}{n+3}, \quad \text{as } n \geq 3$$

and

$$\beta_5 := \left|(n+5) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(Q_{4n}))\right| = \left|\frac{13-n}{n+3}\right| = \begin{cases} \frac{13-n}{n+3}, & \text{when } n \leq 12 \\ \frac{n-13}{n+3}, & \text{when } n \geq 13. \end{cases}$$

Therefore, by (1.1.*f*), we get

$$\begin{aligned} \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(Q_{4n})) &= 1 \times \beta_1 + 1 \times \beta_2 + 1 \times \beta_3 + 2 \times \beta_4 + (n-2) \times \beta_5 \\ &= \begin{cases} \frac{2\left(n^2+15n-12\right)}{n+3}, & \text{when } n \leq 11 \\ \frac{4\left(n^2+7\right)}{n+3}, & \text{when } n \geq 13. \end{cases} \end{aligned}$$

Hence the result follows. □

Theorem 6.1.3. *The DL-spectrum and DL-energy of CCC-graph of the group $U_{(n,m)}$ ($m \geq 3$, $n \geq 2$) are as given below.*

(a) *If m is odd then*

$$\text{DL-spec}(\Gamma_{\text{ccc}}(U_{(n,m)})) =$$
$$\left\{[0]^1, [n(m+2)]^1, [n(m+1)]^{(n-1)}, \left[\frac{n(m+5)}{2}\right]^{\frac{n(m-1)-2}{2}}, \left[\frac{m(m+3)}{2}\right]^n\right\}$$

and

$$\text{E}_{\text{DL}}(\Gamma_{\text{ccc}}(U_{(n,m)}))=\begin{cases} \frac{m^2n+m\left(4n^2+8n-2\right)-4n^2-n-6}{m+3}, & \text{when } m=3,5\,\&\,n\geq 2; \\ & m=7\,\&\,n=2,3,4; \\ & m=9\,\&\,n=2; m=11\,\&\,n=2 \\ \frac{n\left(m^2n-2m(n-2)+n+12\right)}{m+3}, & \text{otherwise.} \end{cases}$$

(b) *If m and $\frac{m}{2}$ are even then*

$$\text{DL-spec}(\Gamma_{\text{ccc}}(U_{(n,m)})) =$$
$$\left\{[0]^1, [n(m+4)]^2, [n(m+3)]^{2(n-1)}, \left[\frac{n(m+10)}{2}\right]^{\frac{n(m-2)-2}{2}}, \left[\frac{n(m+6)}{2}\right]^{2n}\right\}$$

and

$$\text{E}_{\text{DL}}(\Gamma_{\text{ccc}}(U_{(n,m)}))=\begin{cases} \frac{m^2n+2m\left(8n^2+8n-1\right)+4\left(-4n^2+n-3\right)}{m+6}, & \text{when } m=4,8\,\&\,n\geq 2; \\ & m=12\,\&\,n=2 \\ \frac{2n\left(m^2n-2m(n-2)+8(n+3)\right)}{m+6}, & \text{otherwise.} \end{cases}$$

(c) *If m is even and $\frac{m}{2}$ is odd*

$$\text{DL-spec}(\Gamma_{\text{ccc}}(U_{(n,m)})) =$$
$$\left\{[0]^1, [n(m+4)]^1, [n(m+2)]^{2n-1}, \left[\frac{n(m+10)}{2}\right]^{\frac{n(m-2)}{2}-1}, \left[\frac{n(m+6)}{2}\right]^{2n}\right\}.$$

and

$$\text{E}_{\text{DL}}(\Gamma_{\text{ccc}}(U_{(n,m)})) = \begin{cases} \frac{m^2n+2m\left(8n^2+8n-1\right)-4\left(8n^2+n+3\right)}{m+6}, & \text{when } m=6,10\,\&\,n\geq 2; \\ & m=14\,\&\,n=2 \\ \frac{2n\left(m^2n-4m(n-1)+4(n+6)\right)}{m+6}, & \text{otherwise.} \end{cases}$$

Proof. We know that

$$|Z(U_{(n,m)})| = \begin{cases} 2n, & \text{when } m \text{ is even} \\ n, & \text{when } m \text{ is odd.} \end{cases}$$

By Result 1.2.22 and (1.2.a), we have

$$\Gamma_{\text{ccc}}(U_{(n,m)}) = \begin{cases} K_n + (K_n \cup K_{n(\frac{m-1}{2})}), & \text{when } m \text{ is odd} \\ K_{2n} + (2K_n \cup K_{n(\frac{m}{2}-1)}), & \text{when } m \text{ and } \frac{m}{2} \text{ are even} \\ K_{2n} + (K_{2n} \cup K_{n(\frac{m}{2}-1)}), & \text{when } m \text{ is even and } \frac{m}{2} \text{ is odd} \end{cases}$$

(a) If m is odd then by Result 1.1.15, we get

$$\begin{aligned} \text{Ch}_{\text{DL}}(\Gamma_{\text{ccc}}(U_{(n,m)}), x) = x(x-(m-1)n-3n)(x-(m-1)n-2n)^{n-1} \\ \times \left(x - \frac{1}{2}(m-1)n - 3n\right)^{\frac{1}{2}(m-1)n-1} \left(x - \frac{1}{2}(m-1)n - 2n\right)^{n}. \end{aligned}$$

Therefore,

$$\text{DL-spec}(\Gamma_{\text{ccc}}(U_{(n,m)})) = \left\{ [0]^1, [n(m+2)]^1, [n(m+1)]^{(n-1)}, \left[\frac{n(m+5)}{2}\right]^{\frac{n(m-1)-2}{2}}, \left[\frac{m(m+3)}{2}\right]^{n} \right\}.$$

Here $|v(\Gamma_{\text{ccc}}(U_{(n,m)}))| = \frac{1}{2}(m+3)n$ and $\text{tr}(\text{DL}(\Gamma_{\text{ccc}}(U_{(n,m)})) = \frac{1}{4}n\left\{m^2n + 10mn - 2m +5n - 6\right\}$. Therefore, $\Delta_{\text{D}}(\Gamma_{\text{ccc}}(U_{(n,m)})) = \frac{m^2n+10mn-2m+5n-6}{2(m+3)}$. We have

$$\begin{aligned} \alpha_1 := \left|0 - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(U_{(n,m)}))\right| &= \left| - \frac{m^2n + 10mn - 2m + 5n - 6}{2(m+3)} \right| \\ &= \frac{m^2n + 10mn - 2m + 5n - 6}{2(m+3)} \end{aligned}$$

(as $m^2n + 10mn - 2m + 5n - 6 > 0$),

$$\alpha_2 := \left|n(m+2) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(U_{(n,m)}))\right| = \left|\frac{m^2n + 2m + 7n + 6}{2m+6}\right| = \frac{m^2n + 2m + 7n + 6}{2m+6}$$

(as $m^2n + 2m + 7n + 6 > 0$),

$$\alpha_3 := \left|n(m+1) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(U_{(n,m)}))\right| = \left|\frac{m^2n - 2mn + 2m + n + 6}{2m+6}\right|$$

$$= \frac{m^2n - 2mn + 2m + n + 6}{2m + 6}$$

(as $m^2n - 2mn + 2m + n + 6 > 0$) and

$$\alpha_4 := \left|\frac{1}{2}n(m+5) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{-mn + m + 5n + 3}{m + 3}\right|.$$

Let $f_1(m,n) = -mn + m + 5n + 3 = -\frac{1}{2}m(n-2) - \frac{1}{2}n(m-10) + 3$. Clearly, for $m \geq 11$ and $n \geq 3$, $f_1(m,n) < 0$. Again

$$f_1(m,2) = 13 - m \begin{cases} > 0, & \text{when } 3 \leq m \leq 12 \\ \leq 0, & \text{when } m \geq 13, \end{cases}$$

$$f_1(3,n) = 6 + 2n > 0, \quad f_1(5,n) = 8,$$

$$f_1(7,n) = 10 - 2n \begin{cases} > 0, & \text{when } n = 2,3,4 \\ \leq 0, & \text{when } n \geq 5 \end{cases}$$

and

$$f_1(9,n) = 12 - 4n \begin{cases} > 0, & \text{when } n = 2 \\ \leq 0, & \text{when } n \geq 3. \end{cases}$$

Therefore,

$$\alpha_4 = \begin{cases} \frac{-mn+m+5n+3}{m+3}, & \text{when } m = 3,5 \,\&\, n \geq 2;\ m = 7 \,\&\, n = 2,3,4; \\ & m = 9 \,\&\, n = 2;\ m = 11 \,\&\, n = 2 \\ \frac{-(-mn+m+5n+3)}{m+3}, & \text{otherwise.} \end{cases}$$

Also,

$$\alpha_5 := \left|\frac{1}{2}n(m+3) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{-2mn + m + 2n + 3}{m + 3}\right|.$$

Let $f_2(m,n) = -2mn + m + 2n + 3 = n(2-m) + m(1-n) + 3$. Clearly, for $m \geq 3$, $f_2(m,n) < 0$, as $n \geq 2$. Therefore,

$$\alpha_5 = \frac{-(-2mn + m + 2n + 3)}{m + 3}.$$

Therefore, by (1.1.*f*), we get

$$\begin{aligned}\mathrm{E_{DL}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)})) &= 1\times\alpha_1 + 1\times\alpha_2 + (n-1)\times\alpha_3 + \frac{n(m-1)-2}{2}\times\alpha_4 + n\times\alpha_5\\ &= \begin{cases} \frac{m^2n+m\left(4n^2+8n-2\right)-4n^2-n-6}{m+3}, & \text{when } m=3,5\,\&\,n\geq 2;\\ & m=7\,\&\,n=2,3,4;\ m=9\,\&\,n=2;\\ & m=11\,\&\,n=2\\ \frac{n\left(m^2n-2m(n-2)+n+12\right)}{m+3}, & \text{otherwise.}\end{cases}\end{aligned}$$

(b) If m and $\frac{m}{2}$ are even then by Result 1.1.15, we get

$$\begin{aligned}\mathrm{Ch_{DL}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}),x) &= x\left(x-2\left(\frac{m}{2}-1\right)n-6n\right)^2\left(x-2\left(\frac{m}{2}-1\right)n-5n\right)^{2(n-1)}\\ &\quad\times\left(x-\left(\frac{m}{2}-1\right)n-6n\right)^{\left(\frac{m}{2}-1\right)n-1}\left(x-\left(\frac{m}{2}-1\right)n-4n\right)^{2n}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\text{DL-spec}(\Gamma_{\mathrm{ccc}}(U_{(n,m)})) =\\ &\left\{[0]^1, [n(m+4)]^2, [n(m+3)]^{2(n-1)}, \left[\frac{n(m+10)}{2}\right]^{\frac{n(m-2)-2}{2}}, \left[\frac{n(m+6)}{2}\right]^{2n}\right\}.\end{aligned}$$

Here $|v(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))$=$\frac{1}{2}(m+6)n$ and tr(DL($\Gamma_{\mathrm{ccc}}(U_{(n,m)})$)=$\frac{1}{4}n\left\{m^2n+20mn-2m\right.$ $\left.+28n-12\right\}$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)})) = \frac{m^2n+20mn-2m+28n-12}{2(m+6)}$. We have

$$\begin{aligned}\beta_1 := \left|0-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| &= \left|-\frac{m^2n+20mn-2m+28n-12}{2(m+6)}\right|\\ &= \frac{m^2n+20mn-2m+28n-12}{2(m+6)},\end{aligned}$$

since $m^2n+20mn-2m+28n-12>0$ as $m\geq 3$ and $n\geq 2$. We also have

$$\begin{aligned}\beta_2 := \left|(m+4)n-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| &= \left|\frac{m^2n+2m+20n+12}{2m+12}\right|\\ &= \frac{m^2n+2m+20n+12}{2m+12}\end{aligned}$$

and

$$\beta_3 := \left|n(m+3) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{m^2n - 2mn + 2m + 8n + 12}{2m+12}\right| = \frac{m^2n - 2mn + 2m + 8n + 12}{2m+12},$$

since $m^2n \geq 2mn$, as $m \geq 3$, and so $m^2n - 2mn + 2m + 8n + 12 > 0$. Again

$$\beta_4 := \left|\frac{1}{2}n(m+10) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{-2mn + m + 16n + 6}{m+6}\right|.$$

Let $g_1(m,n) = -2mn + m + 16n + 6 = m(1-n) + n(16-m) + 6$. Clearly, for $m \geq 16$, $g_1(m,n) < 0$, as $n \geq 2$. Also, $g_1(4,n) = 8n + 10 > 0$, $g_1(8,n) = 14 > 0$ and

$$g_1(12,n) = 2(9-4n) \begin{cases} > 0, & \text{when } n = 2 \\ < 0, & \text{when } n \geq 3. \end{cases}$$

Therefore,

$$\beta_4 = \begin{cases} \frac{-2mn+m+16n+6}{m+6}, & \text{when } m = 4, 8 \,\&\, n \geq 2; \\ & m = 12 \,\&\, n = 2 \\ -\left(\frac{-2mn+m+16n+6}{m+6}\right), & \text{otherwise.} \end{cases}$$

Also,

$$\beta_5 := \left|\frac{1}{2}n(m+6) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{-4mn + m + 4n + 6}{m+6}\right|.$$

Let $g_2(m,n) = -4mn + m + 4n + 6 = m(1-n) + 2n(2-m) + (6-mn)$. Clearly, for $m \geq 4$, $g_2(m,n) < 0$, as $n \geq 2$. Therefore,

$$\beta_5 = -\left(\frac{-4mn + m + 4n + 6}{m+6}\right).$$

Hence, by (1.1.*f*), we get

$$\begin{aligned} \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)})) &= 1 \times \beta_1 + 2 \times \beta_2 + 2(n-1) \times \beta_3 + \frac{n(m-2)-2}{2} \times \beta_4 + 2n \times \beta_5 \\ &= \begin{cases} \frac{m^2n + 2m\left(8n^2+8n-1\right) + 4\left(-4n^2+n-3\right)}{m+6}, & \text{when } m = 4, 8 \,\&\, n \geq 2; \\ & m = 12 \,\&\, n = 2 \\ \frac{2n\left(m^2n - 2m(n-2) + 8(n+3)\right)}{m+6}, & \text{otherwise.} \end{cases} \end{aligned}$$

(c) If m is even and $\frac{m}{2}$ is odd then by Result 1.1.15, we get

$$\mathrm{Ch}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}),x) = x\left(x-2\left(\frac{m}{2}-1\right)n-6n\right)\left(x-2\left(\frac{m}{2}-1\right)n-4n\right)^{2n-1} \times\left(x-\left(\frac{m}{2}-1\right)n-6n\right)^{\left(\frac{m}{2}-1\right)n-1}\left(x-\left(\frac{m}{2}-1\right)n-4n\right)^{2n}.$$

Therefore,

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(U_{(n,m)})) = \left\{[0]^1,[n(m+4)]^1,[n(m+2)]^{2n-1},\left[\frac{n(m+10)}{2}\right]^{\frac{n(m-2)}{2}-1},\left[\frac{n(m+6)}{2}\right]^{2n}\right\}.$$

Here $|v(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))| = \frac{1}{2}(m+6)n$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\mathrm{ccc}}(U_{(n,m)})) = \frac{1}{4}n\left(m^2n+20mn-2m+20n-12\right)$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)})) = \frac{m^2n+20mn-2m+20n-12}{2(m+6)}$. We have

$$\beta_1' := \left|0-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|-\frac{m^2n+20mn-2m+20n-12}{2(m+6)}\right| = \frac{m^2n+20mn-2m+20n-12}{2(m+6)},$$

as $m^2n+20mn-2m+20n-12>0$. We also have

$$\beta_2' := \left|n(m+4)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{m^2n+2m+28n+12}{2m+12}\right| = \frac{m^2n+2m+28n+12}{2m+12},$$

$$\beta_3' := \left|n(m+2)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{m^2n-4mn+2m+4n+12}{2m+12}\right| = \frac{m^2n-4mn+2m+4n+12}{2m+12},$$

as for $m\geq 6$, $m^2n-4mn+2m+4n+12 = mn(m-4)+4n+2m+12>0$. Again,

$$\beta_4' := \left|\frac{n(m+10)}{2}-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{-2mn+m+20n+6}{m+6}\right|.$$

Let $h_1(m,n) := -2mn+m+20n+6 = n(20-m)-m(n-1)+6$. Clearly for $m\geq 20$, $h_1(m,n)<0$. Again $h_1(6,n) = 8n+12$, $h_1(10,n) = 16$, $h_1(14,n) = 20-8n = 4$ or <0

according as $n = 2$ or $n \geq 3$ and $h_1(18, n) = 24 - 16n < 0$. Hence

$$\beta_4' = \begin{cases} \frac{-2mn+m+20n+6}{m+6}, & \text{when } m = 6, 10 \,\&\, n \geq 2;\ m = 14 \,\&\, n = 2 \\ -\frac{-2mn+m+20n+6}{m+6}, & \text{otherwise.} \end{cases}$$

Also, we have

$$\beta_5' := \left|\frac{n(m+6)}{2} - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)}))\right| = \left|\frac{-4mn+m+8n+6}{m+6}\right| = -\frac{-4mn+m+8n+6}{m+6},$$

as for $m \geq 6$, $-4mn + m + 8n + 6 = n(8 - 3m) + m(1 - n) + 6 < 0$. Therefore, by (1.1.*f*), we get

$$\begin{aligned} \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(U_{(n,m)})) &= 1 \times \beta_1' + 1 \times \beta_2' + (2n-1) \times \beta_3' + \left(\frac{n(m-2)}{2} - 1\right) \times \beta_4' + 2n \times \beta_5' \\ &= \begin{cases} \frac{m^2n+2m\left(8n^2+8n-1\right)-4\left(8n^2+n+3\right)}{m+6}, & \text{when } m = 6, 10 \,\&\, n \geq 2; \\ & m = 14 \,\&\, n = 2 \\ \frac{2n\left(m^2n-4m(n-1)+4(n+6)\right)}{m+6}, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the result follows. □

Theorem 6.1.4. *The DL-spectrum and DL-energy of CCC-graph of the group V_{8n} $(n \geq 2)$ are as given below.*

(a) *If n is even then*

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(V_{8n})) = \left\{[0]^1, [2n+10]^{2n-3}, [2n+6]^4, [4(n+2)]^2, [4n+6]^2\right\}$$

$$\textit{and } \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(V_{8n})) = \begin{cases} \frac{4n^2+94n-34}{n+3}, & \textit{when } 2 \leq n \leq 6 \\ \frac{16\left(n^2+5\right)}{n+3}, & \textit{when } n \geq 8. \end{cases}$$

(b) *If n is odd then*

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(V_{8n})) = \{[0]^1, [2n+5]^{2n-2}, [2n+3]^2, [4(n+1)]^2\}$$

$$\textit{and } \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(V_{8n})) = \begin{cases} \frac{4\left(2n^2+15n-3\right)}{2n+3}, & \textit{when } 3 \leq n \leq 5 \\ \frac{8\left(2n^2+n+4\right)}{2n+3}, & \textit{when } n \geq 7. \end{cases}$$

Proof. We know that

$$|Z(V_{8n})| = \begin{cases} 4, & \text{when } n \text{ is even} \\ 2, & \text{when } n \text{ is odd.} \end{cases}$$

Now, by Result 1.2.24 and (1.2.a), we have

$$\Gamma_{\text{ccc}}(V_{8n}) = \begin{cases} K_4 + (2K_2 \cup K_{2n-2}), & \text{when } n \text{ is even} \\ K_2 + (2K_1 \cup K_{2n-1}), & \text{when } n \text{ is odd.} \end{cases}$$

(a) If n is even then by Result 1.1.15, we get

$$\text{Ch}_{\text{DL}}(\Gamma_{\text{ccc}}(V_{8n}), x) = x(x-2n-10)^{2n-3}(x-2n-6)^4(x-2(2n-2)-12)^2 \times (x-2(2n-2)-10)^2.$$

Therefore, DL-spec$(\Gamma_{\text{ccc}}(V_{8n})) = \{[0]^1, [2n+10]^{2n-3}, [2n+6]^4, [4(n+2)]^2, [4n+6]^2\}$.

Here $|v(\Gamma_{\text{ccc}}(V_{8n}))| = 2(n+3)$ and $\text{tr}(\text{DL}(\Gamma_{\text{ccc}}(V_{8n}))) = 4n^2+38n+22$. Therefore, $\Delta_{\text{D}}(\Gamma_{\text{ccc}}(V_{8n})) = \frac{2n^2+19n+11}{n+3}$. We have

$$\alpha_1 := \left|0 - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(V_{8n}))\right| = \left|-\frac{2n^2+19n+11}{n+3}\right| = \frac{2n^2+19n+11}{n+3},$$

$$\alpha_2 := \left|(2n+10) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(V_{8n}))\right| = \left|\frac{19-3n}{n+3}\right| = \begin{cases} \frac{19-3n}{n+3}, & \text{when } 2 \le n \le 6 \\ \frac{3n-19}{n+3}, & \text{otherwise,} \end{cases}$$

$$\alpha_3 := \left|(2n+6) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(V_{8n}))\right| = \left|\frac{7-7n}{n+3}\right| = \frac{7n-7}{n+3}, \text{ as } n \ge 2,$$

$$\alpha_4 := \left|4(n+2) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(V_{8n}))\right| = \left|\frac{2n^2+n+13}{n+3}\right| = \frac{2n^2+n+13}{n+3}$$

and

$$\alpha_5 := \left|(4n+6) - \Delta_{\text{D}}(\Gamma_{\text{ccc}}(V_{8n}))\right| = \left|\frac{2n^2-n+7}{n+3}\right| = \frac{2n^2-n+7}{n+3}, \quad \text{as } n \ge 2.$$

Hence, by (1.1.*f*), we get

$$\begin{aligned}\mathrm{E_{DL}}(\Gamma_{\mathrm{ccc}}(V_{8n})) &= 1\times\alpha_1+(2n-3)\times\alpha_2+4\times\alpha_3+2\times\alpha_5\\ &= \begin{cases}\frac{4n^2+94n-34}{n+3}, & \text{when } 2\le n\le 6\\ \frac{16\left(n^2+5\right)}{n+3}, & \text{otherwise.}\end{cases}\end{aligned}$$

(b) If n is odd then by Result 1.1.15, we get

$$\mathrm{Ch_{DL}}(\Gamma_{\mathrm{ccc}}(V_{8n}),x)=x(x-2n-5)^{2n-2}(x-2n-3)^2(x-2(2n-1)-6)^2.$$

Therefore, DL-spec$(\Gamma_{\mathrm{ccc}}(V_{8n}))=\left\{[0]^1,[2n+5]^{(2n-2)},[2n+3]^2,[4(n+1)]^2\right\}$.

Here $|v(\Gamma_{\mathrm{ccc}}(V_{8n}))|=2n+3$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\mathrm{ccc}}(V_{8n})))=4n^2+18n+4$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(V_{8n}))=\frac{4n^2+18n+4}{2n+3}$. We have

$$\beta_1:=\left|0-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(V_{8n}))\right|=\left|-\frac{4n^2+18n+4}{2n+3}\right|=\frac{4n^2+18n+4}{2n+3},$$

$$\beta_2:=\left|(2n+5)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(V_{8n}))\right|=\left|\frac{11-2n}{2n+3}\right|=\begin{cases}\frac{11-2n}{2n+3}, & \text{when } 2\le n\le 5\\ \frac{2n-11}{2n+3}, & \text{otherwise,}\end{cases}$$

$$\beta_3:=\left|(2n+3)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(V_{8n}))\right|=\left|\frac{5-6n}{2n+3}\right|=\frac{5-6n}{2n+3},\ \text{as } n\ge 2,$$

and

$$\beta_4:=\left|4(n+1)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(V_{8n}))\right|=\left|\frac{2\left(2n^2+n+4\right)}{2n+3}\right|=\frac{2\left(2n^2+n+4\right)}{2n+3}.$$

Hence, by (1.1.*f*), we get

$$\begin{aligned}\mathrm{E_{DL}}(\Gamma_{\mathrm{ccc}}(V_{8n})) &= 1\times\beta_1+(2n-2)\times\beta_2+2\times\beta_3+2\times\beta_4\\ &= \begin{cases}\frac{4\left(2n^2+15n-3\right)}{2n+3}, & \text{when } 2\le n\le 5\\ \frac{8\left(2n^2+n+4\right)}{2n+3}, & \text{otherwise.}\end{cases}\end{aligned}$$

Hence the result follows. □

Theorem 6.1.5. *The DL-spectrum and DL-energy of CCC-graph of the group SD_{8n} ($n \geq 2$) are as given below.*

(a) *If n is even then*

$$\text{DL-spec}(\Gamma_{\text{ccc}}(SD_{8n})) = \left\{[0]^1, [2n+5]^{2n-2}, [2n+3]^2, [4(n+1)]^2\right\}$$

$$\textit{and} \quad \mathrm{E_{DL}}(\Gamma_{\text{ccc}}(SD_{8n})) = \begin{cases} \frac{4\left(2n^2+15n-3\right)}{2n+3}, & \textit{when } 2 \leq n \leq 4 \\ \frac{8\left(2n^2+n+4\right)}{2n+3}, & \textit{when } n \geq 6. \end{cases}$$

(b) *If n is odd then*

$$\text{DL-spec}(\Gamma_{\text{ccc}}(SD_{8n})) = \left\{[0]^1, [2n+6]^4, [4(n+2)]^1, [4(n+1)]^3, [2(n+5)]^{2n-3}\right\}$$

$$\textit{and} \quad \mathrm{E_{DL}}(\Gamma_{\text{ccc}}(SD_{8n})) = \begin{cases} \frac{4n^2+94n-74}{n+3}, & \textit{when } 3 \leq n \leq 7 \\ \frac{16\left(n^2-n+4\right)}{n+3}, & \textit{when } n \geq 9. \end{cases}$$

Proof. We know that

$$|Z(SD_{8n})| = \begin{cases} 2, & \text{when } n \text{ is even} \\ 4, & \text{when } n \text{ is odd.} \end{cases}$$

Now, by Result 1.2.25 and (1.2.a), we have

$$\Gamma_{\text{ccc}}(SD_{8n}) = \begin{cases} K_2 + (2K_1 \cup K_{2n-1}), & \text{when } n \text{ is even} \\ K_4 + (K_{2n-2} \cup K_4), & \text{when } n \text{ is odd.} \end{cases}$$

(a) If n is even then by Result 1.1.15, we get

$$\mathrm{Ch_{DL}}(\Gamma_{\text{ccc}}(SD_{8n}), x) = x(x-2n-5)^{2n-2}(x-2n-3)^2(x-2(2n-1)-6)^2.$$

Therefore, $\text{DL-spec}(\Gamma_{\text{ccc}}(SD_{8n})) = \left\{[0]^1, [2n+5]^{2n-2}, [2n+3]^2, [4(n+1)]^2\right\}$.

Here $|v(\Gamma_{\text{ccc}}(SD_{8n}))| = 2n+3$ and $\text{tr}(\mathrm{DL}(\Gamma_{\text{ccc}}(SD_{8n}))) = 4n^2+18n+4$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\text{ccc}}(SD_{8n})) = \frac{4n^2+18n+4}{2n+3}$. We have

$$\alpha_1 := \left|0 - \Delta_{\mathrm{D}}(\Gamma_{\text{ccc}}(SD_{8n}))\right| = \left|-\frac{4n^2+18n+4}{2n+3}\right| = \frac{4n^2+18n+4}{2n+3},$$

$$\alpha_2 := \left|(2n+5) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n}))\right| = \left|\frac{11-2n}{2n+3}\right| = \begin{cases} \frac{11-2n}{2n+3}, & \text{when } 2 \leq n \leq 5 \\ \frac{2n-11}{2n+3}, & \text{otherwise,} \end{cases}$$

$$\alpha_3 := \left|(2n+3) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n}))\right| = \left|\frac{5-6n}{2n+3}\right| = \frac{5-6n}{2n+3}, \text{ as } n \geq 2,$$

and

$$\alpha_4 := \left|4(n+1) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n}))\right| = \left|\frac{2\left(2n^2+n+4\right)}{2n+3}\right| = \frac{2\left(2n^2+n+4\right)}{2n+3}.$$

Hence, by (1.1.*f*), we get

$$\begin{aligned} \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(SD_{8n})) &= 1 \times \alpha_1 + (2n-2) \times \alpha_2 + 2 \times \alpha_3 + 2 \times \alpha_4 \\ &= \begin{cases} \frac{4\left(2n^2+15n-3\right)}{2n+3}, & \text{when } 2 \leq n \leq 5 \\ \frac{8\left(2n^2+n+4\right)}{2n+3}, & \text{otherwise.} \end{cases} \end{aligned}$$

(b) If n is odd then by Result 1.1.15, we get

$$\begin{aligned} \mathrm{Ch}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(SD_{8n}), x) = x(x-2n-6)^4(x-2(2n-2)-12)(x-2(2n-2)-8)^3 \\ \times(x+2n-2(2n-2)-14)^{2n-3}. \end{aligned}$$

Therefore, DL-spec$(\Gamma_{\mathrm{ccc}}(SD_{8n})) = \left\{[0]^1, [2n+6]^4, [4(n+2)]^1, [4(n+1)]^3, [2(n+5)]^{2n-3}\right\}$.

Here $|v(\Gamma_{\mathrm{ccc}}(SD_{8n}))| = 2(n+3)$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\mathrm{ccc}}(SD_{8n}))) = 4n^2+38n+14$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n})) = \frac{2n^2+19n+7}{n+3}$. We have

$$\beta_1 := \left|0 - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n}))\right| = \left|-\frac{2n^2+19n+7}{n+3}\right| = \frac{2n^2+19n+7}{n+3},$$

$$\beta_2 := \left|(2n+6) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n}))\right| = \left|\frac{11-7n}{n+3}\right| = \frac{7n-11}{n+3}, \text{ as } n \geq 2,$$

$$\beta_3 := \left|4(n+2) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n}))\right| = \left|\frac{2n^2+n+17}{n+3}\right| = \frac{2n^2+n+17}{n+3},$$

$$\beta_4 := \left|4(n+1) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n}))\right| = \left|\frac{2n^2-3n+5}{n+3}\right| = \frac{2n^2-3n+5}{n+3}$$

and

$$\beta_5 := \left|2(n+5) - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(SD_{8n}))\right| = \left|\frac{23-3n}{n+3}\right| = \begin{cases} \frac{23-3n}{n+3}, & \text{when } 3 \le n \le 7 \\ \frac{3n-23}{n+3}, & \text{otherwise.} \end{cases}$$

Therefore, by (1.1.*f*), we get

$$\begin{aligned} \mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(SD_{8n})) &= 1 \times \beta_1 + 4 \times \beta_2 + 1 \times \beta_3 + 3 \times \beta_4 + (2n-3) \times \beta_5 \\ &= \begin{cases} \frac{4n^2+94n-74}{n+3}, & \text{when } 3 \le n \le 7 \\ \frac{16\left(n^2-n+4\right)}{n+3}, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the result follows. □

Theorem 6.1.6. *Let G be a finite non-abelian group with center $Z(G)$ and $|Z(G)| = z$. If $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is prime and $|Z(G)| \ge 2$ then the DL-spectrum and DL-energy of CCC-graph of the group G are as given below.*

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(G)) = \left\{[0]^1, [z+2n(p+1)]^p, [2n(1+p)+z-n]^{(n-1)(p+1)}, [n(p+1)+z]^z\right\}$$

and $\mathrm{E}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(G)) = \frac{2\left(p^4z(z+2)-p^3\left(z^2+1\right)-p^2\left(z^2+3z+1\right)+p\left(z^2+1\right)+z\right)}{p(p^2+p-1)}$.

Proof. By Result 1.2.17 and (1.2.*a*), we have

$$\Gamma_{\mathrm{ccc}}(G) = K_z + (p+1)K_n, \text{ where } n = \frac{(p-1)z}{p}.$$

Therefore, by Result 1.1.15, we get

$$\mathrm{Ch}_{\mathrm{DL}}(\Gamma_{\mathrm{ccc}}(G), x) = x(x-2n(p+1)-z)^p(x-2n(p+1)+n-z)^{(n-1)(p+1)}$$

$$\times(x-n(p+1)-z)^z$$

and so DL-spec$(\Gamma_{\mathrm{ccc}}(G)) = \{[0]^1, [z+2n(p+1)]^p, [2n(1+p)-n+z]^{(n-1)(p+1)}, [n(p+1)+z]^z\}$.

Here $|v(\Gamma_{\mathrm{ccc}}(G))| = \frac{(p^2+p-1)z}{p}$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\mathrm{ccc}}(G))) = \frac{z\left(2p^4z+p^3(z-1)-p^2(2z+1)-pz+p+z\right)}{p^2}$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G)) = \frac{2p^4z+p^3z-p^3-2p^2z-p^2-pz+p+z}{p(p^2+p-1)}$. We have

$$\begin{aligned}\alpha_1 := \left|0-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| &= \left|-\frac{2p^4z+p^3z-p^3-2p^2z-p^2-pz+p+z}{p\,(p^2+p-1)}\right| \\ &= \frac{2p^4z+p^3z-p^3-2p^2z-p^2-pz+p+z}{p\,(p^2+p-1)},\end{aligned}$$

since $2p^4z+p^3z-p^3-2p^2z-p^2-pz+p+z = \frac{2}{3}p^3\left(pz-\frac{3}{2}\right)+\frac{2}{3}\left(p^3-\frac{3}{2}\right)pz+\frac{2}{3}p^2\left(p^2z-\frac{3}{2}\right)+(p-2)p^2z+p+z > 0$, as $p \geq 2$.

We also have

$$\begin{aligned}\alpha_2 := \left|(2n(p+1)+z)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| &= \left|\frac{2p^3z+p^3-p^2z+p^2-2pz-p+z}{p\,(p^2+p-1)}\right| \\ &= \frac{2p^3z+p^3-p^2z+p^2-2pz-p+z}{p\,(p^2+p-1)},\end{aligned}$$

since $2p^3z+p^3-p^2z+p^2-2pz-p+z = p^3+(p-1)p^2z+\left(p^2-2\right)pz+(p-1)p+z > 0$, as $p \geq 2$. Further,

$$\begin{aligned}\alpha_3 := \left|(2n(p+1)-n+z)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| &= \left|\frac{p^2z+p^2-pz+p-1}{p^2+p-1}\right| \\ &= \frac{p^2z+p^2-pz+p-1}{p^2+p-1} \qquad (\text{as } p^2z+p^2-pz+p-1>0)\end{aligned}$$

and

$$\alpha_4 := \left|(n(p+1)+z)-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| = \left|\frac{-p^3z+p^2z+p^2+pz+p-z-1}{p^2+p-1}\right|.$$

Let $f(p,z) = -p^3z+p^2z+p^2+pz+p-z-1 = -\frac{1}{4}p^2(pz-4)-\frac{1}{4}(p-4)p^2z-\frac{1}{4}\left(p^2-4\right)pz-\frac{1}{4}p\left(p^2z-4\right)-z-1$. Clearly, for $p \geq 4$, $f(p,z) < 0$. Again

$$f(2,z) = 5-3z < 0, \qquad \text{as } z \geq 2.$$

and $f(3,z) = 11-16z < 0$. Therefore,

$$\alpha_4 = -\left(\frac{-p^3z+p^2z+p^2+pz+p-z-1}{p^2+p-1}\right).$$

Now, by (1.1.*f*), we get

$$\begin{aligned}\mathrm{E_{DL}}(\Gamma_{\mathrm{ccc}}(G)) &= 1 \ \times \alpha_1 + p \times \alpha_2 + (n-1)(p+1) \times \alpha_3 + z \times \alpha_4 \\ &= \frac{2\left(p^4 z(z+2) - p^3\left(z^2+1\right) - p^2\left(z^2+3z+1\right) + p\left(z^2+1\right) + z\right)}{p\left(p^2+p-1\right)}.\end{aligned}$$

Hence the result follows. □

Corollary 6.1.7. *If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, where p is a prime and $n \geq 3$ then the DL-spectrum and DL-energy of CCC-graph of G are as given below.*

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(G)) =$$
$$\left\{[0]^1, [p^{n-3}(2p^2-1)]^{(p+1)(p^{n-2}-p^{n-3}-1)}, [p^{n-3}(2p^2+p-2)]^p, [p^{n-3}(p^2+p-1)]^{p^{n-2}}\right\}$$

and

$$\mathrm{E_{DL}}(\Gamma_{\mathrm{ccc}}(G)) = \frac{2\left(p^{2n} + p^{n+1} - 3p^{n+3} + 2p^{n+5} - p^{2n+1} - p^{2n+2} + p^{2n+3} - p^6 - p^5 + p^4\right)}{p^4\left(p^2+p-1\right)}.$$

Proof. Here, $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, by Theorem 6.1.6, we get the required result. □

Theorem 6.1.8. *Let G be a finite non-abelian group with center $|Z(G)| = z$ such that $\frac{G}{Z(G)} \cong D_{2n}$ $(n \geq 3)$. Then the DL-spectrum and DL-energy of CCC-graph of G are as given below.*

(a) *If n is even then*

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(G)) = \left\{[0]^1, [(n+2)z]^2, \left[\frac{(2n+3)z}{2}\right]^{z-2}, \left[\frac{(n+5)z}{2}\right]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}\right]^{z}\right\}$$

and

$$\mathrm{E_{DL}}(\Gamma_{\mathrm{ccc}}(G)) = \begin{cases} \frac{n^2z+4nz^2+8nz-2n-2z^2+z-6}{n+3}, & \textit{when } n = 4 \,\&\, z \geq 2; \\ & n = 6 \,\&\, 2 \leq z \leq 4;\ n = 8, 10 \,\&\, z = 2 \\ \frac{n^2z^2-nz^2+4nz+2z^2+12z}{n+3}, & \textit{otherwise.} \end{cases}$$

(b) *If n is odd then*

$$\text{DL-spec}(\Gamma_{ccc}(G)) = \left\{[0]^1, [(n+2)z]^1, [(1+n)z]^{z-1}, \left[\frac{(n+5)z}{2}\right]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}\right]^{z}\right\}$$

and

$$E_{DL}(\Gamma_{ccc}(G)) = \begin{cases} \frac{16}{3}, & \text{when } n = 3 \,\&\, z = 1 \\ \frac{n^2z+4nz^2+8nz-2n-4z^2-z-6}{n+3}, & \text{when } z = 1 \,\&\, n \geq 5;\ z = 2 \,\&\, n \leq 12; \\ & n = 3 \,\&\, z \geq 2;\ n = 5 \,\&\, z \geq 1; \\ & n = 7 \,\&\, z \leq 4;\ n = 9 \,\&\, z = 1, 2 \\ \frac{z\left(n^2z-2n(z-2)+z+12\right)}{n+3}, & \text{otherwise.} \end{cases}$$

Proof. By Result 1.2.19 and (1.2.a), we have

$$\Gamma_{ccc}(G) = \begin{cases} K_z + (2K_{\frac{z}{2}} \cup K_{\frac{(n-1)z}{2}}), & \text{when } n \text{ is even} \\ K_z + (K_{\frac{(n-1)z}{2}} \cup K_z), & \text{when } n \text{ is odd.} \end{cases}$$

(a) If n is even then by Result 1.1.15, we get

$$\begin{aligned} \text{Ch}_{DL}(\Gamma_{ccc}(G), x) = x(x-(n-1)z-3z)^2 \left(x-(n-1)z-\frac{5z}{2}\right)^{2\left(\frac{z}{2}-1\right)} \\ \times \left(x-\frac{1}{2}(n-1)z-3z\right)^{\frac{1}{2}(n-1)z-1} \left(x-\frac{1}{2}(n-1)z-2z\right)^{z}. \end{aligned}$$

Therefore,

$$\text{DL-spec}(\Gamma_{ccc}(G)) = \left\{[0]^1, [(n+2)z]^2, \left[\frac{(2n+3)z}{2}\right]^{z-2}, \left[\frac{(n+5)z}{2}\right]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}\right]^{z}\right\}.$$

Here $|v(\Gamma_{ccc}(G))| = \frac{1}{2}(n+3)z$ and $\text{tr}(DL(\Gamma_{ccc}(G)) = \frac{1}{4}z\left(n^2z+10nz-2n+7z-6\right)$. Therefore, $\Delta_D(\Gamma_{ccc}(G)) = \frac{n^2z+10nz-2n+7z-6}{2(n+3)}$. We have

$$\alpha_1 := \left|0 - \Delta_D(\Gamma_{ccc}(G))\right| = \left|-\frac{n^2z+10nz-2n+7z-6}{2n+6}\right| = \frac{n^2z+10nz-2n+7z-6}{2n+6}$$

(as $n^2z + 10nz - 2n + 7z - 6 > 0$),

$$\alpha_2 := \left|(n+2)z - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| = \left|\frac{n^2z + 2n + 5z + 6}{2n+6}\right| = \frac{n^2z + 2n + 5z + 6}{2n+6},$$

$$\alpha_3 := \left|\frac{1}{2}(2n+3)z - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| = \left|\frac{n^2z - nz + 2n + 2z + 6}{2n+6}\right| = \frac{n^2z - nz + 2n + 2z + 6}{2n+6}$$

(as $n^2z - nz + 2n + 2z + 6 > 0$) and

$$\alpha_4 := \left|\frac{1}{2}(n+5)z - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| = \left|\frac{-nz + n + 4z + 3}{n+3}\right|.$$

Let $f_1(n,z) = -nz + n + 4z + 3 = -\frac{1}{3}(nz-9) - \frac{1}{3}n(z-3) - \frac{1}{3}(n-12)z$. Clearly, for $n \geq 12$ and $z \geq 3$, $f_1(n,z) < 0$. We have,

$$f_1(n,2) = 11 - n \begin{cases} > 0, & \text{when } n \leq 10 \\ \leq 0, & \text{otherwise.} \end{cases}$$

Also, $f_1(4,z) = 7 > 0$,

$$f_1(6,z) = 9 - 2z \begin{cases} > 0, & \text{when } 2 \leq z \leq 4 \\ < 0, & \text{otherwise,} \end{cases}$$

$$f_1(8,z) = 11 - 4z \begin{cases} > 0, & \text{when } z = 2 \\ < 0, & \text{otherwise} \end{cases}$$

and

$$f_1(10,z) = 13 - 6z \begin{cases} > 0, & \text{when } z = 2 \\ < 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\alpha_4 = \begin{cases} \frac{-nz+n+4z+3}{n+3}, & \text{when } n = 4 \,\&\, z \geq 2;\ n = 6 \,\&\, 2 \leq z \leq 4; \\ & n = 8, 10 \,\&\, z = 2 \\ -\left(\frac{-nz+n+4z+3}{n+3}\right), & \text{otherwise.} \end{cases}$$

Also

$$\alpha_5 := \left|\frac{1}{2}(n+3)z - \Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| = \left|\frac{-2nz+n+z+3}{n+3}\right|.$$

Let $f_2(n,z) = -2nz+n+z+3 = -n(z-1)-z(n-1)+3$. Clearly, for $n \geq 4$, $f_2(n,z) \leq 0$. Therefore,

$$\alpha_5 = -\left(\frac{-2nz+n+z+3}{n+3}\right).$$

Hence, by (1.1.*f*), we get

$$\begin{aligned}\mathrm{E_{DL}}(\Gamma_{\mathrm{ccc}}(G)) &= 1\times\alpha_1 + 2\times\alpha_2 + (z-2)\times\alpha_3 + (\frac{1}{2}(n-1)z-1)\times\alpha_4 + z\times\alpha_5\\ &= \begin{cases} \frac{n^2z+4nz^2+8nz-2n-2z^2+z-6}{n+3}, & \text{when } n=4\,\&\,z\geq 2;\\ & n=6\,\&\,2\leq z\leq 4;\ n=8,10\,\&\,z=2\\ \frac{n^2z^2-nz^2+4nz+2z^2+12z}{n+3}, & \text{otherwise.}\end{cases}\end{aligned}$$

(b) If n is odd then by Result 1.1.15, we get

$$\begin{aligned}\mathrm{Ch_{DL}}(\Gamma_{\mathrm{ccc}}(G),x) &= x(x-(n-1)z-3z)(x-(n-1)z-2z)^{z-1}\\ &\quad\times\left(x-\frac{1}{2}(n-1)z-3z\right)^{\frac{1}{2}(n-1)z-1}\left(x-\frac{1}{2}(n-1)z-2z\right)^{z}.\end{aligned}$$

Therefore,

$$\text{DL-spec}(\Gamma_{\mathrm{ccc}}(G)) = \left\{[0]^1, [(n+2)z]^1, [(1+n)z]^{z-1}, \left[\frac{(n+5)z}{2}\right]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}\right]^{z}\right\}.$$

Here $|v(\Gamma_{\mathrm{ccc}}(G))| = \frac{1}{2}(n+3)z$ and $\mathrm{tr}(\mathrm{DL}(\Gamma_{\mathrm{ccc}}(G))) = \frac{1}{4}z\left(n^2z+10nz-2n+5z-6\right)$. Therefore, $\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G)) = \frac{n^2z+10nz-2n+5z-6}{2(n+3)}$. We have

$$\beta_1 := \left|0-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| = \left|-\frac{n^2z+10nz-2n+5z-6}{2(n+3)}\right| = \frac{n^2z+10nz-2n+5z-6}{2(n+3)}$$

(as $n^2z+10nz-2n+5z-6>0$),

$$\beta_2 := \left|(n+2)z-\Delta_{\mathrm{D}}(\Gamma_{\mathrm{ccc}}(G))\right| = \left|\frac{n^2z+2n+7z+6}{2n+6}\right| = \frac{n^2z+2n+7z+6}{2n+6},$$

$$\beta_3 := \left|(n+1)z - \Delta_{\mathrm{D}}(\Gamma_{ccc}(G))\right| = \left|\frac{n^2z - 2nz + 2n + z + 6}{2n+6}\right| = \frac{n^2z - 2nz + 2n + z + 6}{2n+6}$$

(as $n^2z - 2nz + 2n + z + 6 > 0$) and

$$\beta_4 := \left|\frac{1}{2}(n+5)z - \Delta_{\mathrm{D}}(\Gamma_{ccc}(G))\right| = \left|\frac{-nz + n + 5z + 3}{n+3}\right|.$$

Let $g_1(n, z) = -nz + n + 5z + 3 = -\frac{1}{2}n(z-2) - \frac{1}{2}(n-10)z + 3$. Clearly, for $n \geq 10$ and $z \geq 3$, $g_1(n, z) < 0$. Also, $g_1(n, 1) = 8 > 0$,

$$g_1(n, 2) = 13 - n \begin{cases} > 0, & \text{when } 3 \leq n \leq 12 \\ \leq 0, & \text{otherwise,} \end{cases}$$

$$g_1(3, z) = 6 + 2z > 0, \quad g_1(5, z) = 8 > 0,$$

$$g_1(7, z) = 10 - 2z \begin{cases} > 0, & \text{when } 1 \leq z \leq 4 \\ \leq 0, & \text{otherwise} \end{cases}$$

and

$$g_1(9, z) = 12 - 4z \begin{cases} > 0, & \text{when } z = 1, 2 \\ < 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\beta_4 = \begin{cases} \frac{-nz+n+5z+3}{n+3}, & \text{when } z = 1 \,\&\, n \geq 3;\ z = 2 \,\&\, n \leq 12;\ n = 3, 5 \,\&\, z \geq 1; \\ & n = 7 \,\&\, z \leq 4;\ n = 9 \,\&\, z = 1, 2 \\ -\left(\frac{-nz+n+5z+3}{n+3}\right), & \text{otherwise.} \end{cases}$$

Further,

$$\beta_5 := \left|\frac{1}{2}(n+3)z - \Delta_{\mathrm{D}}(\Gamma_{ccc}(G))\right| = \left|\frac{-2nz + n + 2z + 3}{n+3}\right|.$$

Let $g_2(n, z) = -2nz + n + 2z + 3 = -(n(z-1)) - z(n-2)z + 3$. Clearly, for $n \geq 3$ and $z \geq 2$, $g_2(n, z) < 0$. Also

$$g_2(n, 1) = 5 - n \begin{cases} = 2 > 0, & \text{when } n = 3 \\ \leq 0, & \text{otherwise.} \end{cases}$$

So

$$\beta_5 = \begin{cases} \frac{-2nz+n+2z+3}{n+3} = \frac{1}{3}, & \text{when } n = 3 \,\&\, z = 1 \\ -\left(\frac{-2nz+n+2z+3}{n+3}\right), & \text{otherwise.} \end{cases}$$

Therefore, by (1.1.*f*), we get

$$\begin{aligned} E_{DL}(\Gamma_{ccc}(G)) &= 1 \times \beta_1 + 1 \times \beta_2 + (z-1) \times \beta_3 + (\frac{1}{2}(n-1)z - 1) \times \beta_4 + z \times \beta_5 \\ &= \begin{cases} \frac{16}{3}, & \text{when } n = 3 \,\&\, z = 1 \\ \frac{n^2z+4nz^2+8nz-2n-4z^2-z-6}{n+3}, & \text{when } z = 1 \,\&\, n \geq 3;\ z = 2 \,\&\, n \leq 12; \\ & n = 3 \,\&\, z \geq 1;\ n = 5 \,\&\, z \geq 1;\ n = 7 \,\&\, z \leq 4; \\ & n = 9 \,\&\, z = 1, 2 \\ \frac{z\left(n^2z-2n(z-2)+z+12\right)}{n+3}, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the result follows. □

We conclude this chapter with the following consequence of Theorem 6.1.1–6.1.8.

Theorem 6.1.9. *Let $\Gamma_{ccc}(G)$ be the CCC-graph of a finite non-abelian group G.*

(a) *If G is isomorphic to D_{2m}, Q_{4n}, $U_{(n,m)}$, V_{8n} or SD_{8n} then $\Gamma_{ccc}(G)$ is DL-integral.*

(b) *If $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m} then $\Gamma_{ccc}(G)$ is DL-integral.*