COMMON NEIGHBORHOOD AND DISTANCE SPECTRAL ASPECTS OF CERTAIN GRAPHS DEFINED ON FINITE GROUPS

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Chapter 7

Conclusion and problems for future research

We have begun this thesis with an introductory chapter, in which we have introduced some notations and recalled certain definitions and results related to groups and graphs. We have also recalled certain results related to commuting graphs and commuting conjugacy class graphs of groups that are useful in this thesis.

In Chapter 2, we have introduced the concepts of CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy. We have established relations between these energies and the first Zagreb index of a graph. Additionally, we have introduced the concepts of CNL-integral, CNSL-integral, CNL-hyperenergetic and CNSL-hyperenergetic graphs and showed that a complete bipartite graph is CNL-integral, CNSL-integral but neither CNL- hyperenergetic nor CNSL-hyperenergetic. Furthermore, we have established connections between various graph energies, including energy, Laplacian energy and signless Laplacian energy. Finally, we have obtained several bounds for CNL-energy and CNSL-energies of graphs.

In Chapter 3, we have computed CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of commuting graphs of the groups QD_{2^n}, $PSL(2, 2^k)$, $GL(2, q)$, $A(n, \nu)$, $A(n, p)$, D_{2n} and groups whose central quotient is isomorphic to $Sz(2)$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m}. We have determined when commuting graphs of these groups are CNL(CNSL)-integral and CNL(CNSL)-hyperenergetic. Finally, we have compared CN-energy, CNL-energy and CNSL-energy of commuting graphs of the above-mentioned groups.

In Chapter 4, we have considered the subgraph of CCC-graph of a finite non-abelian group G induced by the set of conjugacy classes of non-central elements of G which is denoted by $\Gamma_{\text{ccc}}^*(G)$. We have computed CN-spectrum, CNL-spectrum, CNSL-spectrum and their corresponding energies of $\Gamma_{\text{ccc}}^*(G)$ for finite non-abelian groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is any prime) or the dihedral group D_{2m} ($m \geq 3$). We have determined whether $\Gamma_{\text{ccc}}^*(G)$ for these groups are CN-, CNL-, CNSL-integral/ hyperenergetic/borderenergetic. We have also characterized the groups $G = D_{2m}$, Q_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n} such that $\Gamma_{\text{ccc}}^*(G)$ is CN-, CNL-, CNSL-integral/hyperenergetic /borderenergetic. Finally, we have compared various common neighborhood energies of $\Gamma_{\text{ccc}}^*(G)$ for the above-mentioned groups and illustrate their closeness graphically.

In Chapter 5, we have considered the complement of $\Gamma_{\text{ccc}}^*(G)$, denoted by $\Gamma_{\text{nccc}}^*(G)$, which is the subgraph of non-commuting conjugacy class graph of a finite non-abelian group G induced by the set of conjugacy classes of non-central elements of G. We have computed distance spectrum, distance Laplacian spectrum, distance signless Laplacian spectrum along with their respective energies and Wiener index of $\Gamma_{\text{nccc}}^*(G)$ for G when the central quotient of G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2m} (for any integer $m \geq 3$). As a consequence, we have computed various distance spectra, energies and Wiener index of $\Gamma_{\text{nccc}}^*(G)$ for the dihedral group, dicyclic group, semidihedral group along with the groups $U_{(n,m)}$, U_{6n} and V_{8n}. We have showed that any perfect square can be realized as Wiener index of $\Gamma_{\text{nccc}}^*(G)$ for certain dihedral groups. We have also characterized the above-mentioned groups such that $\Gamma_{\text{nccc}}^*(G)$ are D-integral, DL-integral and DQ-integral. We have computed distance energy, distance Laplacian energy and distance signless Laplacian energy of $\Gamma_{\text{nccc}}^*(G)$ for the above-mentioned groups using Wiener index. We have also compared various distance energies of $\Gamma_{\text{nccc}}^*(G)$ and characterized the above-mentioned groups subject to the inequalities involving various distance energies.

In Chapter 6, we have considered commuting conjugacy class graph of a group G, denoted by $\Gamma_{\text{ccc}}(G)$, and compute distance Laplacian spectrum and energy of $\Gamma_{\text{ccc}}(G)$ if $G = D_{2m}$, Q_{4n}, $U_{(n,m)}$ and SD_{8n}. We have also considered finite groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2m}. As a consequence, it was shown that the commuting conjugacy class graphs of these groups are D-integral.

7.1 Problems for future research

During our research we have introduced CNL-spectrum, CNL-energy, CNSL-energy and studied various spectral properties of commuting graphs, CCC-graphs and NCCC-graphs of finite groups. We have come up with certain problems for future research. In this section we list all those problems.

In Chapter 3, we have observed that $\Gamma_{\mathrm{c}}(G)$ is CNL(CNSL)-integral for several families of finite non-abelian groups. Therefore, the following problem arises naturally.

Problem 7.1.1. Characterize all finite non-abelian groups G such that $\Gamma_{\mathrm{c}}(G)$ is CNL (CNSL)-inetgral.

It has been observed that the commuting graphs of some AC-groups are CNL(CNSL)-hyperenergetic but some are not CNL(CNSL)-hyperenergetic. Therefore, it will be interesting to find conditions such that commuting graphs of finite AC-groups are CNL(CNSL) -hyperenergetic. More generally, we have the following problem.

Problem 7.1.2. Characterize all finite non-abelian groups G such that $\Gamma_{\mathrm{c}}(G)$ is CNL (CNSL)-hypernergetic or not CNL(CNSL)-hyperenergetic.

In Chapter 4, we have computed CN-/CNL-/CNSL-spectrum and energy of $\Gamma_{\mathrm{ccc}}^{*}(G)$ for certain families of finite non-abelian groups. We have shown that $\Gamma_{\mathrm{ccc}}^{*}(G)$ is CN-, CNL- and CNSL-integral for all the groups considered in Chapter 4. In view of this, the following problem arises naturally.

Problem 7.1.3. Characterize all finite non-abelian groups G such that $\Gamma_{\mathrm{ccc}}^{*}(G)$ is CN-, CNL- and CNSL-integral.

The existence of finite non-abelian groups G such that $\Gamma_{\mathrm{ccc}}^{*}(G)$ is CN-hyperenergetic is not clear. However, there are finite non-abelian groups G such that $\Gamma_{\mathrm{ccc}}^{*}(G)$ is CN-borderenergetic (see Theorem 4.3.4), CNL-hyperenergetic/CNL- borderenergetic and CNSL-hyperenergetic/CNSL-borderenergetic (see Corollary 4.3.6). Thus the following problem is worth considering.

Problem 7.1.4. Characterize all finite non-abelian groups G such that $\Gamma^*_{\text{ccc}}(G)$ is CN-borderenergetic/CN-hyperenergetic/CNL-hyperenergetic/CNL-borderenergetic/CNSL-hyperenergetic/CNSL-borderenergetic.

In Section 4.4, we have found several classes of finite non-abelian groups G such that $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$. Thus, we pose the following problem.

Problem 7.1.5. Characterize all finite non-abelian groups G such that

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)).$$

In Section 4.4, we have also found several classes of finite non-abelian groups G such that $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$. As shown in Result 1.2.26, there are several classes of finite non-abelian groups G such that $\text{E}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}^+(\Gamma^*_{\text{ccc}}(G)) < \text{LE}(\Gamma^*_{\text{ccc}}(G))$. It follows that there exist finite non-abelian groups such that $\text{E}(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}^+(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}(\Gamma^*_{\text{ccc}}(G))$ and $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)), \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)), \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$ behave similarly. Thus, the following problem arises naturally.

Problem 7.1.6. Determine all the finite non-abelian groups G such that $\text{E}(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}^+(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}(\Gamma^*_{\text{ccc}}(G))$ and $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)), \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)), \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$ satisfy similar inequalities.

It is worth noting that problem similar to Problem 7.1.6 can also be asked for any finite graph. For the groups G consider in Chapter 4, we have

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) \leq \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)). \tag{7.1.a}$$

In view of this it is too early to conjecture (similar to Conjecture 1.1.6) that the inequality (7.1.a) holds for any finite non-abelian group. However, one may consider the following problem.

Problem 7.1.7. Determine all the finite non-abelian groups such that the inequality (7.1.a) does not hold. In general, determine all the finite graphs Γ such that the inequality $\text{E}_{\text{CN}}(\Gamma) \leq \text{LE}_{\text{CN}}(\Gamma)$ does not hold.

In Chapter 5, we have solved Inverse Wiener index Problem (Problem 1.1.24) for $\Gamma^*_{\mathrm{nccc}}(G)$ when n is a perfect square (see Reamrk 5.1.6). However, the solution of Inverse Wiener index Problem is not known for $\Gamma^*_{\mathrm{nccc}}(G)$ when $n \neq 2, 5$ is any positive integer. We pose the following problem in general.

Problem 7.1.8. Solve the Inverse Wiener index Problem for various graphs defined on finite groups.

In Chapter 6, we have computed DL-spectrum and DL-energy of CCC-graph of the groups D_{2m}, Q_{4n}, $U_{(n,m)}$, V_{8n} and SD_{8n}. We have also considered the groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m}. Note that CCC-graphs of these groups are DL-integral (see Theorem 6.1.9). Thus we have the following problem.

Problem 7.1.9. Can one determine all finite non-abelian groups such that their CCC-graph is DL-integral?

At this moment we do not have techniques to compute spectrum, Laplacian spectrum, signless Laplacian spectrum, distance spectrum and distance signless Laplacian spectrum of CCC-graph of a finite non-abelian group G. It may be interesting to develop such techniques through which one can compute energy (E), Laplacian energy (LE), signless Laplacian energy (LE$^+$), distance energy ($\mathrm{E_D}$) and distance signless Laplacian energy ($\mathrm{E_{DQ}}$) of CCC-graph of G and check whether

$$\mathrm{E}(\Gamma_{\mathrm{ccc}}(G)) = \mathrm{LE}(\Gamma_{\mathrm{ccc}}(G)) = \mathrm{LE}^+(\Gamma_{\mathrm{ccc}}(G)) = \mathrm{E_D}(\Gamma_{\mathrm{ccc}}(G)) = \mathrm{E_{DL}}(\Gamma_{\mathrm{ccc}}(G)) = \mathrm{E_{DQ}}(\Gamma_{\mathrm{ccc}}(G)).$$

This may lead to answer (partially) Problem 1.1.13. We have seen that the CCC-graphs of the groups considered in Chapter 6 are not complete. Therefore, it is obvious to ask the following question.

Question 7.1.10. For which groups CCC-graphs are complete?

There are certain generalization of CCC-graphs known as nilpotent conjugacy class graph (in short NCC-graph) and solvable conjugacy class graph (in short SCC-graph) introduced in [77] and [15] respectively. The NCC-graph (SCC-graph) of G is a graph whose vertex set is $\mathrm{Cl}(G)$ and two distinct vertices a^G and b^G are adjacent if there exist some elements $x \in a^G$ and $y \in b^G$ such that

$\langle x, y\rangle$ is nilpotent (solvable). Questions similar to Question 7.1.10 are also interesting for NCC- and SCC-graphs of groups. In this regard we prove the following results which are published in our paper [22].

For any subgroup H of G, we have G acts transitively on the set of right cosets of H by right multiplication. Therefore, G contains an element x fixing no coset of H in G. That is, no conjugate of x lies in H. Thus we have the following proposition.

Proposition 7.1.11. *Let H be a proper subgroup of a finite group G. Then G has a conjugacy class disjoint from H.*

The following theorem answers Question 7.1.10.

Theorem 7.1.12. *Let G be a finite group. Then the CCC-graph (resp., the NCC-graph, the SCC-graph) of G is complete if and only if G is abelian (resp. nilpotent, solvable).*

Proof. Let $C_1, C_2, \ldots, C_r$ be the conjugacy classes of G. Suppose that G is abelian (resp. nilpotent, solvable). Then for any two element $x \in C_i$ and $y \in C_j$ $(i \neq j)$ we have that $\langle x, y\rangle$ is abelian (resp. nilpotent, solvable). Hence, any two distinct vertices in CCC-graph (resp. NCC-graph, SCC-graph) of G is complete.

Suppose that the CCC-graph of G is complete. Let $h = h_1 \in C_1$. By the definition of CCC-graph, there exist $h_i \in C_i$ for all i such that h_i commutes with $h = h_1$. Therefore, by Proposition 7.1.11, we have $\langle h_1, h_2, \ldots, h_r\rangle = G$ and so $h \in Z(G)$. Since h is arbitrary, G is abelian.

Suppose that the NCC-graph of G is complete. Let $g, h \in G$ such that g is a p-element and h is a q-element for two distinct primes p and q. Since NCC-graph of G is complete there exists $x \in G$ such that $\langle g, h^x\rangle$ is nilpotent. By Result 1.2.4, it follows that g commutes with h^x and hence G is nilpotent.

Suppose that SCC-graph of G is complete. Let $g, h \in G$. Then there exists an element $x \in G$ such that $\langle g, h^x\rangle$ is solvable. Hence, by Result 1.2.5, it follows that G is solvable. □

A vertex of a graph is called a *dominant vertex* if it is adjacent to all other vertices of the graph. Regarding dominant vertices of CCC-graphs we prove the following result.

Theorem 7.1.13. *The set of dominant vertices $\Gamma_{\text{ccc}}(G)$ is $\mathrm{Cl}(Z(G))$.*

Proof. If $g \in Z(G)$ then g^G is adjacent to all the vertices of $\Gamma_{\mathrm{ccc}}(G)$. Therefore, g^G is dominant vertex of $\Gamma_{\mathrm{ccc}}(G)$.

Suppose that g^G is a dominant vertex of $\Gamma_{\mathrm{ccc}}(G)$. Then for all $h \in G$, there exists $x \in G$ such that g commutes with h^x. Therefore, $C_G(g)$ meets every conjugacy class of G. By Proposition 7.1.11, we have $C_G(g) = G$ and hence $g \in Z(G)$. This completes the proof. □

At this time we do not know about the dominant vertices of NCC- and SCC-graphs. We pose the following problem for future research.

Problem 7.1.14. Describe the dominant vertices of the NCC- and SCC-graph of a finite group.

We conclude this thesis noting that NCC- and SCC-graphs of groups are not studied much and can be studied in the light of similar problems mentioned problems.