Abstract

The energy of a finite simple graph Γ is the absolute sum of the eigenvalues of its adjacency matrix. In 1978, Gutman introduced this notion in order to estimate the total π-electron energy of a molecular graph. Later on, several mathematicians have introduced and studied various graph energies. Among those energies some well-studied energies are Laplacian energy, signless Laplacian energy, distance energy, distance Laplacian energy and distance signless Laplacian energy. In 2011, Alwardi et al. introduced the concept of common neighborhood energy of a graph. In this thesis, we introduce the concepts of common neighborhood Laplacian spectrum, common neighborhood Laplacian energy, common neighborhood signless Laplacian spectrum and common neighborhood signless Laplacian energy of a graph.

Another widely studied topic in Algebraic Graph Theory is the graphs defined on groups. Among those graphs, we consider commuting graph, commuting conjugacy class graph and non-commuting conjugacy class graph and investigate their common neighborhood and distance spectral aspects for certain families of finite non-abelian groups. In particular, we consider the dihedral group $D_{2m} = \langle x, y : x^m = y^2 = 1, y^{-1}xy = x^{-1}\rangle$ (for $m \geq 3$), the dicyclic group $Q_{4n} = \langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1}\rangle$ (for $n \geq 2$), the semidihedral group $SD_{8n} = \langle x, y : x^{4n} = y^2 = 1, y^{-1}xy = x^{2n-1}\rangle$ (for $n \geq 2$), the quasihedral group $QD_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}}\rangle$ (for $n \geq 4$), the Suzuki group (of order 20) $Sz(2) = \langle x, y : x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$, the projective special linear group $PSL(2, 2^k)$ (for $k \geq 2$), the general linear group $GL(2, q)$ (for any prime power $q > 2$), the Hanaki groups $A(n, v)$ and $A(n, p)$ and the groups $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1, x^{-1}yx =$

$y^{-1}\rangle$ (for $m \geq 3$ and $n \geq 2$), $U_{6n} = \langle x, y : x^{2n} = y^3 = 1,\ x^{-1}yx = y^{-1}\rangle$ (for $n \geq 2$) and $V_{8n} = \langle x, y : x^{2n} = y^4 = 1, yx = x^{-1}y^{-1}, y^{-1}x = x^{-1}y\rangle$ (for $n \geq 2$).

In Chapter 1, we recall some definitions, notations and results from Graph Theory and Group Theory that are required in the subsequent chapters. More precisely, we recall certain results and problems on various spectra and energies, Wiener index, first Zagreb index, commuting graph and commuting conjugacy class graph.

In Chapter 2, we introduce the concepts of common neighborhood (signless) Laplacian spectrum and energy. We establish relations between these energies and the first Zagreb index of a graph. Additionally, we introduce the concepts of CNL-integral, CNSL-integral, CNL-hyperenergetic and CNSL-hyperenergetic graphs and show that a complete bipartite graph is CNL-integral, CNSL-integral but neither CNL-hyperenergetic nor CNSL-hyperenergetic. Furthermore, we establish connections between various graph energies, including energy, Laplacian energy and signless Laplacian energy. Finally, we obtain several bounds for common neighborhood Laplacian and signless Laplacian energies of graphs.

In Chapter 3, we compute common neighborhood Laplacian spectrum, common neighborhood signless Laplacian spectrum, common neighborhood Laplacian energy and common neighborhood signless Laplacian energy of commuting graphs of the groups QD_{2^n}, $PSL(2, 2^k)$, $GL(2, q)$, $A(n, \nu)$, $A(n, p)$, D_{2n} and groups whose central quotient is isomorphic to $Sz(2)$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m}. We determine when commuting graphs of these groups are CNL(CNSL)-integral and CNL(CNSL)-hyperenergetic. Finally, we compare common neighborhood energy, common neighborhood Laplacian and common neighborhood signless Laplacian energy of commuting graphs of the above-mentioned groups.

In Chapter 4, we consider the subgraph of commuting conjugacy class graph of a finite non-abelian group G induced by the set of conjugacy classes of non-central elements of G which is denoted by $\Gamma^*_{\text{ccc}}(G)$. We compute common neighborhood spectrum, common neighborhood Laplacian spectrum, common neighborhood signless Laplacian spectrum and their corresponding energies of $\Gamma^*_{\text{ccc}}(G)$ for finite non-abelian groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is any prime) or the dihedral group D_{2m} ($m \geq 3$). We determine whether $\Gamma^*_{\text{ccc}}(G)$ for these groups are CN-, CNL-, CNSL-integral/hyperenergetic/borderenergetic. We also characterize the groups $G = D_{2m}$, Q_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n} such that $\Gamma^*_{\text{ccc}}(G)$ is CN-, CNL-, CNSL-integral/hyperenergetic/borderenergetic. Finally, we compare various common neighborhood energies of $\Gamma^*_{\text{ccc}}(G)$ for the above-mentioned groups and illustrate their closeness graphically.

In Chapter 5, we consider the complement of $\Gamma^*_{\text{ccc}}(G)$, denoted by $\Gamma^*_{\text{nccc}}(G)$, which is the subgraph of non-commuting conjugacy class graph of a finite non-abelian group G induced by the set of conjugacy classes of non-central elements of G. We compute distance spectrum, distance Laplacian spectrum, distance signless Laplacian spectrum along with their respective energies and Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for G when the central quotient of G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2m} (for any integer $m \geq 3$). As a consequence, we compute various distance spectra, energies and Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for the dihedral group, dicyclic group, semidihedral group along with the groups $U_{(n,m)}$, U_{6n} and V_{8n}. We show that any perfect square can be realized as Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for certain dihedral groups. We also characterize the above-mentioned groups such that $\Gamma^*_{\text{nccc}}(G)$ is D-integral, DL-integral and DQ-integral. We compute distance energy, distance Laplacian energy and distance signless Laplacian energy of $\Gamma^*_{\text{nccc}}(G)$ for the above-mentioned groups using Wiener index. We also compare various distance energies of $\Gamma^*_{\text{nccc}}(G)$ and characterize the above-mentioned groups subject to the inequalities involving various distance energies.

In Chapter 6, we consider commuting conjugacy class graph of a group G, denoted by $\Gamma_{\text{ccc}}(G)$, and compute distance Laplacian spectrum and energy of $\Gamma_{\text{ccc}}(G)$ if $G = D_{2m}$, Q_{4n}, $U_{(n,m)}$ and SD_{8n}. We also consider finite groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2m}. It is shown that the commuting conjugacy class graphs of these groups are D-integral. Finally, in Chapter 7, we shall summarize the work of this thesis and present several open problems that can be explored in future studies.