Chapter 1

Introduction and preliminaries

The energy of a finite simple graph Γ is the absolute sum of the eigenvalues of its adjacency matrix. In 1978, Gutman [54] introduced this notion in order to estimate the total π-electron energy of a molecular graph. The study of this notion gets its popularity from the beginning of the 21st century (see [5, 6]). Among many other graph energies some well-studied energies are Laplacian energy, signless Laplacian energy, distance energy, distance Laplacian energy and distance signless Laplacian energy. In 2011, Alwardi et al. [4] introduced the concept of common neighborhood energy of a graph. In our study, we shall introduce the concepts of common neighborhood Laplacian spectrum, common neighborhood Laplacian energy, common neighborhood signless Laplacian spectrum and common neighborhood signless Laplacian energy of a graph.

Another widely studied topic in Algebraic Graph Theory is the graphs defined on groups (see [21, 22, 75]). Among those graphs, we shall consider commuting graph, commuting conjugacy class graph and non-commuting conjugacy class graph and investigate their common neighborhood and distance spectral aspects for certain families of finite non-abelian groups. In particular, we shall consider the dihedral group $D_{2m} = \langle x, y : x^m = y^2 = 1, y^{-1}xy = x^{-1}\rangle$ (for $m \geq 3$), the dicyclic group $Q_{4n} = \langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1}\rangle$ (for $n \geq 2$), the semidihedral group $SD_{8n} = \langle x, y : x^{4n} = y^2 = 1, y^{-1}xy = x^{2n-1}\rangle$ (for $n \geq 2$), the quasihedral group $QD_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}}\rangle$ (for $n \geq 4$), the Suzuki group (of order 20) $Sz(2) = \langle x, y :$

$x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$, the projective special linear group $PSL(2, 2^k)$ (for $k \geq 2$), the general linear group $GL(2, q)$ (for any prime power $q > 2$). the Hanaki groups $A(n, v)$ and $A(n, p)$ and the groups $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1, x^{-1}yx = y^{-1}\rangle$ (for $m \geq 3$ and $n \geq 2$), $U_{6n} = \langle x, y : x^{2n} = y^3 = 1,\ x^{-1}yx = y^{-1}\rangle$ (for $n \geq 2$) and $V_{8n} = \langle x, y : x^{2n} = y^4 = 1, yx = x^{-1}y^{-1}, y^{-1}x = x^{-1}y\rangle$ (for $n \geq 2$).

In this chapter, we recall some definitions, notations and results from Graph Theory and Group Theory that are required in the subsequent chapters. More precisely, we recall certain results and problems on various spectra and energies, Wiener index, first Zagreb index, commuting graph and commuting conjugacy class graph.

In Chapter 2, we shall introduce the concepts of common neighborhood (signless) Laplacian spectrum and energy. We shall establish relations between these energies and the first Zagreb index of a graph. Additionally, we shall introduce the concepts of CNL-integral, CNSL-integral, CNL-hyperenergetic and CNSL-hyperenergetic graphs and show that a complete bipartite graph is CNL-integral, CNSL-integral but neither CNL- hyperenergetic nor CNSL-hyperenergetic. Furthermore, we shall establish connections between various graph energies, including energy, Laplacian energy and signless Laplacian energy. Finally, we shall obtain several bounds for common neighborhood Laplacian and signless Laplacian energies of graphs.

In Chapter 3, we shall compute common neighborhood Laplacian spectrum, common neighborhood signless Laplacian spectrum, common neighborhood Laplacian energy and common neighborhood signless Laplacian energy of commuting graphs of the groups QD_{2^n}, $PSL(2, 2^k)$, $GL(2, q)$, $A(n, \nu)$, $A(n, p)$, D_{2n} and groups whose central quotient is isomorphic to $Sz(2)$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m}. We shall determine when commuting graphs of these groups are CNL(CNSL)-integral and CNL(CNSL)-hyperenergetic. Finally, we shall compare common neighborhood energy, common neighborhood Laplacian and common neighborhood signless Laplacian energy of commuting graphs of the above-mentioned groups.

In Chapter 4, we shall consider the subgraph of commuting conjugacy class graph of a finite non-abelian group G induced by the set of conjugacy classes of non-central elements of G which is denoted by $\Gamma^*_{\text{ccc}}(G)$. We shall compute common neighborhood spectrum, common neighborhood Laplacian spectrum, common neighborhood signless

Laplacian spectrum and their corresponding energies of $\Gamma^*_{\text{ccc}}(G)$ for finite non-abelian groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is any prime) or the dihedral group D_{2m} ($m \geq 3$). We shall determine whether $\Gamma^*_{\text{ccc}}(G)$ for these groups are CN-, CNL-, CNSL-integral/hyperenergetic/borderenergetic. We shall also characterize the groups $G = D_{2m}$, Q_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n} such that $\Gamma^*_{\text{ccc}}(G)$ is CN-, CNL-, CNSL-integral/hyperenergetic/borderenergetic. Finally, we shall compare various common neighborhood energies of $\Gamma^*_{\text{ccc}}(G)$ for the above-mentioned groups and illustrate their closeness graphically.

In Chapter 5, we shall consider the complement of $\Gamma^*_{\text{ccc}}(G)$, denoted by $\Gamma^*_{\text{nccc}}(G)$, which is the subgraph of non-commuting conjugacy class graph of a finite non-abelian group G induced by the set of conjugacy classes of non-central elements of G. We shall compute distance spectrum, distance Laplacian spectrum, distance signless Laplacian spectrum along with their respective energies and Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for G when the central quotient of G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2m} (for any integer $m \geq 3$). As a consequence, we shall compute various distance spectra, energies and Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for the dihedral group, dicyclic group, semidihedral group along with the groups $U_{(n,m)}$, U_{6n} and V_{8n}. We shall show that any perfect square can be realized as Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for certain dihedral groups. We shall also characterize the above-mentioned groups such that $\Gamma^*_{\text{nccc}}(G)$ is D-integral, DL-integral and DQ-integral. We shall compute distance energy, distance Laplacian energy and distance signless Laplacian energy of $\Gamma^*_{\text{nccc}}(G)$ for the above-mentioned groups using Wiener index. We shall also compare various distance energies of $\Gamma^*_{\text{nccc}}(G)$ and characterize the above-mentioned groups subject to the inequalities involving various distance energies.

In Chapter 6, we shall consider commuting conjugacy class graph of a group G, denoted by $\Gamma_{\text{ccc}}(G)$, and compute distance Laplacian spectrum and energy of $\Gamma_{\text{ccc}}(G)$ if $G = D_{2m}$, Q_{4n}, $U_{(n,m)}$ and SD_{8n}. We shall also consider finite groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2m}. Further, we shall show that the commuting conjugacy class graphs of these groups are D-integral. Finally, in Chapter 7, we shall summarize the work done in this thesis and present several open problems that can be explored in future studies.

1.1 Notations and results of Graph Theory

For all the standard notations and basic definitions of Graph Theory we refer to [24]. Throughout the thesis, we write Γ to denote a finite simple undirected graph with vertex set $v(\Gamma)$ and edge set $e(\Gamma)$. The order of Γ is $|v(\Gamma)|$ and the size of Γ is $|e(\Gamma)|$. For any two vertices $u, v \in v(\Gamma)$, we write uv to denote the edge between u and v. The *degree* of a vertex v in Γ, denoted by $\deg(v)$, is the number of vertices adjacent to v. We write $\Delta = \max\{\deg(u) : u \in v(\Gamma)\}$ and $\delta = \min\{\deg(u) : u \in v(\Gamma)\}$ to denote the maximum and minimum degree of Γ respectively. If the degree of the vertices of Γ are listed in a sequence then that sequence is called *degree sequence* of Γ. A graph Γ is called *regular graph* if all the vertices of Γ have same degree. If $\deg(v) = r$ for all $v \in v(\Gamma)$ then Γ is called *r-regular graph*. A $u - v$ *walk* in Γ is a sequence of vertices in Γ beginning with u and ending at v such that consecutive vertices in the sequence are adjacent. If $u = v$ then we say that the walk $u - v$ is closed. A $u - v$ walk in Γ in which no vertices are repeated is a $u - v$ *path*. A path of order n is denoted by P_n. Again, a $u - v$ *trail* in Γ is a $u - v$ walk in which no edge is traversed more than once. A *circuit* in a graph is a closed trail of length 3 or more. A circuit that repeats no vertex, except for the first and last vertex, is a *cycle*. A cycle of order n is denoted by C_n. A cycle in Γ that contains every vertex of Γ is called a *Hamiltonian cycle*. Γ is called *acyclic* if it has no cycle. An acyclic connected graph is known as *tree*. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest $u - v$ path in Γ. If u and v are not connected in Γ then we write $d(u, v) = \infty$. *Derived graph* of Γ, denoted by $\Gamma^{\dagger}$, is the graph with vertex set $v(\Gamma)$, in which two vertices are adjacent if their distance in Γ is two. A graph Γ is said to be *complete graph* if every two distinct vertices are adjacent. A complete graph of order n is denoted by K_n. The size of K_n is $n(n-1)/2$. The *complement* of Γ is denoted by $\overline{\Gamma}$. A graph Γ is a *k-partite graph* if $v(\Gamma)$ can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ (called as partite sets) such that if uv is an edge of Γ then u and v belongs to different partite sets. Additionally, a k-partite graph Γ is said to be *complete k-partite graph* if every two vertices in different partite sets are joined by an edge. We denote a complete k-partite graph by $K_{n_1,n_2,n_3,\ldots,n_k}$ if $n_i = |V_i|$ for $1 \leq i \leq k$. A complete 2-partite graph is called a *complete bipartite graph*. The complete bipartite graph $K_{1,s}$ or $K_{s,1}$, where $s \geq 1$ is known as *star graph*. A graph Γ_1 is called a *subgraph* of a graph

Γ, denoted by $\Gamma_1 \subseteq \Gamma$, if $v(\Gamma_1) \subseteq v(\Gamma)$ and $e(\Gamma_1) \subseteq e(\Gamma)$. Again, a subgraph Γ_2 of a graph Γ is called an *induced subgraph* of Γ if whenever u and v are vertices of Γ_2 and uv is an edge of Γ then uv is an edge of Γ_2 as well. If Γ_2 is an induced subgraph of Γ and S is a non-empty subset of $v(\Gamma)$ such that $S = v(\Gamma_2)$ then we call Γ_2 is induced by S and it is denoted by $\Gamma[S]$.

Let Γ_1 and Γ_2 be two vertex disjoint graphs. Then the *union* of Γ_1 and Γ_2, denoted by $\Gamma_1 \cup \Gamma_2$, is the graph whose vertex set is $v(\Gamma_1) \cup v(\Gamma_2)$ and edge set is $e(\Gamma_1) \cup e(\Gamma_2)$. For subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ ($k \geq 2$) of a graph Γ with mutually disjoint vertex sets, we write $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$, if every vertex and every edge of Γ belongs to exactly one of these subgraphs. In such case, we say that Γ is a union of $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. If $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_k$ then we write $k\Gamma_1 = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$. The *join* of two graphs Γ_1 and Γ_2, denoted by $\Gamma_1 + \Gamma_2$, is the graph consisting $\Gamma_1 \cup \Gamma_2$ and all edges joining a vertex of Γ_1 and a vertex of Γ_2. Again, two graphs Γ_1 and Γ_2 are said to be *isomorphic* if there exists a one to one correspondence ϕ form $v(\Gamma_1)$ to $v(\Gamma_2)$ such that $uv \in e(\Gamma_1)$ if and only if $\phi(u)\phi(v) \in e(\Gamma_2)$. If Γ_1 and Γ_2 are isomorphic then we write $\Gamma_1 \cong \Gamma_2$.

1.1.1 Spectrum, (signless) Laplacian spectrum and their corresponding energies

For any graph Γ, let $\mathrm{A}(\Gamma)$ and $\mathrm{D}(\Gamma)$ be its adjacency matrix and degree matrix respectively. The *Laplacian matrix* and *signless Laplacian matrix* of Γ are given by

$$\mathrm{L}(\Gamma) := \mathrm{D}(\Gamma) - \mathrm{A}(\Gamma) \text{ and } \mathrm{Q}(\Gamma) := \mathrm{D}(\Gamma) + \mathrm{A}(\Gamma)$$

respectively. The *spectrum*, *Laplacian spectrum* and *signless Laplacian spectrum* of Γ are the sets of eigenvalues of $\mathrm{A}(\Gamma)$, $\mathrm{L}(\Gamma)$ and $\mathrm{Q}(\Gamma)$ with multiplicities respectively. Let $\mathrm{Spec}(\Gamma)$, L-spec(Γ) and Q-spec(Γ) be the spectrum, Laplacian spectrum and signless Laplacian spectrum of Γ respectively. Then $\mathrm{Spec}(\Gamma) = \{[\alpha_1]^{a_1}, [\alpha_2]^{a_2}, \ldots, [\alpha_l]^{a_l}\}$, L-spec$(\Gamma) = \{[\beta_1]^{b_1}, [\beta_2]^{b_2}, \ldots, [\beta_m]^{b_m}\}$ and Q-spec$(\Gamma) = \{[\gamma_1]^{c_1}, [\gamma_2]^{c_2}, \ldots, [\gamma_q]^{c_q}\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are the eigenvalues of $\mathrm{A}(\Gamma)$ with multiplicities $a_1, a_2, \ldots, a_l$; $\beta_1, \beta_2, \ldots, \beta_m$ are the eigenvalues of $\mathrm{L}(\Gamma)$ with multiplicities $b_1, b_2, \ldots, b_m$; $\gamma_1, \gamma_2, \ldots, \gamma_q$ are the eigenvalues of $\mathrm{Q}(\Gamma)$ with multiplicities $c_1, c_2, \ldots, c_q$ respectively. Sometimes we also write $\mathrm{Spec}(\Gamma) = \Big\{\underbrace{\alpha_1, \alpha_1, \ldots, \alpha_1}_{a_1\text{-times}},$

$\underbrace{\alpha_2, \alpha_2, \ldots, \alpha_2}_{a_2\text{-times}}, \ldots, \underbrace{\alpha_l, \alpha_l, \ldots, \alpha_l}_{a_l\text{-times}}\}$; similarly for other spectra of Γ. A graph Γ is called *integral*, *L-integral* and *Q-integral* respectively if $\mathrm{Spec}(\Gamma)$, $\text{L-spec}(\Gamma)$ and $\text{Q-spec}(\Gamma)$ contain only integers. The study of integral graphs began with the following question.

Question 1.1.1. Which graphs have integral spectra?

The Question 1.1.1 was posed by Harary and Schwenk [63] in 1973 (also see [12] for more information). Ahmadi et al. [1] highlighted the significance of integral graphs in designing network topologies for perfect state transfer networks. After that Grone and Merris [53] in 1994 and Simic and Stanic [97] in 2008 introduced the notions of L-integral and Q-integral graphs respectively.

Let $\mathrm{E}(\Gamma)$, $\mathrm{LE}(\Gamma)$ and $\mathrm{LE}^+(\Gamma)$ be the *energy*, *Laplacian energy* and *signless Laplacian energy* of Γ respectively. We have

$$\mathrm{E}(\Gamma) := \sum_{\alpha \in \mathrm{Spec}(\Gamma)} |\alpha|, \tag{1.1.a}$$

$$\mathrm{LE}(\Gamma) := \sum_{\beta \in \text{L-spec}(\Gamma)} \left|\beta - \frac{\mathrm{tr}(\mathrm{D}(\Gamma))}{|v(\Gamma)|}\right| \tag{1.1.b}$$

and

$$\mathrm{LE}^+(\Gamma) := \sum_{\gamma \in \text{Q-spec}(\Gamma)} \left|\gamma - \frac{\mathrm{tr}(\mathrm{D}(\Gamma))}{|v(\Gamma)|}\right|, \tag{1.1.c}$$

where $\mathrm{tr}(\mathrm{D}(\Gamma))$ is the *trace* of $\mathrm{D}(\Gamma)$. In 1978, Gutman [54] introduced the notion of $\mathrm{E}(\Gamma)$, which has been studied extensively by many mathematicians over the years (see [58] and the references therein). In 2006, Gutman and Zhou [62] introduced the notion of $\mathrm{LE}(\Gamma)$; and in 2008, Gutman et al. [57] introduced the notion of $\mathrm{LE}^+(\Gamma)$. Some bounds of $\mathrm{LE}(\Gamma)$ and $\mathrm{LE}^+(\Gamma)$ are given below.

Result 1.1.2. [33, Theorem 3.1 and Remark 3.8] Let Γ be a connected graph with maximum degree Δ. Then

(a) $\mathrm{LE}(\Gamma) \geq 2\left(\Delta + 1 - \frac{2|e(\Gamma)|}{|v(\Gamma)|}\right)$, with equality if and only if $\Gamma \cong K_{1,n-1}$.

(b) $\mathrm{LE}(\Gamma) \leq 4|e(\Gamma)|\left(1 - \frac{1}{|v(\Gamma)|}\right)$.

Result 1.1.3. [36, Theorem 5.5] Let Γ be a graph with degree sequence $d_1, d_2, \ldots, d_{|v(\Gamma)|}$. Then

$$\mathrm{LE}(\Gamma) \leq \sum_{i=1}^{|v(\Gamma)|} \sqrt{\left(d_i - \frac{2|e(\Gamma)|}{|v(\Gamma)|}\right)^2 + d_i}.$$

Result 1.1.4. [51, Theorem 3.1] Let Γ be a connected graph of order $n \geq 3$ with m edges and having maximum degree Δ and minimum degree δ. Then $\mathrm{LE}^+(\Gamma) \geq \Delta + \delta + \sqrt{(\Delta - \delta)^2 + 4\Delta - \frac{4m}{n}}$, with equality if and only if $\Gamma \cong K_{1,n-1}$.

Result 1.1.5. [51, Theorem 4.1] Let Γ be a connected graph of order n with m edges and having maximum degree Δ. Then $\mathrm{LE}^+(\Gamma) \leq 2\left(2m + 1 - \Delta - \frac{2m}{n}\right)$, with equality if and only if $\Gamma \cong K_{1,n-1}$.

In [57], Gutamn et al. showed that $\mathrm{E}(\Gamma) = \mathrm{LE}(\Gamma)$ for certain classes of graphs such as $\Gamma = pC_6 \cup qK_2$ (where p, q are positive integers), regular graph etc. Also, they showed that $\mathrm{E}(\Gamma) \leq \mathrm{LE}(\Gamma)$ for the graphs $\Gamma = K_{a,b},\ K_a \cup K_b,\ Kb_n(k)$ (the graph obtained by deleteing k independent edges from the complete graph K_n), $Kc_n(k)$ (the graph obtained by deleting the $(k(k-1))/2$ edges of a complete graph K_k from the complete graph K_n) etc. Based on these observations, they posed the following conjecture.

Conjecture 1.1.6. (E-LE Conjecture) For any finite simple graph Γ,

$$\mathrm{E}(\Gamma) \leq \mathrm{LE}(\Gamma). \tag{1.1.d}$$

Later on Conjecture 1.1.6 was disproved in [73, 99]. However, comparison of various graph energies became interesting due to the E-LE Conjecture.

In [29], Das et al. showed that $\mathrm{LE}(\Gamma) = \mathrm{LE}^+(\Gamma)$ if Γ is a bipartite graph or a regular graph. They further showed that $\mathrm{LE}(\Gamma_1) > \mathrm{LE}^+(\Gamma_1)$, $\mathrm{LE}(\Gamma_2) < \mathrm{LE}^+(\Gamma_2)$ and $\mathrm{LE}(\Gamma_3) = \mathrm{LE}^+(\Gamma_3)$, where the graphs Γ_1, Γ_2 and Γ_3 are as given below.

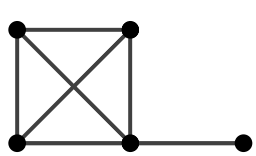

Figure 1.1: The graph Γ_1

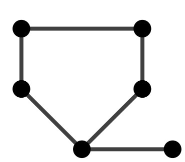

Figure 1.2: The graph Γ_2

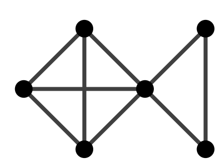

Figure 1.3: The graph Γ_3

These examples of graphs illustrate that $\mathrm{LE}(\Gamma)$ and $\mathrm{LE}^+(\Gamma)$ are not comparable in general. Thus, Das et al. [29] posed the following problem.

Problem 1.1.7. [29, Problem 1] Characterize all the graphs for which $\mathrm{LE}(\Gamma) > \mathrm{LE}^+(\Gamma)$, $\mathrm{LE}(\Gamma) < \mathrm{LE}^+(\Gamma)$ and $\mathrm{LE}(\Gamma) = \mathrm{LE}^+(\Gamma)$.

The concept of hyperenergetic graph was introduced by Walikar et al. [104] and Gutman [55], independently in 1999. A graph Γ is called *hyperenergetic* if $\mathrm{E}(\Gamma) > \mathrm{E}(K_{|v(\Gamma)|})$. In a similar way, L-hyperenergetic and Q-hyperenergetic graphs were defined and introduced in [48]. That is, a graph Γ is called *L-hyperenergetic* and *Q-hyperenergetic* if $\mathrm{LE}(\Gamma) > \mathrm{LE}(K_{|v(\Gamma)|})$ and $\mathrm{LE}^+(\Gamma) > \mathrm{LE}^+(K_{|v(\Gamma)|})$ respectively. Also, Γ is called *borderenergetic*, *L-borderenergetic* and *Q-borderenergetic* if $\mathrm{E}(\Gamma) = \mathrm{E}(K_{|v(\Gamma)|})$, $\mathrm{LE}(\Gamma) = \mathrm{LE}(K_{|v(\Gamma)|})$ and $\mathrm{LE}^+(\Gamma) = \mathrm{LE}^+(K_{|v(\Gamma)|})$ respectively. The concepts of borderenergetic, L-borderenergetic and Q-borderenergetic graphs were introduced by Gong et al. [52], Tura [102] and Tao et al. [100] in the years 2015, 2017 and 2018 respectively. In 1978, Gutman [54] posed the following conjecture.

Conjecture 1.1.8. [54] For any finite graph $\Gamma \ncong K_{|v(\Gamma)|}$, we have $\mathrm{E}(\Gamma) < 2\left(|v(\Gamma)| - 1\right)$. In other words, Γ is not hyperenergetic.

This Conjecture was disproved by several mathematicians (see [56]). In 2004, Balakrishnan [11] posed the following problem.

Problem 1.1.9. [11, Open problem 1] Prove that $K_n - H$, where $v(K_n) = \{1, 2, \ldots, n\}$ and $H = (1\,2\,3\,\cdots\,n)$ a Hamiltonian cycle of K_n, is not hyperenergetic for $n \geq 4$.

Let M be a real square symmetric matrix of size n. Then the spectrum of M, denoted by $\mathrm{Spec}(M)$, is defined as $\mathrm{Spec}(M) = \{\mu_1, \mu_2, \ldots, \mu_n\}$, where $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues (not necessarily distinct) of M. If $\mathrm{Spec}(M) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ then the energy of M, denoted by $\mathrm{E}(M)$, is defined as

$$\mathrm{E}(M) = \sum_{i=1}^{n} |\mu_i|.$$

The following results are useful in our study.

Result 1.1.10. [34, Lemma 2.10] Let M_1 and M_2 be two real square symmetric matrices of order n and let $M = M_1 + M_2$. Then

$$\mathrm{E}(M) \leq \mathrm{E}(M_1) + \mathrm{E}(M_2).$$

Result 1.1.11. [36, Lemma 5.1] Let A be a real symmetric matrix of order n and let $d_1, d_2, \ldots, d_n$ be the diagonal entries of the matrix A^2. Then

$$\mathrm{E}(A) \leq \sum_{i=1}^{n} \sqrt{d_i}.$$

1.1.2 Various distance spectra and energies

Let Γ be a connected graph with $v(\Gamma) = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{D}(\Gamma)$ be the *distance matrix* of Γ. Define $\mathcal{T}(\Gamma)$ as a diagonal matrix whose i-th diagonal entry $\mathcal{T}(\Gamma)_{i,i}$ is the transmission of vertex v_i in Γ given by $\mathcal{T}(\Gamma)_{i,i} = \sum_{j=1}^{n} d_{ij}$, where d_{ij} is the (i,j)-th element of $\mathcal{D}(\Gamma)$. Note that $d_{ij} = d(v_i, v_j)$ is the distance between v_i and v_j. The *distance Laplacian matrix* $\mathrm{DL}(\Gamma)$ and the *distance signless Laplacian matrix* $\mathrm{DQ}(\Gamma)$ are defined as follows:

$$\mathrm{DL}(\Gamma) := \mathcal{T}(\Gamma) - \mathcal{D}(\Gamma) \text{ and } \mathrm{DQ}(\Gamma) := \mathcal{T}(\Gamma) + \mathcal{D}(\Gamma).$$

The *distance spectrum* $\mathrm{D\text{-}spec}(\Gamma)$, *distance Laplacian spectrum* $\mathrm{DL\text{-}spec}(\Gamma)$ and *distance signless Laplacian spectrum* $\mathrm{DQ\text{-}spec}(\Gamma)$ of Γ are the sets of eigenvalues of $\mathcal{D}(\Gamma)$, $\mathrm{DL}(\Gamma)$ and $\mathrm{DQ}(\Gamma)$ respectively, each with their corresponding multiplicities. We write $\mathrm{D\text{-}spec}(\Gamma) = \{[\lambda_1]^{d_1}, [\lambda_2]^{d_2}, \ldots, [\lambda_r]^{d_r}\}$, $\mathrm{DL\text{-}spec}(\Gamma) = \{[\mu_1]^{f_1}, [\mu_2]^{f_2}, \ldots, [\mu_s]^{f_s}\}$ and $\mathrm{DQ\text{-}spec}(\Gamma) = \{[\nu_1]^{g_1}, [\nu_2]^{g_2}, \ldots, [\nu_t]^{g_t}\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the eigenvalues of $\mathcal{D}(\Gamma)$ with multiplicities $d_1, d_2, \ldots, d_r$; $\mu_1, \mu_2, \ldots, \mu_s$ are the eigenvalues of $\mathrm{DL}(\Gamma)$ with multiplicities $f_1, f_2, \ldots, f_s$; $\nu_1, \nu_2, \ldots, \nu_t$ are the eigenvalues of $\mathrm{DQ}(\Gamma)$ with multiplicities $g_1, g_2, \ldots, g_t$ respectively. A connected graph Γ is called *distance integral* (D-integral), *distance Laplacian integral* (DL-integral) and *distance signless Laplacian integral* (DQ-integral) if $\mathrm{D\text{-}spec}(\Gamma)$, $\mathrm{DL\text{-}spec}(\Gamma)$ and $\mathrm{DQ\text{-}spec}(\Gamma)$, respectively, consist solely of integer eigenvalues. The study of $\mathrm{D\text{-}spec}(\Gamma)$ was pioneered by Indulal and Gutman [65] in 2008, while the concepts of $\mathrm{DL\text{-}spec}(\Gamma)$ and $\mathrm{DQ\text{-}spec}(\Gamma)$ were introduced by Aouchiche and Hansen [5] in 2013. For further reading on distance (Laplacian) spectra, we refer to [6, 87] and the references therein. The distance matrices have a wide range of applications across various fields, including the design of communication networks, network flow algorithms, graph embedding theory and even in areas such as molecular stability, branching and model boiling points of an alkane, psychology, phylogenetics, software compression, analysis of internet infrastructures, modeling of traffic and social networks etc. as noted in [6]. However, we shall not address any of these applications in our work.

In analogy to energy $\mathrm{E}(\Gamma)$, Laplacian energy $\mathrm{LE}(\Gamma)$ and signless Laplacian energy $\mathrm{LE}^+(\Gamma)$ of a graph Γ Indulal et al. [65], Gutman et al. [108] and Das et al. [29] introduced *distance energy* $\mathrm{E_D}(\Gamma)$, *distance Laplacian energy* $\mathrm{E_{DL}}(\Gamma)$ and *distance signless Laplacian energy* $\mathrm{E_{DQ}}(\Gamma)$ of Γ as given below:

$$\mathrm{E_D}(\Gamma) := \sum_{\lambda \in \mathrm{D\text{-}spec}(\Gamma)} |\lambda|, \tag{1.1.e}$$

$$\mathrm{E_{DL}}(\Gamma) := \sum_{\mu \in \mathrm{DL\text{-}spec}(\Gamma)} \left| \mu - \frac{\mathrm{tr}(\mathrm{DL}(\Gamma))}{|v(\Gamma)|} \right| \tag{1.1.f}$$

and

$$\mathrm{E_{DQ}}(\Gamma) := \sum_{\nu \in \mathrm{DQ\text{-}spec}(\Gamma)} \left| \nu - \frac{\mathrm{tr}(\mathrm{DQ}(\Gamma))}{|v(\Gamma)|} \right|. \tag{1.1.g}$$

Various results on $\mathrm{E_D}(\Gamma)$ have been obtained by several mathematicians (for example see [6, 65, 98, 106, 109]). Further, several bounds of $\mathrm{E_D}(\Gamma)$, $\mathrm{E_{DL}}(\Gamma)$ and $\mathrm{E_{DQ}}(\Gamma)$ and relations among these three energies have been explored in [29, 37, 108] and the references therein. In 2018, Das et al. [29] posed the following problems.

Problem 1.1.12. [29, Problem 3] Characterize all the graphs for which $\mathrm{E_{DL}}(\Gamma) = \mathrm{E_{DQ}}(\Gamma)$.

Problem 1.1.13. [29, Problem 4] Is there any connected graph $\Gamma\ (\ncong K_n)$ such that $\mathrm{E}(\Gamma) = \mathrm{LE}(\Gamma) = \mathrm{LE}^+(\Gamma) = \mathrm{E_D}(\Gamma) = \mathrm{E_{DL}}(\Gamma) = \mathrm{E_{DQ}}(\Gamma)$?

In our study, we consider Problem 1.1.12 and Problem 1.1.13 for certain subgraphs of non-commuting conjugacy class graphs and obtain graphs satisfying the equalities in Problem 1.1.12 and Problem 1.1.13.

The following result gives characteristic polynomials of $\mathcal{D}(\Gamma)$, $\mathrm{DL}(\Gamma)$ and $\mathrm{DQ}(\Gamma)$ for certain graphs.

Result 1.1.14. Let $\Gamma = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $2 \leq k \leq \sum_{i=1}^{k} n_i - 1$. Then

(a) [72, Lemma 2.5] the characteristic polynomial of $\mathcal{D}(\Gamma)$ is given by

$$\mathrm{Ch}_{\mathrm{D}}(\Gamma, x) = (x+2)^{|v(\Gamma)|-k}\left(\prod_{i=1}^{k}(x-n_i+2) - \sum_{i=1}^{k} n_i \prod_{j=1, j\neq i}^{k}(x-n_j+2)\right).$$

(b) [72, Lemma 2.8] the characteristic polynomial of $\mathrm{DL}(\Gamma)$ is given by

$$\mathrm{Ch}_{\mathrm{DL}}(\Gamma, x) = x(x-|v(\Gamma)|)^{k-1} \prod_{i=1}^{k}(x-|v(\Gamma)|-n_i)^{n_i-1}.$$

(c) [72, Lemma 2.12] the characteristic polynomial $\mathrm{Ch}_{\mathrm{DQ}}(\Gamma, x)$ of $\mathrm{DQ}(\Gamma)$ is given by

$$\prod_{i=1}^{k}(x-|v(\Gamma)|-n_i+4)^{n_i-1}\left(\prod_{i=1}^{k}(x-|v(\Gamma)|-2n_i+4) - \sum_{i=1}^{k} n_i \prod_{j=1, j\neq i}^{k}(x-|v(\Gamma)|-2n_j+4)\right).$$

Result 1.1.15. [101, Corollary 2.2] If $\Gamma = K_a + (bK_c \cup K_d)$ then the distance Laplacian characteristic polynomial of Γ is given by

$$\mathrm{Ch}_{\mathrm{DL}}(\Gamma, x) = x\big(x-(bc+d+a)\big)^{a}\big(x-(2bc-c+2d+a)\big)^{b(c-1)} \times \big(x-(2bc+d+a)\big)^{d-1}\big(x-(2bc+2d+a)\big)^{b}.$$

1.1.3 Common neighborhood spectrum and energy

Let $v(\Gamma) = \{v_1, v_2, \ldots, v_n\}$. The neighborhood of a vertex v_i in Γ, denoted by $N_\Gamma(v_i)$, is the set $\{v_j : v_j$ is adjacent to v_i and $i \neq j\}$. Note that $\deg(v_i) = |N_\Gamma(v_i)|$. Let $m_\Gamma(v_i)$ be the *average degree* of the adjacent vertices of v_i in Γ. If v_i is an *isolated vertex* in Γ then we assume that $m_\Gamma(v_i) = 0$. Hence, $\deg(v_i)\, m_\Gamma(v_i) = \sum_{v_j : v_i v_j \in e(\Gamma)} \deg(v_j)$. The *common neighborhood* of two vertices v_i and v_j, denoted by $N(v_i, v_j)$, is the set containing all the vertices other than v_i and v_j that are adjacent to both v_i and v_j. Thus, $N(v_i, v_j) = N_\Gamma(v_i) \cap N_\Gamma(v_j)$. We have the following result.

Result 1.1.16. [27] Let Γ be a graph of order $|v(\Gamma)|$. Then for each $v_i \in v(\Gamma)$,

$$\sum_{k=1,\, k\neq i}^{|v(\Gamma)|} |N_\Gamma(v_i) \cap N_\Gamma(v_k)| = \sum_{v_j : v_i v_j \in e(\Gamma)} \Big(\deg(v_j) - 1\Big) = \deg(v_i)\, m_\Gamma(v_i) - \deg(v_i).$$

The *common neighborhood matrix* $\mathrm{CN}(\Gamma)$ of Γ (also known as CN-matrix) is given by

$$\mathrm{CN}(\Gamma)_{i,j} = \begin{cases} |N(v_i, v_j)|, & \text{if } i \neq j \\ 0, & \text{if } i = j, \end{cases}$$

where $\mathrm{CN}(\Gamma)_{i,j}$ is the (i, j)-th entry of $\mathrm{CN}(\Gamma)$. The CN-matrix of Γ can be obtained by the following result.

Result 1.1.17. [4, Proposition 2.7] Let Γ be any graph. Then $\mathrm{CN}(\Gamma) = \mathrm{A}(\Gamma)^2 - \mathrm{D}(\Gamma)$.

The *common neighborhood spectrum* (also known as CN-spectrum) of Γ, denoted by $\mathrm{CN\text{-}spec}(\Gamma)$, is the set of eigenvalues of $\mathrm{CN}(\Gamma)$ with multiplicity. We write $\mathrm{CN\text{-}spec}(\Gamma) = \{[\rho_1]^{h_1}, [\rho_2]^{h_2}, \ldots, [\rho_u]^{h_u}\}$ where $\rho_1, \rho_2, \ldots, \rho_u$ are the eigenvalues of $\mathrm{CN}(\Gamma)$ with multiplicities $h_1, h_2, \ldots, h_u$ respectively. The *common neighborhood energy* (also known as CN-energy) of Γ, denoted by $\mathrm{E_{CN}}(\Gamma)$, is defined as

$$\mathrm{E_{CN}}(\Gamma) = \sum_{\rho \in \mathrm{CN\text{-}spec}(\Gamma)} |\rho|. \tag{1.1.h}$$

A graph Γ is called *CN-hyperenergetic* if $\mathrm{E_{CN}}(\Gamma) > \mathrm{E_{CN}}(K_{|v(\Gamma)|})$. The concepts of CN-spectrum and CN-energy are relatively new and not much explored. These concepts were introduced by Alwardi et al. [4] in 2011. The following results are useful in our study.

Result 1.1.18. [49, Theorem 1] Let $\Gamma = l_1K_{m_1} \cup l_2K_{m_2} \cup \cdots \cup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of the complete graphs K_{m_i} on m_i vertices for $i = 1, 2, \ldots, k$. Then

$$\mathrm{CN\text{-}spec}(\Gamma) = \Big\{[-(m_1 - 2)]^{l_1(m_1-1)}, [(m_1 - 1)(m_1 - 2)]^{l_1}, [-(m_2 - 2)]^{l_2(m_2-1)}, [(m_2 - 1)(m_2 - 2)]^{l_2}, \ldots, [-(m_k - 2)]^{l_k(m_k-1)}, [(m_k - 1)(m_k - 2)]^{l_k}\Big\}.$$

Result 1.1.19. [80, Theorem 2] Let $\Gamma = l_1K_{m_1} \cup l_2K_{m_2} \cup \cdots \cup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of the complete graphs K_{m_i} on m_i vertices for $i = 1, 2, \ldots, k$. Then

$$\mathrm{E_{CN}}(\Gamma) = 2\sum_{i=1}^{k} l_i(m_i - 1)(m_i - 2).$$

Using Result 1.1.18 and Result 1.1.19, we get the following result.

Result 1.1.20. Let $\Gamma = l_1K_{m_1} \cup l_2K_{m_2} \cup l_3K_{m_3}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of K_{m_i} for $i = 1, 2, 3$. Then

$$\text{CN-spec}(\Gamma) = \Big\{[-(m_1-2)]^{l_1(m_1-1)}, [(m_1-1)(m_1-2)]^{l_1}, [-(m_2-2)]^{l_2(m_2-1)}, [(m_2-1)(m_2-2)]^{l_2}, [-(m_3-2)]^{l_3(m_3-1)}, [(m_3-1)(m_3-2)]^{l_3}\Big\}$$

and

$$\mathrm{E}_{\mathrm{CN}}(\Gamma) = 2l_1(m_1-1)(m_1-2) + 2l_2(m_2-1)(m_2-2) + 2l_3(m_3-1)(m_3-2).$$

Result 1.1.21. [4, Proposition 2.4] If the graph $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n$ then

$$\mathrm{E}_{\mathrm{CN}}(\Gamma) = \mathrm{E}_{\mathrm{CN}}(\Gamma_1) + \mathrm{E}_{\mathrm{CN}}(\Gamma_2) + \cdots + \mathrm{E}_{\mathrm{CN}}(\Gamma_n).$$

Result 1.1.22. [4, Proposition 2.12] If Γ is a triangle- and quadrangle-free graph then $\mathrm{E}_{\mathrm{CN}}(\Gamma) = \mathrm{E}(\Gamma^\dagger)$.

We conclude this section with the following result.

Result 1.1.23. [4, Corollary 2.13-2.15]

(a) If T is a tree then $\mathrm{E}_{\mathrm{CN}}(T) = \mathrm{E}(T^\dagger)$.

(b) If P_n is a n-vertex path then $\mathrm{E}_{\mathrm{CN}}(P_n) = \mathrm{E}(P_{\lfloor \frac{n}{2} \rfloor}) + \mathrm{E}(P_{\lceil \frac{n}{2} \rceil})$.

(c) For n-vertex cycle C_n, if $n \geq 3$ is odd then $\mathrm{E}_{\mathrm{CN}}(C_n) = \mathrm{E}(C_n)$; if $n = 4$ then $\mathrm{E}_{\mathrm{CN}}(C_n) = 4\,\mathrm{E}(K_2) = 8$; if $n \geq 6$ is even then $\mathrm{E}_{\mathrm{CN}}(C_n) = 2\,\mathrm{E}(C_{\frac{n}{2}})$.

1.1.4 Topological index

Topological indices of a graph are numerical quantities derived from the graph. The oldest topological index is the Wiener index originated from the work of Wiener [105] in 1947. Let Γ be any graph with $v(\Gamma) = \{v_1, v_2, \ldots, v_{|v(\Gamma)|}\}$. The *Wiener index* of Γ, denoted by $W(\Gamma)$, is defined as $W(\Gamma) = \frac{1}{2}\sum_{1 \leq i,j \leq |v(\Gamma)|} d(v_i, v_j)$. For various results on Wiener index we refer to [38, 107] and the references there in. One interesting problem regarding Wiener index is given below.

Problem 1.1.24. (Inverse Wiener index Problem) Given any positive integer n find a graph Γ from a prescribed class such that $W(\Gamma) = n$.

Considering the family of all graphs Gutman et al. [61] solved the Inverse Wiener index Problem for $n \neq 2, 5$. The Inverse Wiener index Problem is also solved for bipartite graphs with some exceptional values of n (see [103] for details). More precisely, we have the following results.

Result 1.1.25. [61, Theorem 1] Let $\mathcal{C}$ be the set of all connected graphs and $\mathcal{W}(\mathcal{C}) := \{W(\Gamma) : \Gamma \in \mathcal{C}\}$. Then $\mathbb{N} \setminus \mathcal{W}(\mathcal{C}) = \{2, 5\}$.

Result 1.1.26. [61, Theorem 3] Let $\mathcal{B}$ be the set of all connected bipartite graphs and $\mathcal{W}(\mathcal{B}) := \{W(\Gamma) : \Gamma \in \mathcal{B}\}$. Then

$$\mathbb{N} \setminus \mathcal{W}(\mathcal{B}) = \{2, 3, 5, 6, 7, 11, 12, 13, 15, 17, 19, 33, 37, 39\}.$$

However, Inverse Wiener index Problem is not solved for graphs defined on groups, in particular for non-commuting conjugacy class graphs. In Chapter 5, we shall consider a subgraph of non-commuting conjugacy class graphs of groups and solved Inverse Wiener index Problem when n is a perfect square.

The first degree based topological index of a graph is the Zagreb index. There are two types of Zagreb indices namely the first Zagreb index and second Zagreb index. These were introduced by Gutman and Trinajstić [60] in 1972. In our study, we have considered only the first Zagreb Index. The *first Zagreb index* of Γ, denoted by $M_1(\Gamma)$, is defined as the sum of the squares of the degrees of the vertices of Γ. Thus,

$$M_1(\Gamma) = \sum_{i=1}^{|v(\Gamma)|} \deg(v_i)^2 = \sum_{v_i v_j \in e(\Gamma)} \Big(\deg(v_i) + \deg(v_j)\Big).$$

It was shown in [30] that

$$M_1(\Gamma) = \sum_{i=1}^{|v(\Gamma)|} \deg(v_i)\, m_\Gamma(v_i). \tag{1.1.i}$$

Mathematical properties on the first Zagreb index was reported in [19, 27, 28, 32]. We shall conclude this section by listing certain relation between (signless) Laplacian energy and first Zagreb index of a graph.

Result 1.1.27. [33, Equation no. (13)–(15)] Let Γ be any graph. Then

(a) $\text{LE}(\Gamma) \leq 2|e(\Gamma)| + M_1(\Gamma) - \frac{4|e(\Gamma)|^2}{|v(\Gamma)|}$.

(b) $\text{LE}(\Gamma) \leq \sqrt{|v(\Gamma)|\left(2|e(\Gamma)| + M_1(\Gamma) - \frac{4|e(\Gamma)|^2}{|v(\Gamma)|}\right)}$.

(c) $\text{LE}(\Gamma) \leq \frac{2|e(\Gamma)|}{|v(\Gamma)|} + \sqrt{(|v(\Gamma)| - 1)\left(2|e(\Gamma)| + M_1(\Gamma) - \frac{4|e(\Gamma)|^2}{|v(\Gamma)|} - \frac{4|e(\Gamma)|^2}{|v(\Gamma)|^2}\right)}$.

Result 1.1.28. [50, Theorem 2.3] Let Γ be a graph of order $|v(\Gamma)| \geq 5$ having maximum degree Δ, second maximum degree Δ' and first Zagreb index $M_1(\Gamma)$. Let v_1 and v_2 be maximum and second maximum degree vertices in Γ. Then

$$\text{LE}^+(\Gamma) \geq \begin{cases} \frac{2M_1(\Gamma)}{|e(\Gamma)|} + 2\Delta' - \frac{8|e(\Gamma)|}{|v(\Gamma)|}, & \text{if } v_1 \text{ and } v_2 \text{ are not adjacent} \\ \frac{2M_1(\Gamma)}{|e(\Gamma)|} + 2\beta - \frac{8|e(\Gamma)|}{|v(\Gamma)|}, & \text{if } v_1 \text{ and } v_2 \text{ are adjacent,} \end{cases}$$

where $\beta = \frac{\Delta + \Delta' - \sqrt{(\Delta - \Delta')^2 + 4}}{2}$ and equality occurs if and only if $\Gamma \cong K_{|v(\Gamma)|-2,2}$.

Result 1.1.29. [51, Theorem 3.3] Let Γ be a connected graph of order $|v(\Gamma)| \geq 3$ having first Zagreb index $M_1(\Gamma)$. Then

$$\text{LE}^+(\Gamma) \geq 2\left(\frac{M_1(\Gamma)}{|e(\Gamma)|} - \frac{2|e(\Gamma)|}{|v(\Gamma)|}\right).$$

1.2 Notations and results of Group Theory

In this section, we recall some definitions, notations and results of Group Theory that are useful in our study. However, for all the standard notations and basic results we refer to [89, 90]. Throughout this thesis, we shall write G to denote a finite non-abelian group with center $Z(G) = \{z \in G : zx = xz \text{ for all } x \in G\}$.

The *centralizer* of an element $x \in G$, denoted by $C_G(x)$, is given by $C_G(x) = \{y \in G : xy = yx\}$. A group G is called an *n-centralizer group* if the number of distinct centralizers of G is n. Below we mention some characterizations of n-centralizer finite groups given by Belcastro and Sherman [14].

Result 1.2.1. [14, Theorem 2] A finite group G is 4-centralizer if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Result 1.2.2. [14, Theorem 4] A finite group G is 5-centralizer if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\frac{G}{Z(G)} \cong D_6$.

A group G is called a *p-group* if the order of every element in G is a power of the prime number p. Ashrafi [10] obtained the following characterization of $(p+2)$-centralizer finite p-group.

Result 1.2.3. [10, Theorem 12] If G is a finite non-abelian p-group then G is $(p+2)$-centralizer if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

A group G is called an *AC-group* if $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$. In our study, we shall consider the following families of AC-groups: dihedral group $D_{2m} = \langle x, y : x^m = y^2 = 1, y^{-1}xy = x^{-1}\rangle$ (for $m \geq 3$), dicyclic group $Q_{4n} = \langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1}\rangle$ (for $n \geq 2$), semidihedral group $SD_{8n} = \langle x, y : x^{4n} = y^2 = 1, y^{-1}xy = x^{2n-1}\rangle$ (for $n \geq 2$), quasihedral group $QD_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}}\rangle$ (for $n \geq 4$), Suzuki group (of order 20) $Sz(2) = \langle x, y : x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$ and the groups $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1, x^{-1}yx = y^{-1}\rangle$ (for $m \geq 3$ and $n \geq 2$), $U_{6n} = \langle x, y : x^{2n} = y^3 = 1,\ x^{-1}yx = y^{-1}\rangle$ (for $n \geq 2$), $V_{8n} = \langle x, y : x^{2n} = y^4 = 1, yx = x^{-1}y^{-1}, y^{-1}x = x^{-1}y\rangle$ (for $n \geq 2$), projective special linear group $PSL(2, 2^k)$ (for $k \geq 2$), general linear group $GL(2, q)$ (for any prime power $q > 2$) and the Hanaki groups $A(n, v)$ and $A(n, p)$.

The *conjugacy class* of $x \in G$, denoted by x^G, is given by $x^G = \{x^g : g \in G\}$, where $x^g := gxg^{-1}$. We write $\mathrm{Cl}(G)$ to denote the set of all the conjugacy classes of G. That is, $\mathrm{Cl}(G) = \{x^G : x \in G\}$. Also, $\mathrm{Cl}(X) = \{x^G : x \in X\}$ for any subset X of G.

A group G is called *nilpotent* if it has a central series, that is normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that G_{i+1}/G_i is contained in the center of G/G_i for all i. We have the following result regarding finite nilpotent group.

Result 1.2.4. [39, Corollary E] Let G be a finite group. Then G is nilpotent if and only if for every pair of distinct primes p and q and for every pair of elements $x, y \in G$ with x a p-element and y a q-element, x and y^g commute for some $g \in G$.

A group G is said to be *solvable* if it has an abelian series that is $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ in which each factor G_{i+1}/G_i is abelian. We would like to mention the following result regarding finite solvable group.

Result 1.2.5. [39, Theorem A] Let G ba a finite group. Then the following are equivalent.

(a) G is solvable.

(b) For all $x, y \in G$, there exists an element $g \in G$ for which $\langle x, y^g \rangle$ is solvable.

(c) For all $x, y \in G$ of prime power order, there exists an element $g \in G$ for which $\langle x, y^g \rangle$ is solvable.

Characterization of finite groups through various graphs defined on it has been an active area of research over the last 50 years. A number of graphs have been defined on groups (see [21]) among which we shall consider the commuting graph, commuting conjugacy class graph and the complement of commuting conjugacy class graph also known as non-commuting conjugacy class graph.

1.2.1 Commuting graph

Let G be a finite non-abelian group with center $Z(G)$. The *commuting graph* of G, denoted by $\Gamma_{\mathrm{c}}(G)$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent if they commute. This graph was originated from the work of Brauer and Fowler [20] published in the year 1955. However, after Neumann's work [82] on its complement (also known as non-commuting graph) in 1976, the commuting graph became popular. Various aspects of commuting graphs of finite non-abelian AC-groups can be found in [2, 13, 66, 79].

Some results on structures of commuting graphs for various families of finite non-abelian groups are given below.

Result 1.2.6. [42, page no. 89] The commuting graph of the quasidihedral group QD_{2^n}, where $n \geq 4$, is given by

$$\Gamma_{\mathrm{c}}(QD_{2^n}) = K_{2^{n-1}-2} \cup 2^{n-2}K_2.$$

Result 1.2.7. [42, page no. 89] The commuting graph of the projective special linear group $PSL(2, 2^k)$, where $k \geq 2$, is given by

$$\Gamma_{\mathrm{c}}(PSL(2, 2^k)) = (2^k + 1)K_{2^k-1} \cup 2^{k-1}(2^k + 1)K_{2^k-2} \cup 2^{k-1}(2^k - 1)K_{2^k}.$$

Result 1.2.8. [42, page no. 90] The commuting graph of the general linear group $GL(2, q)$, where $q = p^n > 2$ and p is a prime, is given by

$$\Gamma_{\rm c}(GL(2,q)) = \frac{q(q+1)}{2}K_{q^2-3q+2} \cup \frac{q(q-1)}{2}K_{q^2-q} \cup (q+1)K_{q^2-2q+1}.$$

Result 1.2.9. [42, page no. 91] Let $F = GF(2^n)$ (where $n \geq 2$) and ν be the Frobenius automorphism of F, that is $\nu(x) = x^2$, for all $x \in F$. Then the commuting graph of the group

$$A(n,\nu) := \left\{ U(a,b) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \nu(a) & 1 \end{pmatrix} \ : \ a, b \in F \right\}$$

under matrix multiplication $U(a,b)U(a',b') := U(a+a', b+b'+a'\nu(a))$ is given by

$$\Gamma_{\rm c}(A(n,\nu)) = (2^n - 1)K_{2^n}.$$

Result 1.2.10. [42, page no. 91] Let $F = GF(p^n)$, where p is a prime. Then the commuting graph of the group

$$A(n,p) := \left\{ V(a,b,c) := \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \ : \ a, b, c \in F \right\}$$

under matrix multiplication $V(a,b,c)V(a',b',c') := V(a+a', b+b'+ca', c+c')$ is given by

$$\Gamma_{\rm c}(A(n,p)) = (p^n + 1)K_{p^{2n}-p^n}.$$

Result 1.2.11. [42, page no. 90] Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong Sz(2)$. Then $\Gamma_{\rm c}(G) = K_{4|Z(G)|} \cup 5K_{3|Z(G)|}$.

Result 1.2.12. [43, page no. 227] Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then $\Gamma_{\rm c}(G) = (p+1)K_{(p-1)|Z(G)|}$.

Result 1.2.13. [43, page no. 228] Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, $m \geq 2$. Then $\Gamma_{\rm c}(G) = K_{(m-1)|Z(G)|} \cup mK_{|Z(G)|}$.

In [42, 43], $\mathrm{Spec}(\Gamma_c(G))$ is computed for various families of finite groups and obtained various groups such that $\Gamma_c(G)$ is integral. In [44], Dutta and Nath have computed $\mathrm{L\text{-}spec}(\Gamma_c(G))$ and $\mathrm{Q\text{-}spec}(\Gamma_c(G))$ for various families of finite groups and obtained various groups such that $\Gamma_c(G)$ is L-integral and Q-integral. In [95, 46, 40], $\mathrm{E}(\Gamma_c(G))$, $\mathrm{LE}(\Gamma_c(G))$ and $\mathrm{LE}^+(\Gamma_c(G))$ have been computed for many families of finite groups and many finite groups have been obtained for which $\Gamma_c(G)$ is hyperenergetic. It was also shown that $\mathrm{E}(\Gamma_c(G)) \leq \mathrm{LE}(\Gamma_c(G))$ for all the groups considered in [40]. In [49], Fasfous et al. have computed CN-spectrum of commuting graphs of several families of finite non-abelian groups and determine certain finite non-abelian groups such that their commuting graphs are CN-integral. In [80], Nath et al. have computed CN-energy of commuting graphs of variuos families of finite non-abelian groups and determine several groups such that their commuting graphs are CN-hyperenergetic. Some of the results of [80] are listed below.

Result 1.2.14. [80, Theorem 3] Let G be a finite group with center $Z(G)$.

(a) If $\frac{G}{Z(G)} \cong Sz(2)$ then $\mathrm{E_{CN}}(\Gamma_c(G)) = 2(61|Z(G)|^2 - 57|Z(G)| + 12)$.

(b) If $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) then $\mathrm{E_{CN}}(\Gamma_c(G)) = 2(p+1)((p-1)|Z(G)| - 1)((p-1)|Z(G)| - 2)$.

(c) If $\frac{G}{Z(G)} \cong D_{2m}$ $(m \geq 2)$ then $\mathrm{E_{CN}}(\Gamma_c(G)) = 2((m^2-m+1)|Z(G)|^2-(6m-3)|Z(G)|+2m+2)$.

Result 1.2.15. [80, Theorem 4] Let G be a finite non-abelian group.

(a) If $G = QD_{2^n}$ $(n \geq 4)$ then $\mathrm{E_{CN}}(\Gamma_c(G)) = 2(2^{n-1} - 3)(2^{n-1} - 4)$.

(b) If $G = PSL(2, 2^k)$ then $\mathrm{E_{CN}}(\Gamma_c(G)) = 2^{4k+1} - 4 \times 2^{3k+1} + 2^{2k+1} + 6 \times 2^{k+1} + 12$.

(c) If $G = GL(2, q)$ then $\mathrm{E_{CN}}(\Gamma_c(G)) = 2q^6 - 6q^5 - 2q^4 + 10q^3 + 6q^2 + 2q$.

Result 1.2.16. [80, Theorem 5] Let G be a finite non-abelian group.

(a) If $G = A(n, \nu)$ then $\mathrm{E_{CN}}(\Gamma_c(G)) = 2(2^n - 1)^2(2^n - 2)$.

(b) If $G = A(n, p)$ then $\mathrm{E_{CN}}(\Gamma_c(G)) = 2(p^n + 1)(p^{2n} - p^n - 1)(p^{2n} - p^n - 2)$.

The energy, Laplacian energy and signless Laplacian energy of non-commuting graphs of various finite groups were computed and compared in [48, 96].

1.2.2 Commuting conjugacy class graph

In our study, we shall also consider the *commuting conjugacy class graph* (abbreviated as CCC-graph) of a finite non-abelian group. The CCC-graph of G is a simple undirected graph, denoted by $\Gamma_{\text{ccc}}(G)$, whose vertex set is $\mathrm{Cl}(G)$ and two distinct vertices a^G and b^G are adjacent if there exist $x \in a^G$ and $y \in b^G$ such that x and y commute. Note that $\Gamma_{\text{ccc}}(G)$ is the compressed version of conjugacy supercommuting graph of G (see [9, 8]). In 2009, Herzog, Longobardi and Maj [64] introduced and studied the concept of CCC-graph of a finite group. In particular, Herzog et al. [64] considered the induced subgraph $\Gamma_{\text{ccc}}(G)[\mathrm{Cl}(G \setminus \{1\})]$; more generally $\Gamma_{\text{ccc}}(G)[\mathrm{Cl}(G \setminus Z(G))]$, where 1 is the identity element of G. We write $\Gamma^*_{\text{ccc}}(G)$ to denote the subgraph $\Gamma_{\text{ccc}}(G)[\mathrm{Cl}(G \setminus Z(G))]$ of $\Gamma_{\text{ccc}}(G)$. Note that

$$\Gamma_{\text{ccc}}(G) = K_{|Z(G)|} + \Gamma^*_{\text{ccc}}(G), \tag{1.2.a}$$

where $\Gamma_1 + \Gamma_2$ is the join of two graphs Γ_1 and Γ_2. In 2016, Mohammadian et al. [78] have characterized finite groups such that $\Gamma^*_{\text{ccc}}(G)$ is triangle-free. Later on, Salahshour and Ashrafi [92, 93] obtained $\Gamma^*_{\text{ccc}}(G)$ for several families of finite AC-groups. Salahshour [91] also obtained $\Gamma^*_{\text{ccc}}(G)$ if G is a finite group such that $\frac{G}{Z(G)}$ is isomorphic to a dihedral group. Characterizations of various classes of finite non-abelian groups G through energy, (signless) Laplacian energy and genus of their $\Gamma^*_{\text{ccc}}(G)$ can be found in [16, 17, 18]. These results on CCC-graphs of finite non-abelian groups can also be found in our paper [22]. Some results on structures of $\Gamma^*_{\text{ccc}}(G)$ are given below.

Result 1.2.17. [92, Theorem 3.1] Let G be a finite non-abelian group with center $Z(G)$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is prime. Then

$$\Gamma^*_{\text{ccc}}(G) = (p+1)K_n,$$

where $n = \frac{(p-1)|Z(G)|}{p}$.

Result 1.2.18. [92, Corollary 3.2] Let G be a finite non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, where p is prime and $n \geq 3$. Then

$$\Gamma^*_{\text{ccc}}(G) = (p+1)K_{p^{n-3}(p-1)}.$$

Result 1.2.19. [91, Theorem 1.2] Let G be a finite group with center $Z(G)$ and $\frac{G}{Z(G)}$ is isomorphic to D_{2m}. Then

$$\Gamma^*_{\text{ccc}}(G) = \begin{cases} K_{\frac{(m-1)|Z(G)|}{2}} \cup 2K_{\frac{|Z(G)|}{2}}, & \text{if } 2 \mid m \\ K_{\frac{(m-1)|Z(G)|}{2}} \cup K_{|Z(G)|}, & \text{if } 2 \nmid m. \end{cases}$$

Result 1.2.20. [93, Proposition 2.1] For the dihedral group D_{2m},

$$\Gamma^*_{\text{ccc}}(D_{2m}) = \begin{cases} K_{\frac{m-1}{2}} \cup K_1, & \text{if } 2 \nmid m \\ K_{\frac{m}{2}-1} \cup 2K_1, & \text{if } 2 \mid m \text{ and } 2 \mid \frac{m}{2} \\ K_{\frac{m}{2}-1} \cup K_2, & \text{if } 2 \mid m \text{ and } 2 \nmid \frac{m}{2}. \end{cases}$$

Result 1.2.21. [93, Proposition 2.2] For the dicyclic group Q_{4n},

$$\Gamma^*_{\text{ccc}}(Q_{4n}) = \begin{cases} K_{n-1} \cup 2K_1, & \text{if } 2 \nmid n \\ K_{n-1} \cup K_2, & \text{if } 2 \nmid n. \end{cases}$$

If $G = U_{(n,m)}$ then $|Z(G)| = n$ or $2n$ according as m is odd or m is even and so $\frac{G}{Z(G)}$ is isomorphic to D_{2m} or $D_{2\times\frac{m}{2}}$ according as m is odd or m is even. Therefore, by Result 1.2.19, we have the following result for $\Gamma^*_{\text{ccc}}(U_{(n,m)})$.

Result 1.2.22. For the group $U_{(n,m)}$,

$$\Gamma^*_{\text{ccc}}(U_{(n,m)}) = \begin{cases} K_{\frac{(m-1)n}{2}} \cup K_n, & \text{if } 2 \nmid m \\ K_{\frac{(m-2)n}{2}} \cup 2K_n, & \text{if } 2 \mid m \text{ and } 2 \mid \frac{m}{2} \\ K_{\frac{(m-2)n}{2}} \cup K_{2n}, & \text{if } 2 \mid m \text{ and } 2 \nmid \frac{m}{2}. \end{cases}$$

Remark 1.2.23. *The cases when m is even and $\frac{m}{2}$ is even or odd were not considered in [93, Proposition 2.3]. Therefore, the structure of $\Gamma^*_{\text{ccc}}(U_{(n,m)})$ given in [93, Proposition 2.3] is not correct.*

Result 1.2.24. [93, Proposition 2.4] For the group V_{8n},

$$\Gamma^*_{\text{ccc}}(V_{8n}) = \begin{cases} K_{2n-2} \cup 2K_2, & \text{if } 2 \mid n \\ K_{2n-1} \cup 2K_1, & \text{if } 2 \nmid n. \end{cases}$$

Result 1.2.25. [93, Proposition 2.5] For the semidihedral group SD_{8n},

$$\Gamma^*_{\text{ccc}}(SD_{8n}) = \begin{cases} K_{2n-1} \cup 2K_1, & \text{if } 2 \mid n \\ K_{2n-2} \cup K_4, & \text{if } 2 \nmid n. \end{cases}$$

We conclude this chapter with the following relations among $\mathrm{E}(\Gamma^*_{\text{ccc}}(G))$, $\mathrm{LE}(\Gamma^*_{\text{ccc}}(G))$ and $\mathrm{LE}^+(\Gamma^*_{\text{ccc}}(G))$ for certain families of groups.

Result 1.2.26. [16, Theorem 4.6] Let G be a finite non-abelian group. Then we have the following.

(a) If G is isomorphic to $D_6, D_8, D_{12}, Q_8, Q_{12}$, $U_{(n,2)}, U_{(n,3)}, U_{(n,4)} (n \geq 2)$, V_{16} or SD_{24} then

$$\mathrm{E}(\Gamma^*_{\text{ccc}}(G)) = \mathrm{LE}^+(\Gamma^*_{\text{ccc}}(G)) = \mathrm{LE}(\Gamma^*_{\text{ccc}}(G)).$$

(b) If G is isomorphic to D_{20}, Q_{20}, $U_{(2,5)}, U_{(3,5)}$ or $U_{(2,6)}$ then

$$\mathrm{LE}^+(\Gamma^*_{\text{ccc}}(G)) < \mathrm{E}(\Gamma^*_{\text{ccc}}(G)) < \mathrm{LE}(\Gamma^*_{\text{ccc}}(G)).$$

(c) If G is isomorphic to $D_{14}, D_{16}, D_{18}, D_{2n} (n \geq 11)$, $Q_{16}, Q_{24}, Q_{4m} (m \geq 8)$, $U_{(n,5)}, (n \geq 4)$, $U_{(n,m)} (m \geq 6 \text{ and } n \geq 3)$, $U_{(n,m)} (m \geq 8 \text{ and } n \geq 2)$, $V_{8n} (n \geq 3)$, SD_{16} or $SD_{8n} (n \geq 4)$ then

$$\mathrm{E}(\Gamma^*_{\text{ccc}}(G)) < \mathrm{LE}^+(\Gamma^*_{\text{ccc}}(G)) < \mathrm{LE}(\Gamma^*_{\text{ccc}}(G)).$$

(d) If G is isomorphic to Q_{28} or $U_{(2,7)}$ then $\mathrm{E}(\Gamma^*_{\text{ccc}}(G)) = \mathrm{LE}^+(\Gamma^*_{\text{ccc}}(G)) < \mathrm{LE}(\Gamma^*_{\text{ccc}}(G))$.

(e) If G is isomorphic to D_{10} then $\mathrm{E}(\Gamma^*_{\text{ccc}}(G)) < \mathrm{LE}^+(\Gamma^*_{\text{ccc}}(G)) = \mathrm{LE}(\Gamma^*_{\text{ccc}}(G))$.