Chapter 2

Common neighborhood energies and their relations with Zagreb index

We have seen that various spectra and energies of a graph based on its adjacency matrix and distance matrix have been studied extensively. However, common neighborhood spectrum and energy of a graph defined by using the common neighborhood matrix are relatively new concept and not much explored. The (signless) Laplacian spectrum and energy of a graph based on its common neighborhood matrix is not defined yet. In this chapter, we introduce the concepts of common neighborhood Laplacian spectrum, common neighborhood signless Laplacian spectrum and their corresponding energies of a graph Γ. We also introduce the concepts of CNL-hyperenergetic and CNSL-hyperenergetic graphs analogous to the concepts of L-hyperenergetic and Q-hyperenergetic graphs respectively. In Section 2.1, we shall establish relations between these energies and the first Zagreb index of a graph. In Section 2.2, we shall compute common neighborhood Laplacian spectrum, common neighborhood signless Laplacian spectrum and their corresponding energies of any finite complete graph and complete bipartite graph. Among other results we shall show that any finite complete bipartite graph is neither CNL-hyperenergetic nor

CNSL-hyperenergetic. In Section 2.3, we shall obtain certain relations between various energies of a graph. Finally, in Section 2.4, we conclude this chapter with several bounds for common neighborhood Laplacian and signless Laplacian energies of a graph. This chapter is based on our paper [81] accepted for publication in *Bulletin Mathematique de la Société des Sciences Mathématiques de Roumanie*.

2.1 Definition and connection with Zagreb index

Let Γ be any graph. First we observe that the (i,j)-th entry of $\mathrm{D}(\Gamma)$ is given by

$$\mathrm{D}(\Gamma)_{i,j} = \begin{cases} \sum\limits_{k=1}^{|v(\Gamma)|} \mathrm{A}(\Gamma)_{i,k}, & \text{if } i = j \text{ and } i = 1, 2, \dots, |v(\Gamma)| \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\mathrm{A}(\Gamma)_{i,k}$ is the (i,k)-th entry of $\mathrm{A}(\Gamma)$. Thus $\mathrm{D}(\Gamma)$ is a diagonal matrix whose diagonal entries are the corresponding row sums of the adjacency matrix of Γ. Similarly, we define common neighborhood row sum matrix (abbreviated as CNRS-matrix) of Γ as given below:

$$\mathrm{CNRS}(\Gamma)_{i,j} = \begin{cases} \sum\limits_{k=1}^{|v(\Gamma)|} \mathrm{CN}(\Gamma)_{i,k}, & \text{if } i = j \text{ and } i = 1, 2, \dots, |v(\Gamma)| \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\mathrm{CNRS}(\Gamma)$ is the CNRS-matrix of Γ and $\mathrm{CNRS}(\Gamma)_{i,j}$ is the (i,j)-th entry of $\mathrm{CNRS}(\Gamma)$. The common neighborhood Laplacian matrix and the common neighborhood signless Laplacian matrix (abbreviated as CNL-matrix and CNSL-matrix) of Γ, denoted by $\mathrm{CNL}(\Gamma)$ and $\mathrm{CNSL}(\Gamma)$, respectively, are defined as

$$\mathrm{CNL}(\Gamma) := \mathrm{CNRS}(\Gamma) - \mathrm{CN}(\Gamma) \text{ and } \mathrm{CNSL}(\Gamma) := \mathrm{CNRS}(\Gamma) + \mathrm{CN}(\Gamma).$$

Note that the matrices $\mathrm{CNL}(\Gamma)$ and $\mathrm{CNSL}(\Gamma)$ are symmetric and positive semidefinite. The set of eigenvalues of $\mathrm{CNL}(\Gamma)$ and $\mathrm{CNSL}(\Gamma)$ with multiplicities are called common neighborhood Laplacian spectrum and common neighborhood signless Laplacian spectrum (abbreviated as CN-Laplacian spectrum and CN-signless Laplacian spectrum) of

Γ, respectively. We write $\text{CNL-spec}(\Gamma)$ and $\text{CNSL-spec}(\Gamma)$ to denote CN-Laplacian spectrum and CN-signless Laplacian spectrum of Γ, respectively. By writing $\text{CNL-spec}(\Gamma) = \{[\nu_1]^{b_1}, [\nu_2]^{b_2}, \ldots, [\nu_\ell]^{b_\ell}\}$ and $\text{CNSL-spec}(\Gamma) = \{[\sigma_1]^{c_1}, [\sigma_2]^{c_2}, \ldots, [\sigma_m]^{c_m}\}$ we mean that $\nu_1, \nu_2, \ldots, \nu_\ell$ are the distinct eigenvalues of $\text{CNL}(\Gamma)$ with corresponding multiplicities $b_1, b_2, \ldots, b_\ell$ and $\sigma_1, \sigma_2, \ldots, \sigma_m$ are the distinct eigenvalues of $\text{CNSL}(\Gamma)$ with corresponding multiplicities $c_1, c_2, \ldots, c_m$. Γ is called CNL-integral and CNSL-integral if $\text{CNL-spec}(\Gamma)$ and $\text{CNSL-spec}(\Gamma)$ contain only integers respectively. Corresponding to CN-Laplacian spectrum and CN-signless Laplacian spectrum of Γ we define common neighborhood Laplacian energy and common neighborhood signless Laplacian energy (abbreviated as CNL-energy and CNSL-energy) of Γ. The CNL-energy and CNSL-energy of Γ, denoted by $\text{LE}_{\text{CN}}(\Gamma)$ and $\text{LE}^+_{\text{CN}}(\Gamma)$, are as defined below:

$$\text{LE}_{\text{CN}}(\Gamma) := \sum_{\nu \in \text{CNL-spec}(\Gamma)} \left| \nu - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} \right| \qquad (2.1.a)$$

and

$$\text{LE}^+_{\text{CN}}(\Gamma) := \sum_{\sigma \in \text{CNSL-spec}(\Gamma)} \left| \sigma - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} \right|. \qquad (2.1.b)$$

This appends two new entries in the list of energies prepared by Gutman and Furtula [59]. Following the concepts of various hyperenergetic graphs [104, 55, 4, 48], we introduce the concepts of CNL-hyperenergetic and CNSL-hyperenergetic graphs. A graph Γ is called CNL-hyperenergetic and CNSL-hyperenergetic if

$$\text{LE}_{\text{CN}}(\Gamma) > \text{LE}_{\text{CN}}(K_{|v(\Gamma)|}) \text{ and } \text{LE}^+_{\text{CN}}(\Gamma) > \text{LE}^+_{\text{CN}}(K_{|v(\Gamma)|}),$$

respectively. Also we call Γ is CNL-borderenergetic and CNSL-borderenergetic if $\text{LE}_{\text{CN}}(\Gamma) = \text{LE}_{\text{CN}}(K_{|v(\Gamma)|})$ and $\text{LE}^+_{\text{CN}}(\Gamma) = \text{LE}^+_{\text{CN}}(K_{|v(\Gamma)|})$ respectively. The following lemma is useful in computing CN-Laplacian spectrum and CN-signless Laplacian spectrum of a graph having disconnected components.

Lemma 2.1.1. *If* $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$, *then*

$$\text{CNL-spec}(\Gamma) = \text{CNL-spec}(\Gamma_1) \cup \text{CNL-spec}(\Gamma_2) \cup \cdots \cup \text{CNL-spec}(\Gamma_k)$$

and

$$\text{CNSL-spec}(\Gamma) = \text{CNSL-spec}(\Gamma_1) \cup \text{CNSL-spec}(\Gamma_2) \cup \cdots \cup \text{CNSL-spec}(\Gamma_k)$$

counting multiplicities.

Now the following lemma and Result 1.1.16 are useful in deriving relations between CN-Laplacian energy, CN-signless Laplacian energy and first Zagreb index.

Lemma 2.1.2. *Let* Γ *be a graph with* $|e(\Gamma)|$ *edges and the first Zagreb index* $M_1(\Gamma)$*. Then* $\operatorname{tr}(\mathrm{CNRS}(\Gamma)) = M_1(\Gamma) - 2\,|e(\Gamma)|$.

Proof. From the definition with Lemma 1.1.16 and (1.1.i), we obtain

$$\begin{aligned}\operatorname{tr}(\mathrm{CNRS}(\Gamma)) &= \sum_{i=1}^{|v(\Gamma)|} \mathrm{CNRS}(\Gamma)_{i,i}\\ &= \sum_{i=1}^{|v(\Gamma)|} \sum_{j=1,\, j\neq i}^{|v(\Gamma)|} |N_\Gamma(v_i) \cap N_\Gamma(v_j)|\\ &= \sum_{i=1}^{|v(\Gamma)|} \Big[\deg(v_i)\, m_\Gamma(v_i) - \deg(v_i)\Big]\\ &= M_1(\Gamma) - 2\,|e(\Gamma)|.\end{aligned}$$

This completes the result. □

We conclude this section with the following relations between $\mathrm{LE_{CN}}(\Gamma)$, $\mathrm{LE^+_{CN}}(\Gamma)$ and $M_1(\Gamma)$ which can be obtained from (2.1.a), (2.1.b) and Lemma 2.1.2.

Theorem 2.1.3. *Let* Γ *be a graph with the first Zagreb index* $M_1(\Gamma)$*. Then*

$$\mathrm{LE_{CN}}(\Gamma) = \sum_{\nu \in \text{CNL-spec}(\Gamma)} \left|\nu - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right|$$

and

$$\mathrm{LE^+_{CN}}(\Gamma) = \sum_{\sigma \in \text{CNSL-spec}(\Gamma)} \left|\sigma - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right|.$$

2.2 CN-(signless) Laplacian spectrum and CN-(signless) Laplacian energy

In this section we compute CN-(signless) Laplacian spectrum and CN-(signless) Laplacian energy of some classes of graphs and discuss their properties.

Example 2.2.1. *For $n = 1$, it is clear that* CNL-spec$(K_1) = \{[0]^1\}$, CNSL-spec$(K_1) = \{[0]^1\}$ *and so* $\mathrm{LE_{CN}}(K_1) = 0$, $\mathrm{LE^+_{CN}}(K_1) = 0$. *Therefore, we consider $n \geq 2$. We have*

$$\mathrm{CN}(K_n) = (n-2)A(K_n) \quad \textit{and} \quad \mathrm{CNRS}(K_n) = \mathrm{diag}[(n-1)(n-2), \dots, (n-1)(n-2)],$$

so $\mathrm{CNL}(K_n) = (n-2)L(K_n)$ *and* $\mathrm{CNSL}(K_n) = (n-2)Q(K_n)$. *Also,* L-spec$(K_n) = \{[0]^1, [n]^{n-1}\}$ *and* Q-spec$(K_n) = \left\{[2(n-1)]^1, [n-2]^{n-1}\right\}$. *Therefore,* CNL-spec$(K_n) = \left\{[0]^1, [n(n-2)]^{n-1}\right\}$ *and* CNSL-spec$(K_n) = \left\{[2(n-1)(n-2)]^1, [(n-2)^2]^{n-1}\right\}$. *We have* $\mathrm{tr}(\mathrm{CNRS}(K_n)) = n(n-1)(n-2)$ *and so* $\frac{\mathrm{tr}(\mathrm{CNRS}(K_n))}{|v(K_n)|} = (n-1)(n-2)$. *Therefore,*

$$\left|0 - \frac{\mathrm{tr}(\mathrm{CNRS}(K_n))}{|v(K_n)|}\right| = (n-1)(n-2),$$

$$\left|n(n-2) - \frac{\mathrm{tr}(\mathrm{CNRS}(K_n))}{|v(K_n)|}\right| = n(n-2) - (n-1)(n-2) = (n-2),$$

and

$$\left|2(n-1)(n-2) - \frac{\mathrm{tr}(\mathrm{CNRS}(K_n))}{|v(K_n)|}\right| = (n-1)(n-2),$$

$$\left|(n-2)^2 - \frac{\mathrm{tr}(\mathrm{CNRS}(K_n))}{|v(K_n)|}\right| = |(n-2)^2 - (n-1)(n-2)| = |-(n-2)| = n-2.$$

Hence, by (2.1.a) and (2.1.b), we obtain

$$\mathrm{LE_{CN}}(K_n) = (n-1)(n-2) + (n-1)(n-2) = 2(n-1)(n-2)$$

and

$$\mathrm{LE^+_{CN}}(K_n) = (n-1)(n-2) + (n-1)(n-2) = 2(n-1)(n-2).$$

Thus

$$\mathrm{LE_{CN}}(K_n) = \mathrm{LE^+_{CN}}(K_n) = 2(n-1)(n-2). \tag{2.2.a}$$

By Result 1.1.21, it follows that if Γ_1 and Γ_2 are two disconnected component of Γ then $\mathrm{E_{CN}}(\Gamma) = \mathrm{E_{CN}}(\Gamma_1) + \mathrm{E_{CN}}(\Gamma_2)$. *However,* $\mathrm{LE_{CN}}(\Gamma) \neq \mathrm{LE_{CN}}(\Gamma_1) + \mathrm{LE_{CN}}(\Gamma_2)$ *and* $\mathrm{LE^+_{CN}}(\Gamma) \neq \mathrm{LE^+_{CN}}(\Gamma_1) + \mathrm{LE^+_{CN}}(\Gamma_2)$, *if $\Gamma = \Gamma_1 \cup \Gamma_2$. For example, if $\Gamma = K_4 \cup K_6$ then by Lemma 2.1.1 with the above result, it follows that*

$$\text{CNL-spec}(\Gamma) = \left\{[0]^2, [8]^3, [24]^5\right\} \quad \textit{and} \quad \text{CNSL-spec}(\Gamma) = \left\{[4]^3, [12]^1, [16]^5, [40]^1\right\}.$$

We have

$$\frac{\operatorname{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|} = \frac{\operatorname{tr}(\mathrm{CNRS}(K_4)) + \operatorname{tr}(\mathrm{CNRS}(K_6))}{10} = \frac{24+120}{10} = \frac{144}{10}.$$

Therefore, by (2.1.a) *and* (2.1.b), *we obtain*

$$\begin{aligned}\mathrm{LE}_{\mathrm{CN}}(\Gamma) &= 2 \times \left|0 - \frac{144}{10}\right| + 3 \times \left|8 - \frac{144}{10}\right| + 5 \times \left|24 - \frac{144}{10}\right| \\ &= 2 \times \frac{144}{10} + 3 \times \frac{64}{10} + 5 \times \frac{96}{10} = 96,\end{aligned}$$

but $\mathrm{LE}_{\mathrm{CN}}(K_4) + \mathrm{LE}_{\mathrm{CN}}(K_6) = 12 + 40 = 52$, *and*

$$\begin{aligned}\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma) &= 3 \times \left|4 - \frac{144}{10}\right| + 1 \times \left|12 - \frac{144}{10}\right| + 5 \times \left|16 - \frac{144}{10}\right| + 1 \times \left|40 - \frac{144}{10}\right| \\ &= 2 \times \frac{104}{10} + \frac{24}{10} + \frac{80}{10} + \frac{256}{10} = \frac{672}{10},\end{aligned}$$

but $\mathrm{LE}^{+}_{\mathrm{CN}}(K_4) + \mathrm{LE}^{+}_{\mathrm{CN}}(K_6) = 12 + 40 = 52$.

Now by using Lemma 2.1.1 and Example 2.2.1, we have the following theorem.

Theorem 2.2.2. *Let* $\Gamma = l_1K_{m_1} \cup l_2K_{m_2} \cup l_3K_{m_3}$, *where* $l_iK_{m_i}$ *denotes the union of* l_i *copies of the complete graphs* K_{m_i} *on* m_i *vertices for* $i = 1, 2, 3$. *Then*

$$\begin{aligned}\text{CNL-spec}(\Gamma) = \Big\{&[0]^{l_1+l_2+l_3}, [m_1(m_1-2)]^{l_1(m_1-1)}, \\ &[m_2(m_2-2)]^{l_2(m_2-1)}, [m_3(m_3-2)]^{l_3(m_3-1)}\Big\}\end{aligned}$$ *and*

$$\begin{aligned}\text{CNSL-spec}(\Gamma) = \Big\{&[2(m_1-1)(m_1-2)]^{l_1}, [(m_1-2)^2]^{l_1(m_1-1)}, [2(m_2-1)(m_2-2)]^{l_2}, \\ &[(m_2-2)^2]^{l_2(m_2-1)}, [2(m_3-1)(m_3-2)]^{l_3}, [(m_3-2)^2]^{l_3(m_3-1)}\Big\}.\end{aligned}$$

Example 2.2.3. *We now compute CN-(signless) Laplacian spectrum and CN-(signless) Laplacian energy of the complete bipartite graph* $K_{m,n}$ *on* $(m+n)$*-vertices. For this, let* $v(K_{m,n}) = \{v_1, v_2, \ldots, v_m, v_{m+1}, v_{m+2}, \ldots, v_{m+n}\}$ *and* $\{v_1, v_2, \ldots, v_m\}$, $\{v_{m+1}, v_{m+2}, \ldots, v_{m+n}\}$ *be two partitions of* $v(K_{m,n})$ *such that every vertex in one set is adjacent to every vertex in the other set. We have*

$$\mathrm{CN}(K_{m,n}) = \begin{bmatrix} n\,A(K_m) & 0 \\ 0 & m\,A(K_n) \end{bmatrix}$$

and

$$\text{CNRS}(K_{m,n}) = \text{diag}\left[\underbrace{(m-1)n, \dots, (m-1)n}_{m\text{-times}}, \underbrace{(n-1)m, \dots, (n-1)m}_{n\text{-times}}\right].$$

Thus we have

$$\text{CNL}(K_{m,n}) = \begin{bmatrix} n\,L(K_m) & 0 \\ 0 & m\,L(K_n) \end{bmatrix} \text{ and } \text{CNSL}(K_{m,n}) = \begin{bmatrix} n\,Q(K_m) & 0 \\ 0 & m\,Q(K_n) \end{bmatrix}.$$

Since L-spec$(K_m) = \{[0]^1, [m]^{m-1}\}$ *and* L-spec$(K_n) = \{[0]^1, [n]^{n-1}\}$, *therefore,*

$$\text{CNL-spec}(K_{m,n}) = \left\{[n \times 0]^1, [n \times m]^{m-1}, [m \times 0]^1, [m \times n]^{n-1}\right\} = \left\{[0]^2, [mn]^{m+n-2}\right\}.$$

We have $\frac{\text{tr}(\text{CNRS}(K_{m,n}))}{|v(K_{m,n})|} = \dfrac{mn(m+n-2)}{m+n}$ *and so*

$$\left|0 - \frac{\text{tr}(\text{CNRS}(K_{m,n}))}{|v(K_{m,n})|}\right| = \frac{mn(m+n-2)}{m+n} \text{ and } \left|mn - \frac{\text{tr}(\text{CNRS}(K_{m,n}))}{|v(K_{m,n})|}\right| = \frac{2mn}{m+n}.$$

Hence, by (2.1.a), *we get*

$$\text{LE}_{\text{CN}}(K_{m,n}) = \frac{2mn(m+n-2)}{m+n} + \frac{2mn(m+n-2)}{m+n} = \frac{4mn(m+n-2)}{m+n}.$$

In particular, for $m = n$, *we obtain* CNL-spec$(K_{m,n}) = \left\{[0]^2,\ [m^2]^{2m-2}\right\}$ *and* $\text{LE}_{\text{CN}}(K_{m,n}) = 4m(m-1)$.

Again since Q-spec$(K_m) = \left\{[2(m-1)]^1,\ [m-2]^{m-1}\right\}$ *and* Q-spec$(K_n) = \left\{[2(n-1)]^1,\ [n-2]^{n-1}\right\}$, *therefore,*

$$\text{CNSL-spec}(K_{m,n}) = \left\{[2n(m-1)]^1,\ [n(m-2)]^{m-1},\ [2m(n-1)]^1,\ [m(n-2)]^{n-1}\right\}.$$

Note that if $m = n = 1$ *then* $K_{1,1} = K_2$. *Hence,* CNSL-spec$(K_{m,n})$ = CNSL-spec(K_2) $= \{[0]^2\}$ *and* $\text{LE}^+_{\text{CN}}(K_{1,1}) = \text{LE}^+_{\text{CN}}(K_2) = 0$. *We now assume that* $n \geq 2$ *or* $m \geq 2$. *Now,*

$$\left|2n(m-1) - \frac{\text{tr}(\text{CNRS}(K_{m,n}))}{|v(K_{m,n})|}\right| = \left|2n(m-1) - \frac{mn(m+n-2)}{m+n}\right|$$

$$= \begin{cases} \dfrac{n(n-1)}{n+1}, & \text{if } m = 1 \\[2ex] \dfrac{n(m+n)(m-2)+2mn}{m+n}, & \text{if } m \geq 2, \end{cases}$$

$$\left| n(m-2) - \frac{\operatorname{tr}(\mathrm{CNRS}(K_{m,n}))}{|v(K_{m,n})|} \right| = \left| n(m-2) - \frac{mn(m+n-2)}{m+n} \right| = \frac{2n^2}{m+n},$$

$$\begin{aligned} \left| 2m(n-1) - \frac{\operatorname{tr}(\mathrm{CNRS}(K_{m,n}))}{|v(K_{m,n})|} \right| &= \left| 2m(n-1) - \frac{mn(m+n-2)}{m+n} \right| \\ &= \begin{cases} \dfrac{m(m-1)}{m+1}, & \text{if } n = 1 \\[2ex] \dfrac{m(m+n)(n-2)+2mn}{m+n}, & \text{if } n \geq 2, \end{cases} \end{aligned}$$

and

$$\left| m(n-2) - \frac{\operatorname{tr}(\mathrm{CNRS}(K_{m,n}))}{|v(K_{m,n})|} \right| = \left| m(n-2) - \frac{mn(m+n-2)}{m+n} \right| = \frac{2m^2}{m+n}.$$

For $m = 1$ and $n \geq 2$, by (2.1.b), we have

$$\mathrm{LE}^{+}_{\mathrm{CN}}(K_{m,n}) = \frac{n(n-1)}{n+1} + \frac{(n+1)(n-2)+2n}{n+1} + \frac{2(n-1)}{n+1} = \frac{2(n-1)(n+2)}{n+1}.$$

For $m \geq 2$ and $n = 1$, by (2.1.b), we have

$$\mathrm{LE}^{+}_{\mathrm{CN}}(K_{m,n}) = \frac{(m+1)(m-2)+2m}{m+1} + \frac{2(m-1)}{m+1} + \frac{m(m-1)}{m+1} = \frac{2(m-1)(m+2)}{m+1}.$$

For $m, n \geq 2$, by (2.1.b), we have

$$\begin{aligned} &\mathrm{LE}^{+}_{\mathrm{CN}}(K_{m,n}) \\ &= \frac{n(m+n)(m-2)+2mn}{m+n} + \frac{2n^2(m-1)}{m+n} + \frac{m(m+n)(n-2)+2mn}{m+n} + \frac{2m^2(n-1)}{m+n} \\ &= \frac{4(m^2(n-1)+n^2(m-1))}{m+n}. \end{aligned}$$

Hence

$$\mathrm{LE}^{+}_{\mathrm{CN}}(K_{m,n}) = \begin{cases} 0, & \text{if } m = 1 \text{ and } n = 1 \\ \dfrac{2(n-1)(n+2)}{n+1}, & \text{if } m = 1 \text{ and } n \geq 2 \\ \dfrac{2(m-1)(m+2)}{m+1}, & \text{if } m \geq 2 \text{ and } n = 1 \\ \dfrac{4(m^2(n-1)+n^2(m-1))}{m+n}, & \text{if } m, n \geq 2. \end{cases}$$

In particular, for $m = n$*, we obtain* $\mathrm{CNSL\text{-}spec}(K_{m,n}) = \left\{[2m(m-1)]^2,\ [m(m-2)]^{2(m-1)}\right\}$ *and* $\mathrm{LE}^{+}_{\mathrm{CN}}(K_{m,n}) = 4m(m-1)$.

Proposition 2.2.4. *Let* $\overline{\Gamma}$ *be the complement of a graph* Γ *and* $\Gamma_1 + \Gamma_2$ *be the join of two graphs* Γ_1 *and* Γ_2.

(a) *If* $\Gamma = K_{n_1} + K_{n_2} + \cdots + K_{n_k}$ *then*

$$\mathrm{CNL\text{-}spec}(\Gamma) = \{[0]^1, [(n_1+n_2+\cdots+n_k)(n_1+n_2+\cdots+n_k-2)]^{(n_1+n_2+\cdots+n_k-1)}\},$$

$$\mathrm{CNSL\text{-}spec}(\Gamma) = \{[2(n_1+n_2+\cdots+n_k-1)(n_1+n_2+\cdots+n_k-2)]^1, [(n_1+n_2+\cdots+n_k-2)^2]^{n-1}\}$$

and

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma) = 2(n_1+n_2+\cdots+n_k-1)(n_1+n_2+\cdots+n_k-2) = \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma).$$

(b) *If* $\Gamma = \overline{K}_m + \overline{K}_n$ *then*

$$\mathrm{CNL\text{-}spec}(\Gamma) = \left\{[0]^2, [mn]^{m+n-2}\right\} \text{ and } \mathrm{LE}_{\mathrm{CN}}(\Gamma) = \frac{4mn(m+n-2)}{m+n},$$

$$\mathrm{CNSL\text{-}spec}(\Gamma) = \left\{[2n(m-1)]^1, [n(m-2)]^{m-1}, [2m(n-1)]^1, [m(n-2)]^{n-1}\right\}$$

and

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma) = \begin{cases} \dfrac{2(n-1)(n+2)}{n+1}, & \text{if } m = 1 \text{ and } n \geq 2 \\ \dfrac{2(m-1)(m+2)}{m+1}, & \text{if } m \geq 2 \text{ and } n = 1 \\ \dfrac{4(m^2(n-1)+n^2(m-1))}{m+n}, & \text{if } m, n \geq 2. \end{cases}$$

Proof. The results follow from Examples 2.2.1 and 2.2.3 noting that $K_{n_1}+K_{n_2}+\cdots+K_{n_k} = K_{n_1+n_2+\cdots+n_k}$ and $\overline{K_m}+\overline{K_n} = K_{m,n}$. □

Proposition 2.2.5. *The graph $K_{m,n}$ is not CNL-hyperenergetic.*

Proof. By Examples 2.2.1 and 2.2.3, we obtain

$$\begin{aligned}\mathrm{LE_{CN}}(K_{m+n}) - \mathrm{LE_{CN}}(K_{m,n}) &= 2(m+n-1)(m+n-2) - \frac{4mn(m+n-2)}{m+n}\\ &= \frac{[m(m-1)+n(n-1)]\,(m+n-2)}{m+n} \geq 0\end{aligned}$$

with equality if and only if $m = n = 1$. Therefore,

$$\mathrm{LE_{CN}}(K_{m+n}) \geq \mathrm{LE_{CN}}(K_{m,n})$$

with equality if and only if $m = n = 1$. Hence, the result follows. □

Corollary 2.2.6. *If S_k denotes the star graph with one internal node and k leaves then*

$$\text{CNL-spec}(S_k) = \left\{[0]^2,\ [k]^{k-1}\right\} \text{ and } \mathrm{LE_{CN}}(S_k) = \frac{4k(k-1)}{k+1}.$$

Moreover, S_k is not CNL-hyperenergetic.

Proof. The result follows from Example 2.2.3 and Proposition 2.2.5, noting that $S_k = K_{1,k}$. □

Proposition 2.2.7. *The graph $K_{m,n}$ is not CNSL-hyperenergetic.*

Proof. If $m = n = 1$ then we have $\mathrm{LE^+_{CN}}(K_{1,1}) = \mathrm{LE^+_{CN}}(K_2) = 0$. Therefore, we consider the case when m, n are not equal to 1 simultaneously. By Example 2.2.1, we have

$$\mathrm{LE^+_{CN}}(K_{m+n}) = 2(m+n-1)(m+n-2). \tag{2.2.b}$$

For $m = 1$ and $n \geq 2$, by (2.2.b) and Example 2.2.3, we have

$$\begin{aligned}\mathrm{LE^+_{CN}}(K_{m+n}) - \mathrm{LE^+_{CN}}(K_{m,n}) &= 2n(n-1) - \frac{2(n-1)(n+2)}{n+1}\\ &= \frac{2(n-1)(n^2-2)}{n+1} > 0.\end{aligned}$$

For $n = 1$ and $m \geq 2$, by (2.2.b) and Example 2.2.3, we have

$$\begin{aligned}\mathrm{LE}^{+}_{\mathrm{CN}}(K_{m+n}) - \mathrm{LE}^{+}_{\mathrm{CN}}(K_{m,n}) &= 2m(m-1) - \frac{2(m-1)(m+2)}{m+1} \\ &= \frac{2(m-1)(m^2-2)}{m+1} > 0.\end{aligned}$$

For $m, n \geq 2$, by (2.2.b) and Example 2.2.3, we have

$$\begin{aligned}&\mathrm{LE}^{+}_{\mathrm{CN}}(K_{m+n}) - \mathrm{LE}^{+}_{\mathrm{CN}}(K_{m,n}) \\ &\qquad = 2(m+n-1)(m+n-2) - \frac{4(m^2(n-1)+n^2(m-1))}{m+n} \\ &\qquad = 2(m^3+n^3+m+n) + (m-n)^2 + 2mn(m+n-2) > 0,\end{aligned}$$

since $m + n - 2 > 0$. Therefore,

$$\mathrm{LE}^{+}_{\mathrm{CN}}(K_{m+n}) \geq \mathrm{LE}^{+}_{\mathrm{CN}}(K_{m,n}).$$

Hence, the result follows. □

Corollary 2.2.8. *If S_k denotes the star graph with one internal node and k leaves then*

$$\text{CNSL-spec}(S_k) = \left\{[0]^1, [2(k-1)]^1, [k-2]^{k-1}\right\} \text{ and } \mathrm{LE}^{+}_{\mathrm{CN}}(S_k) = \frac{2(k-1)(k+2)}{k+1}.$$

Moreover, S_k is not CNSL-hyperenergetic.

Proof. The result follows from Example 2.2.3 and Proposition 2.2.7, noting that $S_k = K_{1,k}$. □

We conclude this section with the following example of CNL-hyperenergetic and CNSL-hyperenergetic graph.

Example 2.2.9. *Let us consider the graph $\Gamma = K_{2n-2} \cup nK_2$, where $n \geq 2$. Here $|v(\Gamma)| = 4n - 2$. By Example 2.2.1, we have*

$$\mathrm{LE}_{\mathrm{CN}}(K_{|v(\Gamma)|}) = \mathrm{LE}^{+}_{\mathrm{CN}}(K_{|v(\Gamma)|}) = 2(4n-3)(4n-4).$$

Again, by Theorem 2.2.2, (2.1.a) and (2.1.b), we have

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma) = \frac{4(n-2)(n-1)(2n-3)(2n+1)}{2n-1} \text{ and } \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma) = \frac{8(n-2)(n-1)n(2n-3)}{2n-1}.$$

Therefore,

$$\mathrm{LE_{CN}}(K_{|v(\Gamma)|}) - \mathrm{LE_{CN}}(\Gamma) = -\frac{4n\left(4n^3 - 32n^2 + 53n - 25\right)}{2n-1} < 0 \text{ if and only if } n \geq 6$$

and

$$\mathrm{LE^{+}_{CN}}(K_{|v(\Gamma)|}) - \mathrm{LE^{+}_{CN}}(\Gamma) = -\frac{8(n-1)^2\left(2n^2 - 13n + 3\right)}{2n-1} < 0 \text{ if and only if } n \geq 7.$$

Hence, $\Gamma = K_{2n-2} \cup nK_2$ *is CNL-hyperenergetic if and only if* $n \geq 6$ *and CNSL- hyperenergetic if and only if* $n \geq 7$.

2.3 Relation between various energies

In this section we derive some relations between $\mathrm{E_{CN}}$, $\mathrm{LE_{CN}}$, $\mathrm{LE^{+}_{CN}}$, E, LE and LE^{+} of a graph Γ.

Theorem 2.3.1. *Let* Γ *be any graph with* $|e(\Gamma)|$ *edges. Then* $\mathrm{E_{CN}}(\Gamma) \leq \mathrm{E}(\Gamma)^2 + 2|e(\Gamma)|$.

Proof. By Results 1.1.17 and 1.1.10, we obtain

$$\begin{aligned}\mathrm{E_{CN}}(\Gamma) &= \mathrm{E}(\mathrm{CN}(\Gamma))\\ &= \mathrm{E}(\mathrm{A}(\Gamma)^2 - \mathrm{D}(\Gamma))\\ &= \mathrm{E}(\mathrm{A}(\Gamma)^2 + (-\mathrm{D}(\Gamma))) \leq \mathrm{E}(\mathrm{A}(\Gamma)^2) + \mathrm{E}(-\mathrm{D}(\Gamma)).\end{aligned} \tag{2.3.a}$$

Let $\mathrm{Spec}(\Gamma) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $n = |v(\Gamma)|$. Then $\mathrm{Spec}(\mathrm{A}(\Gamma)^2) = \{\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2\}$. Therefore,

$$\mathrm{E}(\mathrm{A}(\Gamma)^2) = \sum_{i=1}^{n} |\lambda_i^2| \leq \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \mathrm{E}(\Gamma)^2.$$

Again, let $\mathrm{Spec}(\mathrm{D}(\Gamma)) = \{\deg(v_1), \deg(v_2), \ldots, \deg(v_n)\}$. Then $\mathrm{Spec}(-\mathrm{D}(\Gamma)) = \{-\deg(v_1), -\deg(v_2), \ldots, -\deg(v_n)\}$. Therefore,

$$\mathrm{E}(-\mathrm{D}(\Gamma)) = \sum_{i=1}^{n} |-\deg(v_i)| = \sum_{i=1}^{n} \deg(v_i) = 2 \times |e(\Gamma)| = \mathrm{E}(\mathrm{D}(\Gamma)).$$

Hence, the result follows from (2.3.a). □

Corollary 2.3.2. *Let* Γ *be any graph of order* $|v(\Gamma)|$ *with* $|e(\Gamma)|$ *edges. Then* $\mathrm{E}_{\mathrm{CN}}(\Gamma) \leq 2|e(\Gamma)|\,(|v(\Gamma)|+1)$.

Proof. It is well-known that $\mathrm{E}(\Gamma) \leq \sqrt{2\,|e(\Gamma)|\,|v(\Gamma)|}$. Using the above result with Theorem 2.3.1, we obtain

$$\mathrm{E}_{\mathrm{CN}}(\Gamma) \leq \mathrm{E}(\Gamma)^2 + 2|e(\Gamma)| \leq 2\,|e(\Gamma)|\,|v(\Gamma)| + 2|e(\Gamma)|.$$

Hence the result follows. □

Theorem 2.3.3. *Let* Γ *be any graph with* $|e(\Gamma)|$ *edges and the first Zagreb index* $M_1(\Gamma)$. *Then* $\mathrm{LE}_{\mathrm{CN}}(\Gamma) \leq \mathrm{E}_{\mathrm{CN}}(\Gamma) + 2\,(M_1(\Gamma) - 2\,|e(\Gamma)|)$ *and* $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma) \leq \mathrm{E}_{\mathrm{CN}}(\Gamma) + 2\Big(M_1(\Gamma) - 2\,|e(\Gamma)|\Big)$.

Proof. By Theorem 2.1.3, we have

$$\begin{aligned}\mathrm{LE}_{\mathrm{CN}}(\Gamma) &= \sum_{\nu\in\text{CNL-spec}(\Gamma)} \left|\nu - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right| \\ &\leq \sum_{\nu\in\text{CNL-spec}(\Gamma)} |\nu| + \sum_{\nu\in\text{CNL-spec}(\Gamma)} \left|\frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right| \\ &= \mathrm{E}(\mathrm{CNL}(\Gamma)) + \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|} \sum_{\nu\in\text{CNL-spec}(\Gamma)} 1 \\ &= \mathrm{E}\left(\mathrm{CNRS}(\Gamma) - \mathrm{CN}(\Gamma)\right) + M_1(\Gamma) - 2\,|e(\Gamma)|.\end{aligned}$$

Using Lemmas 1.1.10 & 2.1.2 and the fact that $\mathrm{E}(\mathrm{CN}(\Gamma)) = \mathrm{E}(-\mathrm{CN}(\Gamma))$ & $\mathrm{E}(\mathrm{CNRS}(\Gamma)) = \mathrm{tr}(\mathrm{CNRS}(\Gamma))$, we obtain

$$\begin{aligned}\mathrm{LE}_{\mathrm{CN}}(\Gamma) &\leq \mathrm{E}(\mathrm{CNRS}(\Gamma)) + \mathrm{E}(\mathrm{CN}(\Gamma)) + M_1(\Gamma) - 2\,|e(\Gamma)| \\ &= \mathrm{E}_{\mathrm{CN}}(\Gamma) + 2\,\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right).\end{aligned}$$

Similarly, the bound for $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma)$ follows from Theorem 2.1.3 and Lemma 1.1.10. □

As a consequence of Theorems 2.3.1 and 2.3.3, we get the following relations between $\mathrm{E}(\Gamma)$, $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma)$.

Corollary 2.3.4. *Let Γ be any graph with $|e(\Gamma)|$ edges and the first Zagreb index $M_1(\Gamma)$. Then $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{LE}^+_{\mathrm{CN}}(\Gamma)$ are bounded above by*

$$\mathrm{E}(\Gamma)^2 + 2\left(M_1(\Gamma) - |e(\Gamma)|\right).$$

Remark 2.3.5. *Theorem 2.3.1 gives relation between $\mathrm{E}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{E}(\Gamma)$. Theorem 2.3.3 and Corollary 2.3.4 give relations between $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$, $\mathrm{LE}^+_{\mathrm{CN}}(\Gamma)$, $\mathrm{E}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{E}(\Gamma)$. However, using the facts that*

$$\mathrm{E}(A(\Gamma)^2) = \sum_{i=1}^{n} |\lambda_i^2| = 2|e(\Gamma)| \leq \mathrm{E}(\Gamma)^2$$

and

$$\sum_{\nu \in \text{CNL-spec}(\Gamma)} |\nu| = \mathrm{tr}(\mathrm{CNRS}(\Gamma)) = M_1(\Gamma) - 2\,|e(\Gamma)| = \sum_{\sigma \in \text{CNSL-spec}(\Gamma)} |\sigma|,$$

we get the following better upper bounds for $\mathrm{E}_{\mathrm{CN}}(\Gamma)$, $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{LE}^+_{\mathrm{CN}}(\Gamma)$:

$$\mathrm{E}_{\mathrm{CN}}(\Gamma) \leq 4|e(\Gamma)| \leq 2E(\Gamma)^2, \tag{2.3.b}$$

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma) \leq 2\ \mathrm{tr}(\mathrm{CNRS}(\Gamma)) = 2\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right) \geq \mathrm{LE}^+_{\mathrm{CN}}(\Gamma). \tag{2.3.c}$$

In Section 2.4, we shall obtain more bounds for $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{LE}^+_{\mathrm{CN}}(\Gamma)$.

Recall that the derived graph of Γ, denoted by $\Gamma^\dagger$, is the graph with vertex set $v(\Gamma)$, in which two vertices are adjacent if and only if their distance in Γ is two.

Theorem 2.3.6. *If Γ is a triangle- and quadrangle-free graph then $\mathrm{LE}_{\mathrm{CN}}(\Gamma) = \mathrm{LE}(\Gamma^\dagger)$ and $\mathrm{LE}^+_{\mathrm{CN}}(\Gamma) = \mathrm{LE}^+(\Gamma^\dagger)$, where $\Gamma^\dagger$ is the derived graph of Γ.*

Proof. If Γ is a triangle- and quadrangle-free graph then $\mathrm{CN}(\Gamma) = \mathrm{A}(\Gamma^\dagger)$. Therefore, $\mathrm{CNRS}(\Gamma) = \mathrm{D}(\Gamma^\dagger)$ and so $\mathrm{CNL}(\Gamma) = \mathrm{L}(\Gamma^\dagger)$ and $\mathrm{CNSL}(\Gamma) = \mathrm{Q}(\Gamma^\dagger)$. Hence, CNL-spec$(\Gamma)$ = L-spec$(\Gamma^\dagger)$ and CNSL-spec(Γ) = Q-spec$(\Gamma^\dagger)$. Since $\mathrm{tr}(\mathrm{CNRS}(\Gamma)) = \mathrm{tr}(\mathrm{D}(\Gamma^\dagger))$ and $v(\Gamma) = v(\Gamma^\dagger)$, by (2.1.$a$) and (2.1.$b$), we have

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma) = \sum_{\nu \in \text{L-spec}(\Gamma^\dagger)} \left| \nu - \frac{\mathrm{tr}(\mathrm{D}(\Gamma^\dagger))}{|v(\Gamma)|} \right| = \mathrm{LE}(\Gamma^\dagger)$$

and

$$\mathrm{LE}^+_{\mathrm{CN}}(\Gamma) = \sum_{\sigma \in \text{Q-spec}(\Gamma^\dagger)} \left| \sigma - \frac{\mathrm{tr}(\mathrm{D}(\Gamma^\dagger))}{|v(\Gamma)|} \right| = \mathrm{LE}^+(\Gamma^\dagger).$$

□

Corollary 2.3.7. (a) *If T is a tree then* $\mathrm{LE}_{\mathrm{CN}}(T) = \mathrm{LE}(T^\dagger)$ *and* $\mathrm{LE}^+_{\mathrm{CN}}(T) = \mathrm{LE}^+(T^\dagger)$.

(b) *If P_n is the path on n vertices then* $\mathrm{LE}_{\mathrm{CN}}(P_n) = \mathrm{LE}(P_{\lceil \frac{n}{2} \rceil}) + \mathrm{LE}(P_{\lfloor \frac{n}{2} \rfloor})$ *and* $\mathrm{LE}^+_{\mathrm{CN}}(P_n) = \mathrm{LE}^+(P_{\lceil \frac{n}{2} \rceil}) + \mathrm{LE}^+(P_{\lfloor \frac{n}{2} \rfloor})$.

(c) *Let C_n be the cycle on n vertices.*

(i) *If n is odd and $n \geq 3$ then* $\mathrm{LE}_{\mathrm{CN}}(C_n) = \mathrm{LE}(C_n)$ *and* $\mathrm{LE}^+_{\mathrm{CN}}(C_n) = \mathrm{LE}^+(C_n)$.

(ii) *If n is even and $n > 4$ then* $\mathrm{LE}_{\mathrm{CN}}(C_n) = 2\,\mathrm{LE}(C_{\frac{n}{2}})$ *and* $\mathrm{LE}^+_{\mathrm{CN}}(C_n) = 2\,\mathrm{LE}^+(C_{\frac{n}{2}})$. *Also,* $\mathrm{LE}_{\mathrm{CN}}(C_4) = 2\,\mathrm{LE}(C_4) = \mathrm{LE}^+(C_4) = 2\,\mathrm{LE}^+(C_4) = 8$.

Proof. (a) Follows from Theorem 2.3.6 noting that T is triangle- and quadrangle-free.
(b) Follows from Theorem 2.3.6 noting that P_n is triangle- and quadrangle-free and

$$P_n^\dagger \cong P_{\lceil \frac{n}{2} \rceil} \cup P_{\lfloor \frac{n}{2} \rfloor}.$$

(c) Let C_n be the cycle on n vertices.

(i) If $n = 3$ then $C_3 = K_3$. Therefore, $\mathrm{LE}_{\mathrm{CN}}(C_3) = \mathrm{LE}^+_{\mathrm{CN}}(C_3) = 4 = \mathrm{LE}(C_3) = \mathrm{LE}^+(C_3)$.

If n is odd and $n > 3$ then C_n is triangle- and quadrangle-free. Also, $(C_n)^\dagger \cong C_n$. Hence, the result follows from Theorem 2.3.6.

(ii) If n is even and $n > 4$ then C_n is triangle- and quadrangle-free. Also, $(C_n)^\dagger \cong C_{\frac{n}{2}} \cup C_{\frac{n}{2}}$. Therefore, by Theorem 2.3.6, we get

$$\mathrm{LE}_{\mathrm{CN}}(C_n) = \mathrm{LE}(C_{\frac{n}{2}} \cup C_{\frac{n}{2}}) = 2\,\mathrm{LE}(C_{\frac{n}{2}})$$

and

$$\mathrm{LE}^+_{\mathrm{CN}}(C_n) = \mathrm{LE}^+(C_{\frac{n}{2}} \cup C_{\frac{n}{2}}) = 2\,\mathrm{LE}^+(C_{\frac{n}{2}}).$$

If $n = 4$ then it is easy to see that $\mathrm{CNRS}(C_4) = D(C_4)$, which is a 4×4 diagonal matrix such that every element in the diagonal is equal to 2, and $\text{CNL-spec}(C_4) = \text{CNSL-spec}(C_4) = \{[0]^2, [4]^2\}$. Therefore, by (2.1.$a$) and (2.1.$b$), we have

$$\mathrm{LE}_{\mathrm{CN}}(C_4) = \mathrm{LE}^+_{\mathrm{CN}}(C_4) = 8.$$

Again, $\text{L-spec}(C_4) = \text{Q-spec}(C_4) = \{[0]^1, [2]^2, [4]^1\}$ and so $\mathrm{LE}(C_4) = \mathrm{LE}^+(C_4) = 4$. Thus, $\mathrm{LE}_{\mathrm{CN}}(C_4) = 2\,\mathrm{LE}(C_4)$ and $\mathrm{LE}^+_{\mathrm{CN}}(C_4) = 2\,\mathrm{LE}^+(C_4)$. □

2.4 More bounds for $\mathrm{LE_{CN}}(\Gamma)$ and $\mathrm{LE^{+}_{CN}}(\Gamma)$

In this section we shall obtain several bounds for $\mathrm{LE_{CN}}(\Gamma)$ and $\mathrm{LE^{+}_{CN}}(\Gamma)$. Since the matrices $\mathrm{CNL}(\Gamma)$ and $\mathrm{CNSL}(\Gamma)$ are positive semidefinite, the elements of CNL-spec(Γ) and CNSL-spec(Γ) are non-negative. Thus we may write CNL-spec$(\Gamma) = \{\nu_1, \nu_2, \dots, \nu_{|v(\Gamma)|}\}$ and CNSL-spec$(\Gamma) = \{\sigma_1, \sigma_2, \dots, \sigma_{|v(\Gamma)|}\}$, where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{|v(\Gamma)|}$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{|v(\Gamma)|}$. We have

$$\sum_{i=1}^{|v(\Gamma)|} \nu_i = \sum_{\nu \in \text{CNL-spec}\,(\Gamma)} \nu = \mathrm{tr}(\mathrm{CNRS}(\Gamma)) = \sum_{\sigma \in \text{CNSL-spec}\,(\Gamma)} \sigma = \sum_{i=1}^{|v(\Gamma)|} \sigma_i.$$

Also,

$$\sum_{\nu \in \text{CNL-spec}(\Gamma)} \left(\nu - \frac{\mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|}\right) = \sum_{\sigma \in \text{CNSL-spec}(\Gamma)} \left(\sigma - \frac{\mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|}\right) = 0.$$

Let α, β $(1 \leq \alpha, \beta \leq |v(\Gamma)|)$ be the largest integers such that

$$\nu_\alpha \geq \frac{\mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|} = \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|} \text{ and } \sigma_\beta \geq \frac{\mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|} = \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}. \tag{2.4.a}$$

Let $S_\alpha(\Gamma) = \sum\limits_{i=1}^{\alpha} \nu_i$ and $S^{+}_{\beta}(\Gamma) = \sum\limits_{i=1}^{\beta} \sigma_i$. Then we have the following useful lemmas.

Lemma 2.4.1. *For any graph Γ, we have*

$$\mathrm{LE_{CN}}(\Gamma) = 2S_\alpha(\Gamma) - \frac{2\alpha\ \mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|} = 2S_\alpha(\Gamma) - \frac{2\alpha\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|}$$

and

$$\mathrm{LE^{+}_{CN}}(\Gamma) = 2S^{+}_{\beta}(\Gamma) - \frac{2\beta\ \mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|} = 2S^{+}_{\beta}(\Gamma) - \frac{2\beta\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|},$$

where $|e(\Gamma)|$ is the number of edges and $M_1(\Gamma)$ is the first Zagreb index in Γ.

Lemma 2.4.2. *For any graph Γ, we have*

$$\mathrm{LE_{CN}}(\Gamma) = \max_{1 \leq i \leq |v(\Gamma)|} \left\{ 2S_i(\Gamma) - \frac{2i\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|} \right\}$$

and

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma) = \max_{1\leq i\leq |v(\Gamma)|} \left\{ 2S^{+}_{i}(\Gamma) - \frac{2i\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|} \right\},$$

where $|e(\Gamma)|$ *is the number of edges and* $M_1(\Gamma)$ *is the first Zagreb index in* Γ*.*

Proof. Let k $(1 \leq k \leq |v(\Gamma)|)$ be any integer. For $k < \alpha$, by (2.4.a), we obtain

$$S_{\alpha}(\Gamma) - S_k(\Gamma) = \sum_{i=k+1}^{\alpha} \nu_i \geq \frac{(\alpha - k)\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|}.$$

For $k > \alpha$, we obtain

$$S_k(\Gamma) - S_{\alpha}(\Gamma) = \sum_{i=\alpha+1}^{k} \nu_i < \frac{(k - \alpha)\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|},$$

that is,

$$S_{\alpha}(\Gamma) - S_k(\Gamma) > \frac{(\alpha - k)\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|}.$$

Moreover, $S_{\alpha}(\Gamma) = S_k(\Gamma)$ for $k = \alpha$. Thus for any value of k $(1 \leq k \leq |v(\Gamma)|)$, we obtain

$$S_{\alpha}(\Gamma) - S_k(\Gamma) \geq \frac{(\alpha - k)\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|}$$

and so

$$2S_{\alpha}(\Gamma) - \frac{2\alpha\ \mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|V(\Gamma)|} \geq 2S_k(\Gamma) - \frac{2k\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|}.$$

This gives

$$2S_{\alpha}(\Gamma) - \frac{2\alpha\ \mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|} = \max_{1\leq i\leq |v(\Gamma)|} \left\{ 2S_i(\Gamma) - \frac{2i\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|} \right\}.$$

Similarly, it can be seen that

$$2S^{+}_{\beta}(\Gamma) - \frac{2\beta\ \mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|} = \max_{1\leq i\leq |v(\Gamma)|} \left\{ 2S^{+}_{i}(\Gamma) - \frac{2i\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|} \right\}.$$

Hence, the result follows from Lemma 2.4.1. □

Let $(a) := (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $(b) := (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ be such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Then (a) is said to be majorize (b) if

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \text{ for } 1 \leq k \leq n-1 \quad \text{and} \quad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

It is well-known that the spectrum of any symmetric, positive semidefinite matrix majorizes its main diagonal (see [53]). Since $\mathrm{CNL}(\Gamma)$ and $\mathrm{CNSL}(\Gamma)$ are symmetric and positive semidefinite for any graph Γ, we have the following lemma when the elements of $\mathrm{CNL\text{-}spec}(\Gamma)$, $\mathrm{CNSL\text{-}spec}(\Gamma)$ and main diagonal elements of $\mathrm{CNRS}(\Gamma)$ are arranged in decreasing order.

Lemma 2.4.3. *For any graph Γ, $\mathrm{CNL\text{-}spec}(\Gamma)$ and $\mathrm{CNSL\text{-}spec}(\Gamma)$ majorize main diagonal elements of $\mathrm{CNRS}(\Gamma)$ when the elements of $\mathrm{CNL\text{-}spec}(\Gamma)$, $\mathrm{CNSL\text{-}spec}(\Gamma)$ and main diagonal elements of $\mathrm{CNRS}(\Gamma)$ are arranged in decreasing order.*

We write the main diagonal elements of $\mathrm{CNRS}(\Gamma)$ as $\mathrm{CNRS}(\Gamma)_{i,i}$ for $1 \leq i \leq |v(\Gamma)|$, where $\mathrm{CNRS}(\Gamma)_{1,1} \geq \mathrm{CNRS}(\Gamma)_{2,2} \geq \cdots \geq \mathrm{CNRS}(\Gamma)_{|v(\Gamma)|,|v(\Gamma)|}$. Now we give lower bounds for $\mathrm{LE_{CN}}(\Gamma)$ and $\mathrm{LE^{+}_{CN}}(\Gamma)$ analogous to the bound given in Result 1.1.2 for $\mathrm{LE}(\Gamma)$.

Theorem 2.4.4. *Let Γ be a graph with $|e(\Gamma)|$ edges and the first Zagreb index $M_1(\Gamma)$. Then*

$$\mathrm{LE_{CN}}(\Gamma) \geq 2\left(\Delta\,(\delta - 1) - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right)$$

and

$$\mathrm{LE^{+}_{CN}}(\Gamma) \geq 2\left(\Delta\,(\delta - 1) - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right),$$

where Δ and δ are the maximum degree and the minimum degree in Γ, respectively.

Proof. Let v_1 be the maximum degree vertex in Γ. Then $\deg(v_1) = \Delta$ and $m_\Gamma(v_1) \geq \delta$ as δ is the minimum degree in Γ. As a consequence of Lemma 2.4.3 with Lemma 1.1.16, we obtain

$$\nu_1 \geq \mathrm{CNRS}(\Gamma)_{1,1} = \deg(v_1)\, m_\Gamma(v_1) - \deg(v_1) = \Delta\left(m_\Gamma(v_1) - 1\right) \geq \Delta\,(\delta - 1)$$

and

$$\sigma_1 \geq \mathrm{CNRS}(\Gamma)_{1,1} = \deg(v_1)\, m_\Gamma(v_1) - \deg(v_1) = \Delta\left(m_\Gamma(v_1) - 1\right) \geq \Delta\,(\delta - 1),$$

Using the above result with Lemma 2.4.2, we obtain

$$\begin{aligned}\mathrm{LE}_{\mathrm{CN}}(\Gamma) \geq 2\, S_1(\Gamma) - \frac{2\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|} &= 2\,\nu_1 - \frac{2\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|} \\ &\geq 2\left(\Delta\,(\delta - 1) - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right).\end{aligned}$$

Similarly,

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma) \geq 2\,\sigma_1 - \frac{2\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|} \geq 2\left(\Delta\,(\delta - 1) - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right).$$

□

Theorem 2.4.5. *Let* Γ *be a graph with* $|e(\Gamma)|$ *edges and the first Zagreb index* $M_1(\Gamma)$*. Then*

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma) \geq 2\left(\sum_{i=1}^{\alpha} \mathrm{CNRS}(\Gamma)_{i,i} - \frac{\alpha\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|}\right)$$

and

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma) \geq 2\left(\sum_{i=1}^{\beta} \mathrm{CNRS}(\Gamma)_{i,i} - \frac{\beta\left(M_1(\Gamma) - 2\,|e(\Gamma)|\right)}{|v(\Gamma)|}\right),$$

where α *and* β *are as given in* (2.4.*a*) *and* $\mathrm{CNRS}(\Gamma)_{i,i} = \deg(v_i)\,(m_\Gamma(v_i) - 1)$*.*

Proof. By Lemma 1.1.16, we have $\mathrm{CNRS}(\Gamma)_{i,i} = \deg(v_i)\,(m_\Gamma(v_i) - 1)$. By Lemma 2.4.3, we obtain

$$\sum_{i=1}^{k} \nu_i \geq \sum_{i=1}^{k} \mathrm{CNRS}(\Gamma)_{i,i} \text{ and } \sum_{i=1}^{k} \sigma_i \geq \sum_{i=1}^{k} \mathrm{CNRS}(\Gamma)_{i,i} \text{ for } 1 \leq k \leq |v(\Gamma)|.$$

In particular, we have

$$\sum_{i=1}^{\alpha} \nu_i \geq \sum_{i=1}^{\alpha} \mathrm{CNRS}(\Gamma)_{i,i} \text{ and } \sum_{i=1}^{\beta} \sigma_i \geq \sum_{i=1}^{\beta} \mathrm{CNRS}(\Gamma)_{i,i}.$$

Therefore,

$$S_\alpha(\Gamma) \geq \sum_{i=1}^{\alpha} \mathrm{CNRS}(\Gamma)_{i,i} \text{ and } S^{+}_{\beta}(\Gamma) \geq \sum_{i=1}^{\beta} \mathrm{CNRS}(\Gamma)_{i,i}.$$

Hence, the result follows from Lemma 2.4.1. □

Using Lemma 2.4.1, we also have the following upper bounds for $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma)$ analogous to the bound given in Result 1.1.2(b).

Theorem 2.4.6. *Let Γ be a graph of order $|v(\Gamma)|$ with $|e(\Gamma)|$ edges and the first Zagreb index $M_1(\Gamma)$. Then $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma)$ are bounded above by*

$$2\left(1-\frac{1}{|v(\Gamma)|}\right)\Big(M_1(\Gamma)-2\,|e(\Gamma)|\Big).$$

Proof. We have

$$S_\alpha(\Gamma)\leq S_{|v(\Gamma)|}(\Gamma)=\mathrm{tr}(\mathrm{CNRS}(\Gamma))=M_1(\Gamma)-2\,|e(\Gamma)|$$

and

$$S^{+}_\beta(\Gamma)\leq S^{+}_{|v(\Gamma)|}(\Gamma)=\mathrm{tr}(\mathrm{CNRS}(\Gamma))=M_1(\Gamma)-2\,|e(\Gamma)|.$$

Therefore, by Lemma 2.4.1 with $1\leq\alpha\leq|v(\Gamma)|$, we obtain

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma)\leq 2\left(1-\frac{1}{|v(\Gamma)|}\right)\Big(M_1(\Gamma)-2\,|e(\Gamma)|\Big).$$

Similarly, we get the bound for $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma)$. □

Note that the bounds obtained in Theorem 2.4.6 are better then the bounds obtained in (2.3.c). We conclude this section with another upper bounds for $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$ and $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma)$ analogous to the bound obtained in Result 1.1.3. The Result 1.1.11 is useful in this regard.

Theorem 2.4.7. *Let Γ be a graph of order $|v(\Gamma)|$ with $|e(\Gamma)|$ edges and the first Zagreb index $M_1(\Gamma)$. Then*

(a) $\mathrm{LE}_{\mathrm{CN}}(\Gamma)\leq\sum\limits_{i=1}^{|v(\Gamma)|}\sqrt{\left(\mathrm{CNRS}(\Gamma)_{i,i}-\frac{M_1(\Gamma)-2\,|e(\Gamma)|}{|v(\Gamma)|}\right)^2+\sum\limits_{k=1,\,k\neq i}^{|v(\Gamma)|}|N_\Gamma(v_i)\cap N_\Gamma(v_k)|^2}$,

(b) $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma)\leq\sum\limits_{i=1}^{|v(\Gamma)|}\sqrt{\left(\mathrm{CNRS}(\Gamma)_{i,i}-\frac{M_1(\Gamma)-2\,|e(\Gamma)|}{|v(\Gamma)|}\right)^2+\sum\limits_{k=1,\,k\neq i}^{|v(\Gamma)|}|N_\Gamma(v_i)\cap N_\Gamma(v_k)|^2}$,

where $\mathrm{CNRS}(\Gamma)_{i,i}=\deg(v_i)\Big(m_\Gamma(v_i)-1\Big)$ for $1\leq i\leq|v(\Gamma)|$ are main diagonal elements of $\mathrm{CNRS}(\Gamma)$ such that $\mathrm{CNRS}(\Gamma)_{1,1}\geq\mathrm{CNRS}(\Gamma)_{2,2}\geq\cdots\geq\mathrm{CNRS}(\Gamma)_{|v(\Gamma)|,|v(\Gamma)|}$.

Proof. (a) Let $M = \text{CNL}(\Gamma) - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} I_{|v(\Gamma)|}$, where $I_{|v(\Gamma)|}$ is the identity matrix of size $|v(\Gamma)|$. Then $\text{Spec}(M) = \left\{\nu_i - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} \,:\, 1 \leq i \leq |v(\Gamma)|\right\}$, where CNL-spec$(\Gamma) = \{\nu_1, \nu_2, \ldots, \nu_{|v(\Gamma)|}\}$. We have

$$\begin{aligned} M^2 =& \left(\text{CNL}(\Gamma) - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} I_{|v(\Gamma)|}\right)^2 \\ =& (\text{CNL}(\Gamma))^2 - \frac{2\ \text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} \text{CNL}(\Gamma) + \frac{(\text{tr}(\text{CNRS}(\Gamma)))^2}{|v(\Gamma)|^2} I_{|v(\Gamma)|}. \end{aligned}$$

Therefore, the i-th diagonal element of M^2 is

$$(M^2)_{i,i} = (\text{CNL}(\Gamma))^2_{i,i} - \frac{2\ \text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} (\text{CNRS}(\Gamma))_{i,i} + \frac{(\text{tr}(\text{CNRS}(\Gamma)))^2}{|v(\Gamma)|^2}.$$

We have

$$\begin{aligned} (\text{CNL}(\Gamma))^2 =& \Big(\text{CNRS}(\Gamma) - \text{CN}(\Gamma)\Big)^2 \\ =& \text{CNRS}(\Gamma)^2 - \text{CNRS}(\Gamma)\,\text{CN}(\Gamma) - \text{CN}(\Gamma)\,\text{CNRS}(\Gamma) + \text{CN}(\Gamma)^2. \end{aligned}$$

Therefore,

$$(\text{CNL}(\Gamma))^2_{i,i} = (\text{CNRS}(\Gamma))^2_{i,i} + (\text{CN}(\Gamma))^2_{i,i} = (\text{CNRS}(\Gamma)_{i,i})^2 + \sum_{k=1,\, k\neq i}^{|v(\Gamma)|} (\text{CN}(\Gamma)_{i,k})^2.$$

Hence,

$$(M^2)_{i,i} = \left(\text{CNRS}(\Gamma)_{i,i} - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|}\right)^2 + \sum_{k=1,\, k\neq i}^{|v(\Gamma)|} (\text{CN}(\Gamma)_{i,k})^2.$$

Since $\text{tr}(\text{CNRS}(\Gamma)) = M_1(\Gamma) - 2\,|e(\Gamma)|$, by Result 1.1.11, we obtain

$$\text{E}(M) \leq \sum_{i=1}^{|v(\Gamma)|} \sqrt{\left(\text{CNRS}(\Gamma)_{i,i} - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right)^2 + \sum_{k=1,\, k\neq i}^{|v(\Gamma)|} |N_\Gamma(v_i) \cap N_\Gamma(v_k)|^2}.$$

Hence the result follows noting that

$$\text{LE}_{\text{CN}}(\Gamma) = \text{E}(M).$$

(b) Let $N = \text{CNSL}(\Gamma) - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} I_{|v(\Gamma)|}$, where $I_{|v(\Gamma)|}$ is the identity matrix of size $|v(\Gamma)|$. Then $\text{Spec}(N) = \left\{\sigma_i - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} : 1 \leq i \leq |v(\Gamma)|\right\}$, where CNSL-spec$(\Gamma) = \{\sigma_1, \sigma_2, \ldots, \sigma_{|v(\Gamma)|}\}$. We have

$$\begin{aligned} N^2 =& \left(\text{CNSL}(\Gamma) - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} I_{|v(\Gamma)|}\right)^2 \\ =& (\text{CNSL}(\Gamma))^2 - \frac{2\,\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} \text{CNSL}(\Gamma) + \frac{(\text{tr}(\text{CNRS}(\Gamma)))^2}{|v(\Gamma)|^2} I_{|v(\Gamma)|}. \end{aligned}$$

Therefore, the i-th diagonal element of N^2 is

$$(N^2)_{i,i} = (\text{CNSL}(\Gamma))^2_{i,i} - \frac{2\,\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|} (\text{CNRS}(\Gamma))_{i,i} + \frac{(\text{tr}(\text{CNRS}(\Gamma)))^2}{|v(\Gamma)|^2}.$$

We have

$$\begin{aligned} (\text{CNSL}(\Gamma))^2 =& \Big(\text{CNRS}(\Gamma) + \text{CN}(\Gamma)\Big)^2 \\ =& \text{CNRS}(\Gamma)^2 + \text{CNRS}(\Gamma)\,\text{CN}(\Gamma) + \text{CN}(\Gamma)\,\text{CNRS}(\Gamma) + \text{CN}(\Gamma)^2. \end{aligned}$$

Therefore,

$$(\text{CNSL}(\Gamma))^2_{i,i} = \Big(\text{CNRS}(\Gamma)_{i,i}\Big)^2 + \sum_{k=1}^{|v(\Gamma)|} (\text{CN}(\Gamma)_{i,k})^2.$$

Hence,

$$(N^2)_{i,i} = \left(\text{CNRS}(\Gamma)_{i,i} - \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|}\right)^2 + \sum_{k=1}^{|v(\Gamma)|} (\text{CN}(\Gamma)_{i,k})^2.$$

Since $\text{tr}(\text{CNRS}(\Gamma)) = M_1(\Gamma) - 2\,|e(\Gamma)|$, by Result 1.1.11, we have

$$\text{E}(N) \leq \sum_{i=1}^{|v(\Gamma)|} \sqrt{\left(\text{CNRS}(\Gamma)_{i,i} - \frac{M_1(\Gamma) - 2\,|e(\Gamma)|}{|v(\Gamma)|}\right)^2 + \sum_{k=1}^{|v(\Gamma)|} |N_\Gamma(v_i) \cap N_\Gamma(v_k)|^2}.$$

Hence, the result follows noting that

$$\text{LE}^+_{\text{CN}}(\Gamma) = \text{E}(N).$$

□

We conclude this chapter with the following result.

Theorem 2.4.8. *If* Γ *is a* r*-regular graph of order* $|v(\Gamma)|$ *then* $\mathrm{LE}_{\mathrm{CN}}(\Gamma)$ *and* $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma)$ *are bounded above by*

$$\sum_{i=1}^{|v(\Gamma)|} \sqrt{\sum_{k=1}^{|v(\Gamma)|} |N_\Gamma(v_i) \cap N_\Gamma(v_k)|^2}.$$

Proof. Since Γ is a regular graph, by Lemma 1.1.16, we obtain

$$\begin{aligned}\mathrm{tr}(\mathrm{CNRS}(\Gamma)) &= \sum_{i=1}^{|v(\Gamma)|} \sum_{j=1,\, j\neq i}^{|v(\Gamma)|} |N_\Gamma(v_i) \cap N_\Gamma(v_k)| \\ &= \sum_{i=1}^{|v(\Gamma)|} \deg(v_i) \Big(m_\Gamma(v_i) - 1 \Big) \\ &= \sum_{i=1}^{|v(\Gamma)|} r\,(r-1) = |v(\Gamma)|\, r\,(r-1)\end{aligned}$$

and

$$\mathrm{CNRS}(\Gamma)_{i,i} = r\,(r-1) = \frac{\mathrm{tr}(\mathrm{CNRS}(\Gamma))}{|v(\Gamma)|}, \qquad 1 \leq i \leq |v(\Gamma)|.$$

From Theorem 2.4.7, we get the result. □