Chapter 3

CNL(CNSL)-spectrum and CNL(CNSL)-energy of commuting graphs

In this chapter, we consider commuting graphs of finite non-abelian groups and compute their CNL(CNSL)-spectrum, CNL(CNSL)-energy and determine whether these graphs are CNL(CNSL)-integral and CNL(CNSL)-hyperenergetic. More precisely, in Section 3.1, we shall compute CNL(CNSL)-spectrum and CNL(CNSL)-energy of commuting graphs of several classes of finite AC-groups including QD_{2^n} (quasi dihedral group), $PSL(2, 2^k)$ (projective special linear group), $GL(2, q)$ (general linear group where $q > 2$ is a prime power), $A(n, v)$, $A(n, p)$ (Hanaki groups), D_{2m} (dihedral group), groups whose central quotient is isomorphic to $Sz(2)$ or $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m} along with some other groups. In Section 3.2, we shall determine when commuting graphs of these groups are CNL(CNSL)-integral and CNL(CNSL)-hyperenergetic. Finally in Section 3.3, we shall compare CN-energy, CNL-energy and CNSL-energy of commuting graphs of the above-mentioned groups. This chapter is based on our paper [69] accepted for publication in *Palestine Journal of Mathematics*. It is worth mentioning that CN-spectrum and CN-energy of commuting graphs of certain finite non-abelian groups were computed in [49] and [80] by Fasfous et al. and Nath et al. respectively.

3.1 CNL(CNSL)-spectrum and CNL(CNSL)-energy

In this section, we compute the CNL(CNSL)-spectrum and CNL(CNSL)-energy of commuting graphs of several families of finite non-abelian groups. Throughout this thesis we write $\Delta_{\text{CN}}(\Gamma) = \frac{\text{tr}(\text{CNRS}(\Gamma))}{|v(\Gamma)|}$.

3.1.1 Some families of AC-group

Here we compute the CNL(CNSL)-spectrum and CNL(CNSL)-energy of commuting graphs of the quasihedral groups QD_{2^n} for $n \geq 4$, projective special linear groups $PSL(2, 2^k)$ for $k \geq 2$, general linear groups $GL(2, q)$ for any prime power $q > 2$ and the Hanaki groups $A(n, v)$ and $A(n, p)$. We begin with the commuting graph of QD_{2^n}.

Proposition 3.1.1. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of commuting graphs of the quasihedral groups QD_{2^n}, where $n \geq 4$, are given by*

$$\text{CNL-spec}(\Gamma_{\text{c}}(QD_{2^n})) = \left\{[0]^{2^{n-1}+1}, [(2^{n-1}-2)(2^{n-1}-4)]^{(2^{n-1}-3)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\text{c}}(QD_{2^n})) = \left\{[0]^{2^{n-1}}, [2(2^{n-1}-3)(2^{n-1}-4)]^1, [(2^{n-1}-4)^2]^{(2^{n-1}-3)}\right\},$$

$$\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(QD_{2^n})) = \frac{(2^n-8)(2^n-6)(2^n-4)(2^n+2)}{8(2^n-2)} \text{ and } \text{LE}^+_{\text{CN}}(\Gamma_{\text{c}}(QD_{2^n})) = \frac{2^{n-3}(2^n-8)(2^n-6)(2^n-4)}{2^n-2}.$$

Proof. By Result 1.2.6, we have $\Gamma_{\text{c}}(QD_{2^n}) = K_{2^{n-1}-2} \cup 2^{n-2}K_2$. Therefore, by Theorem 2.2.2, we get

$$\text{CNL-spec}(\Gamma_{\text{c}}(QD_{2^n})) = \left\{[0]^{2^{n-2}+1}, [(2^{n-1}-2)(2^{n-1}-4)]^{(2^{n-1}-3)}, [2(2-2)]^{2^{n-2}(2-1)}\right\}$$

and

$$\text{CNSL-spec}(\Gamma_{\text{c}}(QD_{2^n})) = \left\{[2(2^{n-1}-3)(2^{n-1}-4)]^1, [(2^{n-1}-4)^2]^{(2^{n-1}-3)}, [2(2-1)(2-2)]^{(2^{n-2})}, [(2-2)^2]^{(2^{n-2})(2-1)}\right\}.$$

Thus, we get the required CNL-spec$(\Gamma_{\text{c}}(QD_{2^n}))$ and CNSL-spec$(\Gamma_{\text{c}}(QD_{2^n}))$ on simplification.

Here $|v(\Gamma_{\text{c}}(QD_{2^n}))| = 2^n - 2$ and $\text{tr}(\text{CNRS}(\Gamma_{\text{c}}(QD_{2^n}))) = \frac{1}{8}(2^n-8)(2^n-6)(2^n-4)$. Therefore, $\Delta_{\text{CN}}(\Gamma_{\text{c}}(QD_{2^n})) = \frac{(2^n-8)(2^n-6)(2^n-4)}{8(2^n-2)}$. In order to compute CNL-energy of $\Gamma_{\text{c}}(QD_{2^n})$, we first determine the quantities $|\alpha - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n}))|$, where $\alpha \in$ CNL-spec$(\Gamma_{\mathrm{c}}(QD_{2^n}))$, so that (2.1.$a$) can be used. We have

$$L_1 := |0 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n}))| = \frac{(2^n-8)\,(2^n-6)\,(2^n-4)}{8\,(2^n-2)}$$

and $L_2 := |\left(2^{n-1}-2\right)\left(2^{n-1}-4\right) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n}))| = 2^n(2^{n-3}-1)-1+\frac{6}{2^n-2}$. Hence, by (2.1.$a$), we get

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) = (2^{n-1}+1)L_1 + (2^{n-1}-3)L_2 = \frac{(2^n-8)\,(2^n-6)\,(2^n-4)\,(2^n+2)}{8\,(2^n-2)}.$$

In order to compute CNSL-energy of $\Gamma_{\mathrm{c}}(QD_{2^n})$, we first determine the quantities $|\beta - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n}))|$, where $\beta \in$ CNSL-spec$(\Gamma_{\mathrm{c}}(QD_{2^n}))$, so that (2.1.$b$) can be used. While computing CNL-energy, we have already seen that

$$|0 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n}))| = \frac{(2^n-8)\,(2^n-6)\,(2^n-4)}{8\,(2^n-2)}.$$

For our convenience, we denote this quantity by B_1. We have

$$B_2 := \left|2\left(2^{n-1}-3\right)\left(2^{n-1}-4\right) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n}))\right| = 15 + 2^n(3\times 2^{n-3}-5) + \frac{6}{2^n-2}$$

and $B_3 := \left|\left(2^{n-1}-4\right)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n}))\right| = 7 + 2^{n+1}(2^{n-4}-1) + \frac{6}{2^n-2}$. Hence, by (2.1.$b$), we get

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) = 2^{n-1}B_1 + 1\times B_2 + (2^{n-1}-3)B_3 = \frac{2^{n-3}\,(2^n-8)\,(2^n-6)\,(2^n-4)}{2^n-2}.$$

This completes the proof. □

Proposition 3.1.2. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of projective special linear groups $PSL(2,2^k)$, where $k \geq 2$, are given by*

$$\text{CNL-spec}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) = \Big\{[0]^{2^k+2^{2k}+1}, [(2^k-1)(2^k-3)]^{(2^k+1)(2^k-2)}, [(2^k-2)(2^k-4)]^{2^{k-1}(2^k+1)(2^k-3)}, [2^k(2^k-2)]^{2^{k-1}(2^k-1)^2}\Big\},$$

$$\text{CNSL-spec}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) = \Big\{[2(2^k-2)(2^k-3)]^{2^k+1}, [(2^k-3)^2]^{(2^k+1)(2^k-2)}, [2(2^k-3)(2^k-4)]^{2^{k-1}(2^k+1)}, [(2^k-4)^2]^{2^{k-1}(2^k+1)(2^k-3)}, [2(2^k-1)(2^k-2)]^{2^{k-1}(2^k-1)}, [(2^k-2)^2]^{2^{k-1}(2^k-1)^2}\Big\},$$

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) = \frac{(2^k-2)\left(19\times 2^k - 2^{3k+2} + 3\times 4^{k+1} - 7\times 16^k - 5\times 32^k + 3\times 64^k + 6\right)}{8^k-2^k-1}$$ *and*

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) = \begin{cases} \frac{9260}{59}, & \text{for } k=2 \\ \frac{-5\times 2^{k+2}+7\times 2^{3k+1}+2^{4k+5}-9\times 4^k+32^k-13\times 64^k+3\times 128^k-12}{8^k-2^k-1}, & \text{for } k\geq 3. \end{cases}$$

Proof. By Result 1.2.7, we have

$$\Gamma_{\mathrm{c}}(PSL(2,2^k)) = (2^k+1)K_{2^k-1} \cup 2^{k-1}(2^k+1)K_{2^k-2} \cup 2^{k-1}(2^k-1)K_{2^k}.$$

Therefore, by Theorem 2.2.2, we get
CNL-spec$(\Gamma_{\mathrm{c}}(PSL(2,2^k))) = \Big\{[0]^{2^k+1}, [(2^k-1)(2^k-1-2)]^{(2^k+1)(2^k-1-1)}, [0]^{2^{k-1}(2^k+1)},$
$[(2^k-2)(2^k-2-2)]^{2^{k-1}(2^k+1)(2^k-2-1)}, [0]^{2^{k-1}(2^k-1)}, [2^k(2^k-2)]^{2^{k-1}(2^k-1)(2^k-1)}\Big\}$ and
CNSL-spec$(\Gamma_{\mathrm{c}}(PSL(2,2^k)))$=$\Big\{[2(2^k-1-1)(2^k-1-2)]^{2^k+1}, [(2^k-1-2)^2]^{(2^k+1)(2^k-1-1)},$
$[2(2^k-2-1)(2^k-2-2)]^{2^{k-1}(2^k+1)}, [(2^k-2-2)^2]^{2^{k-1}(2^k+1)(2^k-2-1)},$
$[2(2^k-1)(2^k-2)]^{2^{k-1}(2^k-1)}, [(2^k-2)^2]^{2^{k-1}(2^k-1)(2^k-1)}\Big\}$. After simplification, we get the required CNL-spectrum and CNSL-spectrum.

Here $|v(\Gamma_{\mathrm{c}}(PSL(2,2^k)))| = 8^k - 2^k - 1$ and tr(CNRS$(\Gamma_{\mathrm{c}}(PSL(2,2^k))))$=$\left(2^k-2\right)\left(5\times 2^k -3\times 8^k + 16^k + 3\right)$. So, $\Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) = \frac{\left(2^k-2\right)\left(5\times 2^k-3\times 8^k+16^k+3\right)}{8^k-2^k-1}$. We have

$$L_1 := \left|0 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = \frac{\left(2^k-2\right)\left(5\times 2^k - 3\times 8^k + 16^k + 3\right)}{8^k-2^k-1},$$

$$L_2 := \left|\left(2^k-1\right)\left(2^k-3\right) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = \frac{3 + 8\times 2^k + 4^k(4^k - 4\times 2^k - 2)}{8^k-2^k-1},$$

$$L_3 := \left|\left(2^k-2\right)\left(2^k-4\right) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = \frac{2 - 5\times 2^k + 2^k(2^{2k}(2^k-1))}{8^k-2^k-1}$$

and $L_4 := \left|2^k(2^k-2) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = \frac{6+9\times 2^k+4^k(2\times 4^k-4)+8^k(2\times 2^k-7)}{8^k-2^k-1}$. Therefore, by (2.1.$a$), we get

$$\begin{aligned}\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) = (2^k + 2^{2k} + 1)L_1 + (2^k+1)(2^k-2)L_2 + 2^{k-1}(2^k+1)(2^k-3)L_3 \\ + 2^{k-1}(2^k-1)^2 L_4.\end{aligned}$$

Hence, the expression for $\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))$ is obtained.

Again

$$\begin{aligned}B_1 &:= \left|2\left(2^k-2\right)\left(2^k-3\right) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| \\ &= \frac{5\times 2^k + 2^{3k+2} + 3\times 4^k - 5\times 16^k + 32^k - 6}{8^k-2^k-1},\end{aligned}$$

$$B_2 := \left|\left(2^k-3\right)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = -\frac{-3 + 4\times 2^k - 2^{3k} + 2\times 2^{3k} - (2^k-1)2^{3k}}{8^k-2^k-1},$$

$$B_3 := \left|2\left(2^k-3\right)\left(2^k-4\right)-\Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = \begin{cases} \frac{174}{59}, & \text{for } k=2 \\ \frac{-3\times 2^k+2^{3k+4}+7\times 4^k-9\times 16^k+32^k-18}{8^k-2^k-1}, & \text{for } k\geq 3, \end{cases}$$

$$B_4 := \left|\left(2^k-4\right)^2-\Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = -\frac{-2^k+2^{2k+1}+9\times 8^k-3\times 16^k-10}{8^k-2^k-1},$$

$$B_5 := \left|2\left(2^k-1\right)\left(2^k-2\right)-\Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = \frac{2+9\times 2^k+2^{2k}((2^k-1)2^k\times 2^k-4\times 2^k-1)}{8^k-2^k-1}$$

and $B_6 := \left|\left(2^k-2\right)^2-\Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\right| = \frac{2+7\times 2^k+((2^k-1)-3)8^k+2^{3k}-2^{2k+1}}{8^k-2^k-1}$. Therefore, by (2.1.$b$), we get

$$\begin{aligned}\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) &= (2^k+1)B_1+(2^k+1)(2^k-2)B_2+2^{k-1}(2^k+1)B_3 \\ &\quad + 2^{k-1}(2^k+1)(2^k-3)B_4+2^{k-1}(2^k-1)B_5+2^{k-1}(2^k-1)^2B_6.\end{aligned}$$

Hence, the expression for $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))$ is obtained. □

Proposition 3.1.3. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of general linear groups $GL(2,q)$, where $q=p^n>2$ and p is a prime, are given by*

$$\begin{aligned}\text{CNL-spec}(\Gamma_{\mathrm{c}}(GL(2,q))) = \Big\{&[0]^{q^2+q+1}, [(q^2-3q+2)(q^2-3q)]^{\frac{1}{2}(q^2+q)(q^2-3q+1)}, \\ &[(q^2-q)(q^2-q-2)]^{\frac{1}{2}(q^2-q)(q^2-q-1)}, [(q^2-2q+1)(q^2-2q-1)]^{(q+1)(q^2-2q)}\Big\},\end{aligned}$$

$$\begin{aligned}\text{CNSL-spec}(\Gamma_{\mathrm{c}}(GL(2,q))) = \Big\{&[2(q^2-3q+1)(q^2-3q)]^{\frac{1}{2}(q^2+q)}, [(q^2-3q)^2]^{\frac{1}{2}(q^2+q)(q^2-3q+1)}, \\ &[2(q^2-q-1)(q^2-q-2)]^{\frac{1}{2}(q^2-q)}, [(q^2-q-2)^2]^{\frac{1}{2}(q^2-q)(q^2-q-1)}, \\ &[2(q^2-2q)(q^2-2q-1)]^{q+1}, [(q^2-2q-1)^2]^{(q+1)(q^2-2q)}\Big\},\end{aligned}$$

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) = \frac{(q-2)(q-1)q^4(q+1)(2q-3)((q-1)q-1)}{q^3-q-1} \text{ and}$$

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) = \begin{cases} \frac{(q-2)q(q+1)(q(q(q(q(2q((q-4)q+6)-7)-7)-1)+11)+2)}{q^3-q-1}, & \text{if } q\leq 5 \\ \frac{q(q+1)\left((q(q(q(q(2q-11)+20)-16)+7)+8)q^3-12q-4\right)}{q^3-q-1}, & \text{if } q\geq 6. \end{cases}$$

Proof. By Result 1.2.8, we have

$$\Gamma_{\mathrm{c}}(GL(2,q)) = \frac{q(q+1)}{2}K_{q^2-3q+2}\cup\frac{q(q-1)}{2}K_{q^2-q}\cup(q+1)K_{q^2-2q+1}.$$

Therefore, by Theorem 2.2.2, we get

$$\begin{aligned}\text{CNL-spec}(\Gamma_c(GL(2,q))) = \Big\{&[0]^{\frac{q(q+1)}{2}}, [(q^2-3q+2)(q^2-3q+2-2)]^{(\frac{q(q+1)}{2})(q^2-3q+2-1)},\\ &[0]^{\frac{q(q-1)}{2}}, [(q^2-q)(q^2-q-2)]^{(\frac{q(q-1)}{2})(q^2-q-1)}, [0]^{q+1},\\ &[(q^2-2q+1)(q^2-2q+1-2)]^{(q+1)(q^2-2q+1-1)}\Big\} \text{ and}\end{aligned}$$

$$\begin{aligned}\text{CNSL-spec}(\Gamma_c(GL(2,q))) = \Big\{&[2(q^2-3q+2-1)(q^2-3q+2-2)]^{\frac{q(q+1)}{2}},\\ &[(q^2-3q+2-2)^2]^{(\frac{q(q+1)}{2})(q^2-3q+2-1)}, [2(q^2-q-1)(q^2-q-2)]^{\frac{q(q-1)}{2}},\\ &[(q^2-q-2)^2]^{(\frac{q(q-1)}{2})(q^2-q-1)}, [2(q^2-2q+1-1)(q^2-2q+1-2)]^{q+1},\\ &[(q^2-2q+1-2)^2]^{(q+1)(q^2-2q+1-1)}\Big\}.\end{aligned}$$

Thus we get CNL-spec($\Gamma_c(GL(2,q))$) and CNSL-spec($\Gamma_c(GL(2,q))$) on simplification.

Here $|v(\Gamma_c(GL(2,q)))| = (q-1)(q^3-q-1)$ and $\text{tr}(\text{CNRS}(\Gamma_c(GL(2,q)))) = (q-2)(q-1)q(q+1)\left((q-2)(q-1)q^2+1\right)$. So, $\Delta_{CN}(\Gamma_c(GL(2,q))) = \frac{(q-2)q(q+1)\left((q-2)(q-1)q^2+1\right)}{q^3-q-1}$. We have

$$L_1 := |0 - \Delta_{CN}(\Gamma_c(GL(2,q)))| = \frac{(q-2)q(q+1)((q-2)(q-1)q^2+1)}{q^3-q-1},$$

$$L_2 := \left|\left(q^2-3q+2\right)\left(q^2-3q\right) - \Delta_{CN}(\Gamma_c(GL(2,q)))\right| = \frac{(q-2)q\left((q(2q-3)-1)q^2+4\right)}{q^3-q-1},$$

$$L_3 := \left|\left(q^2-q\right)\left(q^2-q-2\right) - \Delta_{CN}(\Gamma_c(GL(2,q)))\right| = \frac{(q-2)q^3(q+1)(2q-3)}{q^3-q-1},$$

and $L_4 := \left|\left(q^2-2q+1\right)\left(q^2-2q-1\right) - \Delta_{CN}(\Gamma_c(GL(2,q)))\right| = \frac{q(-3+q(3+(-2+q)q))-1}{q^3-q-1}$. Therefore, by (2.1.$a$), we get

$$\begin{aligned}\text{LE}_{CN}(\Gamma_c(GL(2,q))) = &(q^2+q+1)L_1 + \frac{q(q+1)}{2}(q^2-3q+1)L_2 + \frac{q(q-1)}{2}(q^2-q-1)L_3\\ &+ (q+1)(q^2-2q)L_4.\end{aligned}$$

Hence, the expression for $\text{LE}_{CN}(\Gamma_c(GL(2,q)))$ is obtained.

Again

$$B_1 := \left|2\left(q^2-3q+1\right)\left(q^2-3q\right) - \Delta_{CN}(\Gamma_c(GL(2,q)))\right| = \begin{cases} -\frac{q\left(q\left((q-5)(q-3)q^3-5q-13\right)+8\right)}{q^3-q-1}, & \text{for } q \le 5\\ \frac{q\left(q\left((q-5)(q-3)q^3-5q-13\right)+8\right)}{q^3-q-1}, & \text{for } q \ge 6,\end{cases}$$

$$B_2 := \left|\left(q^2-3q\right)^2 - \Delta_{CN}(\Gamma_c(GL(2,q)))\right| = -\frac{q\left(-2q^5+5q^4+q^3-8q+2\right)}{q^3-q-1},$$

$$B_3 := \left|2\left(q^2-q-1\right)\left(q^2-q-2\right) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q)))\right| = \frac{(q-2)(q+1)\left(q^5+q^4-6q^3+3q+2\right)}{q^3-q-1},$$

$$B_4 := \left|\left(q^2-q-2\right)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q)))\right| = \frac{(q-2)(q+1)\left(2q^4-5q^3+2q+2\right)}{q^3-q-1}$$

$$B_5 := \left|2\left(q^2-2q\right)\left(q^2-2q-1\right) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q)))\right| = \frac{(q-2)q\left((q-3)(q+1)q^3+5q+1\right)}{q^3-q-1}$$

and $B_6 := \left|\left(q^2-2q-1\right)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q)))\right| = -\frac{q(q(q(q(3-2q)+6)-5)-3)-1}{q^3-q-1}$. Therefore, by (2.1.$b$), we get

$$\begin{aligned}\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) = {} & \frac{1}{2}(q^2+1)B_1 + \frac{1}{2}(q^2+1)\left(q^2-3q+1\right)B_2 + \frac{1}{2}(q^2-q)B_3 \\ & + \frac{1}{2}(q^2-q)(q^2-q-1)B_4 + (q+1)B_5 + (q+1)(q^2-2q)B_6.\end{aligned}$$

Hence, the expression for $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q)))$ is obtained. □

Proposition 3.1.4. *Let $F = GF(2^n)$ (where $n \geq 2$) and ν be the Frobenius automorphism of F, that is $\nu(x) = x^2$, for all $x \in F$. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the groups*

$$A(n,\nu) = \left\{ U(a,b) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \nu(a) & 1 \end{pmatrix} : a, b \in F \right\}$$

under matrix multiplication are given by

$$\text{CNL-spec}(\Gamma_{\mathrm{c}}(A(n,\nu))) = \left\{[0]^{(2^n-1)}, [2^n(2^n-2)]^{(2^n-1)^2}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\mathrm{c}}(A(n,\nu))) = \left\{[2(2^n-1)(2^n-2)]^{(2^n-1)}, [(2^n-2)^2]^{(2^n-1)^2}\right\}$$

and $\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(A(n,\nu))) = \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(A(n,\nu))) = 2\left(2^n-2\right)\left(2^n-1\right)^2.$

Proof. By Result 1.2.9, we have $\Gamma_{\mathrm{c}}(A(n,\nu)) = (2^n-1)K_{2^n}$. Therefore, by Theorem 2.2.2, we get CNL-spec($\Gamma_{\mathrm{c}}(A(n,\nu))$) and CNSL-spec($\Gamma_{\mathrm{c}}(A(n,\nu))$).

Here $|v(\Gamma_{\mathrm{c}}(A(n,\nu)))| = 4^n - 2^n$ and $\mathrm{tr}(\mathrm{CNRS}(\Gamma_{\mathrm{c}}(A(n,\nu)))) = 2^n\left(2^n-2\right)\left(2^n-1\right)^2$. So, $\Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(A(n,\nu))) = 4^n - 3 \times 2^n + 2$. We have

$L_1 := |0 - \Delta_{\text{CN}}(\Gamma_{\text{c}}(A(n,\nu)))| = 4^n - 3 \times 2^n + 2$ and $L_2 := |2^n(2^n-2) - \Delta_{\text{CN}}(\Gamma_{\text{c}}(A(n,\nu)))| = 2^n - 2$. Hence, by (2.1.$a$), we get

$$\text{LE}_{\text{CN}}(\Gamma_c(A(n,\nu))) = (2^n-1)L_1 + (2^n-1)^2 L_2 = 2(2^n-2)(2^n-1)^2.$$

Again,
$B_1 := |2(2^n-1)(2^n-2) - \Delta_{\text{CN}}(\Gamma_{\text{c}}(A(n,\nu)))| = 4^n - 3 \times 2^n + 2$ and
$B_2 := \left|(2^n-2)^2 - \Delta_{\text{CN}}(\Gamma_{\text{c}}(A(n,\nu)))\right| = 2^n - 2$. Hence, by (2.1.$b$), we get
$\text{LE}^+_{\text{CN}}(\Gamma_c(A(n,\nu))) = (2^n-1)B_1 + (2^n-1)^2 B_2 = 2\left(2^n-2\right)\left(2^n-1\right)^2.$ □

Proposition 3.1.5. *Let $F = GF(p^n)$, where p is a prime. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of commuting graphs of the groups*

$$A(n,p) = \left\{ V(a,b,c) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \ : \ a,b,c \in F \right\}$$

under matrix multiplication $V(a,b,c)V(a',b',c') = V(a+a', b+b'+ca', c+c')$ are given by

$$\text{CNL-spec}(\Gamma_{\text{c}}(A(n,p))) = \left\{[0]^{p^n+1}, [(p^{2n}-p^n)(p^{2n}-p^n-2)]^{(p^n+1)(p^{2n}-p^n-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\text{c}}(A(n,p))) = \left\{[2(p^{2n}-p^n-1)(p^{2n}-p^n-2)]^{(p^n+1)}, [(p^{2n}-p^n-2)^2]^{(p^n+1)(p^{2n}-p^n-1)}\right\}$$

and $\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(A(n,p))) = \text{LE}^+_{\text{CN}}(\Gamma_{\text{c}}(A(n,p))) = 2\left(p^n+1\right)^2\left(-3p^{2n}+p^{3n}+p^n+2\right)$.

Proof. By Result 1.2.10, $\Gamma_{\text{c}}(A(n,p)) = (p^n+1)K_{p^{2n}-p^n}$. Therefore, by Theorem 2.2.2, we get CNL-spec($\Gamma_{\text{c}}(A(n,p))$) and CNSL-spec($\Gamma_{\text{c}}(A(n,p))$).

Here $|v(\Gamma_{\text{c}}(A(n,p)))| = p^n\left(p^{2n}-1\right)$ and $\text{tr}(\text{CNRS}(\Gamma_{\text{c}}(A(n,p)))) = p^n\left(p^n-2\right)\left(p^n-1\right) \times \left(p^n+1\right)^2\left(p^n\left(p^n-1\right)-1\right)$. So, $\Delta_{\text{CN}}(\Gamma_{\text{c}}(A(n,p))) = \left(p^n-2\right)\left(p^n+1\right)\left(p^n\left(p^n-1\right)-1\right)$. We have

$$L_1 := |0 - \Delta_{\text{CN}}(\Gamma_{\text{c}}(A(n,p)))| = \left(p^n-2\right)\left(p^n+1\right)\left(p^n\left(p^n-1\right)-1\right),$$

and $L_2 := \left|(p^{2n}-p^n)(p^{2n}-p^n-2) - \Delta_{\text{CN}}(\Gamma_{\text{c}}(A(n,p)))\right| = (p^n-2)(p^n+1)$. Hence, by (2.1.$a$), we get

$$\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(A(n,p))) = (p^n+1)L_1 + (p^{2n}-p^n-1)(p^n+1)L_2$$

$$= 2(p^n + 1)^2(-3p^{2n} + p^{3n} + p^n + 2).$$

Again

$$B_1 := \left|2(p^{2n} - p^n - 1)(p^{2n} - p^n - 2) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(A(n,p)))\right| = (p^n - 2)\,(p^n + 1)\,(p^n\,(p^n - 1) - 1)$$

and $B_2 := \left|(p^{2n} - p^n - 2)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(A(n,p)))\right| = p^{2n} - p^n - 2$. Therefore, by (2.1.$b$), we get

$$\mathrm{LE}^+_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(A(n,p))) = (p^n + 1)B_1 + \left(p^{2n} - p^n - 1\right)(p^n + 1)\,B_2.$$

Hence, the expression for $\mathrm{LE}^+_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(A(n,p)))$ is obtained. □

We conclude this subsection with the following two results for finite non-abelian AC-group in general.

Theorem 3.1.6. *Let G be a finite non-abelian AC-group and $X_1, X_2, \ldots, X_n$ be the distinct centralizers of non-central elements of G. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of $\Gamma_{\mathrm{c}}(G)$ are given by*
CNL-spec$(\Gamma_{\mathrm{c}}(G)) = \{[0]^n, [(|X_i| - |Z(G)|)(|X_i| - |Z(G)| - 2)]^{(|X_i| - |Z(G)| - 1)}$, *where* $1 \leq i \leq n\}$,
CNSL-spec$(\Gamma_{\mathrm{c}}(G)) = \{[2(|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)| - 2)]^1, [(|X_i| - |Z(G)| - 2)^2]^{(|X_i| - |Z(G)| - 1)}$, *where* $1 \leq i \leq n\}$,
$\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = nL_0 + \sum\limits_{i=1}^{n}(|X_i| - |Z(G)| - 1)L_{X_i}$, *where* $L_0 = |0 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))|$ *and* $L_{X_i} = |(|X_i| - |Z(G)|)(|X_i| - |Z(G)| - 2) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))|$ *and*
$\mathrm{LE}^+_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \sum\limits_{i=1}^{n} B_{X_i} + \sum\limits_{i=1}^{n}(|X_i| - |Z(G)| - 1)B'_{X_i}$, *where* $B_{X_i} = |2(|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)| - 2) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))|$ *and* $B'_{X_i} = |(|X_i| - |Z(G)| - 2)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))|$.

Proof. By [42, Lemma 2.1], we have $\Gamma_{\mathrm{c}}(G) = \cup_{i=1}^{n} K_{|X_i| - |Z(G)|}$. Here $|v(\Gamma_{\mathrm{c}}(G))| = \sum\limits_{i=1}^{n}|X_i| - |Z(G)|$ and $\mathrm{tr}(\mathrm{CNRS}(\Gamma_{\mathrm{c}}(G))) = \sum\limits_{i=1}^{n}(|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)|)(|X_i| - |Z(G)| - 2)$. Therefore,

$$\Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \frac{\sum\limits_{i=1}^{n}(|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)|)(|X_i| - |Z(G)| - 2)}{\sum\limits_{i=1}^{n}|X_i| - |Z(G)|}.$$

Hence, the result follows from the Theorem 2.2.2. □

Theorem 3.1.7. *Let G be a finite non-abelian AC-group and $X_1, X_2, \ldots, X_n$ be the distinct centralizers of non-central elements of G. If A is a finite abelian group then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of $\Gamma_c(G \times A)$ are given by*
$\text{CNL-spec}(\Gamma_c(G\times A)) = \{[0]^n, [|A|(|X_i|-|Z(G)|)(|A|(|X_i|-|Z(G)|)-2)]^{(|A|(|X_i|-|Z(G)|)-1)},$ *where* $1 \leq i \leq n\}$,
$\text{CNSL-spec}(\Gamma_c(G \times A)) = \{[2(|A|(|X_i| - |Z(G)|) - 1)(|A|(|X_i| - |Z(G)|) - 2)]^1, [(|A|(|X_i| - |Z(G)|) - 2)^2]^{(|A|(|X_i|-|Z(G)|)-1)},$ *where* $1 \leq i \leq n\}$,
$LE_{CN}(\Gamma_c(G\times A)) = nL_0 + \sum_{i=1}^{n}(|A|(|X_i|-|Z(G)|)-1)L_{X_i}$, *where* $L_0 = |-\Delta_{CN}(\Gamma_c(G\times A))|$ *and* $L_{X_i} = ||A|(|X_i| - |Z(G)|)(|A|(|X_i| - |Z(G)|) - 2) - \Delta_{CN}(\Gamma_c(G \times A))|$ *and*
$LE^+_{CN}(\Gamma_c(G \times A)) = \sum_{i=1}^{n} S_{X_i} + \sum_{i=1}^{n}(|A|(|X_i| - |Z(G)|) - 1)S'_{X_i}$, *where* $S_{X_i} = |2(|A|(|X_i| - |Z(G)|) - 1)(|A|(|X_i| - |Z(G)|) - 2) - \Delta_{CN}(\Gamma_c(G \times A))|$ *and* $S'_{X_i} = |(|A|(|X_i| - |Z(G)|) - 2)^2 - \Delta_{CN}(\Gamma_c(G \times A))|$.

Proof. We have $Z(G \times A) = Z(G) \times A$ and $X_1 \times A, X_2 \times A, \ldots, X_n \times A$ are the distinct centralizers of non-central elements of $G \times A$. Since G is an AC-group, $G \times A$ is also an AC-group. Therefore,

$$\Gamma_c(G \times A) = \cup_{i=1}^{n} K_{|X_i\times A|-|Z(G)\times A|} = \cup_{i=1}^{n} K_{|A|(|X_i|-|Z(G)|)}.$$

We have, $|v(\Gamma_c(G\times A))| = \sum_{i=1}^{n} |A|(|X_i|-|Z(G)|)$ and $\text{tr}(\text{CNRS}(\Gamma_c(G\times A))) = \sum_{i=1}^{n}(|A|(|X_i|-|Z(G)|) - 1)(|A|(|X_i| - |Z(G)|)(|A|(|X_i| - |Z(G)|) - 2))$. Therefore,
$\Delta_{CN}(\Gamma_c(G \times A)) = \frac{\sum_{i=1}^{n}(|A|(|X_i|-|Z(G)|)-1)(|A|(|X_i|-|Z(G)|)(|A|(|X_i|-|Z(G)|)-2))}{\sum_{i=1}^{n}|A|(|X_i|-|Z(G)|)}$. Hence the result follows from the Theorem 2.2.2. □

3.1.2 Groups whose central quotient is isomorphic to $Sz(2)$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m}

Let us begin with the groups G such that $\frac{G}{Z(G)}$ is isomorphic to the Suzuki group (of order 20) $Sz(2) = \langle x, y : x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$.

Theorem 3.1.8. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong Sz(2)$ and $|Z(G)| = z$. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of $\Gamma_c(G)$ are* *given by*

$$\text{CNL-spec}(\Gamma_c(G)) = \left\{[0]^6, [4z(4z-2)]^{(4z-1)}, [3z(3z-2)]^{5(3z-1)}\right\},$$

$$\begin{aligned}&\text{CNSL-spec}(\Gamma_c(G))\\ &\quad= \left\{[2(4z-1)(4z-2)]^1, [(4z-2)^2]^{(4z-1)}, [2(3z-1)(3z-2)]^5, [(3z-2)^2]^{5(3z-1)}\right\},\end{aligned}$$

$$\text{LE}_{\text{CN}}(\Gamma_c(G)) = \begin{cases} \frac{648}{19}, & \textit{for } z = 1 \\ \frac{2}{19}(4z-1)(z(105z+31)-38), & \textit{for } z \geq 2 \end{cases}$$

and $\text{LE}^{+}_{\text{CN}}(\Gamma_c(G)) = \frac{10}{19}(3z-1)(z(28z+45)-38)$.

Proof. By Result 1.2.11, we have $\Gamma_c(G) = K_{4z} \cup 5K_{3z}$. Therefore, by Theorem 2.2.2, we get

$$\text{CNL-spec}(\Gamma_c(G)) = \left\{[0]^1, [4z(4z-2)]^{(4z-1)}, [0]^5, [3z(3z-2)]^{5(3z-1)}\right\}$$

and $\text{CNSL-spec}(\Gamma_c(G)) = \left\{[2(4z-1)(4z-2)]^1, [(4z-2)^2]^{(4z-1)}, [2(3z-1)(3z-2)]^5,\right.$ $\left.[(3z-2)^2]^{5(3z-1)}\right\}$. Hence, we get the required CNL-spec($\Gamma_c(G)$) and CNSL-spec($\Gamma_c(G)$) on simplification.

Here $|v(\Gamma_c(G))| = 19z$ and $\text{tr}(\text{CNRS}(\Gamma_c(G))) = 199z^3 - 183z^2 + 38z$. So, $\Delta_{\text{CN}}(\Gamma_c(G)) = \frac{199z^2-183z+38}{19}$. We have

$$L_1 := |0 - \Delta_{\text{CN}}(\Gamma_c(G))| = \left|-\frac{199z^2-183z+38}{19}\right| = \frac{199z^2-183z+38}{19},$$

$$L_2 := |4z(4z-2) - \Delta_{\text{CN}}(\Gamma_c(G))| = \frac{105z^2+31z-38}{19}$$

and $L_3 := |3z(3z-2) - \Delta_{\text{CN}}(\Gamma_c(G))| = \left|\frac{z(69-28z)-38}{19}\right| = \begin{cases} \frac{3}{19}, & \text{for } z = 1 \\ -\frac{z(69-28z)-38}{19}, & \text{for } z \geq 2. \end{cases}$

Therefore, by (2.1.a), we get $\text{LE}_{\text{CN}}(\Gamma_c(G)) = 6L_1 + (4z-1)L_2 + 5(3z-1)L_3$. Hence, the expression for $\text{LE}_{\text{CN}}(\Gamma_c(G))$ is obtained.

Again

$$B_1 := |2(4z-1)(4z-2) - \Delta_{\text{CN}}(\Gamma_c(G))| = \frac{409z^2-273z+38}{19},$$

$$B_2 := \left|(4z-2)^2 - \Delta_{\text{CN}}(\Gamma_{\text{c}}(G))\right| = \frac{105z^2 - 121z + 38}{19},$$

$$B_3 := |2(3z-1)(3z-2) - \Delta_{\text{CN}}(\Gamma_{\text{c}}(G))| = \frac{143z^2 - 159z + 38}{19}$$

and $B_4 := |(3z-2)^2 - \Delta_{\text{CN}}(\Gamma_{\text{c}}(G))| = \frac{28z^2+45z-38}{19}$. Therefore, by (2.1.$b$), we get

$$\text{LE}^{+}_{\text{CN}}(\Gamma_{\text{c}}(G)) = 1 \times B_1 + (4z-1)B_2 + 5 \times B_3 + 5(3z-1)B_4.$$

Hence, the expression for $\text{LE}^{+}_{\text{CN}}(\Gamma_{\text{c}}(G))$ is obtained. □

Theorem 3.1.9. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime and $|Z(G)| \geq 2$. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_{\text{c}}(G)) = \left\{[0]^{(p+1)}, [((p-1)z)((p-1)z-2)]^{(p+1)((p-1)z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\text{c}}(G)) = \left\{[2((p-1)z-1)((p-1)z-2)]^{(p+1)}, [((p-1)z-2)^2]^{(p+1)((p-1)z-1)}\right\}$$

and

$$\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(G)) = \text{LE}^{+}_{\text{CN}}(\Gamma_{\text{c}}(G)) = 2(p+1)((p-1)z-2)((p-1)z-1)$$

where $z = |Z(G)|$.

Proof. From Result 1.2.12, we have $\Gamma_{\text{c}}(G) = (p+1)K_{(p-1)z}$. Therefore, by Theorem 2.2.2, we get $\text{CNL-spec}(\Gamma_{\text{c}}(G))$ and $\text{CNSL-spec}(\Gamma_{\text{c}}(G))$.

Here $|v(\Gamma_{\text{c}}(G))| = (p^2-1)\,z$ and $\text{tr}(\text{CNRS}(\Gamma_{\text{c}}(G))) = (p-1)(p+1)z((p-1)z-2)((p-1)z-1)$. So, $\Delta_{\text{CN}}(\Gamma_{\text{c}}(G)) = ((p-1)z-2)((p-1)z-1)$. We have

$$L_1 := |0 - \Delta_{\text{CN}}(\Gamma_{\text{c}}(G))| = ((p-1)z-2)((p-1)z-1)$$

and $L_2 := |((p-1)z)((p-1)z-2) - \Delta_{\text{CN}}(\Gamma_{\text{c}}(G))| = (p-1)z-2$. Therefore, by (2.1.$a$), we get

$$\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(G)) = (p+1)L_1 + (p+1)((p-1)z-1)L_2.$$

Hence, the expression for $\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(G))$ is obtained.

Again

$$B_1 := |2((p-1)z-1)((p-1)z-2) - \Delta_{\text{CN}}(\Gamma_{\text{c}}(G))| = ((p-1)z-2)((p-1)z-1)$$

and $B_2 := |((p-1)z-2)^2 - \Delta_{\text{CN}}(\Gamma_{\text{c}}(G))| = pz - z - 2$.
Therefore, by (2.1.b), we get

$$\text{LE}^+_{\text{CN}}(\Gamma_{\text{c}}(G)) = (p+1)B_1 + (p+1)((p-1)z-1)B_2.$$

Hence, the expression for $\text{LE}^+_{\text{CN}}(\Gamma_{\text{c}}(G))$ is obtained. □

Corollary 3.1.10. *Let G be a non-abelian group of order p^3, where p is a prime. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_{\text{c}}(G)) = \left\{[0]^{(p+1)}, [p(p-1)(p(p-1)-2)]^{(p+1)((p-1)p-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\text{c}}(G)) = \left\{[2((p-1)p-1)((p-1)p-2)]^{(p+1)}, [((p-1)p-2)^2]^{(p+1)((p-1)p-1)}\right\}$$

and $\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(G)) = \text{LE}^+_{\text{CN}}(\Gamma_{\text{c}}(G)) = 2(p+1)((p-1)p-2)((p-1)p-1)$.

Proof. We know that for a non-abelian group G of order p^3, $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from the Theorem 3.1.9. □

Corollary 3.1.11. *Let G be a finite 4-centralizer group. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_{\text{c}}(G)) = \left\{[0]^3, [(z(z-2)]^{3((p-1)z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\text{c}}(G)) = \left\{[2(z-1)(z-2)]^3, [(z-2)^2]^{3(z-1)}\right\} \text{ and}$$

$$\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(G)) = \text{LE}^+_{\text{CN}}(\Gamma_{\text{c}}(G)) = 6(z-2)(z-1),$$

where $z = |Z(G)|$.

Proof. The results follows from Result 1.2.1 and Theorem 3.1.9. □

Corollary 3.1.12. *Let G be a finite $(p+2)$-centralizer p-group. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_{\text{c}}(G)) = \left\{[0]^{(p+1)}, [((p-1))((p-1)|Z(G)|-2)]^{(p+1)((p-1)|Z(G)|-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\text{c}}(G)) = \left\{[2((p-1)|Z(G)|-1)((p-1)|Z(G)|-2)]^{(p+1)}, [((p-1)|Z(G)|-2)^2]^{(p+1)((p-1)|Z(G)|-1)}\right\} \text{ and}$$

$$\text{LE}_{\text{CN}}(\Gamma_{\text{c}}(G)) = \text{LE}^+_{\text{CN}}(\Gamma_{\text{c}}(G)) = 2(p+1)((p-1)|Z(G)|-2)((p-1)|Z(G)|-1).$$

Proof. The result follows from Result 1.2.3 and Theorem 3.1.9. □

Theorem 3.1.13. *If G is a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, $m \geq 3$ then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_c(G)) = \left\{[0]^{m+1}, [(m-1)z((m-1)z-2)]^{((m-1)z-1)}, [z(z-2)]^{m(z-1)}\right\},$$
$$\text{CNSL-spec}(\Gamma_c(G)) = \Big\{[2((m-1)z-1)((m-1)z-2)]^1, [((m-1)z-2)^2]^{((m-1)z-1)}, [2(z-1)(z-2)]^m, [(z-2)^2]^{m(z-1)}\Big\},$$

$$\text{LE}_{\text{CN}}(\Gamma_c(G)) = \frac{2((m-1)z-1)(m(z((m-2)mz-m+3)-4)+z+2)}{2m-1}$$

and $$\text{LE}^+_{\text{CN}}(\Gamma_c(G)) = \begin{cases} \frac{-2m^4+6m^3+3m^2-13m+7}{1-2m}, & \text{for } m=3 \,\&\, z=1 \\ \frac{2(m-2)(m-1)mz^2(mz-3)}{2m-1}, & \text{otherwise} \end{cases}$$

where $z = |Z(G)|$.

Proof. From Result 1.2.13, we have $\Gamma(G) = K_{(m-1)z} \cup mK_z$. Therefore, by Theorem 2.2.2, we get CNL-spec$(\Gamma_c(G))$ and CNSL-spec$(\Gamma_c(G))$.

Here $|v(\Gamma_c(G))| = (2m-1)z$ and $\text{tr}(\text{CNRS}(\Gamma_c(G))) = m(z-2)(z-1)z + (m-1)z((m-1)z-2)((m-1)z-1)$. So, $\Delta_{\text{CN}}(\Gamma_c(G)) = \frac{(m((m-3)m+4)-1)z^2-3((m-1)m+1)z+4m-2}{2m-1}$. We have

$$L_1 := |0 - \Delta_{\text{CN}}(\Gamma_c(G))| = -\frac{(2-4m)+3z(1-m)+z^2(1-4m)+m^2z^2(3-\frac{m}{2})+m^2z(3-\frac{mz}{2})}{2m-1},$$

$$L_2 := |((m-1)z)((m-1)z-2) - \Delta_{\text{CN}}(\Gamma_c(G))| = \frac{m(z((m-2)mz-m+3)-4)+z+2}{2m-1}$$

and $$L_3 := |z(z-2) - \Delta_{\text{CN}}(\Gamma_c(G))| = -\frac{m(-z((m-2)(m-1)z-3m+7)-4)+5z+2}{2m-1}.$$

Therefore, by (2.1.a), we get

$$\text{LE}_{\text{CN}}(\Gamma_c(G)) = (m+1)L_1 + ((m-1)z-1)L_2 + m(z-1)L_3.$$

Hence, the expression for $\text{LE}_{\text{CN}}(\Gamma_c(G))$ is obtained.

Again,

$$B_1 := |2((m-1)z-1)((m-1)z-2) - \Delta_{\text{CN}}(\Gamma_c(G))|$$
$$= \frac{((m-1)m(3m-4)-1)z^2 - 3(m(3m-5)+1)z+4m-2}{2m-1},$$

$$B_2 := \left|((m-1)z-2)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))\right| = \frac{m(z((m-2)mz-5m+9)+4)-z-2}{2m-1},$$

$$B_3 := |2(z-1)(z-2) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))| = -\frac{-\left(\left((m-3)m^2+1\right)z^2\right)+3((m-5)m+3)z+4m-2}{2m-1}$$

and

$$B_4 := \left|(z-2)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))\right| = \begin{cases} \frac{m(4-z((m-2)(m-1)z-3m+11))+7z-2}{2m-1}, & \text{for } m=3 \,\&\, z=1 \\ -\frac{m(4-z((m-2)(m-1)z-3m+11))+7z-2}{2m-1}, & \text{otherwise.} \end{cases}$$

Therefore, by (2.1.b), we get

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = 1 \times B_1 + ((m-1)z-1)\ B_2 + m\ B_3 + m(z-1)\ B_4.$$

Hence, the expression for $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))$ is obtained. □

Corollary 3.1.14. *Let G be a finite 5-centralizer group. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_{\mathrm{c}}(G)) = \left\{[0]^4, [2z(2z-2)]^{4(2z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\mathrm{c}}(G)) = \left\{[2(2z-1)(2z-2)]^4, [(2z-2)^2]^{4(z-1)}\right\}$$

and $\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = 8(2z-2)(2z-1)$, *where* $z = |Z(G)|$*;*

or

$$\text{CNL-spec}(\Gamma_{\mathrm{c}}(G)) = \left\{[0]^4, [4z(z-1)]^{2z-1}, [z(z-2)]^{3(z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{\mathrm{c}}(G)) = \left\{[4(z-1)(2z-1)]^1, [4(z-1)^2]^{2z-1}, [2(z-1)(z-2)]^3, [(z-2)^2]^{3(z-1)}\right\},$$

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \tfrac{2}{5}(2z-1)\left(9z^2+z-10\right) \text{ and } \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \tfrac{36}{5}(z-1)z^2,$$

where $z = |Z(G)|$.

Proof. The result follows from Result 1.2.2, Theorem 3.1.13 and Theorem 3.1.9. □

Corollary 3.1.15. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the group* $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1,\ x^{-1}yx = y^{-1}\rangle$, *where* $m \geq 3$ *and* $n \geq 2$, *are given by*

$\text{CNL-spec}(\Gamma_{\mathrm{c}}(U_{(n,m)})) =$

$$\begin{cases} \left\{[0]^{m+1}, [(m-1)n((m-1)n-2)]^{((m-1)n-1)}, [n(n-2)]^{m(n-1)}\right\}, & \text{if } 2 \nmid m \\ \left\{[0]^{\frac{m}{2}+1}, [(\frac{m}{2}-1)2n((\frac{m}{2}-1)2n-2)]^{((\frac{m}{2}-1)2n-1)}, [2n(2n-2)]^{\frac{m}{2}(2n-1)}\right\}, & \text{if } 2 \mid m, \end{cases}$$

CNSL-spec$(\Gamma_c(U_{(n,m)}))=$

$$\begin{cases}\Big\{[2((m-1)n-1)((m-1)n-2)]^1, [((m-1)n-2)^2]^{((m-1)n-1)}, \\ \qquad [2(n-1)(n-2)]^m, [(n-2)^2]^{m(n-1)}\Big\}, & \text{if } 2 \nmid m \\ \Big\{[2((\frac{m}{2}-1)2n-1)((\frac{m}{2}-1)2n-2)]^1, [((\frac{m}{2}-1)2n-2)^2]^{((\frac{m}{2}-1)2n-1)}, \\ \qquad [2(2n-1)(2n-2)]^{\frac{m}{2}}, [(2n-2)^2]^{\frac{m}{2}(2n-1)}\Big\}, & \text{if } 2 \mid m,\end{cases}$$

$$\mathrm{LE_{CN}}(\Gamma_c(U_{(n,m)}))=\begin{cases}\frac{2((m-1)n-1)(m(n((m-2)mn-m+3)-4)+n+2)}{2m-1}, & \text{if } 2 \nmid m \\ \frac{((m-2)n-1)(m(n((m-4)mn-m+6)-4)+4(n+1))}{m-1}, & \text{if } 2 \mid m\end{cases}$$

and

$$\mathrm{LE^+_{CN}}(\Gamma_c(U_{(n,m)}))=\begin{cases}\frac{2(m-2)(m-1)mn^2(mn-3)}{2m-1}, & \text{if } 2 \nmid m \\ \frac{(m-4)(m-2)mn^2(mn-3)}{m-1}, & \text{if } 2 \mid m.\end{cases}$$

Proof. We have

$$Z(U_{(n,m)})=\begin{cases}\langle x^2\rangle, & \text{if } 2 \nmid m \\ \langle x^2\rangle \cup y^{\frac{m}{2}}\langle x^2\rangle, & \text{if } 2 \mid m\end{cases}$$

and so $|Z(U_{(n,m)})| = n$ and $2n$ according as $2 \nmid m$ and $2 \mid m$. Therefore,

$$\frac{U_{(n,m)}}{Z(U_{(n,m)})} \cong \begin{cases}D_{2m}, & \text{if } 2 \nmid m \\ D_{2\times 2}, & \text{if } m = 4 \\ D_m, & \text{if } m \geq 6 \text{ and } 2 \mid m.\end{cases}$$

Hence, the result follows from Theorem 3.1.9 and Theorem 3.1.13. □

Corollary 3.1.16. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the dihedral groups,* $D_{2m} = \langle x, y : x^{2m} = y^2 = 1, yxy^{-1} = x^{-1}\rangle$, $m \geq 3$, *are given by*

$$\text{CNL-spec}(\Gamma_c(D_{2m}))=\begin{cases}\Big\{[0]^{m+1}, [(m-1)(m-3)]^{(m-2)}\Big\}, & \text{if } 2 \nmid m \\ \Big\{[0]^{(m+1)}, [(m-2)(m-4)]^{(m-3)}\Big\}, & \text{if } 2 \mid m,\end{cases}$$

$$\text{CNSL-spec}(\Gamma_c(D_{2m})) = \begin{cases} \left\{[0]^m, [2(m-2)(m-3)]^1, [(m-3)^2]^{(m-2)}\right\}, & \textit{if } 2 \nmid m \\ \left\{[0]^m, [2(m-3)(m-4)]^1, [(m-4)^2]^{(m-3)}\right\}, & \textit{if } 2 \mid m, \end{cases}$$

$$\text{LE}_{\text{CN}}(\Gamma_c(D_{2m})) = \begin{cases} \frac{2(m-3)(m-2)(m-1)(m+1)}{2m-1}, & \textit{if } 2 \nmid m \\ \frac{(m-4)(m-3)(m-2)(m+1)}{m-1}, & \textit{if } 2 \mid m \end{cases}$$

and $\text{LE}^+_{\text{CN}}(\Gamma_c(D_{2m})) = \begin{cases} \frac{2(m-3)(m-2)(m-1)m}{2m-1}, & \textit{if } 2 \nmid m \\ \frac{(m-4)(m-3)(m-2)m}{m-1}, & \textit{if } 2 \mid m. \end{cases}$

Proof. We know that $|Z(D_{2m})| = \begin{cases} 1, & \text{if } 2 \nmid m \\ 2, & \text{if } 2 \mid m \end{cases}$ and

$$\frac{D_{2m}}{Z(D_{2m})} = \begin{cases} D_{2m}, & \text{if } 2 \nmid m \\ D_{2\times 2}, & \text{if } m = 4 \\ D_m, & \text{if } m \geq 6 \text{ and } 2 \mid m. \end{cases}$$

Therefore, by using Theorem 3.1.9 and Theorem 3.1.13 we get the required result. □

Corollary 3.1.17. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the groups* $U_{6n} = \langle x, y : x^{2n} = y^3 = 1,\ x^{-1}yx = y^{-1}\rangle$ *are given by*

$$\text{CNL-spec}(\Gamma_c(U_{6n})) = \left\{[0]^4, [2n(2n-2)]^{2n-1}, [n(n-2)]^{3(n-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_c(U_{6n})) = \left\{[(2n-1)(2n-2)]^1, [(2n-2)^2]^{2n-1}, [2(n-1)(n-2)]^3, [(n-2)^2]^{3(n-1)}\right\},$$

$$\text{LE}_{\text{CN}}(\Gamma_c(U_{6n})) = \tfrac{2}{5}(n-1)(2n-1)(9n+10) \textit{ and } \text{LE}^+_{\text{CN}}(\Gamma_c(U_{6n})) = \tfrac{36}{5}(n-1)n^2.$$

Proof. We have $Z(U_{6n}) = \langle x^2 \rangle$ and $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$ with $|Z(U_{6n})| = n$. Therefore, we get the required result by putting $m = 3$ and $z = n$ in Theorem 3.1.13. □

Corollary 3.1.18. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the dicyclic groups,* $Q_{4n} = \langle x, y :\ y^{2n} = 1,\ x^2 = y^n,\ xyx^{-1} = y^{-1}\rangle$, $n \geq 2$, *are given by*

$$\text{CNL-spec}(\Gamma_c(Q_{4n})) = \{[0]^{2n+1}, [(2n-2)(2n-4)]^{2n-3}\},$$
$$\text{CNSL-spec}(\Gamma_c(Q_{4n})) = \{[0]^{2n}, [2(2n-3)(2n-4)]^1, [(2n-4)^2]^{2n-3}\},$$
$$\text{LE}_{\text{CN}}(\Gamma_c(Q_{4n})) = \tfrac{4(n-2)(n-1)(2n-3)(2n+1)}{2n-1} \textit{ and } \text{LE}^+_{\text{CN}}(\Gamma_c(Q_{4n})) = \tfrac{8(n-2)(n-1)n(2n-3)}{2n-1}.$$

Proof. We have $Z(Q_{4n}) = \{1, x^n\}$ and $\frac{Q_{4n}}{Z(Q_{4n})} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or D_{2n} according as $n = 2$ or $n \geq 3$. Therefore, by using Theorem 3.1.9 and Theorem 3.1.13, we get the required result. □

3.2 CNL(CNSL)-integral and CNL(CNSL)-hyperenergetic graphs

We begin this section with the following theorem which follows from the results obtained in Section 3.1.

Theorem 3.2.1. *Let G be a finite non-abelian group.*

(a) *If G is isomorphic to QD_{2^n} (where $n \geq 4$), $PSL(2, 2^k)$ (where $k \geq 2$), $GL(2, q)$ (where $q > 2$ is a prime power), $A(n, \nu)$ and $A(n, p)$ then $\Gamma_c(G)$ is CNL(CNSL)-integral.*

(b) *If $\frac{G}{Z(G)}$ is isomorphic to $Sz(2)$, D_{2m} and $\mathbb{Z}_p \times \mathbb{Z}_p$ then $\Gamma_c(G)$ is CNL(CNSL)-integral.*

(c) *If G is a 4, 5-centralizer finite group or a $(p+2)$-centralizer finite p-group then $\Gamma_c(G)$ is CNL(CNSL)-integral.*

(d) *If G is isomorphic to $U_{(n,m)}$, D_{2m}, U_{6n} and Q_{4n} then $\Gamma_c(G)$ is CNL(CNSL)-integral.*

(e) *If G is an AC-group then $\Gamma_c(G)$ is CNL(CNSL)-integral.*

Theorem 3.2.2. *The commuting graph of the quasihedral group QD_{2^n} ($n \geq 4$) is*

(a) *CNL(CNSL)-hyperenergetic if $n \geq 5$.*

(b) *not CNL(CNSL)-hyperenergetic if $n = 4$.*

Proof. We have $|v(\Gamma_c(QD_{2^n}))| = 2^n - 2$. Using (2.2.$a$) and Proposition 3.1.1, we get

$$\text{LE}_{\text{CN}}(K_{|v(\Gamma_c(QD_{2^n}))|} - \text{LE}_{\text{CN}}(\Gamma_c(QD_{2^n})) = -\frac{2^{(-3+n)}(-4+2^n)(100 - 7 \times 2^{2+n} + 4^n)}{-2+2^n}.$$

Note that $\mathrm{LE_{CN}}(K_{|v(\Gamma_c(QD_{2^n}))|} - \mathrm{LE_{CN}}(\Gamma_c(QD_{2^n})) < 0$ or > 0 according as $n \geq 5$ or $n = 4$. Also,

$$\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma_c(QD_{2^n}))|} - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(QD_{2^n})) = -\frac{(-4+2^n)^2(24 - 13 \times 2^{1+n} + 4^n)}{8(-2+2^n))}.$$

Note that $\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma_c(QD_{2^n}))|} - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(QD_{2^n})) > 0$ or < 0 according as $n \geq 5$ or $n = 4$. Hence, the result follows. □

Proposition 3.2.3. *The commuting graph of the projective special linear group $PSL(2,2^k)$, $k \geq 2$, is not CNL(CNSL)-hyperenergetic.*

Proof. We have $|v(\Gamma_c(PSL(2, 2^k)))| = -2^k + 2^{3k} - 1$. Using (2.2.$a$) and Proposition 3.1.2, we get

$$\begin{aligned}&\mathrm{LE_{CN}}(K_{|v(\Gamma_c(PSL(2,2^k)))|}) - \mathrm{LE_{CN}}(\Gamma_c(PSL(2, 2^k)))\\ &\quad = \frac{2^k\left(7 \times 2^{3k+1} - 7 \times 2^k + 2 \times 2^{8k} - 2^{5k} - 9 \times 2^{6k} + 3 \times 2^{4k}\right)}{8^k - 2^k - 1} > 0.\end{aligned}$$

Therefore, $\Gamma_{PSL(2,2^k)}$ is not CNL-hyperenergetic.

For $k \geq 3$, we have

$$\begin{aligned}&\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma_c(PSL(2,2^k)))|}) - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(PSL(2, 2^k)))\\ &\quad = \frac{2^k\left(-3 \times 2^k + 3 \times 2^{2k+1} + 2^{8k+1} - 8^{k+1} + 5 \times 16^k + 32^k - 9 \times 64^k - 2\right)}{-2^k + 8^k - 1} > 0.\end{aligned}$$

Also, for $k = 2$, $\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma_c(PSL(2,2^k)))|}) - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(PSL(2, 2^k))) = \frac{380848}{59} > 0$. Therefore, $\Gamma_c(PSL(2, 2^k))$ is not CNSL-hyperenergetic. □

Theorem 3.2.4. *The commuting graph of the general linear group $GL(2, q)$, where $q = p^n > 2$ and p is prime, is not CNL(CNSL)-hyperenergetic.*

Proof. We have $|v(\Gamma_c(GL(2, q)))| = q^4 - q^3 - q^2 + 1$. Using (2.2.$a$) and Proposition 3.1.3, we get

$$\begin{aligned}&\mathrm{LE_{CN}}(K_{|v(\Gamma_c(GL(2,q)))|}) - \mathrm{LE_{CN}}(\Gamma_c(GL(2, q)))\\ &\quad = \frac{q^2(q+1)((q-1)q-1)\left(q^2(q-1)(5q-8+2q^2(q-2))+2\right)}{q^3 - q - 1} > 0.\end{aligned}$$

Therefore, $\Gamma_c(GL(2,q))$ is not CNL-hyperenergetic.

Also,

$$\begin{aligned}
&\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma_c(GL(2,q)))|}) - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(GL(2,q))) \\
&\quad = \begin{cases} \frac{2q^{11}-6q^{10}+6q^9-10q^8+9q^7+24q^6-22q^5-28q^4+3q^3+22q^2+4q}{q^3-q-1} > 0, & \text{if } q \le 5 \\ \frac{2q^{11}-6q^{10}+5q^9-3q^8+2q^7+9q^6-17q^5-10q^4+8q^3+14q^2+4q}{q^3-q-1} > 0, & \text{if } q \ge 6. \end{cases}
\end{aligned}$$

Therefore, $\Gamma_c(GL(2,q))$ is not CNSL-hyperenergetic. □

Theorem 3.2.5. *Let G be a finite group such that $\frac{G}{Z(G)} \cong Sz(2)$. Then the commuting graph of G is*

(a) *CNL-hyperenergetic if $|Z(G)| \geq 17$.*

(b) *not CNL-hyperenergetic if $1 \leq |Z(G)| \leq 16$.*

(c) *CNSL-hyperenergetic if $|Z(G)| \geq 16$.*

(d) *not CNSL-hyperenergetic if $1 \leq |Z(G)| \leq 15$.*

Proof. We have $|v(\Gamma_c(G))| = 3 \times 5z + 4z = 19z$, where $z = |Z(G)|$. Using (2.2.a) and Theorem 3.1.8, for $z \geq 2$, we get

$$\mathrm{LE}_{\mathrm{CN}}(K_{|v(\Gamma_c(G))|}) - \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G)) = -\frac{120}{19}z(z(7z-114)+15).$$

Note that $\mathrm{LE}_{\mathrm{CN}}(K_{|v(\Gamma_c(G))|}) - \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G)) < 0$ or > 0 according as $z \geq 17$ or $1 \leq z \leq 16$. Therefore, $\Gamma_c(G)$ is CNL-hyperenergetic and not CNL-hyperenergetic according as $z \geq 17$ and $1 \leq z \leq 16$.

Again,

$$\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma_c(G))|}) - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G)) = -\frac{8}{19}(3z(z(35z-527)+24)+38).$$

Note that $\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma_c(G))|}) - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G)) < 0$ or > 0 according as $z \geq 16$ or $1 \leq z \leq 15$. Therefore, $\Gamma_c(G)$ is CNSL hyperenergetic and not CNSL hyperenergetic according as $z \geq 16$ and $1 \leq z \leq 15$. □

Theorem 3.2.6. *Let G be a finite group with $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then the commuting graph of G is not CNL(CNSL)-hyperenergetic.*

Proof. We have $|v(\Gamma_c(G))| = (p^2-1)z$, where $z = |Z(G)|$. By (2.2.a) and Theorem 3.1.9, we get

$$\mathrm{LE}_{\mathrm{CN}}(K_{|v(\Gamma_c(G))|}) - \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G)) = 2p((p-1)^2(p+1)z^2 - 2) > 0.$$

Hence, $\Gamma_c(G)$ is not CNL-hyperenergetic. Again, $\mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G)) = \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G))$. Therefore, $\Gamma_c(G)$ is also not CNSL-hyperenergetic. □

As a corollary of Theorem 3.2.6 we get the following result.

Corollary 3.2.7. *The commuting graph of G is not CNL(CNSL)-hyperenergetic if G is a finite*

(a) *non-abelian group of order p^3, where p is a prime.*

(b) 4-*centralizer group.*

(c) $(p+2)$-*centralizer* p-*group.*

(d) *non-abelian group such that the maximal size of the set of pairwise non-commuting elements of G is* 3 *or* 4.

Proof. Parts (a)-(d) are follow from Theorem 3.2.6. Under the hypothesis in part (e), G is either 4-centralizer or 5-centralizer finite group. Hence, the result follows from parts (b) and (c). □

Theorem 3.2.8. *Let G be a finite group with $\frac{G}{Z(G)} \cong D_{2m}$, where $m \geq 3$ and $|Z(G)| = z$. Then the commuting graph of G is*

(a) *not CNL-hyperenergetic if $m = 3 \,\&\, z \leq 6$; $m = 4 \,\&\, z = 2, 3$; $m = 5 \,\&\, z = 1, 2$; and $m = 7, 9 \,\&\, z = 1$. Otherwise, it is CNL-hyperenergetic.*

(b) *not CNSL-hyperenergetic if $m = 3 \,\&\, 1 \leq z \leq 7$; $m = 4 \,\&\, z = 2, 3$; $m = 5 \,\&\, z = 1, 2$; $m = 6 \,\&\, z = 2$ and $m = 7, 9, 11 \,\&\, z = 1$. Otherwise, it is CNSL-hyperenergetic.*

Proof. We have $|v(\Gamma_c(G))| = 2mz - z$, where $z = |Z(G)|$.

By (2.2.a) and Theorem 3.1.13, we get

$$\mathrm{LE_{CN}}(K_{|v(\Gamma_c(G))|}) - \mathrm{LE_{CN}}(\Gamma_c(G)) = -\frac{2(m-1)mz(z(m^2z - 2m(z+5)+8)+9)}{2m-1}.$$

It can be seen that $\mathrm{LE_{CN}}(K_{|v(\Gamma_c(G))|}) - \mathrm{LE_{CN}}(\Gamma_c(G)) > 0$ if if $m = 3$ and $z \leq 6$; $m = 4$ and $z = 2, 3$; $m = 5$ and $z = 1, 2$; and $m = 7, 9$ and $z = 1$. Otherwise $\mathrm{LE_{CN}}(K_{|v(\Gamma_c(G))|}) - \mathrm{LE_{CN}}(\Gamma_c(G)) < 0$. Hence, $\Gamma_c(G)$ is not CNL-hyperenergetic if $m = 3$ and $z \leq 6$; $m = 4$ and $z = 2, 3$; $m = 5$ and $z = 1, 2$; and $m = 7, 9$ and $z = 1$. Otherwise it is CNL-hyperenergetic.

Now we determine whether $\Gamma_c(G)$ is CNSL-hyperenergetic by considering the following cases.

Case 1. $m = 3 \,\&\, z = 1$

By (2.2.a) and Theorem 3.1.13, we get

$$\mathrm{LE^+_{CN}}(K_{|v(\Gamma_c(G))|} - \mathrm{LE^+_{CN}}(\Gamma_c(G)) = 24 > 0.$$

Therefore, $\Gamma_c(G)$ is not CNSL-hyperenergetic.

Case 2. $m = 3 \,\&\, z \geq 2$; $m = 4 \,\&\, z \geq 2$; and $m \geq 5 \,\&\, z \geq 1$

By (2.2.a) and Theorem 3.1.13, we get

$$\begin{aligned}\mathrm{LE^+_{CN}}&(K_{|v(\Gamma_c(G))|} - \mathrm{LE^+_{CN}}(\Gamma_c(G)) \\ &= \frac{2(2m-1)((2m-1)z-2)((2m-1)z-1) - 2(m-2)(m-1)mz^2(mz-3)}{2m-1}.\end{aligned}$$

It can be seen that $\mathrm{LE^+_{CN}}(K_{|v(\Gamma_c(G))|} - \mathrm{LE^+_{CN}}(\Gamma_c(G)) > 0$ if $m = 3$ and $2 \leq z \leq 7$; $m = 4$ and $z = 2, 3$; $m = 5$ and $z = 1, 2$; $m = 6$ and $z = 2$; and $m = 7, 9, 11$ and $z = 1$. Otherwise $\mathrm{LE^+_{CN}}(K_{|v(\Gamma_c(G))|} - \mathrm{LE^+_{CN}}(\Gamma_c(G)) < 0$. Hence, $\Gamma_c(G)$ is not CNSL-hyperenergetic if $m = 3$ and $2 \leq z \leq 7$; $m = 4$ and $z = 2, 3$; $m = 5$ and $z = 1, 2$; $m = 6$ and $z = 2$; and $m = 7, 9, 11$ and $z = 1$. Otherwise it is CNSL-hyperenergetic. Hence, the result follows. □

As a consequences of Theorem 3.2.8 we get the following results.

Corollary 3.2.9. *Suppose that G is isomorphic to the group $U_{(n,m)}$, where $m \geq 3$ and $n \geq 2$.*

(a) *If m is even then*

(i) $\Gamma_c(G)$ *is not CNL-hyperenergetic whenever* $m = 4 \,\&\, n \geq 2$; $m = 6 \,\&\, n = 2, 3$. *Otherwise,* $\Gamma_c(G)$ *is CNL-hyperenergetic.*

(ii) $\Gamma_c(G)$ *is not CNSL-hyperenergetic whenever* $m = 4 \,\&\, n \geq 2$; $m = 6 \,\&\, n = 2, 3$. *Otherwise,* $\Gamma_c(G)$ *is CNSL-hyperenergetic.*

(b) *If* m *is odd then*

(i) $\Gamma_c(G)$ *is not CNL-hyperenergetic whenever* $m = 3 \,\&\, n \leq 6$; $m = 5 \,\&\, n = 2$. *Otherwise,* $\Gamma_c(G)$ *is CNL-hyperenergetic.*

(ii) $\Gamma_c(G)$ *is not CNSL-hyperenergetic whenever* $m = 3 \,\&\, n \leq 7$. *Otherwise,* $\Gamma_c(G)$ *is CNSL-hyperenergetic.*

Corollary 3.2.10. *Suppose that* G *is isomorphic to the dihedral group* D_{2m}, *where* $m \geq 3$.

(a) *If* m *is even then*

(i) $\Gamma_c(G)$ *is not CNL-hyperenergetic whenever* $4 \leq m \leq 10$. *Otherwise,* $\Gamma_c(G)$ *is CNL-hyperenergetic.*

(ii) $\Gamma_c(G)$ *is not CNSL-hyperenergetic whenever* $4 \leq m \leq 12$. *Otherwise,* $\Gamma_c(G)$ *is CNSL-hyperenergetic.*

(b) *If* m *is odd then*

(i) $\Gamma_c(G)$ *is not CNL-hyperenergetic whenever* $3 \leq m \leq 9$. *Otherwise,* $\Gamma_c(G)$ *is CNL-hyperenergetic.*

(ii) $\Gamma_c(G)$ *is not CNSL-hyperenergetic whenever* $3 \leq m \leq 11$. *Otherwise,* $\Gamma_c(G)$ *is CNSL-hyperenergetic.*

Corollary 3.2.11. *Let* G *be a finite non-abelian group.*

(a) *If* G *is isomorphic to the dicyclic group* Q_{4n}, *where* $n \geq 2$ *then*

(i) $\Gamma_c(G)$ *is not CNL-hyperenergetic when* $n = 2, 3, 4, 5$. *Otherwise,* $\Gamma_c(G)$ *is CNL-hyperenergetic.*

(ii) $\Gamma_c(G)$ *is not CNSL-hyperenergetic when* $n = 2, 3, 4, 5, 6$. *Otherwise,* $\Gamma_c(G)$ *is CNSL-hyperenergetic.*

(b) *If G is isomorphic to U_{6n}, where $n \geq 2$ then*

(i) $\Gamma_c(G)$ *is not CNL-hyperenergetic when* $n \leq 6$. *Otherwise,* $\Gamma_c(G)$ *is CNL-hyperenergetic.*

(ii) $\Gamma_c(G)$ *is not CNSL-hyperenergetic when* $n \leq 7$. *Otherwise,* $\Gamma_c(G)$ *is CNSL-hyperenergetic.*

As a consequence of Result 1.2.2, Theorem 3.2.8 and Theorem 3.2.6 we have the following result.

Corollary 3.2.12. *If G is a 5-centralizer group then*

(a) $\Gamma_c(G)$ *is not CNL(CNSL)-hyperenergetic whenever* $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

(b) $\Gamma_c(G)$ *is not CNL-hyperenergetic for* $|Z(G)| \leq 6$*; otherwise CNL-hyperenergetic and not CNSL-hyperenergetic for* $|Z(G)| \leq 7$*; otherwise CNSL-hyperenergetic whenever* $\frac{G}{Z(G)} \cong D_6$.

We conclude this section with the following proposition.

Proposition 3.2.13. *If G is one of the groups considered in Propositions 3.1.4-3.1.5 then commuting graph of G is not CNL(CNSL)-hyperenergetic.*

3.3 Comparison of various CN-energies

In this section, we compare CN-energy, CNL-energy and CNSL-energy of commuting graphs of the groups considered in Section 3.1. We begin with the commuting graphs of the QD_{2^n}.

Theorem 3.3.1. *For the group QD_{2^n} (where $n \geq 4$),* $\mathrm{E_{CN}}(\Gamma_c(QD_{2^n})) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(QD_{2^n})) < \mathrm{LE_{CN}}(\Gamma_c(QD_{2^n}))$.

Proof. Using Proposition 3.1.1 and Result 1.2.15(a), we have

$$\mathrm{LE_{CN}}(\Gamma_c(QD_{2^n})) - \mathrm{E_{CN}}(\Gamma_c(QD_{2^n})) = \frac{2^{n-3}\,(2^n-8)\,(2^n-6)^2}{2^n-2} > 0,$$

since $(2^n - 8) > 0$, $(2^n - 6) > 0$ and $2^n - 2 > 0$, as $n \geq 4$. Therefore,

$$\mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) > \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})).$$

Also,

$$\mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) - \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) = \frac{(2^n - 8)\,(2^n - 6)\,(-2^{n+3} + 4^n + 8)}{8\,(2^n - 2)} > 0,$$

since $(2^n - 8) > 0$, $(2^n - 6) > 0$, $\left(-2^{n+3} + 4^n + 8\right) = 2^{2n} - 2^{n+3} + 8 > 0$, as $n \geq 4$ and $(2^n - 2) > 0$, as $n \geq 4$. Therefore,

$$\mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) > \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})).$$

Further,

$$\mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) - \mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) = -\frac{(2^n - 8)\,(2^n - 6)\,(2^n - 4)}{4\,(2^n - 2)} < 0,$$

since $(2^n - 8) > 0$, $(2^n - 6) > 0$, $(2^n - 4) > 0$ and $(2^n - 2) > 0$, as $n \geq 4$. Therefore,

$$\mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})) < \mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(QD_{2^n})).$$

Hence the result follows. □

Theorem 3.3.2. *For the group $PSL(2, 2^k)$ (where $k \geq 2$),* $\mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(PSL(2, 2^k))) < \mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(PSL(2, 2^k))) < \mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(PSL(2, 2^k)))$.

Proof. Using Proposition 3.1.2 and Result 1.2.15(b), we have

$$\begin{aligned}\mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(PSL(2, 2^k))) &- \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(PSL(2, 2^k)))\\ &= \frac{2^k\left(9 \times 2^k + 2^{2k+1} - 8^{k+1} + 3 \times 16^k - 3 \times 32^k + 64^k - 8\right)}{-2^k + 8^k - 1} > 0,\end{aligned}$$

since $9 \times 2^k + 2^{2k+1} - 8^{k+1} + 3 \times 16^k - 3 \times 32^k + 64^k - 8 = (9 \times 2^k - 8) + (3 \times 2^k \times 8^k - 8 \times 8^k) + (2^k \times 32^k - 3 \times 32^k) + 2^{2k+1} > 0$ and $-2^k + 8^k - 1 > 0$, as $k \geq 2$. Therefore,

$$\mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(PSL(2, 2^k))) > \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(PSL(2, 2^k))).$$

Also,

$$\begin{aligned}&\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) - \mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\\ &\qquad= \begin{cases} \frac{3832}{59}, & \text{for } k=2\\ \frac{2^k\left(5\times 2^k+7\times 2^{3k+1}-4^{k+1}+16^k-5\times 32^k+64^k+4\right)}{-2^k+8^k-1} > 0, & \text{for } k\geq 3,\end{cases}\end{aligned}$$

since $5\times 2^k+7\times 2^{3k+1}-4^{k+1}+16^k-5\times 32^k+64^k+4 = 5\times 2^k+7\times 2^{3k+1}+(4^k\times 4^k - 4\times 4^k)+(2^k\times 32^k-5\times 32^k)+4>0$, as $k\geq 3$. Therefore,

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) > \mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))).$$

Further,

$$\begin{aligned}&\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) - \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k)))\\ &\qquad= \begin{cases} -\frac{1528}{59}, & \text{for } k=2\\ -\frac{2^{k+1}\left(2^{k+1}+3\times 4^k-11\times 8^k+16^k+32^k-6\right)}{-2^k+8^k-1} < 0, & \text{for } k\geq 3\end{cases}\end{aligned}$$

since $2^{k+1}+3\times 4^k-11\times 8^k+16^k+32^k-6 = 2^{k+1}+3\times 4^k+4^k\times 8^k-11\times 8^k+16^k-6>0$. Therefore,

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(PSL(2,2^k))).$$

Hence the result follows. □

Theorem 3.3.3. *For the group $GL(2,q)$ where $q=p^n>2$ and p is prime*

$$\mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) < \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))).$$

Proof. Using Proposition 3.1.3 and Result 1.2.15(c), we have

$$\begin{aligned}&\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) - \mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q)))\\ &\qquad= \frac{q\left(2q^9-11q^8+15q^7+14q^6-31q^5-15q^4+12q^3+16q^2+8q+2\right)}{q^3-q-1}.\end{aligned}$$

Let $f_1(q) := 2q^9 - 11q^8 + 15q^7 + 14q^6 - 31q^5 - 15q^4 + 12q^3 + 16q^2 + 8q + 2$. Then $f_1(q) = (2q-11)q^8 + 12q^3 + 16q^2 + (15q^2 - 31)\, q^5 + (14q^2 - 15)\, q^4 + 8q + 2 > 0$, for $q \geq 6$. Again, $f_1(3) = 1952$; $f_1(4) = 71970$ and $f_1(5) = 895692$. Therefore,

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) > \mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))).$$

Also,

$$\begin{aligned}\mathrm{LE}^{+}_{\mathrm{CN}}&(\Gamma_{\mathrm{c}}(GL(2,q))) - \mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q)))\\ &= \begin{cases} \frac{q\left(2q^9-12q^8+22q^7+q^6-38q^5+6q^4+32q^3+9q^2-16q-2\right)}{q^3-q-1}, & \text{for } q \leq 5\\ \frac{q\left(2q^9-11q^8+15q^7+8q^6-23q^5+q^4+14q^3+4q^2-8q-2\right)}{q^3-q-1}, & \text{for } q \geq 6.\end{cases}\end{aligned}$$

Let $f_2(q) := 2q^9 - 12q^8 + 22q^7 + q^6 - 38q^5 + 6q^4 + 32q^3 + 9q^2 - 16q - 2$ and $f_3(q) := 2q^9 - 11q^8 + 15q^7 + 8q^6 - 23q^5 + q^4 + 14q^3 + 4q^2 - 8q - 2$. We have, $f_2(3) = 1624$; $f_2(4) = 67150$ and $f_2(5) = 842268$. Therefore, $f_2(q) > 0$. Again, $f_3(q) = (2q-11)q^8 + 8q^6 + q^4 + 14q^3 + (15q^2 - 23)\, q^5 + 4(q-2)q - 2 > 0$ for $q \geq 6$. Therefore,

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) > \mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))).$$

Further,

$$\begin{aligned}\mathrm{LE}^{+}_{\mathrm{CN}}&(\Gamma_{\mathrm{c}}(GL(2,q))) - \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q)))\\ &= \begin{cases} -\frac{q\left(q^8-7q^7+13q^6+7q^5-21q^4-20q^3+7q^2+24q+4\right)}{q^3-q-1}, & \text{for } q \leq 5\\ -\frac{2q\left(3q^6-4q^5-8q^4-q^3+6q^2+8q+2\right)}{q^3-q-1}, & \text{for } q \geq 6.\end{cases}\end{aligned}$$

Let $f_4(q) := q^8 - 7q^7 + 13q^6 + 7q^5 - 21q^4 - 20q^3 + 7q^2 + 24q + 4$ and $f_5(q) := 3q^6 - 4q^5 - 8q^4 - q^3 + 6q^2 + 8q + 2$. We have, $f_4(3) = 328$; $f_4(4) = 4820$ and $f_4(5) = 53424$. Therefore, $f_4(q) > 0$. Again, $f_5(q) = (q-4)q^5 + (q^3 - 1)\, q^3 + 6q^2 + (q^2 - 8)\, q^4 + 8q + 2 > 0$ for $q \geq 6$. Therefore,

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(GL(2,q))).$$

Hence the result follows. □

In view of Theorem 3.1.4-Theorem 3.1.5 and Result 1.2.16, we have the following results.

Theorem 3.3.4. *Let G be the Hanaki group $A(n,\nu)$ or $A(n,p)$. Then* $\mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(G)) = \mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(G)) = \mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(G))$.

Theorem 3.3.5. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong Sz(2)$ and $|Z(G)| = z$. Then*

$$\mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(G)) < \mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(G)) < \mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(G)).$$

Proof. Using Proposition 3.1.8 and Result 1.2.14(a), we have

$$\mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(G)) - \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(G)) = \begin{cases} \frac{40}{19}, & \text{for } z = 1 \\ \frac{20}{19}\left(42z^3 - 114z^2 + 90z - 19\right), & \text{for } z \geq 2. \end{cases}$$

Let $f_1(z) := 42z^3 - 114z^2 + 90z - 19$. Then $f_1(z) = z^2(42z - 114) + 90z - 19 > 0$, for $z \geq 3$. Again $f_1(2) = 41$. Therefore,

$$\mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(G)) > \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(G)).$$

Also,

$$\mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(G)) - \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(G)) = \frac{4}{19}\left(210z^3 - 312z^2 + 144z - 19\right).$$

Let $f_2(z) := 210z^3 - 312z^2 + 144z - 19$. Then $f_2(z) = (210z - 312)z^2 + 144z - 19 > 0$, for $z \geq 2$. Again $f_2(1) = 23$. Therefore,

$$\mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(G)) > \mathrm{E_{CN}}(\Gamma_{\mathrm{c}}(G)).$$

Further,

$$\mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(G)) - \mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(G)) = \begin{cases} \frac{52}{19}, & \text{for } z = 1 \\ \frac{8}{19}\left(z(129z - 153) + 38\right) > 0, & \text{for } z \geq 2. \end{cases}$$

Therefore,

$$\mathrm{LE^{+}_{CN}}(\Gamma_{\mathrm{c}}(G)) > \mathrm{LE_{CN}}(\Gamma_{\mathrm{c}}(G)).$$

Hence the result follows. □

In view of Theorem 3.1.9 and Result 1.2.14(b), we get the following result.

Theorem 3.3.6. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then* $\mathrm{E_{CN}}(\Gamma_c(G)) = \mathrm{LE_{CN}}(\Gamma_c(G)) = \mathrm{LE^+_{CN}}(\Gamma_c(G))$.

As a consequence we get the following corollary.

Corollary 3.3.7. (a) *If G is 4-centralizer group then* $\mathrm{E_{CN}}(\Gamma_c(G)) = \mathrm{LE_{CN}}(\Gamma_c(G)) = \mathrm{LE^+_{CN}}(\Gamma_c(G))$.

(b) *If G is $(p+2)$-centralizer group then* $\mathrm{E_{CN}}(\Gamma_c(G)) = \mathrm{LE_{CN}}(\Gamma_c(G)) = \mathrm{LE^+_{CN}}(\Gamma_c(G))$.

Theorem 3.3.8. *If G is a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, $m \geq 3$ then*

(a) *for $m = 3\,\&\, z = 1$,* $\mathrm{E_{CN}}(\Gamma_c(G)) = \mathrm{LE^+_{CN}}(\Gamma_c(G)) = \mathrm{LE_{CN}}(\Gamma_c(G))$.

(b) *for all other cases* $\mathrm{E_{CN}}(\Gamma_c(G)) < \mathrm{LE^+_{CN}}(\Gamma_c(G)) < \mathrm{LE_{CN}}(\Gamma_c(G))$.

Proof. Using Theorem 3.1.13 and Result 1.2.14(c), we have

$$\mathrm{LE_{CN}}(\Gamma_c(G)) - \mathrm{E_{CN}}(\Gamma_c(G)) = \frac{2m\left(m^3z^3 - 3m^2z^3 - 4m^2z^2 + 2mz^3 + 9mz^2 + 9mz - 4m - 5z^2 - 9z + 2\right)}{2m-1}.$$

Let $f_1(m,z) := m^3z^3 - 3m^2z^3 - 4m^2z^2 + 2mz^3 + 9mz^2 + 9mz - 4m - 5z^2 - 9z + 2$. Then $f_1(m,z) = \frac{1}{2}(m-6)m^2z^3 + \frac{1}{2}m^2z^2(mz-8) + (9m-5)z^2 + z\left(2mz^2 - 9\right) + m(9z-4) + 2 > 0$, for $m \geq 8$. We have, $f_1(3,z) = 12(z^2(3z-7) + 9z - 5)$. Clearly for $z \geq 3$, $f_1(3,z) > 0$. Again, $f_1(3,1) = 0$ and $f_1(3,2) = 108$. Therefore, $f_1(3,z) = 0$ or > 0 according as $z = 1$ or $z \geq 2$. We have $f_1(4,z) = 8(z^2(24z-33) + 27z - 14) > 0$, for $z \geq 2$. Also we have $f_1(5,z) = 60(10z^2(z-1) + 6z - 3) > 0$; $f_1(6,z) = 12(z^2(120z - 95) + 45z - 22) > 0$ and $f_1(7,z) = 28(z^2(105z - 69) + 27z - 13) > 0$, as $z \geq 1$. Therefore, $f_1(m,z) = 0$ or > 0 according as $m = 3\,\&\, z = 1$ or otherwise. Hence $\mathrm{LE_{CN}}(\Gamma_c(G)) \geq \mathrm{E_{CN}}(\Gamma_c(G))$ and equality holds if and only if $m = 3\,\&\, z = 1$. Also,

$$\mathrm{LE^+_{CN}}(\Gamma_c(G)) - \mathrm{E_{CN}}(\Gamma_c(G)) = \begin{cases} 0, & \text{for } m = 3\,\&\, z = 1 \\ \frac{2\left(m^4z^3 - 3m^3z^3 - 5m^3z^2 + 2m^2z^3 + 12m^2z^2 + 12m^2z - 4m^2 - 9mz^2 - 12mz - 2m + z^2 + 3z + 2\right)}{2m-1}, & \text{otherwise.} \end{cases}$$

Let $f_2(m,z) := m^4z^3 - 3m^3z^3 - 5m^3z^2 + 2m^2z^3 + 12m^2z^2 + 12m^2z - 4m^2 - 9mz^2 - 12mz - 2m + z^2 + 3z + 2$. Then $f_2(m,z) = \frac{1}{2}(m-6)m^3z^3 + \frac{1}{2}m^3z^2(mz-10) + m^2\left(z^3-4\right) + m\left(mz^3-2\right) + (12m-9)mz^2 + 12(m-1)mz + z^2 + 3z + 2 > 0$, for $m \geq 6$ and $z \geq 2$. Now we have $f_2(m,1) = (m-3)(m-2)((m-3)m+1) > 0$, as in this case $m \geq 5$. Now we need to check for $m \geq 3$ and $z \geq 2$. We have $f_2(3,z) = (z-1)(z(18z-35)+40) > 0$; $f_2(4,z) = z(z(96z-163)+147) - 70 > 0$; $f_2(5,z) = 3(z^2(100z-123)+81z-36) > 0$. Therefore, $f_2(m,z) > 0$. Hence, $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) \geq \mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))$ and equality holds if and only if $m = 3 \,\&\, z = 1$. Further,

$$\begin{aligned}&\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) - \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))\\ &\qquad = \begin{cases} 0, & \text{for } m = 3 \,\&\, z = 1\\ \frac{2\left(-m^3z^2+3m^2z^2+3m^2z-4mz^2-3mz-4m+z^2+3z+2\right)}{2m-1}, & \text{otherwise.}\end{cases}\end{aligned}$$

Let $f_3(m,z) := -m^3z^2 + 3m^2z^2 + 3m^2z - 4mz^2 - 3mz - 4m + z^2 + 3z + 2$. Then $f_3(m,z) = -\frac{1}{2}(m-6)m^2z^2 - \frac{1}{2}m^2z(mz-6) - (4m-1)z^2 - (2m-3)z - (mz-2) - 4m < 0$, for $m \geq 6$. Now we have $f_3(3,z) = -z(11z-21) - 10 < 0$ as in this case $z \geq 2$; $f_3(4,z) = -z(31z-39) - 14 < 0$ as in this case $z \geq 2$ and $f_3(5,z) = -z(69z-63) - 18 < 0$ as $z \geq 1$. Therefore, $f_3(m,z) < 0$. Hence $\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) \leq \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))$ and equality holds if and only if $m = 3 \,\&\, z = 1$.

Therefore, the result follows. □

Corollary 3.3.9. *Let G be a 5-centralizer group. Then*

(a) $\mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))$ *whenever* $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

(b) $\mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) < \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))$ *whenever* $\frac{G}{Z(G)} \cong D_6$.

Corollary 3.3.10. *Let G be $U_{(n,m)}$, D_{2m}, U_{6n} and Q_{4n}. Then*

(a) $\mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) = \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))$ *if and only if* $G \cong U_{(n,4)}$ *or* D_6 *or* D_8 *or* Q_8.

(b) $\mathrm{E}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) < \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_{\mathrm{c}}(G))$ *if and only if* $G \cong U_{(n,m)}$ *(for $m \neq 4$) or D_{2m} (for $m \neq 3, 4$) or U_{6n} (for $n \geq 2$) or Q_{4n} (for $n \neq 2$).*

If CN-energy, CNL-energy and CNSL-energy of $\Gamma_c(G)$ are not equal then the closeness among them are not clear from the above results. In the following figures we describe the closeness among CN-energy, CNL-energy and CNSL-energy of commuting graphs of D_{2m}, QD_{2^n}, $PSL(2, 2^k)$, $GL(2, q)$, Q_{4n}, U_{6n} and G such that $\frac{G}{Z(G)} \cong Sz(2)$ graphically.

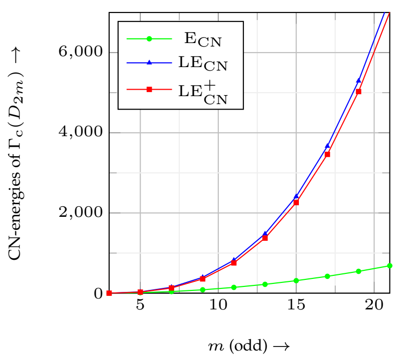

Figure 3.1: CN-energies of $\Gamma_c(D_{2m})$, m is odd

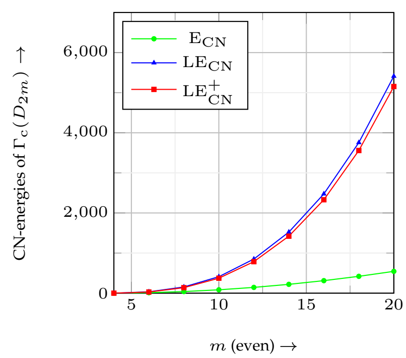

Figure 3.2: CN-energies of $\Gamma_c(D_{2m})$, m is even

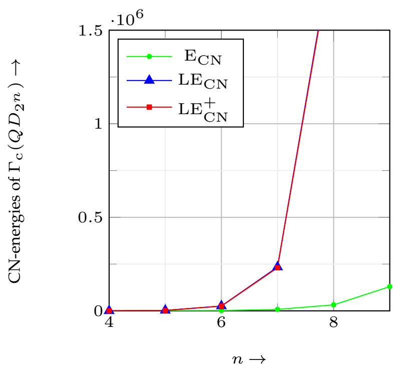

Figure 3.3: CN-energies of $\Gamma_c(QD_{2^n})$, $(n \geq 4)$

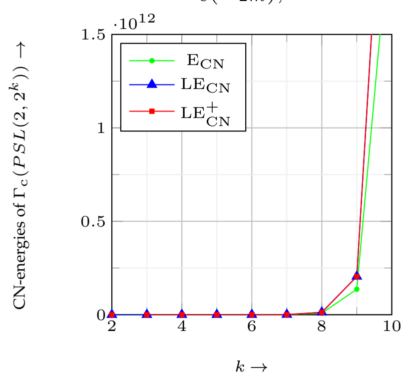

Figure 3.4: CN-energies of $\Gamma_c(PSL(2,2^k))$, $(k \geq 2)$

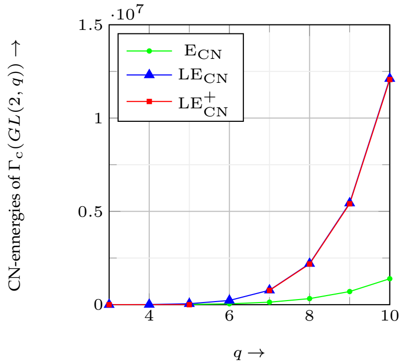

Figure 3.5: CN-energies of $\Gamma_c(GL(2,q))$, $(q > 2)$

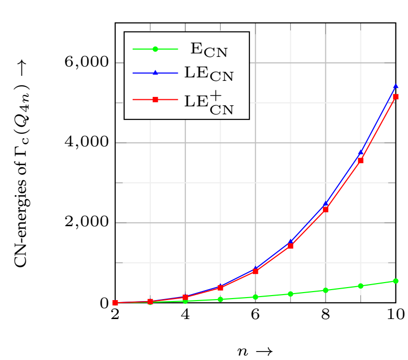

Figure 3.6: CN-energies of $\Gamma_c(Q_{4n})$, $(n \geq 2)$

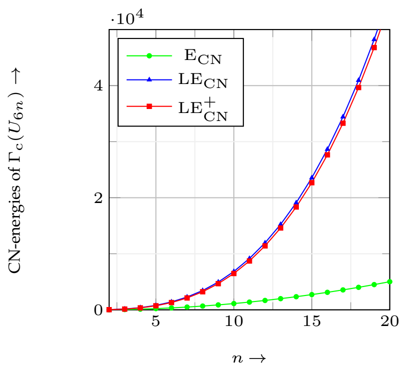

Figure 3.7: CN-energies of $\Gamma_c(U_{6n})$, $(n \geq 2)$

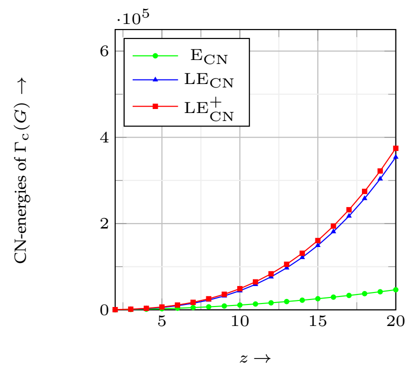

Figure 3.8: CN-energies of $\Gamma_c(G)$ $(\frac{G}{Z(G)} \cong Sz(2))$

In [40], Dutta et al. compared $\mathrm{E}(\Gamma_c(G))$, $\mathrm{LE}(\Gamma_c(G))$ and $\mathrm{LE}^+(\Gamma_c(G))$ for $G = D_{2m}$, QD_{2^n}, $PSL(2, 2^k)$, $GL(2,q)$, Q_{4n}, $U_{(n,m)}$, U_{6n}, $A(n,\nu)$, $A(n,p)$ and G such that $\frac{G}{Z(G)} \cong Sz(2)$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. We observe that $\mathrm{E}(\Gamma_c(G))$, $\mathrm{LE}(\Gamma_c(G))$, $\mathrm{LE}^+(\Gamma_c(G))$ and $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G))$, $\mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G))$, $\mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G))$ satisfy similar (sometimes different) inequalities for the above mentioned groups. Comparing our results and the results obtained in [40], we get the following results.

Theorem 3.3.11. *Let $G = D_{2m}$, Q_{4n}, $U_{(n,m)}$, $A(n,\nu)$, $A(n,p)$ or G is any group such that $\frac{G}{Z(G)} \cong Sz(2)$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. Then*

(a) $\mathrm{E}(\Gamma_c(G)) = \mathrm{LE}(\Gamma_c(G)) = \mathrm{LE}^+(\Gamma_c(G))$ *as well as* $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G)) = \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G))$ *if and only if $G = D_8$, Q_8, $U_{(n,4)}$, $A(n,\nu)$, $A(n,p)$ and G such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

(b) $\mathrm{E}(\Gamma_c(G)) < \mathrm{LE}^+(\Gamma_c(G)) < \mathrm{LE}(\Gamma_c(G))$ *as well as* $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G))$ *if and only if* $G = D_{2m}\,(m \text{ (odd) } \geq 5; m \text{ (even) } \geq 8)$, $Q_{4m}\,(m \geq 4)$, $U_{6n}\,(n = 3, 4)$ *and* $U_{(n,m)}\,(m \neq 3, 4\,\&\, n \geq 2;\ m = 6\,\&\, n \geq 3)$.

(c) $\mathrm{E}(\Gamma_c(G)) < \mathrm{LE}(\Gamma_c(G)) < \mathrm{LE}^+(\Gamma_c(G))$ *as well as* $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G))$ *if and only if G such that* $\frac{G}{Z(G)} \cong Sz(2)\,(|Z(G)| \geq 2)$.

Theorem 3.3.12. *Let* $G = D_{2m}$, Q_{4n}, $U_{(n,m)}$, QD_{2^n}, $PSL(2, 2^k)$, $GL(2, q)$ *or G is any group such that* $\frac{G}{Z(G)} \cong Sz(2)$. *Then*

(a) $\mathrm{E}(\Gamma_c(G)) < \mathrm{LE}(\Gamma_c(G)) = \mathrm{LE}^+(\Gamma_c(G))$ and $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G))$ if and only if $G = U_{(5,3)}$.

(b) $\mathrm{E}(\Gamma_c(G)) < \mathrm{LE}(\Gamma_c(G)) = \mathrm{LE}^+(\Gamma_c(G))$ and $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) = \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G))$ if and only if $G = D_6$.

(c) $\mathrm{E}(\Gamma_c(G)) < \mathrm{LE}(\Gamma_c(G)) = \mathrm{LE}^+(\Gamma_c(G))$ and $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G))$ if and only if $G = U_{30}$.

(d) $\mathrm{LE}^+(\Gamma_c(G)) < \mathrm{E}(\Gamma_c(G)) < \mathrm{LE}(\Gamma_c(G))$ and $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G))$ if and only if $G = U_{(2,3)}$, D_{12}, Q_{12}, $PSL(2, 2^k)\,(k \geq 2)$ and $GL(2, q)\,(q = p^3)$.

(e) $\mathrm{LE}^+(\Gamma_c(G)) < \mathrm{E}(\Gamma_c(G)) < \mathrm{LE}(\Gamma_c(G))$ and $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G))$ if and only if G such that $\frac{G}{Z(G)} \cong Sz(2)\,(|Z(G))| = 1)$.

(f) $\mathrm{E}(\Gamma_c(G)) < \mathrm{LE}(\Gamma_c(G)) < \mathrm{LE}^+(\Gamma_c(G))$ and $\mathrm{E}_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma_c(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma_c(G))$ if and only if $G = U_{6n}\,(n \geq 6)$, $QD_{2^n}\,(n \geq 4)$ and $GL(2, q)\,(q = p^n,\ n \neq 3)$, $U_{(2,6)}$.