Chapter 4

Various CN-spectra and energies of subgraphs of CCC-graphs

In this chapter, we consider the subgraph $\Gamma_{\text{ccc}}^{*}(G)$ of CCC-graphs $\Gamma_{\text{ccc}}(G)$ of finite non-abelian groups G induced by $\text{Cl}(G \setminus Z(G))$. In Sections 4.1–4.2, we shall compute CN-spectrum, CNL-spectrum and CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of $\Gamma_{\text{ccc}}^{*}(G)$, where G is group whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) and the dihedral group D_{2m} ($m \geq 3$) respectively. In Section 4.3, we shall determine whether $\Gamma_{\text{ccc}}^{*}(G)$, where G is the above mentioned groups, are CN-integral, CNL-integral, CNSL-integral, CN-hyperenergetic, CNL-hyperenergetic, CNSL-hyperenergetic, CN-borderenergetic, CNL-borderenergetic and CNSL-borderenergetic. As a consequence, we characterize the groups $G = D_{2m}$, Q_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n} such that $\Gamma_{\text{ccc}}^{*}(G)$ is CN-, CNL-, CNSL-integral/hyperenergetic/borderenergetic. In Section 4.4, we shall compare various CN-energies of $\Gamma_{\text{ccc}}^{*}(G)$ for the groups G considered in Sections 4.1–4.2 and show that $\text{E}_{\text{CN}}(\Gamma_{\text{ccc}}^{*}(G)) = \text{LE}_{\text{CN}}(\Gamma_{\text{ccc}}^{*}(G)) = \text{LE}_{\text{CN}}^{+}(\Gamma_{\text{ccc}}^{*}(G))$ or $\text{E}_{\text{CN}}(\Gamma_{\text{ccc}}^{*}(G)) < \text{LE}_{\text{CN}}^{+}(\Gamma_{\text{ccc}}^{*}(G)) < \text{LE}_{\text{CN}}(\Gamma_{\text{ccc}}^{*}(G))$. We shall also characterize the groups $G = D_{2n}$, Q_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n} such that $\Gamma_{\text{ccc}}^{*}(G)$ satisfy above mentioned equality/inequality. For the groups satisfying the inequality $\text{E}_{\text{CN}}(\Gamma_{\text{ccc}}^{*}(G)) < \text{LE}_{\text{CN}}^{+}(\Gamma_{\text{ccc}}^{*}(G)) < \text{LE}_{\text{CN}}(\Gamma_{\text{ccc}}^{*}(G))$, the closeness of various CN-energies of $\Gamma_{\text{ccc}}^{*}(G)$ are depicted graphically in Figures 4.1 – 4.8. This chapter is based on our papers [67] published in *Journal of Algebraic Systems* and [68] published in *Boletim da Sociedade Paranaense de Matemática*.

4.1 Groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$

The structure of $\Gamma^*_{\text{ccc}}(G)$ for this class of groups have been obtained by Salahshour and Ashrafi (see Result 1.2.17). Various spectra and energies based on the adjacency matrix of $\Gamma^*_{\text{ccc}}(G)$ of this class of groups have been obtained in [16]. The commuting and non-commuting graphs of this class of groups are also studied in [40, 41, 43, 44, 45, 49, 47, 80]. In the following theorem, we derive CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of $\Gamma^*_{\text{ccc}}(G)$ for this class of groups.

Theorem 4.1.1. *Let G be a finite non-abelian group with $|Z(G)| = z \geq 2$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of $\Gamma^*_{\text{ccc}}(G)$ are given by*

(a) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(G)) = \left\{[-(n-2)]^{(p+1)(n-1)}, [(n-1)(n-2)]^{(p+1)}\right\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = 2(p+1)(n-1)(n-2).$$

(b) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(G)) = \left\{[0]^{p+1}, \left[\frac{1}{p^2}(pz-z)(pz-z-2p)\right]^{\frac{(p+1)}{p}(pz-z-p)}\right\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \begin{cases} \frac{3}{2}, & \text{for } p = 2 \,\&\, z = 3 \\ \frac{4(p-2)(p+1)}{p^2}, & \text{for } p \geq 2 \,\&\, z = 2 \\ \frac{2(p+1)(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{otherwise.} \end{cases}$$

(c) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(G)) =$
$\left\{\left[\frac{2}{p^2}(pz-z-p)(pz-z-2p)\right]^{p+1}, \left[\frac{1}{p^2}(pz-z-2p)^2\right]^{\frac{(p+1)}{p}(pz-z-p)}\right\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \begin{cases} \frac{3}{2}, & \text{for } p = 2 \,\&\, z = 3 \\ \frac{4(p-2)(p+1)}{p^2}, & \text{for } p \geq 2 \,\&\, z = 2 \\ \frac{2(p+1)(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{otherwise.} \end{cases}$$

Proof. From Result 1.2.17, we have $\Gamma^*_{\text{ccc}}(G) = (p+1)K_n$, where $n = \frac{(p-1)z}{p}$.
(a) By Theorem 1.1.20, we get CN-spec$(\Gamma^*_{\text{ccc}}(G)) = \Big\{[-(n-2)]^{(p+1)(n-1)}$, $[(n-1)(n-2)]^{(p+1)}\Big\}$ and $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = 2(p+1)(n-1)(n-2)$.

(b) By Theorem 2.2.2, we get

$$\text{CNL-spec}(\Gamma^*_{\text{ccc}}(G)) = \left\{[0]^{p+1}, \left[\frac{1}{p^2}(pz-z)(pz-z-2p)\right]^{\frac{(p+1)}{p}(pz-z-p)}\right\}.$$

Here $|v(\Gamma^*_{\text{ccc}}(G))| = \frac{(p^2-1)z}{p}$ and $\text{tr}(\text{CNRS}(\Gamma^*_{\text{ccc}}(G))) = \frac{(p-1)(p+1)z(p(z-2)-z)(p(z-1)-z)}{p^3}$. Therefore, $\Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \frac{(p(z-2)-z)(p(z-1)-z)}{p^2}$.

We have

$$L_1 := |0 - \Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))| = \left|-\frac{(p(z-2)-z)(p(z-1)-z)}{p^2}\right|.$$

Let $\alpha_1(p,z) = -(p(z-2)-z)(p(z-1)-z)$. Then $\alpha_1(p,z) = -2p^2 - 3pz - z^2 + \frac{1}{2}p^2z(6-z) + \frac{1}{2}pz^2(4-p) < 0$ for $p \geq 4$ and $z \geq 6$. It can be seen that $\alpha_1(2,z) = -z(z-6) - 8 = 1$ or ≤ 0 according as $z = 3$ or $z \neq 3$; $\alpha_1(3,z) = -2(z-3)(2z-3) = 2$ or ≤ 0 according as $z = 2$ or $z \neq 2$; $\alpha_1(p,2) = 2p - 4 \geq 0$; $\alpha_1(p,3) = -2p^2 + 9p - 9 = 1$ or ≤ 0 according as $p = 2$ or $p \neq 2$; $\alpha_1(p,4) = -6p^2 + 20p - 16 \leq 0$ and $\alpha_1(p,5) = -12p^2 + 35p - 25 \leq 0$. Therefore,

$$L_1 = \begin{cases} -\frac{(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{for } p = 2 \,\&\, z = 3; p \geq 2 \,\&\, z = 2 \\ \frac{(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{otherwise.} \end{cases}$$

Also,

$$L_2 := \left|\frac{(pz-z)(pz-2p-z)}{p^2} - \Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))\right| = \left|\frac{pz-2p-z}{p}\right|.$$

Let $\alpha_2(p,z) = pz - 2p - z$. Then $\alpha_2(p,z) = (z-2)p - z \geq z - 4 \geq 0$ for all $z \geq 4$ since $p \geq 2$. It can be seen that $\alpha_2(p,2) = -2 < 0$ and $\alpha_2(p,3) = p - 3 \geq 0$ or < 0 according as $p \geq 3$ or $p = 2$. Therefore,

$$L_2 = \begin{cases} -\frac{pz-2p-z}{p}, & \text{for } p = 2 \,\&\, z = 3; p \geq 2 \,\&\, z = 2 \\ \frac{pz-2p-z}{p}, & \text{otherwise.} \end{cases}$$

Hence, by (2.1.a), we get

$$\begin{aligned}\mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) &= (p+1)\times L_1 + \frac{p+1}{p}(pz-z-p)\times L_2\\ &= \begin{cases} \frac{3}{2}, & \text{for } p=2\,\&\,z=3\\ \frac{4(p-2)(p+1)}{p^2}, & \text{for } p\geq 2\,\&\,z=2\\ \frac{2(p+1)(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{otherwise.}\end{cases}\end{aligned}$$

(c) By Theorem 2.2.2, we get

$$\mathrm{CNSL\text{-}spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \left\{\left[\frac{2}{p^2}(pz-z-p)(pz-z-2p)\right]^{p+1}, \left[\frac{1}{p^2}(pz-z-2p)^2\right]^{(p+1)\frac{1}{p}(pz-z-p)}\right\}.$$

We have

$$\begin{aligned}B_1 &:= \left|\frac{2(pz-p-z)(pz-2p-z)}{p^2} - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))\right| = \left|\frac{(pz-2p-z)(pz-p-z)}{p^2}\right|\\ &= -L_1 = \begin{cases} -\frac{(pz-2p-z)(pz-p-z)}{p^2}, & \text{for } p=2\,\&\,z=3; p\geq 2\,\&\,z=2\\ \frac{(pz-2p-z)(pz-p-z)}{p^2}, & \text{otherwise.}\end{cases}\end{aligned}$$

Also,

$$\begin{aligned}B_2 &:= \left|\frac{(pz-2p-z)^2}{p^2} - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))\right| = \left|\frac{-pz+2p+z}{p}\right| = -L_2\\ &= \begin{cases} \frac{2p+z-pz}{p}, & \text{for } p=2\,\&\,z=3; p\geq 2\,\&\,z=2\\ -\frac{2p+z-pz}{p}, & \text{otherwise.}\end{cases}\end{aligned}$$

Hence, by (2.1.b), we get

$$\begin{aligned}\mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) &= (p+1)\times B_1 + \frac{p+1}{p}(pz-z-p)\times B_2\\ &= \begin{cases} \frac{3}{2}, & \text{for } p=2\,\&\,z=3\\ \frac{4(p-2)(p+1)}{p^2}, & \text{for } p\geq 2\,\&\,z=2\\ \frac{2(p+1)(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{otherwise.}\end{cases}\end{aligned}$$

Hence the result follows. □

As a corollary of the above theorem we get the following result.

Corollary 4.1.2. *Let G be a non-abelian group of order p^n with $|Z(G)| = p^{n-2}$. Then CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of $\Gamma^*_{\text{ccc}}(G)$ are given by*

(a) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(G)) = \{[-(p^{n-2} - p^{n-3} - 2)]^{(p+1)(p^{n-2}-p^{n-3}-1)},$
$\left[(p^{n-2} - p^{n-3} - 1)(p^{n-2} - p^{n-3} - 2)\right]^{(p+1)}\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = 2(p+1)(p^{n-2} - p^{n-3} - 1)(p^{n-2} - p^{n-3} - 2).$$

(b) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(G)) = \left\{[0]^{p+1}, \left[\frac{1}{p^2}(p^{n-1} - p^{n-2})(p^{n-1} - p^{n-2} - 2p)\right]^{\frac{(p+1)}{p}(p^{n-1}-p^{n-2}-p)}\right\}$
and

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \begin{cases} 0, & \text{for } p = 2 \,\&\, n = 3 \\ \frac{2(p+1)\left(p\left(p^{n-2}-2\right)-p^{n-2}\right)\left(p\left(p^{n-2}-1\right)-p^{n-2}\right)}{p^2}, & \text{otherwise.} \end{cases}$$

(c) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(G)) = \left\{ \left[\frac{2}{p^2}(p^{n-1} - p^{n-2} - p)(p^{n-1} - p^{n-2} - 2p)\right]^{p+1}, \right.$
$\left.\left[\frac{1}{p^2}(p^{n-1} - p^{n-2} - 2p)^2\right]^{\frac{(p+1)}{p}(p^{n-1}-p^{n-2}-p)}\right\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \begin{cases} 0, & \text{for } p = 2 \,\&\, n = 3 \\ \frac{2(p+1)\left(p\left(p^{n-2}-2\right)-p^{n-2}\right)\left(p\left(p^{n-2}-1\right)-p^{n-2}\right)}{p^2}, & \text{otherwise.} \end{cases}$$

Proof. Here $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 4.1.1. □

4.2 Groups whose central quotient is isomorphic to D_{2m}

The structure of $\Gamma^*_{\text{ccc}}(G)$ for this class of graphs have been obtained by Salahshour (see 1.2.19). Various spectra and energy based on adjacency matrix of $\Gamma^*_{\text{ccc}}(G)$ of this class of groups have been obtained in [16]. The spectrum, L-spectrum and Q-spectrum along with their respective energies of commuting and non-commuting graphs of this class of groups were studied in [40, 41, 43, 44, 45, 47]. In the following theorem, we derive various CN-spectra and CN-energies of $\Gamma^*_{\text{ccc}}(G)$ for this class of groups.

Theorem 4.2.1. *Let G be a finite group such that $|Z(G)| = z$ and $\frac{G}{Z(G)} \cong D_{2m}$ (where $m \geq 3$). Then CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of $\Gamma^*_{\mathrm{ccc}}(G)$ are as given below:*

(a) *If m is even then*

(i) $\text{CN-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \Big\{ \Big[-\Big(\frac{(m-1)z}{2} - 2\Big)\Big]^{\left(\frac{(m-1)z}{2}-1\right)}, \Big[\Big(\frac{(m-1)z}{2}\Big)^2 - \frac{3(m-1)z}{2} + 2\Big]^1,$
$[-(\frac{z}{2} - 2)]^{2\left(\frac{z}{2}-1\right)}, \Big[(\frac{z}{2})^2 - \frac{3z}{2} + 2\Big]^2 \Big\}$ *and*

$$\mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \frac{m^2z^2}{2} - mz^2 - 3mz + \frac{3z^2}{2} - 3z + 12.$$

(ii) $\text{CNL-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \Big\{[0]^3, [\frac{1}{4}(mz - z)(mz - z - 4)]^{\frac{1}{2}(mz-z-2)}, [\frac{1}{4}z(z-4)]^{z-2}\Big\}$ *and*

$$\mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \frac{((m-1)z - 2)(m(z+1)((m-2)z-4) + 11z - 4)}{2(m+1)}.$$

(iii)
$\text{CNSL-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \Big\{ \Big[\frac{1}{2}(mz - z - 2)(mz - z - 4)\Big]^1, \Big[\frac{1}{4}(mz - z - 4)^2\Big]^{\frac{1}{2}(mz-z-2)},$
$\Big[\frac{1}{2}(z-2)(z-4)\Big]^2, \Big[\frac{1}{4}(z-4)^2\Big]^{z-2} \Big\}$ *and*

$$\mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \begin{cases} \frac{28}{5}, & \textit{for } m = 4 \,\&\, z = 2 \\ \frac{3}{5}z^2(4z-6), & \textit{for } m = 4 \,\&\, z \geq 3 \\ \frac{(m-2)(m-1)z^2(mz-6)}{2(m+1)}, & \textit{otherwise.} \end{cases}$$

(b) *If m is odd then*

(i) $\text{CN-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \Big\{ \Big[-\Big(\frac{(m-1)z}{2} - 2\Big)\Big]^{\left(\frac{(m-1)z}{2}-1\right)}, \Big[\Big(\frac{(m-1)z}{2}\Big)^2 - \frac{3(m-1)z}{2} + 2\Big]^1,$
$[-(z-2)]^{(z-1)}, [z^2 - 3z + 2]^1 \Big\}$ *and*

$$\mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \frac{m^2z^2}{2} - mz^2 - 3mz + \frac{5z^2}{2} - 3z + 8.$$

(ii) $\text{CNL-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \Big\{[0]^2, \Big[\frac{1}{4}(mz - z)(mz - z - 4)\Big]^{\frac{1}{2}(mz-z-2)}, [z(z-2)]^{z-1}\Big\}$

and

$$\mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \begin{cases} 0, & \textit{for } m = 3 \,\&\, z = 1 \\ 4(z-1)(z-2), & \textit{for } m = 3 \,\&\, z \geq 2 \\ \frac{((m-1)z-2)\big((m-3)(m+1)z^2+((m-6)m+17)z-4(m+1)\big)}{2(m+1)}, & \textit{otherwise.} \end{cases}$$

(iii)

$$\text{CNSL-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \Big\{ \left[\tfrac{1}{2}(mz-z-2)(mz-z-4)\right]^1, \left[\tfrac{1}{4}(mz-z-4)^2\right]^{\frac{1}{2}(mz-z-2)}, [2(z-1)(z-2)]^1, [(z-2)^2]^{z-1} \Big\}$$

$$\textit{and } \mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \begin{cases} 0, & \textit{for } m = 3, 5 \,\&\, z = 1 \\ 4(z-1)(z-2), & \textit{for } m = 3 \,\&\, z \geq 2 \\ \frac{(m-5)(m-3)(m+3)}{2(m+1)}, & \textit{for } m \geq 7 \,\&\, z = 1 \\ \frac{(m-3)(m-1)z^2(mz+z-6)}{2(m+1)}, & \textit{otherwise.} \end{cases}$$

Proof. From Result 1.2.19, we have $\Gamma^*_{\mathrm{ccc}}(G) = K_{\frac{(m-1)z}{2}} \cup 2K_{\frac{z}{2}}$ or $K_{\frac{(m-1)z}{2}} \cup K_z$ according as m is even or odd.

(a)(i) If m is even then by Theorem 1.1.20, we get the required CN-spec($\Gamma^*_{\mathrm{ccc}}(G)$) and $\mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$.

(ii) If m is even then by Theorem 2.2.2, we get

$$\text{CNL-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \left\{ [0]^1, \left[\frac{(m-1)z}{2}\left(\frac{(m-1)z}{2} - 2\right)\right]^{\frac{(m-1)z}{2}-1}, [0]^2, \left[\frac{z}{2}\left(\frac{z}{2}-2\right)\right]^{2(\frac{z}{2}-1)} \right\}.$$

Here $|v(\Gamma^*_{\mathrm{ccc}}(G))| = \frac{1}{2}(m+1)z$ and $\mathrm{tr}(\mathrm{CNRS}(\Gamma_G)) = \frac{1}{8}z((m((m-3)m+3)+1)z^2 - 6((m-2)m+3)z + 8(m+1))$. So, $\Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \frac{(m((m-3)m+3)+1)z^2-6((m-2)m+3)z+8(m+1)}{4(m+1)}$. Note that $z \geq 2$. We have

$$L_1 := |0 - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))| = \left| -\frac{(m((m-3)m+3)+1)z^2 - 6((m-2)m+3)z + 8(m+1)}{4(m+1)} \right|.$$

Let $\alpha_1(m,z) = (m((m-3)m+3)+1)z^2 - 6((m-2)m+3)z + 8(m+1)$. Then $\alpha_1(m,z) = 8 + 8m + 6z(2m-3) + z^2 + 3mz^2 + \frac{m^2z}{2}(mz-12) + \frac{m^2z^2}{2}(m-6) > 0$ for $m \geq 12$. Also, $\alpha_1(4,z) = 29z^2 - 66z + 40 \geq 0$, $\alpha_1(6,z) = 127z^2 - 162z + 56 \geq 0$, $\alpha_1(8,z) = 345z^2 - 306z + 72 \geq 0$ and $\alpha_1(10,z) = 731z^2 - 498z + 88 \geq 0$. Therefore,

$$L_1 = \frac{(m((m-3)m+3)+1)z^2 - 6((m-2)m+3)z + 8(m+1)}{4(m+1)}.$$

We have

$$L_2 := \left|\frac{1}{4}(mz-z)(mz-z-4) - \Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))\right| = \left|\frac{m(z+1)((m-2)z-4)+11z-4}{2(m+1)}\right|.$$

Let $\alpha_2(m,z) = \{m(z+1)((m-2)z-4)+11z-4\}$. Then $\alpha_2(m,z) > 0$ for $m \geq 6$, since $m-2 \geq 4 \implies z(m-2)-4 \geq 0 \implies m(z+1)(z(m-2)-4) \geq 0$. Also, $\alpha_2(4,z) = 8z^2+3z-20 \geq 0$. Therefore,

$$L_2 = \frac{m(z+1)((m-2)z-4)+11z-4}{2(m+1)}.$$

We have

$$L_3 := \left|\frac{1}{4}z(z-4) - \Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))\right| = \left|\frac{14z-8-8m-mz(2(z+8)+m(-6+(m-3)z))}{4(m+1)}\right|.$$

Let $\alpha_3(m,z) = 14z-8-8m-mz(2(z+8)+m(-6+(m-3)z))$. For $m \geq 10$, $m((m-3)z-6) > 0$ and $2(z+8) > 0$. So, $\alpha_3(m,z) < 0$ for all $m \geq 10$. Also, $\alpha_3(4,z) = -24z^2+46z-40 \leq 0$, $\alpha_3(6,z) = -120z^2+134z-56 \leq 0$ and $\alpha_3(8,z) = -336z^2+270z-72 \leq 0$. Therefore,

$$L_3 = -\frac{14z-8-8m-mz(2(z+8)+m(-6+(m-3)z))}{4(m+1)}.$$

Hence, by (2.1.a), we get

$$\begin{aligned}\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) &= 3\times L_1 + \frac{1}{2}(mz-z-2)\times L_2 + (z-2)\times L_3\\ &= \frac{((m-1)z-2)(m(z+1)((m-2)z-4)+11z-4)}{2(m+1)}.\end{aligned}$$

(a)(iii) If m is even then by Theorem 2.2.2, we get

$$\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(G)) = \left\{\left[2\left(\frac{(m-1)z}{2}-1\right)\left(\frac{(m-1)z}{2}-2\right)\right]^1, \left[\left(\frac{(m-1)z}{2}-2\right)^2\right]^{\frac{(m-1)z}{2}-1},\right.$$
$$\left.\left[\frac{1}{2}(z-2)(z-4)\right]^2, \left[\frac{1}{4}(z-4)^2\right]^{z-2}\right\}.$$

Here $|v(\Gamma^*_{\text{ccc}}(G))| = \frac{1}{2}(m+1)z$ and $\text{tr}(\text{CNRS}(\Gamma_G) = \frac{1}{8}z((m((m-3)m+3)+1)z^2-6((m-2)m+3)z+8(m+1))$. So, $\Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \frac{(m((m-3)m+3)+1)z^2-6((m-2)m+3)z+8(m+1)}{4(m+1)}$. Note that $z \geq 2$. We have

$$B_1 := \left|\frac{1}{2}(mz-z-2)(mz-z-4) - \Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))\right|$$

$$= \left| \frac{\left(m\left(m^2+m-5\right)+1\right)z^2-6m(m+2)z+8m+30z+8}{4(m+1)} \right|.$$

Let $\beta_1(m,z) = (m(m^2+m-5)+1)z^2-6m(m+2)z+8m+30z+8$. Then $\beta_1(m,z) = 8+8m+30z+z^2+mz^2(m-5)+\frac{mz}{2}(m^2z-24)+\frac{m^2z}{2}(mz-12)$. Clearly for $m \geq 12$, $\beta_1(m,z) > 0$, as $m^2z-24 \geq 0$ and $mz-12 \geq 0$. It can be seen that $\beta_1(4,z) = 61z^2-114z+40 \geq 0$, $\beta_1(6,z) = 223z^2-258z+56 \geq 0$, $\beta_1(8,z) = 537z^2-450z+72 \geq 0$ and $\beta_1(10,z) = 1051z^2-690z+88 \geq 0$. Therefore,

$$B_1 = \frac{\left(m\left(m^2+m-5\right)+1\right)z^2-6m(m+2)z+8m+30z+8}{4(m+1)}.$$

We have

$$B_2 := \left|\frac{1}{4}(mz-z-4)^2 - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))\right| = \left|\frac{m((m-2)z^2-(m+6)z+4)+13z+4}{2(m+1)}\right|.$$

Let $\beta_2(m,z) = m((m-2)z^2-(m+6)z+4)+13z+4$. Then $\beta_2(m,z) = 4+4m+13z+\frac{mz}{3}(mz-18)+\frac{m^2z}{3}(z-3)+\frac{mz^2}{3}(m-6) > 0$ for $m \geq 6$ and $z \geq 3$. It can be seen that $\beta_2(4,z) = 8z^2-27z+20 = -2$ or ≥ 0 according as $z=2$ or $z \geq 3$ and $\beta_2(m,2) = 2m(m-8)+30 = -2$ or ≥ 0 according as $m=4$ or $m \geq 6$. Therefore,

$$B_2 = \begin{cases} \frac{1}{5}, & \text{for } m=4 \,\&\, z=2 \\ \frac{m((m-2)z^2-(m+6)z+4)+13z+4}{2(m+1)}, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} B_3 &:= \left|\frac{1}{2}(z-2)(z-4) - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))\right| \\ &= \left|\frac{-\left(\left(m^3-3m^2+m-1\right)z^2\right)+6(m-4)mz+8m+6z+8}{4(m+1)}\right|. \end{aligned}$$

Let $\beta_3(m,z) = -((m^3-3m^2+m-1)z^2)+6(m-4)mz+8m+6z+8$. Then $\beta_3(m,z) = 8(1-mz)+8m(1-z)+2z(3-4m)+z^2(1-m)+\frac{m^2z}{2}(12-mz)+\frac{m^2z^2}{2}(6-m) < 0$ for $m \geq 12$. It can be seen that $\beta_3(4,z) = -19z^2+6z+40 \leq 0$, $\beta_3(6,z) = -113z^2+78z+56 \leq 0$, $\beta_3(8,z) = -327z^2+198z+72 \leq 0$ and $\beta_3(10,z) = -709z^2+366z+88 \leq 0$. Therefore,

$$B_3 = -\frac{-\left(\left(m^3-3m^2+m-1\right)z^2\right)+6(m-4)mz+8m+6z+8}{4(m+1)}.$$

We have

$$B_4 := \left|\frac{1}{4}(z-4)^2 - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))\right| = \left|\frac{m(8-z(m((m-3)z-6)+2(z+10)))+10z+8}{4(m+1)}\right|.$$

Let $\beta_4(m,z) = m(8 - z(m((m-3)z-6) + 2(z+10))) + 10z + 8$. Then $\beta_4(m,z) = 8m - 10mz + 10z - 10mz + 8 - 2mz^2 + \frac{m^2z}{2}(12 - mz) + \frac{m^2z^2}{2}(6-m) < 0$ for $m \geq 12$. It can be seen that $\beta_4(4,z) = -24z^2+26z+40 \leq 0$, $\beta_4(6,z) = -120z^2+106z+56 \leq 0$, $\beta_4(8,z) = -336z^2+234z+72 \leq 0$ and $\beta_4(10,z) = -720z^2+410z+88 \leq 0$. Therefore,

$$B_4 = -\frac{m(8 - z(m((m-3)z-6) + 2(z+10))) + 10z + 8}{4(m+1)}.$$

Hence, by $(2.1.b)$, we get

$$\begin{aligned} \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) &= 1 \times B_1 + \frac{1}{2}(mz - z - 2) \times B_2 + 2 \times B_3 + (z-2) \times B_4 \\ &= \begin{cases} \frac{28}{5}, & \text{for } m = 4 \,\&\, z = 2 \\ \frac{3}{5}z^2(4z-6), & \text{for } m = 4 \,\&\, z \geq 3 \\ \frac{(m-2)(m-1)z^2(mz-6)}{2(m+1)}, & \text{otherwise.} \end{cases} \end{aligned}$$

(b) (i) If m is odd then by Theorem 1.1.20, we get the required CN-spec$(\Gamma^*_{\mathrm{ccc}}(G))$ and $\mathrm{E}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$.

(ii) If m is odd then by Theorem 2.2.2, we get

$$\text{CNL-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \left\{ [0]^1, \left[\frac{(m-1)z}{2}\left(\frac{(m-1)z}{2} - 2\right)\right]^{\frac{(m-1)z}{2}-1}, [0]^1, [z(z-2)]^{z-1} \right\}.$$

Here $|v(\Gamma^*_{\mathrm{ccc}}(G))| = \frac{1}{2}(m+1)z$ and $\mathrm{tr}(\mathrm{CNRS}(\Gamma^*_{\mathrm{ccc}}(G)) = \frac{1}{8}z(mz+z-4)(((m-4)m+7)z-2(m+1))$. So, $\Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \frac{(mz+z-4)(((m-4)m+7)z-2(m+1))}{4(m+1)}$. We have

$$L'_1 := |0 - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))| = \left| -\frac{(mz+z-4)(((m-4)m+7)z-2(m+1))}{4(m+1)} \right|.$$

Let $\alpha'_1(m,z) = (mz+z-4)(((m-4)m+7)z-2(m+1))$. Then $\alpha'_1(m,z) = (mz+z-4)(7z - 2 + \frac{mz}{2}(m-8) + \frac{m}{2}(mz-4)) > 0$ for $m \geq 8$, since $z \geq 1$. Again $\alpha'_1(3,z) = 16z^2 - 48z + 32 \geq 0$, $\alpha'_1(5,z) = 72z^2 - 120z + 48 \geq 0$ and $\alpha'_1(7,z) = 224z^2 - 240z + 64 \geq 0$, as $z \geq 1$. Therefore,

$$L'_1 = \frac{(mz+z-4)(((m-4)m+7)z-2(m+1))}{4(m+1)}.$$

We have

$$\begin{aligned} L'_2 &:= \left| \frac{1}{4}(mz-z)(mz-z-4) - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) \right| \\ &= \left| \frac{m^2z^2 + m^2z - 2mz^2 - 6mz - 4m - 3z^2 + 17z - 4}{2(m+1)} \right|. \end{aligned}$$

Let $\alpha_2'(m,z) = m^2z^2+m^2z-2mz^2-6mz-4m-3z^2+17z-4$. Then $\alpha_2'(m,z) = 17z-4+\frac{m}{2}(mz-8)+\frac{mz}{2}(m-12)+\frac{z^2}{2}(m^2-6)+\frac{mz^2}{2}(m-4) > 0$ for $m \geq 8$. It can be seen that $\alpha_2'(3,z) = 8z-16 = -8$ or ≥ 0 according as $z = 1$ or $z \geq 2$; $\alpha_2'(5,z) = 12z^2+12z-24 \geq 0$ and $\alpha_2'(7,z) = 32z^2+24z-32 \geq 0$, as $z \geq 1$. Therefore,

$$L_2' = \begin{cases} 1, & \text{for } m = 3 \,\&\, z = 1 \\ \frac{m^2z^2+m^2z-2mz^2-6mz-4m-3z^2+17z-4}{2(m+1)}, & \text{otherwise.} \end{cases}$$

We have

$$L_3' := |z(z-2) - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))| = \left| \frac{-m^3z^2+3m^2z^2+6m^2z+mz^2-20mz-8m-3z^2+22z-8}{4(m+1)} \right|.$$

Let $\alpha_3'(m,z) = -m^3z^2+3m^2z^2+6m^2z+mz^2-20mz-8m-3z^2+22z-8$. Then $\alpha_3'(m,z) = -8-8m-2z(10m-11)-3z^2-\frac{m^2z}{3}(mz-18)-\frac{mz^2}{3}(m^2-3)-\frac{m^2z^2}{3}(m-9) < 0$ for $m \geq 19$. It can be seen that $\alpha_3'(3,z) = 16z-32 = -16$ or ≥ 0 according as $z = 1$ or $z \geq 2$, $\alpha_3'(5,z) = -48z^2+72z-48 \leq 0$, $\alpha_3'(7,z) = -192z^2+176z-64 \leq 0$, $\alpha_3'(9,z) = -480z^2+328z-80 \leq 0$, $\alpha_3'(11,z) = -960z^2+528z-96 \leq 0$, $\alpha_3'(13,z) = -1680z^2+776z-112 \leq 0$, $\alpha_3'(15,z) = -2688z^2+1072z-128 \leq 0$ and $\alpha_3'(17,z) = -4032z^2+1416z-144 \leq 0$. Therefore,

$$L_3' = \begin{cases} z-2, & \text{for } m = 3 \,\&\, z \geq 2 \\ -\frac{-m^3z^2+3m^2z^2+6m^2z+mz^2-20mz-8m-3z^2+22z-8}{4(m+1)}, & \text{otherwise.} \end{cases}$$

Hence, by $(2.1.a)$, we get

$$\begin{aligned} \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) &= 2 \times L_1' + \frac{1}{2}(mz-z-2) \times L_2' + (z-1) \times L_3' \\ &= \begin{cases} 0, & \text{for } m = 3 \,\&\, z = 1 \\ 4(z-1)(z-2), & \text{for } m = 3 \,\&\, z \geq 2 \\ \frac{((m-1)z-2)\big((m-3)(m+1)z^2+((m-6)m+17)z-4(m+1)\big)}{2(m+1)}, & \text{otherwise.} \end{cases} \end{aligned}$$

(b)(iii) If m is odd then by Theorem 2.2.2, we get

$$\begin{aligned} \text{CNSL-spec}(\Gamma^*_{\mathrm{ccc}}(G)) = \Bigg\{ &\left[2\left(\frac{(m-1)z}{2}-1\right)\left(\frac{(m-1)z}{2}-2\right)\right]^1, \\ &\left[\left(\frac{(m-1)z}{2}-2\right)^2\right]^{\frac{(m-1)z}{2}-1}, [2(z-1)(z-2)]^1, \left[(z-2)^2\right]^{z-1} \Bigg\}. \end{aligned}$$

Here $|v(\Gamma_{\mathrm{ccc}}^{*}(G))| = \frac{1}{2}(m+1)z$ and $\operatorname{tr}(\mathrm{CNRS}(\Gamma_{\mathrm{ccc}}^{*}(G)) = \frac{1}{8}z(mz+z-4)(((m-4)m+7)z-2(m+1))$. So, $\Delta_{\mathrm{CN}}(\Gamma_{\mathrm{ccc}}^{*}(G)) = \frac{(mz+z-4)(((m-4)m+7)z-2(m+1))}{4(m+1)}$. We have

$$\begin{aligned} B_1' &:= \left|\frac{1}{2}(mz-z-2)(mz-z-4) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{ccc}}^{*}(G))\right| \\ &= \left|\frac{m^3z^2+m^2z^2-6m^2z-5mz^2-12mz+8m-5z^2+42z+8}{4(m+1)}\right|. \end{aligned}$$

Let $\beta_1'(m,z) = m^3z^2+m^2z^2-6m^2z-5mz^2-12mz+8m-5z^2+42z+8$. Then $\beta_1'(m,z) = 8+8m+42z+z^2(m(m-5)-5)+mz(m(mz-6)-12)$. For $m \geq 9$ we have $mz-6 \geq 3$ which gives $m(mz-6)-12 > 0$ and $m(m-5)-5 > 0$. Thus, $\beta_1'(m,z) > 0$. Again $\beta_1'(3,z) = 16z^2-48z+32 \geq 0$, $\beta_1'(5,z) = 120z^2-168z+48 \geq 0$ and $\beta_1'(7,z) = 352z^2-336z+64 \geq 0$, as $z \geq 1$. Therefore,

$$B_1' = \frac{m^3z^2+m^2z^2-6m^2z-5mz^2-12mz+8m-5z^2+42z+8}{4(m+1)}.$$

We have

$$B_2' := \left|\frac{1}{4}(mz-z-4)^2 - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{ccc}}^{*}(G))\right| = \left|\frac{m^2z^2-m^2z-2mz^2-6mz+4m-3z^2+19z+4}{2(m+1)}\right|.$$

Let $\beta_2'(m,z) = m^2z^2-m^2z-2mz^2-6mz+4m-3z^2+19z+4$. Then $\beta_2'(m,z) = 4+4m+19z+\frac{mz}{4}(mz-24)+\frac{m^2z}{4}(z-4)+\frac{z^2}{4}(m^2-12)+\frac{mz^2}{4}(m-8) > 0$ for $m \geq 9$ and $z \geq 5$. It can be seen that $\beta_2'(3,z) = 16-8z = 8$ or ≤ 0 according as $z=1$ or $z \geq 2$; $\beta_2'(5,z) = 12z^2-36z+24 \geq 0$; $\beta_2'(7,z) = 32z^2-72z+32 = -8$ or ≥ 0 according as $z=1$ or $z \geq 2$; $\beta_2'(m,1) = 20-4m \geq 0$ or < 0 according as $m = 1,3,5$ or $m \geq 7$; $\beta_2'(m,2) = 2m(m-8)+30 \geq 0$ for all m; $\beta_2'(m,3) = 6m^2-32m+34 = -8$ or ≥ 0 according as $m=3$ or $m \geq 5$ and $\beta_2'(m,4) = 12m^2-52m+32 = -16$ or ≥ 0 according as $m=3$ or $m \geq 5$. Therefore,

$$B_2' = \begin{cases} z-2, & \text{for } m=3 \,\&\, z \geq 2 \\ -\frac{m^2z^2-m^2z-2mz^2-6mz+4m-3z^2+19z+4}{2(m+1)}, & \text{for } m \geq 7 \,\&\, z=1 \\ \frac{m^2z^2-m^2z-2mz^2-6mz+4m-3z^2+19z+4}{2(m+1)}, & \text{otherwise.} \end{cases}$$

We have

$B_3' := |2(z-1)(z-2) - \Delta_{\mathrm{CN}}(\Gamma_{\mathrm{ccc}}^{*}(G))| = \left|\frac{-m^3z^2+3m^2z^2+6m^2z+5mz^2-36mz+8m+z^2+6z+8}{4(m+1)}\right|$.

Let $\beta_3'(m,z) = -m^3z^2+3m^2z^2+6m^2z+5mz^2-36mz+8m+z^2+6z+8$. Then $\beta_3'(m,z) = -8mz+8-8mz+8m-20mz+6z-\frac{m^2z}{4}(mz-24)-\frac{z^2}{4}(m^3-4)-\frac{mz^2}{4}(m^2-5)-\frac{m^2z^2}{4}(m-12) < 0$ for $m \geq 25$. Again, $\beta_3'(3,z) = 16z(z-3)+32 \geq 0$, $\beta_3'(5,z) = -24z^2-24z+48 \leq 0$, $\beta_3'(7,z) =$

$-160z^2 + 48z + 64 \leq 0$, $\beta_3'(9, z) = -440z^2 + 168z + 80 \leq 0$, $\beta_3'(11, z) = -912z^2 + 336z + 96 \leq 0$, $\beta_3'(13, z) = -1624z^2 + 552z + 112 \leq 0$, $\beta_3'(15, z) = -2624z^2 + 816z + 128 \leq 0$, $\beta_3'(17, z) = -3960z^2 + 1128z + 144 \leq 0$, $\beta_3'(19, z) = -5680z^2 + 1488z + 160 \leq 0$, $\beta_3'(21, z) = -7832z^2 + 1896z + 176 \leq 0$ and $\beta_3'(23, z) = -10464z^2 + 2352z + 192 \leq 0$. Therefore,

$$B_3' = \begin{cases} z(z-3) + 2, & \text{for } m = 3 \,\&\, z \geq 1 \\ -\frac{m^3(-z^2) + 3m^2z^2 + 6m^2z + 5mz^2 - 36mz + 8m + z^2 + 6z + 8}{4(m+1)}, & \text{for } m \geq 5 \,\&\, z \geq 1. \end{cases}$$

We have

$$B_4' := \left|(z-2)^2 - \Delta_{\text{CN}}(\Gamma_{\text{ccc}}^*(G))\right| = \left|\frac{-m^3z^2 + 3m^2z^2 + 6m^2z + mz^2 - 28mz + 8m - 3z^2 + 14z + 8}{4(m+1)}\right|.$$

Let $\beta_4'(m, z) = -m^3z^2 + 3m^2z^2 + 6m^2z + mz^2 - 28mz + 8m - 3z^2 + 14z + 8$. Then $\beta_4'(m, z) = -8m(z-1) - 14z(m-1) + (8 - 6mz - 3z^2) - \frac{m^2z}{3}(mz - 18) - \frac{mz^2}{3}(m^2 - 3) - \frac{m^2z^2}{3}(m-9) < 0$ for $m \geq 19$. It can be seen that $\beta_4'(3, z) = 32 - 16z = 16$ or ≤ 0 according as $z = 1$ or $z \geq 2$; $\beta_4'(5, z) = 24z(1-2z) + 48 = 24$ or ≤ 0 according as $z = 1$ or $z \geq 2$; $\beta_4'(7, z) = -192z^2 + 112z + 64 \leq 0$; $\beta_4'(9, z) = -480z^2 + 248z + 80 \leq 0$; $\beta_4'(11, z) = -960z^2 + 432z + 96 \leq 0$; $\beta_4'(13, z) = -1680z^2 + 664z + 112 \leq 0$; $\beta_4'(15, z) = -2688z^2 + 944z + 128 \leq 0$ and $\beta_4'(17, z) = -4032z^2 + 1272z + 144 \leq 0$, as $z \geq 1$. Therefore,

$$B_4' = \begin{cases} \frac{-m^3z^2 + 3m^2z^2 + 6m^2z + mz^2 - 28mz + 8m - 3z^2 + 14z + 8}{4(m+1)}, & \text{for } m = 3, 5 \,\&\, z = 1 \\ -\frac{-m^3z^2 + 3m^2z^2 + 6m^2z + mz^2 - 28mz + 8m - 3z^2 + 14z + 8}{4(m+1)}, & \text{otherwise.} \end{cases}$$

Hence, by $(2.1.b)$, we get

$$\begin{aligned} \text{LE}_{\text{CN}}^+(\Gamma_{\text{ccc}}^*(G)) &= 1 \times B_1' + \frac{1}{2}(mz - z - 2) \times B_2' + 1 \times B_3' + (z-1) \times B_4' \\ &= \begin{cases} 0, & \text{for } m = 3, 5 \,\&\, z = 1 \\ 4(z-1)(z-2), & \text{for } m = 3 \,\&\, z \geq 2 \\ \frac{(m-5)(m-3)(m+3)}{2(m+1)}, & \text{for } m \geq 7 \,\&\, z = 1 \\ \frac{(m-3)(m-1)z^2(mz+z-6)}{2(m+1)}, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the result follows. □

As a corollary of the above Theorem 4.2.1, we get the following results.

Corollary 4.2.2. *The CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of* $\Gamma^*_{\text{ccc}}(D_{2m})$ *(where* $m \geq 3$*) are as given below:*

(a) *If* m *is odd then*

(i) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(D_{2m})) = \left\{[0]^1, \left[-\frac{1}{2}(m-5)\right]^{\frac{1}{2}(m-3)}, \left[\frac{1}{4}(m-3)(m-5)\right]^1\right\}$ *and*

$$E_{\text{CN}}(\Gamma^*_{\text{ccc}}(D_{2m})) = \frac{1}{2}(m-3)(m-5).$$

(ii) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(D_{2m})) = \left\{[0]^2, \left[\frac{1}{4}(m-1)(m-5)\right]^{\frac{1}{2}(m-3)}\right\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(D_{2m})) = \frac{(m-5)(m-3)(m-1)}{m+1}.$$

(iii) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(D_{2m})) = \left\{[0]^1, \left[\frac{1}{2}(m-3)(m-5)\right]^1, \left[\frac{1}{4}(m-5)^2\right]^{\frac{1}{2}(m-3)}\right\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(D_{2m})) = \frac{(m-5)(m-3)(m+3)}{2(m+1)}.$$

(b) *If* m *is even then*

(i) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(D_{2m})) = \left\{[0]^2, \left[-\frac{1}{2}(m-6)\right]^{\frac{1}{2}(m-4)}, \left[\frac{1}{4}(m-4)(m-6)\right]^1\right\}$ *and*

$$E_{\text{CN}}(\Gamma^*_{\text{ccc}}(D_{2m})) = \frac{1}{2}(m-4)(m-6).$$

(ii) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(D_{2m})) = \left\{[0]^3, \left[\frac{1}{4}(m-2)(m-6)\right]^{\frac{1}{2}(m-4)}\right\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(D_{2m})) = \frac{3(m-6)(m-4)(m-2)}{2(m+2)}.$$

(iii) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(D_{2m})) = \left\{[0]^2, \left[\frac{1}{2}(m-4)(m-6)\right]^1, \left[\frac{1}{4}(m-6)^2\right]^{\frac{1}{4}(m-4)}\right\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(D_{2m})) = \begin{cases} \frac{28}{5}, & \textit{for } m = 8 \\ \frac{(m-6)(m-4)(m-2)}{m+2}, & \textit{for } m \neq 8. \end{cases}$$

Proof. We know that $\frac{D_{2m}}{Z(D_{2m})} \cong D_{2\times\frac{m}{2}}$ or D_{2m} according as m is even or odd. Therefore, by Theorem 4.2.1, we get the required result. ◻

Corollary 4.2.3. *The CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of* $\Gamma^*_{\text{ccc}}(Q_{4n})$ *(where* $n \geq 2$*) are as given below:*

(a) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(Q_{4n})) = \left\{[-(n-3)]^{(n-2)}, [(n-2)(n-3)]^1, [0]^2\right\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(Q_{4n})) = 2(n-2)(n-3).$$

(b) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(Q_{4n})) = \{[0]^3, [(n-1)(n-3)]^{n-2}\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(Q_{4n})) = \frac{6(n-3)(n-2)(n-1)}{n+1}.$$

(c) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(Q_{4n})) = \{[0]^2, [2(n-2)(n-3)]^1, [(n-3)^2]^{n-2}\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(Q_{4n})) = \begin{cases} \frac{28}{5}, & \text{for } n = 4 \\ \frac{4(n-3)(n-2)(n-1)}{n+1}, & \text{for } n \neq 4. \end{cases}$$

Proof. We know that $\frac{Q_{4n}}{Z(Q_{4n})} \cong D_{2n}$. Therefore, by Theorem 4.1.1 (for the case $n = 2$) and Theorem 4.2.1, we get the required result. □

Corollary 4.2.4. *The CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of* $\Gamma^*_{\text{ccc}}(U_{6n})$ *(where* $n \geq 2$*) are as given below:*

(a) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(U_{6n})) = \left\{[-(n-2)]^{2(n-1)}, [(n-1)(n-2)]^2\right\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(U_{6n})) = 4(n-1)(n-2).$$

(b) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(U_{6n})) = \left\{[0]^2, [n(n-2)]^{2(n-1)}\right\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(U_{6n})) = 4(n-2)(n-1).$$

(c) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(U_{6n})) = \left\{[2(n-1)(n-2)]^2, [(n-2)^2]^{2(n-1)}\right\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(U_{6n})) = 4(n-2)(n-1).$$

Proof. We know that $\frac{U_{6n}}{Z(U_{6n})} = D_{2\times 3}$. Therefore, by Theorem 4.2.1, we get the required result. □

Corollary 4.2.5. *The CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of $\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})$ (where $m \geq 3$ and $n \geq 2$) are as given below:*

(a) *If m is odd then*

(i) $\text{CN-spec}\,(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \Big\{[-(n-2)]^{n-1}, [(n-1)(n-2)]^1, \left[-\frac{1}{2}(mn-n-4)\right]^{\frac{1}{2}(mn-n-4)}, \left[\frac{1}{4}(mn-n-2)(mn-n-4)\right]^1\Big\}$ *and*

$$\mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \frac{1}{2}(mn-n-2)(mn-n-4) + 2(n-1)(n-2).$$

(ii) $\text{CNL-spec}\,(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \left\{[0]^2, [n(n-2)]^{n-1}, \left[\frac{1}{4}(nm-n)(nm-n-4)\right]^{\frac{1}{2}(nm-n-2)}\right\}$ *and*

$$\mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \begin{cases} 4(n-1)(n-2), & \text{for } m = 3 \,\&\, n \geq 2 \\ \frac{((m-1)n-2)\left((m-3)(m+1)n^2+((m-6)m+17)n-4(m+1)\right)}{2(m+1)}, & \text{otherwise.} \end{cases}$$

(iii) $\text{CNSL-spec}\,(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \Big\{[2(n-1)(n-2)]^1, [(n-2)^2]^{n-1}, \left[\frac{1}{2}(nm-n-2)(nm-n-4)\right]^1, \left[\frac{1}{4}(nm-n-4)^2\right]^{\frac{1}{2}(nm-n-2)}\Big\}$ *and*

$$\mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \begin{cases} 4(n-1)(n-2), & \text{for } m = 3 \,\&\, n \geq 2 \\ \frac{(m-3)(m-1)n^2(mn+n-6)}{2(m+1)}, & \text{otherwise.} \end{cases}$$

(b) *If m and $\frac{m}{2}$ are even then*

(i) $\text{CN-spec}(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \Big\{[-(n-2)]^{2(n-1)}, [(n-1)(n-2)]^2, \left[-\frac{1}{2}(mn-2n-4)\right]^{\frac{1}{2}(mn-2n-2)}, \left[\frac{1}{4}(mn-2n-2)(mn-2n-4)\right]\Big\}$ *and*

$$\mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \frac{1}{2}(mn-2n-2)(mn-2n-4) + 4(n-1)(n-2).$$

(ii) $\text{CNL-spec}\,(\Gamma^*_{\mathrm{ccc}}(U_{(n,m)})) = \Big\{[0]^3, [(n-2)n]^{2(n-1)}, \left[\frac{1}{4}(m-2)n((m-2)n-4)\right]^{\frac{1}{2}(mn-2n-2)}\Big\}$ *and*

$$\mathrm{LE_{CN}}((\Gamma^*_{\mathrm{ccc}}(U_{(n,m)}))) = \begin{cases} 6(n-1)(n-2), & \text{for } m = 4 \;\&\; n \geq 2 \\ \frac{((m-2)n-2)\left(m^2n(2n+1)-4m\left(2n^2+3n+1\right)+44n-8\right)}{2(m+2)}, & \text{otherwise.} \end{cases}$$

(iii) CNSL-spec $(\Gamma^*_{ccc}(U_{(n,m)})) = \Big\{\left[\frac{1}{2}(mn-2n-4)(mn-2n-2)\right]^1,$
$\left[\frac{1}{4}(mn-2n-4)^2\right]^{\frac{1}{2}(mn-2n-2)}, [2(n-2)(n-1)]^2, \left[(n-2)^2\right]^{2(n-1)}\Big\}$ *and*

$$\mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{ccc}(U_{(n,m)})) = \begin{cases} 6(n-1)(n-2), & \textit{for } m=4 \;\&\; n\geq 2 \\ \frac{24}{5}n^2(4n-3), & \textit{for } m=8 \;\&\; n\geq 2 \\ \frac{(m-4)(m-2)n^2(mn-6)}{m+2}, & \textit{otherwise.} \end{cases}$$

(c) *If m is even and $\frac{m}{2}$ is odd then*
(i) CN-spec$(\Gamma^*_{ccc}(U_{(n,m)})) = \{[-2(n-1)]^{2n-1}, [2(2n-1)(n-1)]^1,$
$\left[-\frac{1}{2}(mn-2n-4)\right]^{\frac{1}{2}(mn-2n-2)}, \left[\frac{1}{4}(mn-2n-2)(mn-2n-4)\right]\Big\}$ *and*

$$\mathrm{E}_{\mathrm{CN}}(\Gamma^*_{ccc}(U_{(n,m)})) = \frac{1}{2}(mn-n-2)(mn-n-4)+4(2n-1)(n-1).$$

(ii) CNL-spec $(\Gamma^*_{ccc}(U_{(n,m)})) = \Big\{[0]^2, \left[\frac{1}{4}(mn-2n-4)(mn-2n)\right]^{\frac{1}{2}(mn-2n-2)},$
$[4(n-1)n]^{2n-1}\}$ *and*

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{ccc}(U_{(n,m)})) = \begin{cases} 8(n-1)(2n-1), & \textit{for } m=6 \,\&\, n\geq 2 \\ \frac{((m-2)n-2)\left(m^2n(2n+1)-4m\left(2n^2+3n+1\right)-24n^2+68n-8\right)}{2(m+2)}, & \textit{otherwise.} \end{cases}$$

(ii) CNSL-spec $(\Gamma^*_{ccc}(U_{(n,m)})) = \Big\{\left[\frac{1}{2}(mn-2n-4)(mn-2n-2)\right]^1,$
$\left[\frac{1}{4}(mn-2n-4)^2\right]^{\frac{1}{2}(mn-2n-2)}, [4(n-1)(2n-1)]^1, [4(n-1)^2]^{2n-1}\Big\}$ *and*

$$\mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{ccc}(U_{(n,m)})) = \begin{cases} 8(n-1)(2n-1), & \textit{for } m=6 \;\&\; n\geq 2 \\ \frac{(m-6)(m-2)n^2((m+2)n-6)}{m+2}, & \textit{otherwise.} \end{cases}$$

Proof. We know that $\frac{U_{(n,m)}}{Z(U_{(n,m)})}$ is isomorphic to $D_{2\times\frac{m}{2}}$ or D_{2m} according as m is even or odd. Therefore, by Theorem 4.2.1, we get the required result. □

Corollary 4.2.6. *The CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of $\Gamma^*_{ccc}(SD_{8n})$ (where $n\geq 2$) are as given below:*

(a) *If n is even then*

(i) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \left\{[-(2n-3)]^{(2n-2)}, [(2n-2)(2n-3)]^1, [0]^2\right\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(SD_{8n})) = 2(2n-2)(2n-3).$$

(ii) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \{[0]^3, [(2n-1)(2n-3)]^{2n-2}\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \frac{12(n-1)(4(n-2)n+3)}{2n+1}.$$

(iii) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \{[0]^2, [2(2n-2)(2n-3)]^1, [(2n-3)^2]^{2n-2}\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \begin{cases} \frac{28}{5}, & \textit{for } n = 2 \\ \frac{8(n-1)(2n-3)(2n-1)}{2n+1}, & \textit{for } n \geq 4. \end{cases}$$

(b) *If n is odd then*

(i) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \left\{[-(2n-4)]^{(2n-3)}, [(2n-3)(2n-4)]^1, [-2]^3, [6]^1\right\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(SD_{8n})) = 2(2n-3)(2n-4) + 12.$$

(ii) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \{[0]^2, [8]^3, [(2n-2)(2n-4)]^{2n-3}\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \begin{cases} 24, & \textit{for } n = 3 \\ \frac{4(2n-3)(5(n-3)n+4)}{n+1}, & \textit{for } n \geq 5. \end{cases}$$

(iii) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \{[2(2n-3)(2n-4)]^1, [(2n-4)^2]^{2n-3}, [12]^1, [4]^3\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(SD_{8n})) = \begin{cases} 24, & \textit{for } n = 3 \\ \frac{16(n-3)(n-1)(2n-1)}{n+1}, & \textit{for } n \geq 5. \end{cases}$$

Proof. We know that $\frac{SD_{8n}}{Z(SD_{8n})}$ is isomorphic to $D_{2\times 2n}$ or D_{2n} according as n is even or odd. Therefore, by Theorem 4.2.1, we get the required result. □

Corollary 4.2.7. *The CN-spectrum, CNL-spectrum, CNSL-spectrum, CN-energy, CNL-energy and CNSL-energy of $\Gamma^*_{\text{ccc}}(V_{8n})$ (where $n \geq 2$) are as given below:*

(a) *If n is even then*

(i) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(V_{8n})) = \{[-(2n-4)]^{(2n-3)}, [(2n-3)(2n-4)]^1, [0]^4\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n})) = 2(2n-3)(2n-4).$$

(ii) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(V_{8n})) = \{[0]^5, [(2n-2)(2n-4)]^{2n-3}\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n})) = \frac{20(n-2)(n-1)(2n-3)}{n+1}.$$

(iii) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(V_{8n})) = \{[0]^4, [2(2n-3)(2n-4)]^1, [(2n-4)^2]^{2n-3}\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n})) = \frac{16(n-2)(n-1)(2n-3)}{n+1}.$$

(b) *If n is odd then*

(i) $\text{CN-spec}(\Gamma^*_{\text{ccc}}(V_{8n})) = \{[-(2n-3)]^{(2n-2)}, [(2n-2)(2n-3)]^1, [0]^2\}$ *and*

$$\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n})) = 2(2n-2)(2n-3).$$

(ii) $\text{CNL-spec}(\Gamma^*_{\text{ccc}}(V_{8n})) = \{[0]^3, [(2n-1)(2n-3)]^{2n-2}\}$ *and*

$$\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n})) = \frac{12(n-1)(4(n-2)n+3)}{2n+1}.$$

(iii) $\text{CNSL-spec}(\Gamma^*_{\text{ccc}}(V_{8n})) = \{[0]^2, [2(2n-2)(2n-3)]^1, [(2n-3)^2]^{2n-2}\}$ *and*

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n})) = \frac{8(n-1)(2n-3)(2n-1)}{2n+1}.$$

Proof. (a) If n is even then by 1.2.24, we have $\Gamma^*_{\text{ccc}}(V_{8n}) = K_{2n-2} \cup 2K_2$.

(i) By Theorem 2.2.2, we get

$$\text{CNL-spec}(\Gamma^*_{\text{ccc}}(V_{8n})) = \left\{[0]^5, [(2n-2)(2n-4)]^{2n-3}\right\}.$$

Here $|v(\Gamma^*_{\text{ccc}}(V_{8n}))| = 2(n+1)$ and $\text{tr}(\text{CNRS}(\Gamma^*_{\text{ccc}}(V_{8n}))) = 4(n-2)(n-1)(2n-3)$. So, $\Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n})) = \frac{2(n-2)(n-1)(2n-3)}{n+1}$. We have

$$L_1 := |0 - \Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n}))| = \left|-\frac{2(n-2)(n-1)(2n-3)}{n+1}\right| = \frac{2(n-2)(n-1)(2n-3)}{n+1},$$

since $-2(n-2)(n-1)(2n-3) < 0$, as $n \geq 2$, so $2n-3 > 0$, $n-2 \geq 0$ and $n-1 > 0$. Also

$$L_2 := |(2n-2)(2n-4) - \Delta_{\text{CN}}(\Gamma^*_{\text{ccc}}(V_{8n}))| = \left|\frac{10(n-2)(n-1)}{n+1}\right|$$

$$= \frac{10(n-2)(n-1)}{n+1}, \text{ as } n \geq 2.$$

Therefore, by (2.1.a), we get

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n})) = 5 \times L_1 + (2n-3) \times L_2 = \frac{20(n-2)(n-1)(2n-3)}{n+1}.$$

(ii) By Theorem 2.2.2, we get

$$\text{CNSL-spec}(\Gamma^*_{\mathrm{ccc}}(V_{8n})) = \left\{[0]^4, [2(2n-3)(2n-4)]^1, [(2n-4)^2]^{2n-3}\right\}.$$

We have

$$B_1 := |0 - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n}))| = L_1,$$

$$\begin{aligned} B_2 := |2(2n-3)(2n-4) - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n}))| &= \left|\frac{2(n-2)(n+3)(2n-3)}{n+1}\right| \\ &= \frac{2(n-2)(n+3)(2n-3)}{n+1}, \qquad \text{as } n \geq 2, \end{aligned}$$

and

$$B_3 := \left|(2n-4)^2 - \Delta_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n}))\right| = \left|\frac{2(n-2)(3n-7)}{n+1}\right| = \frac{2(n-2)(3n-7)}{n+1},$$

as $n \geq 2$, so $2(n-2)(3n-7) \geq 0$. Therefore, by (2.1.b), we get

$$\mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n})) = 4 \times B_1 + 1 \times B_2 + (2n-3) \times B_3 = \frac{16(n-2)(n-1)(2n-3)}{n+1}.$$

(b) If n is odd then by Result 1.2.24, we have $\Gamma^*_{\mathrm{ccc}}(V_{8n}) = K_{2n-1} \cup 2K_1 = \Gamma^*_{\mathrm{ccc}}(D_{2\times 4n})$. Hence, the result follows from Corollary 4.2.2. □

4.3 Some consequences

In this section, we discuss some consequences of the results obtained in Sections 4.1–4.2. Looking at the CN-spectrum, CNL-spectrum and CNSL-spectrum of $\Gamma^*_{\mathrm{ccc}}(G)$ for the groups considered in Sections 4.1–4.2, we get the following result.

Theorem 4.3.1. *Let G be a finite non-abelian group with center $Z(G)$. Then $\Gamma^*_{\rm ccc}(G)$ is CN-,CNL- and CNSL-integral if*

(a) $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(b) $\frac{G}{Z(G)} \cong D_{2m}$.

(c) *G is isomorphic to D_{2m}, Q_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n}.*

Now we shall determine whether $\Gamma^*_{\rm ccc}(G)$ are CN-, CNL- and CNSL-hyperenergetic for these groups.

Theorem 4.3.2. *Let G be a finite non-abelian group and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then $\Gamma^*_{\rm ccc}(G)$ is not CN-,CNL- and CNSL-hyperenergetic.*

Proof. Let $|Z(G)| = z$. Then $z \geq 2$ and $|v(\Gamma^*_{\rm ccc}(G))| = np + n = \frac{(p^2-1)z}{p}$.

By Theorem 4.1.1 and (2.2.a), we get

$$\begin{aligned} &\mathrm{E}_{\rm CN}(K_{|v(\Gamma^*_{\rm ccc}(G))|}) - \mathrm{E}_{\rm CN}(\Gamma^*_{\rm ccc}(G)) \\ &\qquad = 2(np+n-1)(np+n-2) - 2(n-1)(n-2)(p+1) \\ &\qquad = 2n^2p + 2p(n^2p-2) > 0, \end{aligned}$$

since $p \geq 2$ and n is a positive integer. Hence, $\Gamma^*_{\rm ccc}(G)$ is not CN-hyperenergetic.

By Theorem 4.1.1 and (2.2.a), we get

$$\mathrm{LE}_{\rm CN}(K_{|v(\Gamma^*_{\rm ccc}(G))|}) - \mathrm{LE}_{\rm CN}(\Gamma^*_{\rm ccc}(G)) = \begin{cases} \frac{4\left((p-2)p\left(2p^2+p-2\right)+4\right)}{p^2}, & \text{for } p \geq 2 \,\&\, z = 2 \\ 16, & \text{for } p = 2 \,\&\, z = 3 \\ \frac{2p^3z^2-2p^2z^2-4p^2-2pz^2+2z^2}{p}, & \text{otherwise.} \end{cases}$$

Let $f_1(p) = 4\left((p-2)p\left(2p^2+p-2\right)+4\right)$ and $f_2(p,z) = 2p^3z^2 - 2p^2z^2 - 4p^2 - 2pz^2 + 2z^2$, where $z \geq 3$. Then $f_1(p) > 0$. Also, $f_2(p,z) = \frac{2}{3}(p-3)p^2z^2 + \frac{2}{3}p^2\left(pz^2-6\right) + \frac{2}{3}\left(p^2-3\right)pz^2 + 2z^2 > 0$ for $p \geq 3$. For $p = 2$ we have $f_2(p,z) = 6z^2 - 16 > 0$. Hence, $\mathrm{LE}_{\rm CN}(K_{|v(\Gamma^*_{\rm ccc}(G))|}) - \mathrm{LE}_{\rm CN}(\Gamma^*_{\rm ccc}(G)) > 0$. By Theorem 4.1.1, we also have $\mathrm{LE}_{\rm CN}(\Gamma^*_{\rm ccc}(G)) = \mathrm{LE}^+_{\rm CN}(\Gamma^*_{\rm ccc}(G))$. Therefore, $\mathrm{LE}^+_{\rm CN}(K_{|v(\Gamma^*_{\rm ccc}(G))|}) - \mathrm{LE}^+_{\rm CN}(\Gamma^*_{\rm ccc}(G)) > 0$. Hence the result follows. □

An immediate corollary of the above theorem is given below.

Corollary 4.3.3. *Let G be a non-abelian group of order p^n and center $|Z(G)| = p^{n-2}$. Then $\Gamma^*_{\mathrm{ccc}}(G)$ is not CN-,CNL- and CNSL-hyperenergetic.*

Theorem 4.3.4. *Let G be a finite non-abelian group and $\frac{G}{Z(G)} \cong D_{2m}$ (where $m \geq 3$). Then $\Gamma^*_{\mathrm{ccc}}(G)$ is*

(a) *not CN-hyperenergetic. In particular CN-borderenergetic if and only if $G \cong D_6$.*

(b) *CNL-borderenergetic if $m = 3, 11$ & $z = 1$.*

(c) *CNL-hyperenergetic except for $m = 4, 6$ & $z = 2$; $m = 4$ & $z = 3$; $m = 4$ & $z = 4$; $m = 3$ & $z \geq 2$; $m = 5, 7, 9$ & $z = 1$ and $m = 5$ & $z = 2, 3$.*

(d) *CNSL-borderenergetic for $m = 3$ & $z = 1$.*

(e) *CNSL-hyperenergetic except for $m = 4$ & $z = 2, 3, 4, 5$; $m = 6, 8$ & $z = 2$; $m = 6$ & $z = 3$; $m = 5$ & $z = 1$; $m = 3$ & $z \geq 2$; $m \geq 7$ (m is odd) & $z = 1$; $m = 5, 7, 9$ & $z = 2$ and $m = 5$ & $z = 3, 4$.*

Proof. We have

$$|v(\Gamma^*_{\mathrm{ccc}}(G))| = \begin{cases} \frac{1}{2}(m+1)z, & \text{for } m \text{ is even} \\ \frac{1}{2}(m+1)z, & \text{for } m \text{ is odd.} \end{cases}$$

Case 1. m is even

By Theorem 4.2.1 and Result 1.1.20, we get

$$\begin{aligned} \mathrm{E_{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) &= 2\left(\frac{mz}{2} + \frac{z}{2} - 1\right)\left(\frac{mz}{2} + \frac{z}{2} - 2\right) \\ &\quad - \left(\frac{m^2z^2}{2} - mz^2 - 3mz + \frac{3z^2}{2} - 3z + 12\right) \\ &= z^2(2m-1) - 8 > 0, \quad \text{since } m \geq 3 \text{ and } z \geq 2. \end{aligned}$$

In this case $z \geq 2$. By Theorem 4.2.1 and (2.2.a), we have

$$\mathrm{LE_{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$$

$$= \frac{-m^3z^3 + 3m^2z^3 + 12m^2z^2 - 2mz^3 - 18mz^2 - 24mz + 12z^2 + 12z}{2(m+1)}.$$

Let $f_1(m,z) = -m^3z^3 + 3m^2z^3 + 12m^2z^2 - 2mz^3 - 18mz^2 - 24mz + 12z^2 + 12z$. Then $f_1(m,z) = \frac{1}{2}m^2z^3(6-m) + \frac{1}{2}m^2z^2(24-mz) - 2mz^3 + 6z^2(2-3m) + 12z(1-2m) < 0$ for $m \geq 6$ and $z \geq 4$. We have $f_1(m,2) = 8m^2(9-m) + 72 - 136m < 0$ for $m \geq 10$. Also $f_1(4,2) = 168$, $f_1(6,2) = 120$ and $f_1(8,2) = -504$. Therefore, $f_1(m,2) > 0$ or < 0 according as $m = 4, 6$ or $m \geq 8$. We have $f_1(m,3) = 27m^2(7-m) + 144 - 288m < 0$ for $m \geq 8$.

Also $f_1(4,3) = 288$ and $f_1(6,3) = -612$. Therefore, $f_1(4,3) > 0$ or < 0 according as $m = 4$ or $m \geq 6$.

Now we need to check for $m = 4$ and $z \geq 4$. We have $f_1(4,z) = 12z^2(11-2z) - 84z < 0$ for $z \geq 6$. Also $f_1(4,4) = 240$ and $f_1(4,5) = -120$. Therefore, $f_1(4,z) > 0$ or < 0 according as $z = 4$ or $z \geq 5$. Hence, $\mathrm{LE_{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) > 0$ for $m = 4, 6$ & $z = 2$ and $m = 4$ & $z = 3, 4$. Otherwise, $\mathrm{LE_{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < 0$.

By Theorem 4.2.1 and (2.2.a), we also have

$$\mathrm{LE^+_{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$$
$$= \begin{cases} \frac{92}{5}, & \text{for } m = 4 \,\&\, z = 2 \\ \frac{1}{10}z\left(-24z^2 + 161z - 150\right) + 4, & \text{for } m = 4 \,\&\, z \geq 3 \\ \frac{-m^3z^3+m^3z^2+3m^2z^3+9m^2z^2-6m^2z-2mz^3-15mz^2-12mz+8m+13z^2-6z+8}{2(m+1)}, & \text{otherwise.} \end{cases}$$

Let $f_2(z) = \frac{1}{10}z\left(-24z^2 + 161z - 150\right) + 4$ and $f_3(m,z) = -m^3z^3 + m^3z^2 + 3m^2z^3 + 9m^2z^2 - 6m^2z - 2mz^3 - 15mz^2 - 12mz + 8m + 13z^2 - 6z + 8$. Then $f_2(z) = \frac{1}{10}z\left(z\left(161 - 24z\right) - 150\right) + 4 > 0$ or < 0 according as $z = 3, 4, 5$ or $z \geq 6$. Also, $f_3(m,z) = \frac{1}{3}m^3(3-z)z^2 + \frac{1}{3}(9-m)m^2z^3 + \frac{1}{3}m^2z^2(27-mz) - 6m^2z + z^2(13-2mz) + (8m - 15mz^2) + (8 - 12mz) - 6z < 0$ for $m \geq 10$ and $z \geq 3$.

We have $f_3(m,2) = 4m^2(12-m) + 48 - 92m < 0$ for $m \geq 12$. Also, $f_3(6,2) = 360$, $f_3(8,2) = 336$ and $f_3(10,2) = -72$. Therefore, $f_3(m,2) > 0$ or < 0 according as $m = 6, 8$ or $m \geq 10$.

We have $f_3(6,z) = z^2(463 - 120z) - 294z + 56 < 0$ for $z \geq 4$. Also, $f_3(6,3) = 101$. Therefore, $f_3(6,z) > 0$ or < 0 according as $z = 3$ or $z \geq 4$. We have $f_3(8,z) = z^2(981 -$

$336z) - 486z + 72 < 0$ for $z \geq 3$. Therefore, $f_3(m, z) > 0$ if $m = 6, 8 \,\&\, z = 2$ and $m = 6 \,\&\, z = 3$. Otherwise, $f_3(m, z) < 0$. Hence, $\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) > 0$ if $m = 4 \,\&\, z = 2, 3, 4, 5$; $m = 6, 8 \,\&\, z = 2$ and $m = 6 \,\&\, z = 3$. Otherwise, $\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < 0$.

Case 2. m is odd

By Theorem 4.2.1 and Result 1.1.20, we get

$$\begin{aligned}\mathrm{E}_{\mathrm{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{E}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) &= 2\left(\frac{mz}{2} + \frac{z}{2} - 1\right)\left(\frac{mz}{2} + \frac{z}{2} - 2\right) \\ &\quad - \left(\frac{n^2z^2}{2} - mz^2 - 3mz + \frac{5z^2}{2} - 3z + 8\right) \\ &= 2z^2(m-1) - 4.\end{aligned}$$

We have

$$2z^2(m-1) - 4 \begin{cases} = 0, & \text{for } m = 3, z = 1 \\ > 0, & \text{otherwise.} \end{cases}$$

Hence the result follows.

By Theorem 4.2.1 and (2.2.a), we have

$$\begin{aligned}&\mathrm{LE}_{\mathrm{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) \\ &= \begin{cases} 0, & \text{for } m = 3 \,\&\, z = 1 \\ 4z^2 - 4, & \text{for } m = 3 \,\&\, z \geq 2 \\ \frac{24z - 24mz + 12z^2 - 24mz^2 + 12m^2z^2 - 3z^3 + mz^3 + 3m^2z^3 - m^3z^3}{2(m+1)}, & \text{otherwise.} \end{cases}\end{aligned}$$

Clearly $4z^2 - 4 > 0$ for $z \geq 2$. Let $f_4(m, z) = 24z - 24mz + 12z^2 - 24mz^2 + 12m^2z^2 - 3z^3 + mz^3 + 3m^2z^3 - m^3z^3$. Then $f_4(m, z) = \frac{1}{3}m^2z^3(9 - m) + \frac{1}{3}mz^3(3 - m^2) + \frac{1}{3}m^2z^2(36 - mz) - 3z^3 + 12z^2(1 - 2m) + 24z(1 - m) < 0$ for $m \geq 9$ and $z \geq 4$. We have $f_4(m, 1) = m^2(15 - m) + 33 - 47m < 0$ for $m \geq 15$. Also, $f_4(5, 1) = 48$, $f_4(7, 1) = 96$, $f_4(9, 1) = 96$,

$f_4(11,1) = 0$ and $f_4(13,1) = -240$. Therefore,

$$f_4(m,1) \begin{cases} = 0, & \text{for } m = 11 \\ > 0, & \text{for } m = 5,7,9 \\ < 0, & \text{for } m \geq 13. \end{cases}$$

We have $f_4(m,2) = 8m^2(9-m) + 72 - 136m < 0$ for $m \geq 9$, $f_4(5,2) = 192$ and $f_4(7,2) = -96$. Therefore, $f_4(m,2) > 0$ or < 0 according as $m = 5$ or $m \geq 7$.

We have $f_4(m,3) = 27m^2(7-m) + 99 - 261m < 0$ for $m \geq 7$ and $f_4(5,3) = 144$. Therefore, $f_4(m,3) > 0$ or < 0 according as $m = 5$ or $m \geq 7$.

Again, we have $f_4(5,z) = 48z^2(4-z) - 96z < 0$ and $f_4(7,z) = 48z^2(9-4z) - 144z < 0$ for $z \geq 4$. Therefore,

$$f_4(m,z) \begin{cases} = 0, & \text{for } m = 11 \,\&\, z = 1 \\ > 0, & \text{for } m = 5,7,9 \,\&\, z = 1; m = 5 \,\&\, z = 3 \\ < 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} &\text{LE}_{\text{CN}}(K_{|v(\Gamma^*_{\text{ccc}}(G))|}) - \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) \\ &\qquad \begin{cases} = 0, & \text{for } m = 3, 11 \,\&\, z = 1 \\ > 0, & \text{for } m = 3 \,\&\, z \geq 2; \\ & m = 5,7,9 \,\&\, z = 1;\; m = 5 \;\&\; z = 2,3 \\ < 0, & \text{otherwise.} \end{cases} \end{aligned}$$

By Theorem 4.2.1 and (2.2.a), we also have

$\text{LE}^+_{\text{CN}}(K_{|v(\Gamma^*_{\text{ccc}}(G))|}) - \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$

$$= \begin{cases} 0, & \text{for } m = 3 \,\&\, z = 1 \\ 4, & \text{for } m = 5 \,\&\, z = 1 \\ 4(z^2-1), & \text{for } m = 3 \,\&\, z \geq 2 \\ \frac{m^2+4m-21}{m+1}, & \text{for } m \geq 7 \,\&\, z = 1 \\ \frac{-m^3z^3+m^3z^2+3m^2z^3+9m^2z^2-6m^2z+mz^3-21mz^2-12mz+8m-3z^3+19z^2-6z+8}{2(m+1)}, & \text{otherwise.} \end{cases}$$

Clearly $4(z^2-1) > 0$ for $z \geq 2$ and $m^2+4m-21 > 0$ for $m \geq 7$. Let $f_5(m,z) = -m^3z^3+m^3z^2+3m^2z^3+9m^2z^2-6m^2z+mz^3-21mz^2-12mz+8m-3z^3+19z^2-6z+8$. Then $f_5(m,z) = \frac{1}{4}m^3z^2(4-z) + \frac{1}{4}m^2z^3(12-m) + \frac{1}{4}m^2z^2(36-mz) + \frac{1}{4}mz^3(4-m^2) - 21mz^2 + (8m-12mz) - 6m^2z + z^2(19-3z) + (8-6z) < 0$ for $m \geq 13$ and $z \geq 7$. We have $f_5(n,2) = 4n^2(12-n) + 48 - 92n < 0$ for $n \geq 13$. Again $f_5(5,2) = 288$, $f_5(7,2) = 384$, $f_5(9,2) = 192$ and $f_5(11,2) = -480$. Therefore, $f_5(m,2) > 0$ or < 0 according as $m = 5,7,9$ or $m \geq 11$.

We have $f_5(m,3) = 18m^2(8-m) + 80 - 190m < 0$ for $m \geq 9$. Again, $f_5(5,3) = 480$ and $f_5(7,3) = -368$. Therefore, $f_5(m,3) > 0$ or < 0 according as $m = 5$ or $m \geq 7$.

We have $f_5(m,4) = 24m^2(13-2m) + 96 - 312m < 0$ for $m \geq 7$. Also, $f_5(5,4) = 336$. Therefore, $f_5(m,4) > 0$ or < 0 according as $m = 5$ or $m \geq 7$.

We have $f_5(m,5) = 10m^2(57-10m) + 78 - 452m < 0$ for $m \geq 7$. Also, $f_5(5,5) = -432$. Therefore, $f_5(m,5) < 0$.

We have $f_5(m,6) = 36m^2(26-5m) + 8 - 604m < 0$ for $m \geq 7$. Also, $f_5(5,6) = -2112$. Therefore, $f_5(m,6) < 0$.

Now we shall check for $m = 5,7,9,11$ and $z \geq 7$. We have $f_5(5,z) = 24z^2(11-2z)+48-216z < 0$, $f_5(7,z) = 16z^2(41-12z)+64-384z < 0$, $f_5(9,z) = 8z^2(161-60z)+80-600z < 0$ and $f_5(11,z) = 96z^2(23-10z) + 96 - 864z < 0$, as $z \geq 7$. Thus, $f_5(m,z) > 0$ if $m = 5,7,9$ & $z = 2$ and $m = 5$ & $z = 3,4$. Otherwise, $f_5(m,z) < 0$.

Hence

$$\mathrm{LE}^{+}_{\mathrm{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(G))|}) - \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$$

$$\begin{cases} = 0, & \text{for } m = 3 \,\&\, z = 1 \\ > 0, & \text{for } m = 5 \;\&\; z = 1;\; m = 3 \;\&\; z \geq 2; \\ & m \geq 7 \;\&\; z = 1;\; m = 5, 7, 9 \;\&\; z = 2;\; m = 5 \;\&\; z = 3, 4 \\ < 0, & \text{otherwise.} \end{cases}$$

Hence the result follows. □

As a corollary of the above theorem we get the following results.

Corollary 4.3.5. *$\Gamma^*_{\mathrm{ccc}}(U_{6n})$ $(n \geq 2)$ is not CN-, CNL- and CNSL- hyperenergetic.*

Corollary 4.3.6. *Let $G = D_{2m}, Q_{4n}, U_{6n}$, SD_{8n} or $U_{(n,m)}$. Then*

(a) *$\Gamma^*_{\mathrm{ccc}}(G)$ is not CN-hyperenergetic.*

(b) *$\Gamma^*_{\mathrm{ccc}}(G)$ is CNL-borderenergetic if and only if $G = D_6$ and D_{22}.*

(c) *$\Gamma^*_{\mathrm{ccc}}(G)$ is CNL-hyperenergetic if and only if $G = D_{2m}$ for $m \geq 13$; Q_{4n} for $n \geq 7$; SD_{8n} for $n \geq 4$ and $U_{(n,m)}$ except for $m = 3 \,\&\, n \geq 2$, $m = 5 \,\&\, n = 2, 3$, $m = 4 \,\&\, n \geq 2$, $m = 8 \,\&\, n = 2$ and $m = 6 \,\&\, n \geq 2$.*

(d) *$\Gamma^*_{\mathrm{ccc}}(G)$ is CNSL-borderenergetic if and only if $G = D_6$.*

(e) *$\Gamma^*_{\mathrm{ccc}}(G)$ is CNSL-hyperenergetic if and only if $G = D_{2m}$ for m is even and $m \geq 20$; Q_{4n} for $n \geq 10$; SD_{8n} for $n \geq 6$ and $U_{(n,m)}$ except for $m = 3 \,\&\, n \geq 2$, $m = 5, 7, 9 \,\&\, n = 2$, $m = 5 \,\&\, n = 3, 4$, $m = 4 \,\&\, n \geq 2$, $m = 8 \,\&\, n = 2$, $m = 6 \,\&\, n \geq 2$ and $m = 10 \,\&\, n = 2$.*

Corollary 4.3.7. *$\Gamma^*_{\mathrm{ccc}}(V_{8n})$ $(n \geq 2)$ is not CN-hyperenergetic but CNL-hyperenergetic for $n \geq 6$ and CNSL-hyperenergetic for $n \geq 4$.*

Proof. **Case 1.** n is even

We have $|v(\Gamma^*_{\mathrm{ccc}}(V_{8n}))| = (2n + 2)$. By Theorem 4.2.7 and Result 1.1.20, we get

$$\mathrm{E_{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(V_{8n}))|}) - \mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n})) = 4n(2n + 1) - 2(2n - 3)(2n - 4).$$

Since $n \geq 2$ we have $4n(2n+1) - 2(2n-3)(2n-4) = 8(4n-3) > 0$. Therefore, $E_{\mathrm{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(V_{8n}))|}) > E_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n}))$ and so $\Gamma^*_{\mathrm{ccc}}(V_{8n})$ is not CN-hyperenergetic.

By Theorem 4.2.7 and (2.2.a), we get

$$\mathrm{LE}_{\mathrm{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(V_{8n}))|}) - \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n})) = \frac{120 - 32(n-4)(n-2)n}{n+1} \begin{cases} > 0, & \text{for } 2 \leq n \leq 4 \\ < 0, & \text{for } n \geq 6. \end{cases}$$

$$\mathrm{LE}^+_{\mathrm{CN}}(K_{|v(\Gamma^*_{\mathrm{ccc}}(V_{8n}))|}) - \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(V_{8n})) = -\frac{6(n-1)(n(5n-19)+16)}{n+1} \begin{cases} > 0, & \text{for } n = 2 \\ < 0, & \text{for } n \geq 4. \end{cases}$$

Thus, $\Gamma^*_{\mathrm{ccc}}(V_{8n})$ is not CNL-hyperenergetic if $n = 2, 4$ and $\Gamma^*_{\mathrm{ccc}}(V_{8n})$ is CNL-hyperenergetic if $n \geq 6$. Also, it is not CNSL-hyperenergetic if $n = 2$ and $\Gamma^*_{\mathrm{ccc}}(V_{8n})$ is CNSL-hyperenergetic if $n \geq 4$.

Case 2. n is odd

We have $\Gamma^*_{\mathrm{ccc}}(V_{8n}) = K_{2n-1} \cup 2K_1 = \Gamma_{\mathrm{ccc}}(D_{2\times 4n})$. Then, by Corollary 4.3.6, we have that $\Gamma^*_{\mathrm{ccc}}(V_{8n})$ is CNL-hyperenergetic if $n \geq 4$ and CNSL-hyperenergetic if $n \geq 5$. Hence, the result follows. □

4.4 Comparing various CN-energies

In this section, we compare CN-energy, CNL-energy and CNSL-energy of $\Gamma^*_{\mathrm{ccc}}(G)$ of the groups considered in Sections 4.1–4.2.

Theorem 4.4.1. *Let G be a finite group such that $|Z(G)| = z \geq 2$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. If $p = 2 \,\&\, z = 3$ or $p \geq 3 \,\&\, z = 2$ then*

$$E_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)).$$

For all other cases, $E_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$.

Proof. In view of Theorem 4.1.1, it is sufficient to compare $E_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$ and $\mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$. By Theorem 4.1.1, we have

$$\mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) - E_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \begin{cases} 3, & \text{for } p = 2 \,\&\, z = 3 \\ \frac{8(p-2)(p+1)}{p^2}, & \text{for } p \geq 2 \,\&\, z = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $8(p-2)(p+1) = 0$ or > 0 according as $p = 2$ or $p > 2$. Hence, the result follows. □

As a corollary to Theorem 4.4.1 we have the following result.

Corollary 4.4.2. *Let G be a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, where p is a prime and $n \geq 3$. Then* $\mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$.

Theorem 4.4.3. *Let G be a finite group and $\frac{G}{Z(G)} \cong D_{2m}$ $(m \geq 3)$. If $m = 3 \,\&\, z \geq 1$ or $m = 5 \,\&\, z = 1$ (where $|Z(G)| = z$) then*

$$\mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)).$$

For all other cases, $\mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$.

Proof. **Case 1.** m is even

In this case $z \geq 2$. By Theorem 4.2.1, we have

$$\begin{aligned}
&\mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) - \mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) \\
&= \begin{cases} \frac{8}{5}, & \text{for } m = 4 \,\&\, z = 2 \\ \frac{z^2(24z-91)+150z-120}{10}, & \text{for } m = 4 \,\&\, z \geq 3 \\ \frac{(m-2)(m-1)mz^3-(m(m(m+5)-17)+15)z^2+6(m+1)^2z-24(m+1)}{2(m+1)}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Let $f_1(z) = z^2(24z-91)+150z-120$ and $f_2(m,z) = (m-2)(m-1)mz^3 - (m(m(m+5)-17)+15)z^2 + 6(m+1)^2z - 24(m+1)$. Then $f_1(z) > 0$ for $z \geq 4$. Also, $f_1(3) = \frac{159}{10}$. Therefore, $f_1(z) > 0$ for $z \geq 3$. It can be seen that $f_2(m,z) = \frac{1}{3}m^3(z-3)z^2 + \frac{1}{3}(m-9)m^2z^3 + \frac{1}{3}m^2z^2(mz-15) + 6\left(m^2z-4\right) + 2mz^3 + (17m-15)z^2 + 12m(z-2) + 6z > 0$ for $m \geq 10$ and $z \geq 3$. Also, $f_2(m,2) = 4(m-2)(m-3)^2 > 0$ for $m \geq 6$. We have $f_2(6,z) = z^2(120z-309)+294z-168 > 0$ and $f_2(8,z) = z^2(336z-711)+486z-216 > 0$ for $z \geq 3$. Therefore, $f_2(m,z) > 0$ for $m \geq 6$ and $z \geq 2$. Hence,

$$\mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) - \mathrm{E_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) > 0.$$

Again

$$\mathrm{LE^+_{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) - \mathrm{LE_{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$$

$$= \begin{cases} -\frac{8}{5}, & \text{for } m = 4 \,\&\, z = 2 \\ \frac{-z(29z-66)-40}{10}, & \text{for } m = 4 \,\&\, z \geq 3 \\ -\frac{(m((m-3)m+3)+1)z^2-6((m-2)m+3)z+8(m+1)}{2(m+1)}, & \text{otherwise.} \end{cases}$$

Clearly, $-z(29z-66)-40 < 0$ for $z \geq 3$. Let $f_3(m,z) = -((m((m-3)m+3)+1)z^2 - 6((m-2)m+3)z+8(m+1))$. Then $f_3(m,z) = -\frac{1}{2}(m-6)m^2z^2 - \frac{1}{2}m^2z(mz-12) - 3mz^2 - 6(2m-3)z - 8m - z^2 - 8 < 0$ for $m \geq 6$ and $z \geq 2$. Therefore,

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) - \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < 0.$$

Hence, $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$.

Case 2. m is odd

By Theorem 4.2.1, we have

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) - \text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$$

$$= \begin{cases} 0, & \text{for } m = 3, 5 \,\&\, z = 1 \\ 0, & \text{for } m = 3 \,\&\, z \geq 2 \\ \frac{(m-5)(m-3)}{m+1}, & \text{for } m \geq 7 \,\&\, z = 1 \\ \frac{z\left((m-3)(m-1)(m+1)z^2-(m(m(m+5)-21)+23)z+6(m+1)^2\right)-16(m+1)}{2(m+1)}, & \text{otherwise.} \end{cases}$$

Clearly $(m-5)(m-3) > 0$ for $m \geq 7$. Let $f_4(m,z) = z\left((m-3)(m-1)(m+1)z^2 - (m(m(m+5)-21)+23)z + 6(m+1)^2\right) - 16(m+1)$, where $m \geq 5$ and $z \geq 2$. Then $f_4(m,z) = \frac{1}{4}m^3(z-4)z^2 + \frac{1}{4}(m-12)m^2z^3 + \frac{1}{4}\left(m^2-4\right)mz^3 + \frac{1}{4}m^2z^2(mz-20) + 2\left(3m^2z-8\right) + (21m-23)z^2 + 4m(3z-4) + 3z^3 + 6z > 0$ for $m \geq 13$ and $z \geq 4$. We have $f_4(m,2) = 4(m-2)(m-3)^2 > 0$ as $m \geq 5$; $f_4(m,3) = 18m^2(m-6) + 182m - 124 > 0$ for $m \geq 7$; and $f_4(5,3) = 336$. Therefore, $f_4(m,3) > 0$. Further, for $z \geq 4$ we have $f_4(5,z) = 24z^2(2z-7) + 216z - 96 > 0$; $f_4(7,z) = 16z^2(12z-29) + 384z - 128 > 0$; $f_4(9,z) = 8z^2(60z-121) + 600z - 160 > 0$ and $f_4(11,z) = 192z^2(5z-9) + 864z - 192 > 0$. Therefore, $f_4(m,z) > 0$. Thus,

$$\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) - \text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) \begin{cases} = 0, & \text{for } m = 3 \,\&\, z \geq 1;\, m = 5 \,\&\, z = 1 \\ > 0, & \text{otherwise.} \end{cases}$$

Again

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) - \mathrm{LE}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) = \begin{cases} 0, & \text{for } m = 3 \,\&\, z = 1 \\ 0, & \text{for } m = 3 \,\&\, z \geq 2 \\ 0, & \text{for } m = 5 \,\&\, z = 1 \\ -\frac{(m-5)^2(m-3)}{2(m+1)}, & \text{for } m \geq 7 \,\&\, z = 1 \\ -\frac{(mz+z-4)(((m-4)m+7)z-2(m+1))}{2(m+1)}, & \text{otherwise.} \end{cases}$$

Clearly, $-(m-5)^2(m-3) < 0$ for $m \geq 7$. Let $f_5(m,z) = -(mz+z-4)(((m-4)m+7)z-2(m+1))$. Then $f_5(m,z) = -\frac{1}{2}(m-6)m^2z^2 - \frac{1}{2}m^2z(mz-12) - 3mz^2 - 6(2m-5)z - 8m - 7z^2 - 8 < 0$ for $m \geq 7$ and $z \geq 2$. For $z \geq 2$ we have $f_5(5,z) = -24(z-1)(3z-2) < 0$. Therefore, $f_5(m,z) < 0$. Thus,

$$\mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) - \mathrm{LE}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) \begin{cases} = 0, & \text{for } m = 3 \,\&\, z \geq 1;\ m = 5 \,\&\, z = 1 \\ < 0, & \text{otherwise.} \end{cases}$$

Hence, $\mathrm{E}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) = \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G))$, if $m = 3 \,\&\, z \geq 1$ or $m = 5 \,\&\, z = 1$. For all other cases, $\mathrm{E}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) < \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G))$. This completes the proof. □

Using Theorem 4.4.3, we have the following corollary.

Corollary 4.4.4. *Let* $G = D_{2m}, Q_{4n}, U_{6n}, SD_{8n}, V_{8n}$ *or* $U_{(n,m)}$. *Then*

(a) $\mathrm{E}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) = \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G))$ *if and only if* $G = D_6, D_8, D_{10}, D_{12}, Q_8, Q_{12}, SD_{24}, V_{16}, U_{6n}$ *for* $n \geq 2$ *and* $U_{(n,m)}$ *for* $m = 3, 4, 6$ *and* $n \geq 2$.

(b) $\mathrm{E}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) < \mathrm{LE}^{+}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma^{*}_{\mathrm{ccc}}(G))$ *if and only* G *is not among the groups listed in* (a).

We conclude this chapter by showing the closeness of various CN-energies of $\Gamma^{*}_{\mathrm{ccc}}(G)$ if $G = D_{2m}, Q_{4n}, U_{6n}, SD_{8n}, V_{8n}$ and $U_{(n,m)}$ graphically in the following figures.

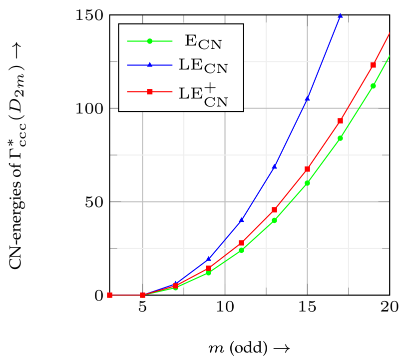

Figure 4.1: CN-energies of $\Gamma^*_{\text{ccc}}(D_{2m})$, m is odd

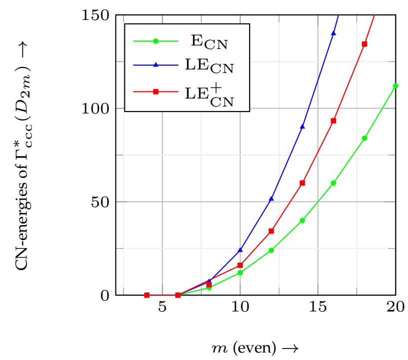

Figure 4.2: CN-energies of $\Gamma^*_{\text{ccc}}(D_{2m})$, m is even

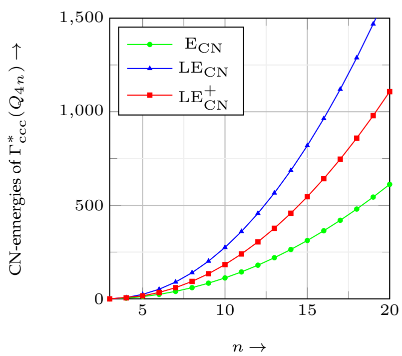

Figure 4.3: CN-energies of $\Gamma^*_{\text{ccc}}(Q_{4n})$

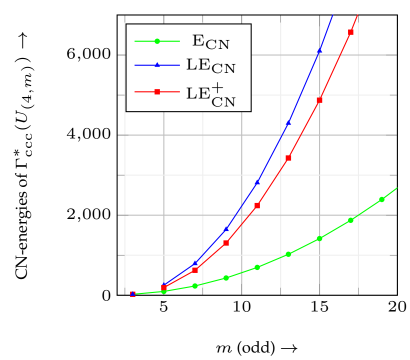

Figure 4.4: CN-energies of $\Gamma^*_{\text{ccc}}(U_{(4,m)})$, m is odd

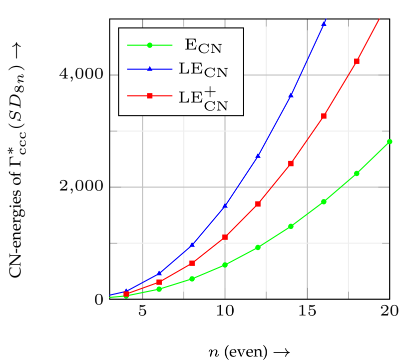

Figure 4.5: CN-energies of $\Gamma^*_{\text{ccc}}(SD_{8n})$, n is even

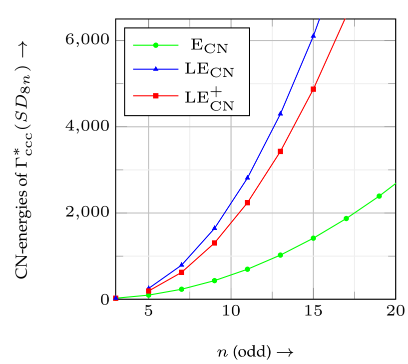

Figure 4.6: CN-energies of $\Gamma^*_{\text{ccc}}(SD_{8n})$, n is odd

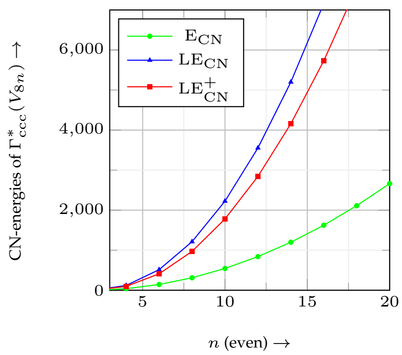

Figure 4.7: CN-energies of $\Gamma^*_{\text{ccc}}(V_{8n})$, n is even

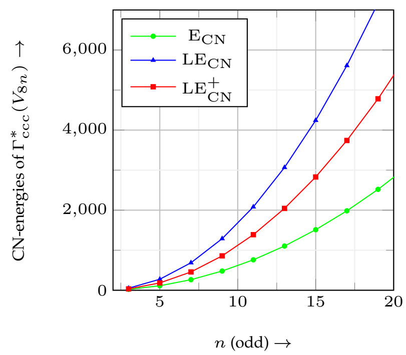

Figure 4.8: CN-energies of $\Gamma^*_{\text{ccc}}(V_{8n})$, n is odd

Considering the group $G = D_{2m}, Q_{4n}, U_{(n,m)}, V_{8n}$ and SD_{8n}, in [16], Bhowal and Nath compare $\text{E}(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}(\Gamma^*_{\text{ccc}}(G))$ and $\text{LE}^+(\Gamma^*_{\text{ccc}}(G))$. We observe that $\text{E}(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}^+(\Gamma^*_{\text{ccc}}(G))$ and $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$, $\text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$ satisfy similar (sometimes different) inequalities for the above mentioned groups. Comparing our results and the results obtained in [16], we get the following results.

Theorem 4.4.5. *Let $G = D_{2m}, Q_{4n}, U_{(n,m)}, V_{8n}$ and SD_{8n}. Then*

(a) $\text{E}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}^+(\Gamma^*_{\text{ccc}}(G))$ *as well as* $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$ *if and only if* $G = D_6, D_8, D_{12}, Q_8, Q_{12}, V_{16}, SD_{24}$ *and* $U_{(n,m)}$ *for* $m = 3, 4$ *and* $n \geq 2$.

(b) $\text{E}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}^+(\Gamma^*_{\text{ccc}}(G)) < \text{LE}(\Gamma^*_{\text{ccc}}(G))$ *as well as* $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$ *if and only* $G = D_{14}, D_{16}, D_{18}, D_{2n} (n \geq 11)$, $Q_{16}, Q_{24}, Q_{4m} (m \geq 8)$, $U_{(n,5)} \, (n \geq 4)$, $U_{(n,m)} \, (m \geq 7$ *and* $n \geq 3)$, $U_{(n,m)} (m \geq 8$ *and* $n \geq 2)$, $V_{8n} (n \geq 3)$, SD_{16} *and* $SD_{8n} (n \geq 4)$.

Theorem 4.4.6. *Let $G = D_{2m}, Q_{4n}, U_{(n,m)}, V_{8n}$ and SD_{8n}. Then*

(a) $\text{LE}^+(\Gamma^*_{\text{ccc}}(G)) < \text{E}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}(\Gamma^*_{\text{ccc}}(G))$ *and* $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$ *if and only if* $G = D_{20}$, Q_{20}, $U_{(2,5)}$ *and* $U_{(3,5)}$.

(b) $\text{LE}^+(\Gamma^*_{\text{ccc}}(G)) < \text{E}(\Gamma^*_{\text{ccc}}(G)) < \text{LE}(\Gamma^*_{\text{ccc}}(G))$ *and* $\text{E}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}^+_{\text{CN}}(\Gamma^*_{\text{ccc}}(G)) = \text{LE}_{\text{CN}}(\Gamma^*_{\text{ccc}}(G))$ *if and only if* $G = U_{(2,6)}$.

(c) $\mathrm{E}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}^+(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}(\Gamma^*_{\mathrm{ccc}}(G))$ *and* $\mathrm{E}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$ *if and only if* $G = Q_{28}$ *and* $U_{(2,7)}$.

(d) $\mathrm{E}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}^+(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}(\Gamma^*_{\mathrm{ccc}}(G))$ *and* $\mathrm{E}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$ *if and only if* $G = D_{10}$.

(e) $\mathrm{E}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}^+(\Gamma^*_{\mathrm{ccc}}(G))$ *and* $\mathrm{E}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$ *if and only if* $G = U_{(n,2)}$.

(f) $\mathrm{E}(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}^+(\Gamma^*_{\mathrm{ccc}}(G)) < \mathrm{LE}(\Gamma^*_{\mathrm{ccc}}(G))$ *and* $\mathrm{E}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}^+_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G)) = \mathrm{LE}_{\mathrm{CN}}(\Gamma^*_{\mathrm{ccc}}(G))$ *if and only if* $G = U_{(n,6)}$ $(n \geq 3)$.