Chapter 5

Various distance spectra, energies and Wiener index of the complement of $\Gamma^*_{\text{CCC}}(G)$

The non-commuting conjugacy class graph (abbreviated as NCCC-graph) of a finite non-abelian group G, denoted by $\Gamma_{\text{nccc}}(G)$, is a simple undirected graph whose vertex set is $\text{Cl}(G)$ and two distinct vertices a^G and b^G are adjacent if $a'b' \neq b'a'$ for all $a' \in a^G$ and $b' \in b^G$. Thus, $\Gamma_{\text{nccc}}(G)$ is the complement of $\Gamma_{\text{ccc}}(G)$. In this chapter, we consider the subgraph $\Gamma_{\text{nccc}}(G)[\text{Cl}(G \setminus Z(G))]$ of $\Gamma_{\text{nccc}}(G)$ induced by $\text{Cl}(G \setminus Z(G))$. For notational convenience we write $\Gamma^*_{\text{nccc}}(G)$ to denote the graph $\Gamma_{\text{nccc}}(G)[\text{Cl}(G \setminus Z(G))]$. Note that $\Gamma^*_{\text{nccc}}(G)$ is the complement of the graph $\Gamma^*_{\text{ccc}}(G)$ considered in Chapter 4. In Section 5.1, we shall compute distance spectrum, distance Laplacian spectrum, distance signless Laplacian spectrum and Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for the groups when $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2n} (for any integer $n \geq 3$). As a consequence, we shall compute the above-mentioned graph parameters of $\Gamma^*_{\text{nccc}}(G)$ when G is the dihedral group D_{2n} (for $n \geq 3$), the dicyclic group Q_{4n} (for $n \geq 2$), the semidihedral group SD_{8n} (for $n \geq 2$) and the groups $U_{(n,m)}$ (for $m \geq 3$ and $n \geq 2$), U_{6n} (for $n \geq 2$) and V_{8n} (for $n \geq 2$). We shall show that any perfect square can be realized as Wiener index of $\Gamma^*_{\text{nccc}}(G)$

for certain dihedral groups. We shall also characterize the above-mentioned groups such that $\Gamma^*_{\mathrm{nccc}}(G)$ are D-integral, DL-integral and DQ-integral. In Section 5.2, we shall compute distance energy, distance Laplacian energy and distance signless Laplacian energy of $\Gamma^*_{\mathrm{nccc}}(G)$ for the above mentioned groups using Wiener index. Further, in Section 5.3, we shall compare various distance energies of $\Gamma^*_{\mathrm{nccc}}(G)$ and characterize the above-mentioned groups subject to the inequalities involving various distance energies. In Sections 5.2–5.3, we shall also consider Problems 1.1.12–1.1.13 and obtain graphs satisfying the equalities in Problem 1.1.12–1.1.13 through $\Gamma^*_{\mathrm{nccc}}(G)$ for the above mentioned groups. This chapter is based on our paper [70] accepted for publication in *Journal of Algebra Combinatorics Discrete Structures and Applications*.

5.1 Distance spectra and Wiener index

In this section, we compute distance spectrum, distance Laplacian spectrum, distance signless Laplacian spectrum and Wiener index of $\Gamma^*_{\mathrm{nccc}}(G)$ for the groups when $\frac{G}{Z(G)}$ is isomorphic to

(a) $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is any prime.

(b) D_{2n}, where $n \geq 3$ is any integer.

As consequences, we get various distance spectra and Wiener index of $\Gamma^*_{\mathrm{nccc}}(G)$ if $G = D_{2n}, Q_{4n}, SD_{8n}, U_{(n,m)}, U_{6n}$ and V_{8n}. The following simple-minded result is very useful in computing Wiener index of any finite graph. However, this relation was neglected while computing Wiener index of various graphs (see [76, 3, 94, 107, 38]).

Lemma 5.1.1. *Let Γ be any graph having n vertices. Then*

$$W(\Gamma) = \frac{1}{2} \sum_{\beta \in \text{DL-spec}(\Gamma)} \beta = \frac{1}{2} \sum_{\gamma \in \text{DQ-spec}(\Gamma)} \gamma.$$

Proof. From the definitions of $\mathrm{DL}(\Gamma)$ and $\mathrm{DQ}(\Gamma)$ we have $\mathrm{tr}(\mathrm{DL}(\Gamma))=\mathrm{tr}(\mathcal{T}(\Gamma))=\mathrm{tr}(\mathrm{DQ}(\Gamma))$. Also,

$$\mathrm{tr}(\mathcal{T}(\Gamma)) = \sum_{1 \leq i,j \leq n} d_{ij} = \sum_{1 \leq i,j \leq n} d(v_i, v_j).$$

Therefore, $\text{tr}(\mathcal{T}(\Gamma)) = 2W(\Gamma)$ and so

$$\text{tr}(\text{DL}(\Gamma)) = \text{tr}(\text{DQ}(\Gamma)) = 2W(\Gamma). \tag{5.1.a}$$

Since trace of a square matrix is equal to the sum of its eigenvalues we have

$$\sum_{\beta \in \text{DL-spec}(\Gamma)} \beta = \text{tr}(\text{DL}(\Gamma)) = \text{tr}(\text{DQ}(\Gamma)) = \sum_{\gamma \in \text{DQ-spec}(\Gamma)} \gamma.$$

Hence, the result follows. □

The following theorem gives various distance spectra and Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for the groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Theorem 5.1.2. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is any prime and $|Z(G)| \geq 2$. If $n = \frac{(p-1)z}{p}$, where $z = |Z(G)|$ then*

$$\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \{[-2]^{(n-1)(p+1)}, [n-2]^p, [np+2n-2]^1\},$$

$$\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \{[0]^1, [n(p+1)]^p, [n(p+1)+n]^{(p+1)(n-1)}\},$$

$$\text{DQ-spec}(\Gamma^*_{\text{nccc}}(G)) = \{[np+2n-4]^{(p+1)(n-1)}, [np+3n-4]^p, [2np+4n-4]^1\}$$

and $W(\Gamma^*_{\text{nccc}}(G)) = \frac{n(p+1)(n(p+2)-2)}{2}$.

Proof. By Result 1.2.17, we have $\Gamma^*_{\text{nccc}}(G) = K_{n_1,n_2,\dots,n_{p+1}}$, where $n_1 = n_2 = \cdots = n_{p+1} = n = \frac{(p-1)z}{p}$. Here, $|v(\Gamma^*_{\text{nccc}}(G))| = (p+1)n$. Therefore, by Result 1.1.14(a), we have

$$\begin{aligned}\text{Ch}_{\text{D}}(\Gamma^*_{\text{nccc}}(G), x) &= (x+2)^{(n-1)(p+1)} \left(\prod_{i=1}^{p+1} (x - n_i + 2) - \sum_{i=1}^{p+1} n_i \prod_{j=1, j\neq i}^{p+1} (x - n_j + 2) \right) \\ &= (x+2)^{(n-1)(p+1)} \left((x-n+2)^p (x - np - 2n + 2) \right).\end{aligned}$$

Hence, $\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \left\{[-2]^{(n-1)(p+1)}, [n-2]^p, [np+2n-2]^1\right\}$.

By Result 1.1.14(b), we have

$$\begin{aligned}\text{Ch}_{\text{DL}}(\Gamma^*_{\text{nccc}}(G), x) &= x(x - n(p+1))^{(p+1)-1} \prod_{i=1}^{p+1} (x - (p+1)n - n_i)^{n_i - 1} \\ &= x(x-n(p+1))^p (x - n(p+1) - n)^{(p+1)(n-1)}.\end{aligned}$$

Therefore, DL-spec$(\Gamma^*_{\text{nccc}}(G)) = \Big\{[0]^1, [n(p+1)]^p, [n(p+1)+n]^{(p+1)(n-1)}\Big\}$.

By Result 1.1.14(c), we have

$$\begin{aligned}\text{Ch}_{\text{DQ}}(\Gamma^*_{\text{nccc}}(G), x) &= \prod_{i=1}^{p+1}(x - n(p+1) - n_i + 4)^{n_i - 1}\left(\prod_{i=1}^{p+1}(x - n(p+1) - 2n_i + 4) - \right.\\ &\left.\qquad \sum_{i=1}^{p+1} n_i \prod_{j=1, j\neq i}^{p+1}(x - n(p+1) - 2n_j + 4)\right)\\ &= (x - np - 2n + 4)^{(p+1)(n-1)}(x - np - 3n + 4)^p(x - 2np - 4n + 4).\end{aligned}$$

Therefore, DQ-spec$(\Gamma^*_{\text{nccc}}(G)) = \Big\{[np+2n-4]^{(p+1)(n-1)}, [np+3n-4]^p, [2np+4n-4]^1\Big\}$. The expression for $W(\Gamma^*_{\text{nccc}}(G))$ follows from Lemma 5.1.1. □

If G is a non-abelian group of order p^n with $|Z(G)| = p^{n-2}$, where p is prime and $n \geq 3$ then $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, we have the following corollary.

Corollary 5.1.3. *Let G be a non-abelian group of order p^n with $|Z(G)| = p^{n-2}$, where p is prime and $n \geq 3$. Then*

$$\begin{aligned}\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{&[-2]^{(p+1)\left((p-1)p^{n-3}-1\right)}, \Big[(p-1)p^{n-3} - 2\Big]^p,\\ &\Big[2(p-1)p^{n-3} + (p-1)p^{n-2} - 2\Big]^1\Big\},\end{aligned}$$

$$\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \left\{[0]^1, \Big[\Big(p^2-1\Big)p^{n-3}\Big]^p, \Big[\Big(p^2+p-2\Big)p^{n-3}\Big]^{(p+1)\left((p-1)p^{n-3}-1\right)}\right\}$$

$$\begin{aligned}\text{DQ-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{&\Big[-2p^{n-3} + p^{n-2} + p^{n-1} - 4\Big]^{(p+1)\left((p-1)p^{n-3}-1\right)},\\ &\Big[-3p^{n-3} + 2p^{n-2} + p^{n-1} - 4\Big]^p, \Big[2\Big(p^2+p-2\Big)p^{n-3} - 4\Big]^1\Big\}.\end{aligned}$$

and $W(\Gamma^*_{\text{nccc}}(G)) = \frac{(p-1)(p+1)p^{n-3}\left((p-1)(p+2)p^{n-3}-2\right)}{2}$.

The following theorem gives various distance spectra and Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for finite groups whose central quotient is isomorphic to a dihedral group.

Theorem 5.1.4. *Let G be a finite non-abelian group with $|Z(G)| = z$ and $\frac{G}{Z(G)} \cong D_{2n}$, (where $n \geq 3$).*

(a) *If n is even then*

$$\text{D-spec}(\Gamma^*_{nccc}(G)) = \Big\{[-2]^{\frac{1}{2}(n+1)z-3}, \left[\tfrac{z}{2}-2\right]^1, \left[\tfrac{1}{4}\left(-\sqrt{4n^2-12n+17}z+2nz+z-8\right)\right]^1, \left[\tfrac{1}{4}\left(\sqrt{4n^2-12n+17}z+2nz+z-8\right)\right]^1\Big\},$$

$$\text{DL-spec}(\Gamma^*_{nccc}(G)) = \left\{[0]^1, \left[\frac{(n+1)z}{2}\right]^2, [nz]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+2)z}{2}\right]^{z-2}\right\},$$

$$\text{DQ-spec}(\Gamma^*_{nccc}(G)) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+2)z}{2}-4\right]^{z-2}, \left[\frac{(n+3)z}{2}-4\right]^1, \left[\tfrac{1}{4}\left(-\sqrt{9n^2-34n+41}z+5nz+3z-16\right)\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9n^2-34n+41}z+5nz+3z-16\right)\right]^1\Big\}$$

and $W(\Gamma^*_{nccc}(G)) = \frac{1}{4}z\left(n^2z-2n+2z-2\right)$.

(b) *If* n *is odd then*

$$\text{D-spec}(\Gamma^*_{nccc}(G)) = \left\{[-2]^{\frac{(n+1)z}{2}-2}, \left[\tfrac{1}{2}\left(-\sqrt{n^2-4n+7}z+nz+z-4\right)\right]^1, \left[\tfrac{1}{2}\left(\sqrt{n^2-4n+7}z+nz+z-4\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma^*_{nccc}(G)) = \left\{[0]^1, \left[\frac{(n+1)z}{2}\right]^1, [nz]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}\right]^{z-1}\right\},$$

$$\text{DQ-spec}(\Gamma^*_{nccc}(G)) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}-4\right]^{z-1}, \left[\tfrac{1}{4}\left(-\sqrt{9n^2-46n+73}z+5nz+5z-16\right)\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9n^2-46n+73}z+5nz+5z-16\right)\right]^1\Big\}$$

and $W(\Gamma^*_{nccc}(G)) = \frac{1}{4}z\left(n^2z-2n+3z-2\right)$.

Proof. (a) If n is even then by Result 1.2.19, we have $\Gamma^*_{nccc}(G) = K_{\frac{(n-1)z}{2},\frac{z}{2},\frac{z}{2}}$. Here, $|v(\Gamma^*_{nccc}(G))| = \frac{(n+1)z}{2}$.

Using Result 1.1.14(a), we get

$$\begin{aligned}\text{Ch}_{\text{D}}(\Gamma^*_{nccc}(G), x) &= (x+2)^{\frac{(n+1)z}{2}-3}\left(\prod_{i=1}^{3}(x-n_i+2) - \sum_{i=1}^{3} n_i \prod_{j=1, j\neq i}^{3}(x-n_j+2)\right)\\ &= (x+2)^{\frac{1}{2}(n+1)z-3}(x-\frac{z}{2}+2)\Bigg((x-\frac{(n-1)z}{2}+2)(x-\frac{z}{2}+2)\\ &\quad -\frac{(n-1)z}{2}(x-\frac{z}{2}+2) - z(x-\frac{(n-1)z}{2}+2)\Bigg).\end{aligned}$$

Therefore, $\text{D-spec}(\Gamma^*_{nccc}(G)) = \Big\{[-2]^{\frac{1}{2}(n+1)z-3}, [\frac{z}{2}-2]^1,$ $[\frac{1}{4}\left(2nz+z-8+z\sqrt{4n^2-12n+17}\right)]^1, [\frac{1}{4}\left(2nz+z-8-z\sqrt{4n^2-12n+17}\right)]^1\Big\}.$

Using Result 1.1.14(b), we get

$$\begin{aligned}\mathrm{Ch}_{\mathrm{DL}}(\Gamma^*_{\mathrm{nccc}}(G),x) &= x\left(x-\frac{(n+1)z}{2}\right)^{3-1}\prod_{i=1}^{3}\left(x-\frac{(n+1)z}{2}-n_i\right)^{n_i-1}\\ &= x\left(x-\frac{(n+1)z}{2}\right)^{2}(x-nz)^{\frac{(n-1)z}{2}-1}\left(x-\frac{(n+2)z}{2}\right)^{z-2}.\end{aligned}$$

Therefore, DL-spec$(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[0]^1, \left[\frac{(n+1)z}{2}\right]^2, [nz]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+2)z}{2}\right]^{z-2}\right\}$.

Using Result 1.1.14(c), we get

$$\begin{aligned}\mathrm{Ch}_{\mathrm{DQ}}(\Gamma^*_{\mathrm{nccc}}(G),x) &= \prod_{i=1}^{3}\left(x-\frac{(n+1)z}{2}-n_i+4\right)^{n_i-1}\left(\prod_{i=1}^{3}\left(x-\frac{(n+1)z}{2}-2n_i+4\right)\right.\\ &\qquad\left.-\sum_{i=1}^{3}n_i\prod_{j=1,j\neq i}^{3}\left(x-\frac{(n+1)z}{2}-2n_j+4\right)\right)\end{aligned}$$

$$\begin{aligned}&= (x-nz+4)^{\frac{(n-1)z}{2}-1}\left(x-\frac{(n+2)z}{2}+4\right)^{z-2}\\ &\left(-\frac{1}{2}(n-1)z\left(-\frac{1}{2}(n+1)z+x-z+4\right)^2+\left(-\frac{1}{2}(n+1)z-(n-1)z+x+4\right)\right.\\ &\left(-\frac{1}{2}(n+1)z+x-z+4\right)^2-z\left(-\frac{1}{2}(n+1)z-(n-1)z+x+4\right)\\ &\left.\left(-\frac{1}{2}(n+1)z+x-z+4\right)\right).\end{aligned}$$

Therefore, DQ-spec$(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+2)z}{2}-4\right]^{z-2}, \left[\frac{(n+3)z}{2}-4\right]^{1},\right.$ $\left.\left[\frac{1}{4}\left(5nz+3z-16-z\sqrt{9n^2-34n+41}\right)\right]^1, \left[\frac{1}{4}\left(5nz+3z-16+z\sqrt{9n^2-34n+41}\right)\right]^1\right\}$. The expression for $W(\Gamma^*_{\mathrm{nccc}}(G))$ follows from Lemma 5.1.1.

(b) If n is odd then by Result 1.2.19, we have $\Gamma^*_{\mathrm{nccc}}(G) = K_{\frac{(n-1)z}{2},z}$. Here, $|v(\Gamma^*_{\mathrm{nccc}}(G))| = \frac{(n+1)z}{2}$.

Using Result 1.1.14(a), we get

$$\begin{aligned}\mathrm{Ch}_{\mathrm{D}}(\Gamma^*_{\mathrm{nccc}}(G),x) &= (x+2)^{\frac{(n+1)z}{2}-2}\left(\prod_{i=1}^{2}(x-n_i+2)-\sum_{i=1}^{2}n_i\prod_{j=1,j\neq i}^{3}(x-n_j+2)\right)\\ &= (x+2)^{\frac{(n+1)z}{2}-2}\left(\left(x-\frac{(n-1)z}{2}+2\right)(x-z+2)-\frac{(n-1)z}{2}(x-z+2)\right.\\ &\qquad\left.-z\left(x-\frac{(n-1)z}{2}+2\right)\right).\end{aligned}$$

Therefore, $\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[-2]^{\frac{(n+1)z}{2}-2}, \left[\frac{1}{2}(nz+z-4-z\sqrt{n^2-4n+7})\right]^1,$

$\left[\frac{1}{2}(nz+z-4+z\sqrt{n^2-4n+7})\right]^1\Big\}.$

Using Result 1.1.14(b), we get

$$\begin{aligned}\text{Ch}_{\text{DL}}(\Gamma^*_{\text{nccc}}(G), x) &= x\left(x - \frac{(n+1)z}{2}\right)^{2-1}\prod_{i=1}^{2}\left(x - \frac{(n+1)z}{2} - n_i\right)^{n_i-1}\\ &= x\left(x - \frac{(n+1)z}{2}\right)(x-nz)^{\frac{(n-1)z}{2}-1}\left(x - \frac{(n+3)z}{2}\right)^{z-1}.\end{aligned}$$

Therefore, $\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[0]^1, \left[\frac{(n+1)z}{2}\right]^1, [nz]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}\right]^{z-1}\Big\}.$

Using Result 1.1.14(c), we get

$$\begin{aligned}\text{Ch}_{\text{DQ}}(\Gamma^*_{\text{nccc}}(G), x) = \prod_{i=1}^{2}\left(x - \frac{(n+1)z}{2} - n_i + 4\right)^{n_i-1}&\left(\prod_{i=1}^{2}\left(x - \frac{(n+1)z}{2} - 2n_i + 4\right)\right.\\ &\left.-\sum_{i=1}^{2} n_i \prod_{j=1, j\neq i}^{2}\left(x - \frac{(n+1)z}{2} - 2n_j + 4\right)\right)\\ = (x-nz+4)^{\frac{(n-1)z}{2}-1}\left(x - \frac{(n+3)z}{2}+4\right)^{z-1}&\left(\left(x - \frac{(3n-1)z}{2}+4\right)\right.\\ \left(x - \frac{(n+5)z}{2}+4\right) - \frac{(n-1)z}{2}\left(x - \frac{(n+5)z}{2}+4\right) &\left.- z\left(x - \frac{(3n-1)z}{2}+4\right)\right).\end{aligned}$$

Therefore, $\text{DQ-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}-4\right]^{z-1},$ $\left[\frac{1}{4}(5nz+5z-16-z\sqrt{9n^2-46n+73})\right]^1, \left[\frac{1}{4}(5nz+5z-16+z\sqrt{9n^2-46n+73})\right]^1\Big\}$. The expression for $W(\Gamma^*_{\text{nccc}}(G))$ follows from Lemma 5.1.1. □

Corollary 5.1.5. *Let G be the dihedral group D_{2n}, where $n \geq 3$.*

(a) *If n is odd then*

$\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[-2]^{\frac{n-3}{2}}, \left[\frac{1}{2}\left(-\sqrt{n^2-4n+7}+n-3\right)\right]^1,$

$\left[\frac{1}{2}\left(\sqrt{n^2-4n+7}+n-3\right)\right]^1\Big\},$

$\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[0]^1, \left[\frac{n+1}{2}\right]^1, [n]^{\frac{n-3}{2}}\Big\},$

$\text{DQ-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[n-4]^{\frac{n-3}{2}}, \left[\frac{1}{4}\left(-\sqrt{9n^2-46n+73}+5n-11\right)\right]^1,$

$\left[\frac{1}{4}\left(\sqrt{9n^2-46n+73}+5n-11\right)\right]^1\Big\}$

and $W(\Gamma^*_{nccc}(G)) = \frac{(n-1)^2}{4}$.

(b) *If* n *and* $\frac{n}{2}$ *are even then*

$$\text{D-spec}(\Gamma^*_{nccc}(G)) = \left\{[-2]^{\frac{n-4}{2}}, [-1]^1, \left[\tfrac{1}{2}\left(-\sqrt{n^2-6n+17}+n-3\right)\right]^1, \left[\tfrac{1}{2}\left(\sqrt{n^2-6n+17}+n-3\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma^*_{nccc}(G)) = \left\{[0]^1, \left[\tfrac{n+2}{2}\right]^2, [n]^{\frac{n-4}{2}}\right\},$$

$$\text{DQ-spec}(\Gamma^*_{nccc}(G)) = \left\{[n-4]^{\frac{n-4}{2}}, \left[\tfrac{n-2}{2}\right]^1, \left[\tfrac{1}{4}\left(-\sqrt{9n^2-68n+164}+5n-10\right)\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9n^2-68n+164}+5n-10\right)\right]^1\right\}$$

and $W(\Gamma^*_{nccc}(G)) = \frac{(n^2-2n+4)}{4}$.

(c) *If* n *is even and* $\frac{n}{2}$ *is odd then*

$$\text{D-spec}(\Gamma^*_{nccc}(G)) = \left\{[-2]^{\frac{n-2}{2}}, \left[\tfrac{1}{2}\left(-\sqrt{n^2-8n+28}+n-2\right)\right]^1, \left[\tfrac{1}{2}\left(\sqrt{n^2-8n+28}+n-2\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma^*_{nccc}(G)) = \left\{[0]^1, \left[\tfrac{n+2}{2}\right]^1, [n]^{\frac{n-4}{2}}, \left[\tfrac{n+6}{2}\right]^1\right\},$$

$$\text{DQ-spec}(\Gamma^*_{nccc}(G)) = \left\{[n-4]^{\frac{n-4}{2}}, \left[\tfrac{n-2}{2}\right]^1, \left[\tfrac{1}{4}\left(-\sqrt{9n^2-92n+292}+5n-6\right)\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9n^2-92n+292}+5n-6\right)\right]^1\right\}$$

and $W(\Gamma^*_{nccc}(G)) = \frac{(n^2-2n+8)}{4}$.

Proof. We know that

$$|Z(G)| = \begin{cases} 1, & \text{for } n \text{ is odd} \\ 2, & \text{for } n \text{ is even} \end{cases}$$

and

$$\frac{G}{Z(G)} \cong \begin{cases} D_{2n}, & \text{for } n \text{ is odd} \\ D_{2\times 2}, & \text{for } n = 4 \\ D_{2\times \frac{n}{2}}, & \text{for } n \text{ is even and } n \geq 6. \end{cases}$$

Now, by using Theorem 5.1.2 and Theorem 5.1.4, we get the required result. □

Remark 5.1.6. *Given any perfect square k^2 (where $k \geq 1$) if we consider the group $G = D_{2(2k+1)}$ then by Corollary 5.1.5(a) we have $W(\Gamma^*_{\text{nccc}}(G)) = \frac{(2k+1-1)^2}{4} = k^2$. This shows that every perfect square can be viewed as Wiener index of $\Gamma^*_{\text{nccc}}(G)$ for some dihedral groups. Hence, Inverse Wiener index Problem is solved for $\Gamma^*_{\text{nccc}}(G)$ when n is a perfect square. However, it may be challenging to solve Inverse Wiener index Problem in general for $\Gamma^*_{\text{nccc}}(G)$.*

Corollary 5.1.7. *Let G be the group $U_{(n,m)}$, where $m \geq 3$ and $n \geq 2$.*

(a) *If m is odd then*

$$\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \left\{[-2]^{\frac{1}{2}(mn+n-4)}, \left[\tfrac{1}{2}\left(-\sqrt{m^2-4m+7}n+mn+n-4\right)\right]^1, \left[\tfrac{1}{2}\left(\sqrt{m^2-4m+7}n+mn+n-4\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \left\{[0]^1, \left[\tfrac{1}{2}(m+1)n\right]^1, [mn]^{\frac{1}{2}(m-1)n-1}, \left[\tfrac{1}{2}(m+3)n\right]^{n-1}\right\},$$

$$\text{DQ-spec}(\Gamma^*_{\text{nccc}}(G)) = \left\{[mn-4]^{\frac{1}{2}(m-1)n-1}, \left[\tfrac{1}{2}(m+3)n-4\right]^{n-1}, \left[\tfrac{1}{4}\left(-\sqrt{9m^2-46m+73}n+5mn+5n-16\right)\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9m^2-46m+73}n+5mn+5n-16\right)\right]^1\right\}$$

and $W(\Gamma^*_{\text{nccc}}(G)) = \frac{1}{4}n\left(m^2n-2m+3n-2\right)$.

(b) *If m and $\frac{m}{2}$ are even then*

$$\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \left\{[-2]^{\frac{mn}{2}+n-3}, [n-2]^1, \left[\tfrac{1}{4}\left(-2\sqrt{m^2-6m+17}n+2mn+2n-8\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{m^2-6m+17}n+2mn+2n-8\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \left\{[0]^1, \left[\tfrac{1}{2}(m+2)n\right]^2, [mn]^{\frac{1}{2}(m-2)n-1}, \left[\tfrac{1}{2}(m+4)n\right]^{2(n-1)}\right\},$$

$$\text{DQ-spec}(\Gamma^*_{\text{nccc}}(G)) = \left\{[mn-4]^{\frac{1}{2}(m-2)n-1}, \left[\tfrac{1}{2}(m+4)n-4\right]^{2n-2}, \left[\tfrac{1}{2}(m+6)n-4\right]^1, \left[-\tfrac{1}{4}\left(\sqrt{9m^2-68m+164}-5m-6\right)n-4\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9m^2-68m+164}+5m+6\right)n-4\right]^1\right\}$$

and $W(\Gamma^*_{\text{nccc}}(G)) = \frac{1}{4}n\left(m^2n-2m+8n-4\right)$.

(c) *If m is even and $\frac{m}{2}$ is odd then*

$$\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[-2]^{\frac{mn}{2}+n-2}, \left[-\tfrac{1}{2}\left(\sqrt{m^2-8m+28}-m-2\right)n-2\right]^1, \left[\tfrac{1}{2}\left(\sqrt{m^2-8m+28}+m+2\right)n-2\right]^1\Big\},$$

$$\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[0]^1, \left[\tfrac{1}{2}(m+2)n\right]^1, [mn]^{\frac{1}{2}(m-2)n-1}, \left[\tfrac{1}{2}(m+6)n\right]^{2n-1}\Big\},$$

$$\text{DQ-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[mn-4]^{\frac{1}{2}(m-2)n-1}, \left[\tfrac{1}{2}(m+6)n-4\right]^{2n-1}, \left[-\tfrac{1}{4}\left(\sqrt{9m^2-92m+292}-5m-10\right)n-4\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9m^2-92m+292}+5m+10\right)n-4\right]^1\Big\}$$

and $W(\Gamma^*_{\text{nccc}}(G)) = \frac{1}{4}n\left(m^2n-2m+12n-4\right)$.

Proof. We know that

$$|Z(G)| = \begin{cases} n, & \text{for } m \text{ is odd} \\ 2n, & \text{for } m \text{ is even} \end{cases}$$

and

$$\frac{G}{Z(G)} \cong \begin{cases} D_{2m}, & \text{for } m \text{ is odd} \\ D_{2\times 2}, & \text{for } m = 4 \\ D_{2\times \frac{m}{2}}, & \text{for } m \text{ is even and } m \geq 6. \end{cases}$$

Hence, by using Theorem 5.1.2 and Theorem 5.1.4, we get the required result. □

Corollary 5.1.8. *Let G be the group Q_{4n}, where $n \geq 2$.*

(a) *If n is even then*

$$\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[-2]^{n-2}, [-1]^1, \left[\tfrac{1}{4}\left(-2\sqrt{4n^2-12n+17}+4n-6\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{4n^2-12n+17}+4n-6\right)\right]^1\Big\},$$

$$\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \{[0]^1, [n+1]^2, [2n]^{n-2}\},$$

$$\text{DQ-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[2n-4]^{n-2}, [n-1]^1, \left[\tfrac{1}{4}\left(-2\sqrt{9n^2-34n+41}+10n-10\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{9n^2-34n+41}+10n-10\right)\right]^1\Big\}$$

and $W(\Gamma^*_{\text{nccc}}(G)) = n^2-n+1$.

(b) *If n is odd then*

$$\text{D-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[-2]^{n-1}, \left[\tfrac{1}{2}\left(-2\sqrt{n^2-4n+7}+2n-2\right)\right]^1, \left[\tfrac{1}{2}\left(2\sqrt{n^2-4n+7}+2n-2\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[0]^1, [n+1]^1, [2n]^{n-2}, [n+3]^1\right\},$$

$$\text{DQ-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[2(n-2)]^{n-2}, [n-1]^1, \left[\tfrac{1}{4}\left(-2\sqrt{9n^2-46n+73}+10n-6\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{9n^2-46n+73}+10n-6\right)\right]^1\right\}$$

and $W(\Gamma^*_{\mathrm{nccc}}(G)) = n^2 - n + 2$.

Proof. We know that $|Z(G)| = 2$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or D_{2n} according as $n = 2$ or $n \geq 3$. Hence, by using Theorem 5.1.2 and Theorem 5.1.4, we get the required result. □

Corollary 5.1.9. *Let G be the semidihedral group SD_{8n}, where $n \geq 2$.*

(a) *If n is even then*

$$\text{D-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[-2]^{2n-2}, [-1]^1, \left[\tfrac{1}{4}\left(-2\sqrt{16n^2-24n+17}+8n-6\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{16n^2-24n+17}+8n-6\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[0]^1, [2n+1]^2, [4n]^{2n-2}\right\},$$

$$\text{DQ-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[4n-4]^{2n-2}, [2n-1]^1, \left[\tfrac{1}{4}\left(-2\sqrt{36n^2-68n+41}+20n-10\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{36n^2-68n+41}+20n-10\right)\right]^1\right\}$$

and $W(\Gamma^*_{\mathrm{nccc}}(G)) = 4n^2 - 2n + 1$.

(b) *If n is odd then*

$$\text{D-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[-2]^{2n}, \left[\tfrac{1}{2}\left(4n-4\sqrt{n^2-4n+7}\right)\right]^1, \left[\tfrac{1}{2}\left(4\sqrt{n^2-4n+7}+4n\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[0]^1, [2(n+1)]^1, [4n]^{2n-3}, [2(n+3)]^3\right\},$$

$$\text{DQ-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[4(n-1)]^{2n-3}, [2(n+1)]^3, \left[\tfrac{1}{4}\left(-4\sqrt{9n^2-46n+73}+20n+4\right)\right]^1, \left[\tfrac{1}{4}\left(4\sqrt{9n^2-46n+73}+20n+4\right)\right]^1\right\}$$

and $W(\Gamma^*_{\mathrm{nccc}}(G)) = 4n^2 - 2n + 10$.

Proof. We know that

$$|Z(G)| = \begin{cases} 2, & \text{for } n \text{ is even} \\ 4, & \text{for } n \text{ is odd} \end{cases}$$

and

$$\frac{G}{Z(G)} \cong \begin{cases} D_{4n}, & \text{for } n \text{ is even} \\ D_{2n}, & \text{for } n \text{ is odd.} \end{cases}$$

Hence, by using Theorem 5.1.4, we get required result. □

Corollary 5.1.10. *Let G be the group U_{6n}, where $n \geq 2$. Then*
D-spec$(\Gamma^*_{nccc}(G)) = \left\{[-2]^{2n-2}, [n-2]^1, [3n-2]^1\right\}$, DL-spec$(\Gamma^*_{nccc}(G)) = \{[0]^1, [2n]^1, [3n]^{2(n-1)}\}$, DQ-spec$(\Gamma^*_{nccc}(G)) = \left\{[3n-4]^{2(n-1)}, [4(n-1)]^1, [6n-4]^1\right\}$ *and* $W(\Gamma^*_{nccc}(G)) = n(3n-2)$.

Proof. We know that $|Z(G)| = n$ and $\frac{G}{Z(G)} \cong D_{2\times 3}$. Hence, by using Theorem 5.1.4, we get the required result. □

We conclude this section with the following result.

Theorem 5.1.11. *Let G be the group V_{8n}, where $n \geq 2$.*

(a) *If n is even then*

$$\text{D-spec}(\Gamma^*_{nccc}(G)) = \Big\{[0]^1, [-2]^{2n-1}, [-\sqrt{4n^2-12n+17}+2n-1]^1, [\sqrt{4n^2-12n+17}+2n-1]^1\Big\},$$

$$\text{DL-spec}(\Gamma^*_{nccc}(G)) = \{[0]^1, [2n+2]^2, [4n]^{2n-3}, [2n+4]^2\},$$

$$\text{DQ-spec}(\Gamma^*_{nccc}(G)) = \Big\{[4n-4]^{2n-3}, [2n]^2, [2n+2]^1, \big[-\sqrt{9n^2-34n+41}+5n-1\big]^1 \big[\sqrt{9n^2-34n+41}+5n-1\big]^1\Big\}$$

and $W(\Gamma^*_{nccc}(G)) = 4n^2 - 2n + 6$.

(b) *If n is odd then*

$$\text{D-spec}(\Gamma^*_{nccc}(G)) = \Big\{[-1]^1, [-2]^{2n-2}, \big[\tfrac{1}{2}\big(-\sqrt{16n^2-24n+17}+4n-3\big)\big]^1, \big[\tfrac{1}{2}\big(\sqrt{16n^2-24n+17}+4n-3\big)\big]^1\Big\},$$

$$\text{DL-spec}(\Gamma^*_{nccc}(G)) = \{[0]^1, [4n]^{2n-2}, [2n+1]^2\},$$

$$\text{DQ-spec}(\Gamma^*_{nccc}(G)) = \Big\{[4n-4]^{2n-2}, [2n-1]^1, \big[\tfrac{1}{2}\big(-\sqrt{36n^2-68n+41}+10n-5\big)\big]^1, \big[\tfrac{1}{2}\big(\sqrt{36n^2-68n+41}+10n-5\big)\big]^1\Big\}$$

and $W(\Gamma^*_{nccc}(G)) = 4n^2 - 2n + 1$.

Proof. (a) If n is even then by Result 1.2.24, we have $\Gamma^*_{\text{nccc}}(G) = K_{2n-2,2,2}$. Here, $|v(\Gamma^*_{\text{nccc}}(G))| = 2(n+1)$.

Using Result 1.1.14(a), we get

$$\begin{aligned}\text{Ch}_{\text{D}}(\Gamma^*_{\text{nccc}}(G), x) &= (x+2)^{2n-1}\Big[\prod_{i=1}^{3}(x-n_i+2) - \sum_{i=1}^{3} n_i \prod_{j=1, j\neq i}^{3}(x-n_j+2)\Big]\\ &= (x+2)^{2n-1}x\big[x(x-2n+4)-(2n-2)x-4(x-2n+4)\big].\end{aligned}$$

Therefore, D-spec$(\Gamma^*_{\text{nccc}}(G)) = \Big\{[0]^1, [-2]^{2n-1}, \big[2n-1-\sqrt{4n^2-12n+17}\big]^1,$ $\big[2n-1+\sqrt{4n^2-12n+17}\big]^1\Big\}$.

Using Result 1.1.14(b), we get

$$\begin{aligned}\text{Ch}_{\text{DL}}(\Gamma^*_{\text{nccc}}(G), x) &= x(x-(2n+2))^{3-1}\prod_{i=1}^{3}(x-(2n+2)-n_i)^{n_i-1}\\ &= x(x-2n-2)^2(x-4n)^{2n-3}(x-2n-4)^2.\end{aligned}$$

Therefore, DL-spec$(\Gamma^*_{\text{nccc}}(G)) = \Big\{[0]^1, [2n+2]^2, [4n]^{2n-3}, [2n+4]^2\Big\}$.

Using Result 1.1.14(c), we also get

$$\begin{aligned}\text{Ch}_{\text{DQ}}(\Gamma^*_{\text{nccc}}(G), x) = \prod_{i=1}^{3}(x-(2n+2)-n_i+4)^{n_i-1}\Bigg(&\prod_{i=1}^{3}(x-(2n+2)-2n_i+4)\\ &-\sum_{i=1}^{3} n_i \prod_{j=1, j\neq i}^{3}(x-(2n+2)-2n_j+4)\Bigg)\\ = (x-4n+4)^{2n-3}(x-2n)^2(x-2n-2)\,&((x-6n+6)(x-2n-2)\\ &-(2n-2)(x-2n-2)-4(x-6n+6)).\end{aligned}$$

Thus DQ-spec$(\Gamma^*_{\text{nccc}}(G)) = \Big\{[4n-4]^{2n-3}, [2n]^2, [2n+2]^1, [5n-1-\sqrt{9n^2-34n+41}]^1,$ $[5n-1+\sqrt{9n^2-34n+41}]^1\Big\}$.

(b) If n is odd then by Result 1.2.24, we have $\Gamma^*_{\text{nccc}}(G) = K_{2n-1,1,1} = \Gamma^*_{\text{nccc}}(D_{2\times 4n})$. Hence, the result follows from Corollary 5.1.5. □

In the rest part of this section, we characterize various groups considered above such that $\Gamma^*_{\text{nccc}}(G)$ is D-integral, DL-integral and DQ-integral. By Theorem 5.1.2 and Corollary 5.1.10, the following result follows immediately.

Theorem 5.1.12. *Let G be a finite non-abelian group. Then $\Gamma^*_{\text{nccc}}(G)$ is D-integral, DL-integral and DQ-integral if*

(a) $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, *where p is any prime.*

(b) *G is isomorphic to U_{6n}.*

In view of the expressions for DL-spec$(\Gamma^*_{\text{nccc}}(G))$, in Corollary 5.1.5 – Corollary 5.1.10 and Theorem 5.1.11, the following result follows.

Theorem 5.1.13. *Let $G = D_{2n}$ (where $n \geq 3$), $U_{(n,m)}$ (where $m \geq 3$ and $n \geq 2$), Q_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$) and V_{8n} (where $n \geq 2$). Then $\Gamma^*_{\text{nccc}}(G)$ is DL-integral.*

The following lemma is useful in characterizing the groups considered in Theorem 5.1.13 such that $\Gamma^*_{\text{nccc}}(G)$ are D-integral and DQ-integral.

Lemma 5.1.14. *Let n be any positive integer. Then*

(a) $n^2 - 4n + 7$ *is perfect square if and only if* $n = 1, 3$.

(b) $n^2 - 8n + 28$ *is perfect square if and only if* $n = 2, 6$.

(c) $4n^2 - 12n + 17$ *is perfect square if and only if* $n = 1, 2$.

(d) $9n^2 - 46n + 73$ *is perfect square if and only if* $n = 1, 3, 6$.

(e) $9n^2 - 34n + 41$ *is perfect square if and only if* $n = 1, 2, 4$.

(f) $9n^2 - 68n + 164$ *is perfect square if and only if* $n = 2, 4, 5, 8$.

(g) $9n^2 - 92n + 292$ *is perfect square if and only if* $n = 2, 6, 12$.

Proof. (a) Let $n^2 - 4n + 7$ be a perfect square. Then there exist integers k such that $n^2 - 4n + 7 = k^2$ which gives $(k + n - 2)(k - n + 2) = 3$. Therefore, we have the following cases.

Case 1. $k + n - 2 = 1$ and $k - n + 2 = 3$

In this case, we have $k + n = 3$ and $k - n = 1$ which gives $k = 2$ and $n = 1$.

Case 2. $k + n - 2 = -1$ and $k - n + 2 = -3$

In this case, we have $k+n=1$ and $k-n=-5$ which gives $k=-2$ and $n=3$.

Case 3. $k+n-2=3$ and $k-n+2=1$

In this case, we have $k+n=5$ and $k-n=-1$ which gives $k=2$ and $n=3$.

Case 4. $k+n-2=-3$ and $k-n+2=-1$

In this case, we have $k+n=-1$ and $k-n=-3$ which gives $k=-2$ and $n=1$. Hence, the result follows.

(b) If $n^2-8n+28$ is a perfect square then there exist integers k such that $n^2-8n+28=k^2$ which gives $(k+n-4)(k-n+4)=12$. By considering various cases as above we get $n=2,6$. Hence, the result follows.

(c) If $4n^2-12n+17$ is a perfect square then there exist integers k such that $4n^2-12n+17=k^2$ which gives $(k+2n-3)(k-2n+3)=8$. By considering various cases as above we get $n=1,2$. Hence, the result follows.

(d) Let $9n^2-46n+73$ be a perfect square. Then there exist integers k such that $9n^2-46n+73=k^2$ which implies $9n^2-46n+(73-k^2)=0$. Since n is a positive integer the discriminant $\Omega=(-46)^2-4\times 9\times(73-k^2)=36k^2-512$ of the quadratic equation must be a perfect square. Let $36k^2-512=a^2$ for some integers a. Then we get $(6k+a)(6k-a)=512$. If $(6k+a)=1$ then $(6k-a)=512$ and so $k=\frac{513}{12}$ and $a=\frac{-511}{2}$; a contradiction. If $6k+a=2$ then $6k-a=256$ and so $k=\frac{258}{12}$ and $a=-127$; a contradiction. Similarly, it can be seen that the cases when $(6k+a)=-1$, $(6k-a)=-512$ and $6k+a=-2$, $6k-a=256$ are not possible. Therefore, without loss of generality, we consider the following cases.

Case 1. $6k+a=4$ and $6k-a=128$. In this case, we get $k=11$ and $a=-62$.

Case 2. $6k+a=-4$ and $6k-a=-128$. In this case, we get $k=-11$ and $a=62$.

Case 3. $6k+a=8$ and $6k-a=64$. In this case, we get $k=6$ and $a=-28$.

Case 4. $6k+a=-8$ and $6k-a=-64$. In this case, we get $k=-6$ and $a=28$.

Case 5. $6k+a=16$ and $6k-a=32$. In this case, we get $k=4$ and $a=-8$.

Case 6. $6k+a=-16$ and $6k-a=-32$. In this case, we get $k=-4$ and $a=8$.

Thus the possible values of k are $\pm 4, \pm 6$ and ± 11. Therefore, $9n^2-46n+73=16$, $9n^2-46n+73=36$ and $9n^2-46n+73=121$. On solving these equations we get $n=1,3,6$.

Therefore, $9n^2 - 46n + 73$ is perfect square if and only if $n = 1, 3, 6$.

(e) If $9n^2 - 34n + 41$ is a perfect square then there exist integers k such that $9n^2 - 34n + 41 = k^2$. The discriminant of this quadratic equation is $\Omega = 36k^2 - 320$. Let $36k^2 - 320 = a^2$ for some integer a. Then $(6k + a)(6k - a) = 320$. Now by considering various cases as in the proof of part (d) we get $k = \pm 3, \pm 4, \pm 7$. Therefore, $9n^2 - 34n + 41 = 9$, $9n^2 - 34n + 41 = 16$ and $9n^2 - 34n + 41 = 49$. On solving these equations we get $n = 1, 2, 4$. Therefore, $9n^2 - 34n + 41$ is a perfect square if and only if $n = 1, 2, 4$.

(f) If $9n^2 - 68n + 164$ is a perfect square then there exist integers k such that $9n^2 - 68n + 164 = k^2$. The discriminant of this quadratic equation is $\Omega = 36k^2 - 1280$. Let $36k^2 - 1280 = a^2$ for some integer a. Then $(6k + a)(6k - a) = 1280$. Now by considering various cases as above we get $k = \pm 6, \pm 7, \pm 8, \pm 14, \pm 27$. Therefore, $9n^2 - 68n + 164 = 36$, $9n^2 - 68n + 164 = 49$, $9n^2 - 68n + 164 = 64$, $9n^2 - 68n + 164 = 196$ and $9n^2 - 68n + 164 = 729$. On solving these equations we get $n = 2, 4, 5, 8$. Therefore, $9n^2 - 68n + 164$ is a perfect square if and only if $n = 2, 4, 5, 8$.

(g) If $9n^2 - 92n + 292$ is a perfect square then there exist integers k such that $9n^2 - 92n + 292 = k^2$. The discriminant of this quadratic equation is $\Omega = 36k^2 - 2048$. Let $36k^2 - 2048 = a^2$ for some integer a. Then $(6k + a)(6k - a) = 2048$. Now by considering various cases as above we get $k = \pm 8, \pm 12, \pm 22, \pm 43$. Therefore, $9n^2 - 92n + 292 = 64$, $9n^2 - 92n + 292 = 144$, $9n^2 - 92n + 292 = 484$ and $9n^2 - 92n + 292 = 1849$. On solving these equations we get $n = 2, 6, 12$. Therefore, $9n^2 - 68n + 164$ is perfect square if and only if $n = 2, 6, 12$. □

We conclude this section with the following characterization.

Theorem 5.1.15. *Let $G = D_{2n}$ (where $n \geq 3$), $U_{(n,m)}$ (where $m \geq 3$ and $n \geq 2$), Q_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$) and V_{8n} (where $n \geq 2$). Then*

(a) *$\Gamma^*_{\mathrm{nccc}}(G)$ is D-integral if and only if $G = D_6$, D_8, D_{12}, $U_{(n,3)}$, $U_{(n,4)}$, $U_{(n,6)}$, T_8, T_{12}, SD_{24}, V_{16} and U_{6n} for $n \geq 2$.*

(b) *$\Gamma^*_{\mathrm{nccc}}(G)$ is DQ-integral if and only if $G = D_6$, D_8, D_{12}, D_{16}, $U_{(n,3)}$, $U_{(n,4)}$, $U_{(n,6)}$, $U_{(n,8)}$, T_8, T_{12}, T_{16}, SD_{16}, SD_{24}, V_{16}, V_{32} and U_{6n} for $n \geq 2$.*

Proof. (a) Consider the following cases.

Case 1. $G = D_{2n}$, where $n \geq 3$.

If n is odd then by Corollary 5.1.5(a), it is sufficient to show that $\frac{1}{2}(-\sqrt{n^2-4n+7}+n-3)$ and $\frac{1}{2}(\sqrt{n^2-4n+7}+n-3)$ are integers. By Lemma 5.1.14(a), we have $n = 3$ and so $\frac{1}{2}(-\sqrt{n^2-4n+7}+n-3) = -1$ and $\frac{1}{2}(\sqrt{n^2-4n+7}+n-3) = 1$. Therefore, if n is odd then $\Gamma^*_{\text{nccc}}(G)$ is D-integral if and only if $G = D_6$.

If n and $\frac{n}{2}$ are even then in view of Corollary 5.1.5(b), it is sufficient to show that $\frac{1}{2}(-\sqrt{n^2-6n+17}+n-3)$ and $\frac{1}{2}\left(\sqrt{n^2-6n+17}+n-3\right)$ are integers. Putting $n = \frac{n}{2}$ in Lemma 5.1.14(c) we get that $n^2-6n+17$ is a perfect square if and only if $n = 4$. Therefore, $\frac{1}{2}(-\sqrt{n^2-6n+17}+n-3) = -1$ and $\frac{1}{2}\left(\sqrt{n^2-6n+17}+n-3\right) = 2$. So, in this case $\Gamma^*_{\text{nccc}}(G)$ is D-integral if and only if $G = D_8$.

If n is even and $\frac{n}{2}$ is odd then in view of Corollary 5.1.5(c), it is sufficient to show that $\frac{1}{2}(-\sqrt{n^2-8n+28}+n-2)$ and $\frac{1}{2}(\sqrt{n^2-8n+28}+n-2)$ are integers. By Lemma 5.1.14(b) we have $n = 6$ and so $\frac{1}{2}(-\sqrt{n^2-8n+28}+n-2) = 0$ and $\frac{1}{2}(\sqrt{n^2-8n+28}+n-2) = 4$. Therefore, If n is even and $\frac{n}{2}$ is odd then $\Gamma^*_{\text{nccc}}(G)$ is D-integral if and only if $G = D_{12}$.

Case 2. $G = U_{(n,m)}$, where $m \geq 3$ and $n \geq 2$.

If m is odd then by Corollary 5.1.7(a), it is sufficient to show that $\frac{1}{2}(mn+n-4-\sqrt{m^2-4m+7}n)$ and $\frac{1}{2}(mn+n-4+\sqrt{m^2-4m+7}n)$ are integers. By Lemma 5.1.14(a), we have $m = 3$ and so $\frac{1}{2}\left(-\sqrt{m^2-4m+7}n+mn+n-4\right) = n-2$ and $\frac{1}{2}\left(\sqrt{m^2-4m+7}n+mn+n-4\right) = 3n-2$. Therefore, if m is odd then $\Gamma^*_{\text{nccc}}(G)$ is D-integral if and only if $G = U_{(n,3)}$.

If m and $\frac{m}{2}$ are even then in view of Corollary 5.1.7(b), it is sufficient to show that $\frac{1}{4}\left(-2\sqrt{m^2-6m+17}n+2mn+2n-8\right)$ and $\frac{1}{4}\left(2\sqrt{m^2-6m+17}n+2mn+2n-8\right)$ are integers. Putting $n = \frac{m}{2}$ in Lemma 5.1.14(c) we get that $m^2-6m+17$ is a perfect square if and only if $m = 4$. Therefore, $\frac{1}{4}\left(-2\sqrt{m^2-6m+17}n+2mn+2n-8\right) = n-2$ and $\frac{1}{4}\left(2\sqrt{m^2-6m+17}n+2mn+2n-8\right) = 4n-2$. So, in this case $\Gamma^*_{\text{nccc}}(G)$ is D-integral if and only if $G = U_{(n,4)}$.

If m is even and $\frac{m}{2}$ is odd then in view of Corollary 5.1.7(c), it is sufficient to show that $-\frac{1}{2}\left(\sqrt{m^2-8m+28}-m-2\right)n-2$ and $\frac{1}{2}\left(\sqrt{m^2-8m+28}+m+2\right)n-2$ are integers. By Lemma 5.1.14(b) we have $m = 6$ and so $-\frac{1}{2}\left(\sqrt{m^2-8m+28}-m-2\right)n-2 = 2n-2$ and $\frac{1}{2}\left(\sqrt{m^2-8m+28}+m+2\right)n-2 = 6n-2$. Therefore, If m is even and $\frac{m}{2}$ is odd then $\Gamma^*_{\mathrm{nccc}}(G)$ is D-integral if and only if $G = U_{(n,6)}$.

Case 3. $G = Q_{4n}$, where $n \geq 2$.

If n is even then by Corollary 5.1.8(a), it is sufficient to show that $\frac{1}{4}(-2\sqrt{4n^2-12n+17} +4n-6)$ and $\frac{1}{4}(2\sqrt{4n^2-12n+17}+4n-6)$ are integers. By Lemma 5.1.14(c), we have $n = 2$ and so $\frac{1}{4}(-2\sqrt{4n^2-12n+17}+4n-6) = -1$ and $\frac{1}{4}(2\sqrt{4n^2-12n+17}+4n-6) = 2$. Therefore, if n is even then $\Gamma^*_{\mathrm{nccc}}(G)$ is D-integral if and only if $G = T_8$.

If n is odd then in view of Corollary 5.1.8(b), it is sufficient to show that $\frac{1}{2}(2n-2-2\sqrt{n^2-4n+7})$ and $\frac{1}{2}(2n-2+2\sqrt{n^2-4n+7})$ are integers. By Lemma 5.1.14(a), we have $n = 3$ and so $\frac{1}{2}\left(-2\sqrt{n^2-4n+7}+2n-2\right) = 0$ and $\frac{1}{2}\left(2\sqrt{n^2-4n+7}+2n-2\right) = 4$. So, in this case $\Gamma^*_{\mathrm{nccc}}(G)$ is D-integral if and only if $G = T_{12}$.

Case 4. $G = SD_{8n}$, where $n \geq 2$.

If n is even then by Corollary 5.1.9(a), it is sufficient to show that $\frac{1}{4}(-2\sqrt{16n^2-24n+17} +8n-6)$ and $\frac{1}{4}(8n-6+2\sqrt{16n^2-24n+17})$ are integers. Putting $n = 2n$ in Lemma 5.1.14(c), we get that $16n^2-24n+17$ is a perfect square if and only if $n = 1$. Therefore, if n is even then $\Gamma^*_{\mathrm{nccc}}(G)$ is not D-integral.

If n is odd then in view of Corollary 5.1.9(b), it is sufficient to show that $\frac{1}{2}(4n-4\sqrt{n^2-4n+7})$ and $\frac{1}{2}(4\sqrt{n^2-4n+7}+4n)$ are integers. By Lemma 5.1.14(a), we have $n = 3$ and so $\frac{1}{2}(4n-4\sqrt{n^2-4n+7}) = 2$ and $\frac{1}{2}(4\sqrt{n^2-4n+7}+4n) = 10$. So, in this case $\Gamma^*_{\mathrm{nccc}}(G)$ is D-integral if and only if $G = SD_{24}$.

Case 5. $G = V_{8n}$, where $n \geq 2$.

If n is even then in view of Corollary 5.1.11(a), it is sufficient to show that $2n-1-\sqrt{4n^2-12n+17}$ and $2n-1+\sqrt{4n^2-12n+17}$ are integers. By Lemma 5.1.14(c), we have $n = 2$ and so $-\sqrt{4n^2-12n+17}+2n-1 = 0$ and $\sqrt{4n^2-12n+17}+2n-1 = 6$. So, in this case $\Gamma^*_{\mathrm{nccc}}(G)$ is D-integral if and only if $G = V_{16}$.

If n is odd then by Corollary 5.1.11(b), it is sufficient to show that $\frac{1}{2}(4n-3-\sqrt{16n^2-24n+17})$ and $\frac{1}{2}(4n-3+\sqrt{16n^2-24n+17})$ are integers. Putting $n = 2n$ in Lemma 5.1.14(c), we get that $16n^2-24n+17$ is a perfect square if and only if $n = 1$. Therefore, if n is odd then $\Gamma^*_{\mathrm{nccc}}(G)$ is not D-integral.

Case 6. $G = U_{6n}$, where $n \geq 2$.

By Corollary 5.1.10, it follows that $\Gamma^*_{\mathrm{nccc}}(G)$ is D-integral for $n \geq 2$.

(b) Consider the following cases.

Case 1. $G = D_{2n}$, where $n \geq 3$.

If n is odd then by Corollary 5.1.5(a), it is sufficient to show that $\frac{1}{4}(-9\sqrt{9n^2 - 46n + 73} + 5n - 11)$ and $\frac{1}{4}(9\sqrt{9n^2 - 46n + 73} + 5n - 11)$ are integers. By Lemma 5.1.14(d), we have $n = 3$ and so $\frac{1}{4}(-9\sqrt{9n^2 - 46n + 73} + 5n - 11) = 0$ and $\frac{1}{4}(9\sqrt{9n^2 - 46n + 73} + 5n - 11) = 0$. If n is odd then $\Gamma^*_{nccc}(G)$ is DQ-integral if and only if $G = D_6$.

Note that $\frac{n-2}{2}$ is an integer if n is even. If n and $\frac{n}{2}$ are even then in view of Corollary 5.1.5(b), it is sufficient to show that $\frac{1}{4}(5n - 10 - \sqrt{9n^2 - 68n + 164})$ and $\frac{1}{4}(5n - 10 + \sqrt{9n^2 - 68n + 164})$ are integers. By Lemma 5.1.14(f), we have $n = 4, 8$ and so $\frac{1}{4}(-\sqrt{9n^2 - 68n + 164} + 5n - 10) = 1$ or 4 and $\frac{1}{4}(\sqrt{9n^2 - 68n + 164} + 5n - 10) = 4$ or 11 according as $n = 4$ or $n = 8$. Therefore, if n and $\frac{n}{2}$ are even then $\Gamma^*_{nccc}(G)$ is DQ-integral if and only if $G = D_8, D_{16}$.

If n is even and $\frac{n}{2}$ is odd then in view of Corollary 5.1.5(c), it is sufficient to show that $\frac{1}{4}(-\sqrt{9n^2 - 92n + 292} + 5n - 6)$ and $\frac{1}{4}(\sqrt{9n^2 - 92n + 292} + 5n - 6)$ are integers. By Lemma 5.1.14(g), we have $n = 6$ and so $\frac{1}{4}(-\sqrt{9n^2 - 92n + 292} + 5n - 6) = 4$ and $\frac{1}{4}(\sqrt{9n^2 - 92n + 292} + 5n - 6) = 8$. Therefore, $\Gamma^*_{nccc}(G)$ is DQ-integral if and only if $G = D_{12}$.

Case 2. $G = U_{(n,m)}$, where $m \geq 3$ and $n \geq 2$.

If m is odd then by Corollary 5.1.7(a), it is sufficient to show that $\frac{1}{4}(5mn + 5n - 16 - \sqrt{9m^2 - 46m + 73}n)$ and $\frac{1}{4}(5mn + 5n - 16 + \sqrt{9m^2 - 46m + 73}n)$ are integers. By Lemma 5.1.14(d), we have $m = 3$ and so $\frac{1}{4}(5mn + 5n - 16 - \sqrt{9m^2 - 46m + 73}n) = 4(n-1)$ and $\frac{1}{4}(5mn + 5n - 16 + \sqrt{9m^2 - 46m + 73}n) = 6n - 4$. Again for m is odd, $\frac{(m+3)n}{2} - 4$ is integer. So if m is odd then $\Gamma^*_{nccc}(G)$ is DQ-integral if and only if $G = U_{(n,3)}$.

Note that $\frac{(m+4)n}{2} - 4$ and $\frac{(m+6)n}{2} - 4$ are integers if m is even. If m and $\frac{m}{2}$ are even then in view of Corollary 5.1.7(b), it is sufficient to show that $-\frac{1}{4}(\sqrt{9m^2 - 68m + 164} - 5m - 6)n - 4$ and $\frac{1}{4}\left(\sqrt{9m^2 - 68m + 164} + 5m + 6\right)n - 4$ are integers. By Lemma 5.1.14(f), we have $m = 4, 8$ and so $-\frac{1}{4}(\sqrt{9m^2 - 68m + 164} - 5m - 6)n - 4 = 5n - 4$ or $8n - 4$ and $\frac{1}{4}\left(\sqrt{9m^2 - 68m + 164} + 5m + 6\right)n - 4 = 8n - 4$ or $15n - 4$ according as $m = 4$ or $m = 8$. Therefore, if m and $\frac{m}{2}$ are even then $\Gamma^*_{nccc}(G)$ is DQ-integral if and only if $G = U_{(n,4)}, U_{(n,8)}$.

If m is even and $\frac{m}{2}$ is odd then in view of Corollary 5.1.7(c), it is sufficient to show that $-\frac{1}{4}\left(\sqrt{9m^2 - 92m + 292} - 5m - 10\right)n - 4$ and $\frac{1}{4}\left(\sqrt{9m^2 - 92m + 292} + 5m + 10\right)n - 4$ are integers. By Lemma 5.1.14(g), we have $m = 6$ and so $-\frac{1}{4}(\sqrt{9m^2 - 92m + 292} - 5m - 10)n - 4 = 8n - 4$ and $\frac{1}{4}(5m + 10 + \sqrt{9m^2 - 92m + 292})n - 4 = 12n - 4$. Again for m is even $\frac{(m+6)n}{2} - 4$ is an integer. Therefore, $\Gamma^*_{\text{nccc}}(G)$ is DQ-integral if and only if $G = U_{(n,6)}$.

Case 3. $G = Q_{4n}$, where $n \geq 2$.

If n is even then by Corollary 5.1.8(a), it is sufficient to show that $\frac{1}{4}(-2\sqrt{9n^2 - 34n + 41} + 10n - 10)$ and $\frac{1}{4}(2\sqrt{9n^2 - 34n + 41} + 10n - 10)$ are integers. By Lemma 5.1.14(e), we have $n = 2, 4$ and so $\frac{1}{4}(-2\sqrt{9n^2 - 34n + 41} + 10n - 10) = 1$ or 4 and $\frac{1}{4}(2\sqrt{9n^2 - 34n + 41} + 10n - 10) = 4$ or 11 according as $n = 2$ or $n = 4$. If n is even then $\Gamma^*_{\text{nccc}}(G)$ is DQ-integral if and only if $G = T_8, T_{16}$.

If n is odd then by Corollary 5.1.8(b), it is sufficient to show that $\frac{1}{4}(10n - 6 - 2\sqrt{9n^2 - 46n + 73})$ and $\frac{1}{4}(10n - 6 + 2\sqrt{9n^2 - 46n + 73})$ are integers. By Lemma 5.1.14(d), we have $n = 3$ and so $\frac{1}{4}(-2\sqrt{9n^2 - 46n + 73} + 10n - 6) = 4$ and $\frac{1}{4}(10n - 6 + 2\sqrt{9n^2 - 46n + 73}) = 8$. If n is odd then $\Gamma^*_{\text{nccc}}(G)$ is DQ-integral if and only if $G = T_{12}$.

Case 4. $G = SD_{8n}$, where $n \geq 2$.

If n is even then by Corollary 5.1.9(a), it is sufficient to show that $\frac{1}{4}(-2\sqrt{36n^2 - 68n + 41} + 20n - 10)$ and $\frac{1}{4}(2\sqrt{36n^2 - 68n + 41} + 20n - 10)$ are integers. Putting $n = 2n$ in Lemma 5.1.14(e), we have $n = 2$ and so $\frac{1}{4}(20n - 10 - 2\sqrt{36n^2 - 68n + 41}) = 4$ and $\frac{1}{4}(20n - 10 + 2\sqrt{36n^2 - 68n + 41}) = 11$. If n is even then $\Gamma^*_{\text{nccc}}(G)$ is DQ-integral if and only if $G = SD_{16}$.

If n is odd then by Corollary 5.1.9(b), it is sufficient to show that $\frac{1}{4}(20n + 4 - 4\sqrt{9n^2 - 46n + 73})$ and $\frac{1}{4}(20n + 4 + 4\sqrt{9n^2 - 46n + 73})$ are integers. By Lemma 5.1.14(d), we have $n = 3$ and so $\frac{1}{4}(20n + 4 - 4\sqrt{9n^2 - 46n + 73}) = 12$ and $\frac{1}{4}(20n + 4 - 4\sqrt{9n^2 - 46n + 73}) = 20$. If n is odd then $\Gamma^*_{\text{nccc}}(G)$ is DQ-integral if and only if $G = SD_{24}$.

Case 5. $G = V_{8n}$, where $n \geq 2$.

If n is even then by Corollary 5.1.11(a), it is sufficient to show that $-\sqrt{9n^2 - 34n + 41} + 5n - 1$ and $\sqrt{9n^2 - 34n + 41} + 5n - 1$ are integers. By Lemma 5.1.14(e), we have $n = 2, 4$ and so $-\sqrt{9n^2 - 34n + 41} + 5n - 1 = 6$ or 12 and $\sqrt{9n^2 - 34n + 41} + 5n - 1 = 12$ or 26 according as $n = 2$ or $n = 4$. If n is even then $\Gamma^*_{\text{nccc}}(G)$ is DQ-integral if and only if $G = V_{16}, V_{32}$.

If n is odd then by Corollary 5.1.11(b), it is sufficient to show that $\frac{1}{2}(10n - 5 - \sqrt{36n^2 - 68n + 41})$ and $\frac{1}{2}(10n - 5 + \sqrt{36n^2 - 68n + 41})$ are integers. Putting $n = 2n$

in Lemma 5.1.14(e), we get that $36n^2 - 68n + 41$ is a perfect square if and only if $n = 1$. Therefore, if n is odd then $\Gamma^*_{\text{nccc}}(G)$ is not DQ-integral.

Case 6. $G = U_{6n}$, where $n \geq 2$.

By Corollary 5.1.10, it follows that $\Gamma^*_{\text{nccc}}(G)$ is DQ-integral for $n \geq 2$. □

5.2 Various distance energies

In this section we compute various distance energies of $\Gamma^*_{\text{nccc}}(G)$ for the groups considered in Section 5.1. In view of (5.1.a), we have $\frac{\text{tr}(\text{DL}(\Gamma))}{|v(\Gamma)|} = \frac{2W(\Gamma)}{|v(\Gamma)|} = \frac{\text{tr}(\text{DQ}(\Gamma))}{|v(\Gamma)|}$. Therefore,

$$\text{E}_{\text{DL}}(\Gamma) = \sum_{\beta \in \text{DL-spec}(\Gamma)} |\beta - \Delta_{\text{D}}(\Gamma)| \text{ and } \text{E}_{\text{DQ}}(\Gamma) = \sum_{\gamma \in \text{DQ-spec}(\Gamma)} |\gamma - \Delta_{\text{D}}(\Gamma)|,$$

where $\Delta_{\text{D}}(\Gamma) = \frac{\text{tr}(\text{DL}(\Gamma))}{|v(\Gamma)|} = \frac{2W(\Gamma)}{|v(\Gamma)|}$. Thus, $W(\Gamma)$ plays a crucial role in computing distance Laplacian and signless Laplacian energies of Γ.

We begin with the class of finite groups whose central quotients are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for any prime p.

Theorem 5.2.1. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is any prime and $|Z(G)| \geq 2$. If $n = \frac{(p-1)z}{p}$, where $z = |Z(G)|$, then*

$$\text{E}_{\text{D}}(\Gamma^*_{\text{nccc}}(G)) = \text{E}_{\text{DL}}(\Gamma^*_{\text{nccc}}(G)) = \text{E}_{\text{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 4, & \text{for } n = 1 \\ 4(n-1)(p+1), & \text{for } n \geq 2. \end{cases}$$

Proof. By Theorem 5.1.2 we have $\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \{[-2]^{(n-1)(p+1)}, [n-2]^p, [np+2n-2]^1\}$. Therefore,

$$\begin{aligned} \text{E}_{\text{D}}(\Gamma^*_{\text{nccc}}(G)) &= (n-1)(p+1) \times |-2| + p \times |n-2| + 1 \times |np+2n-2| \\ &= 2(n-1)(p+1) + p \times |n-2| + 1 \times (np+2n-2). \end{aligned}$$

Hence, $\text{E}_{\text{D}}(\Gamma^*_{\text{nccc}}(G)) = 4$ or $4(n-1)(p+1)$ according as $n = 1$ or $n \geq 2$, noting that

$$|n-2| = \begin{cases} 1, & \text{for } n = 1 \\ n-2, & \text{for } n \geq 2. \end{cases}$$

By Theorem 5.1.2 we have DL-spec$(\Gamma^*_{\text{nccc}}(G)) = \{[0]^1, [n(p+1)]^p, [n(p+1)+n]^{(p+1)(n-1)}\}$ and $W(\Gamma^*_{\text{nccc}}(G)) = \frac{n(p+1)(n(p+2)-2)}{2}$. Therefore, $\Delta_{\text{D}}(\Gamma^*_{\text{nccc}}(G)) = n(p+2)-2$. We have

$$\begin{aligned}\text{E}_{\text{DL}}(\Gamma^*_{\text{nccc}}(G)) &= 1 \times |0 - \Delta_{\text{D}}(\Gamma^*_{\text{nccc}}(G))| + p \times |n(p+1) - \Delta_{\text{D}}(\Gamma^*_{\text{nccc}}(G))| \\ &\quad + (p+1)(n-1) \times |n(p+1) + n - \Delta_{\text{D}}(\Gamma^*_{\text{nccc}}(G))| \\ &= |np + 2n - 2| + p \times |2 - n| + (p+1)(n-1) \times |2|.\end{aligned}$$

Hence, $\text{E}_{\text{DL}}(\Gamma^*_{\text{nccc}}(G)) = 4$ or $4(n-1)(p+1)$ according as $n = 1$ or $n \geq 2$.

By Theorem 5.1.2 we also have DQ-spec$(\Gamma^*_{\text{nccc}}(G)) = \{[np+2n-4]^{(p+1)(n-1)}, [np+3n-4]^p, [2np+4n-4]^1\}$. We have

$$\begin{aligned}\text{E}_{\text{DQ}}(\Gamma^*_{\text{nccc}}(G)) &= (n-1)(p+1) \times |(np+2n-4) - \Delta_{\text{D}}(\Gamma^*_{\text{nccc}}(G))| \\ &\quad + p \times |(np+3n-4) - \Delta_{\text{D}}(\Gamma^*_{\text{nccc}}(G))| + 1 \times |(2np+4n-4) - \Delta_{\text{D}}(\Gamma^*_{\text{nccc}}(G))| \\ &= (n-1)(p+1) \times |-2| + p \times |n-2| + |np+2n-2|.\end{aligned}$$

Therefore, $\text{E}_{\text{DQ}}(\Gamma^*_{\text{nccc}}(G)) = 4$ or $4(n-1)(p+1)$ according as $n = 1$ or $n \geq 2$. Hence, the result follows. □

Corollary 5.2.2. *Let G be a non-abelian group of order p^n with $|Z(G)| = p^{n-2}$, where p is prime and $n \geq 3$. Then*

$$\text{E}_{\text{D}}(\Gamma^*_{\text{nccc}}(G)) = \text{E}_{\text{DL}}(\Gamma^*_{\text{nccc}}(G)) = \text{E}_{\text{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \frac{4(p+1)\left(p^{n+1} - p^n - p^3\right)}{p^3}.$$

Remark 5.2.3. *It is noteworthy that the first couple of equalities in Problem 1.1.13 were obtained in [23] for $\Gamma^*_{\text{nccc}}(G)$ where G is a group whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. We attempt to solve Problem 1.1.13 by computing various distance energies of $\Gamma^*_{\text{nccc}}(G)$ for this class of groups. Unfortunately, the third equality in Problem 1.1.13 does not hold though the last couple of equalities hold.*

We now consider the class of finite groups whose central quotient is isomorphic to the dihedral group D_{2n} for $n \geq 3$. This class of groups includes the well-known groups viz. D_{2n} (where $n \geq 3$), $U_{(n,m)}$ (where $m \geq 3$ and $n \geq 2$), Q_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$) and U_{6n} (where $n \geq 2$).

Theorem 5.2.4. *Let G be a finite non-abelian group with $|Z(G)| = z$ and $\frac{G}{Z(G)} \cong D_{2n}$, (where $n \geq 3$).*
(a) *If n is even then*

$$\mathrm{E}_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 2n - 3 + \sqrt{4n^2 - 12n + 17}, & \text{for } z = 2 \\ 6n - 5, & \text{for } z = 3 \\ 2(nz + z - 6), & \text{for } z \geq 4. \end{cases}$$

$$\mathrm{E}_{\mathrm{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{2\left(2n^2z - 2n(z+3) + 5z - 6\right)}{n+1}, & \text{for } n = 4 \ \&\ z = 2, 3 \\ \frac{n^2z^2 + 2n^2z - 3nz^2 - 2nz - 4n + 2z^2 + 2z - 4}{n+1}, & \text{otherwise.} \end{cases}$$

$$\mathrm{E}_{\mathrm{DQ}}(\Gamma^*_{\text{nccc}}(G)) =$$

$$\begin{cases} \frac{n^2z + n\left(\left(\sqrt{9n^2-34n+41}+8\right)z - 8\right) + \left(\sqrt{9n^2-34n+41}-5\right)z - 8}{2(n+1)}, & \text{for } z = 2 \ \&\ n \geq 4; \\ & z = 3 \ \&\ n = 4, 6; \\ & z = 4 \ \&\ n = 4 \\ \frac{1}{5}\left(6z^2 + 9z - 20\right), & \text{for } n = 4 \ \&\ z \geq 5 \\ \frac{z\left(n^2(2z-3) + n\left(\sqrt{9n^2-26n+33} - 6z + 4\right) + \sqrt{9n^2-26n+33} + 4z + 7\right)}{2(n+1)}, & \text{otherwise.} \end{cases}$$

(b) *If n is odd then*

$$\mathrm{E}_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{1}{2}\left(4\sqrt{n^2 - 4n + 7} + n - 3\right), & \text{for } z = 1 \\ \frac{5}{2}(nz + z - 4), & \text{for } z \geq 2. \end{cases}$$

$$\mathrm{E}_{\mathrm{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{2n^2z - 4n + 6z - 4}{n+1}, & \text{for } n = 3, 5 \ \&\ z = 1 \\ \frac{3n^2z - 2nz - 8n + 11z - 8}{n+1}, & \text{for } n = 7 \ \&\ z = 1;\ n = 3 \ \&\ z \geq 2; \\ & n = 5 \ \&\ z = 2 \\ \frac{n^2z^2 + 2n^2z - 4nz^2 - 2nz - 4n + 3z^2 + 4z - 4}{n+1}, & \text{otherwise.} \end{cases}$$

$$E_{DQ}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{n^2z+10nz-8n-7z-8}{n+1}, & \textit{for } n=3 \ \& \ z\geq 2; \\ & n=5 \ \& \ z=3,4,5 \\ \frac{1}{2}\left(n+\sqrt{n(9n-46)+73}-\frac{16}{n+1}+9\right)z-4, & \textit{for } n=3 \ \& \ z=1 \\ \frac{(z-1)(n((n-4)z+4)+3z+4)}{n+1}, & \textit{for } n=5 \ \& \ z\geq 6; \\ & n=7 \ \& \ z\geq 56 \\ \frac{1}{2}\left(n+\sqrt{n(9n-46)+73}-\frac{16}{n+1}+9\right)z-4, & \textit{for } n=5 \ \& \ z=1,2; \\ & n=7 \ \& \ z=1,2,3 \\ \frac{1}{2}z\left(2\left(\frac{8}{n+1}-5\right)z+n(2z-3)+\sqrt{n(9n-46)+73}+9\right), & \textit{otherwise.} \end{cases}$$

Proof. (a) If n is even then $z\neq 1$. By Theorem 5.1.4(a), we have

$$\text{D-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[-2]^{\frac{1}{2}(n+1)z-3}, [\tfrac{z}{2}-2]^1, [\tfrac{1}{4}\left(2nz+z-8+z\sqrt{4n^2-12n+17}\right)]^1, [\tfrac{1}{4}\left(2nz+z-8-z\sqrt{4n^2-12n+17}\right)]^1\Big\}.$$

We have $A_1 := |-2| = 2$; $A_2 := |\frac{z}{2}-2| = 2-\frac{z}{2}$ or $\frac{z}{2}-2$ according as $z=2,3$ or $z\geq 4$; $A_3 := \left|\frac{1}{4}\left(2nz+z-8+z\sqrt{4n^2-12n+17}\right)\right| = \frac{1}{4}\left(2nz+z-8+z\sqrt{4n^2-12n+17}\right)$ and

$$A_4 := \left|\frac{1}{4}\left(2nz+z-8-z\sqrt{4n^2-12n+17}\right)\right| = \begin{cases} -\frac{2nz+z-8-z\sqrt{4n^2-12n+17}}{4}, & \text{for } z=1,2 \\ \frac{2nz+z-8-z\sqrt{4n^2-12n+17}}{4}, & \text{for } z\geq 3. \end{cases}$$

Hence,

$$E_D(\Gamma^*_{\text{nccc}}(G)) = \left(\frac{(n+1)z}{2}-3\right)\times A_1 + 1\times A_2 + 1\times A_3 + 1\times A_4$$
$$= \begin{cases} 2n-3+\sqrt{4n^2-12n+17}, & \text{for } z=2 \\ 6n-5, & \text{for } z=3 \\ 2(nz+z-6), & \text{for } z\geq 4. \end{cases}$$

By Theorem 5.1.4(a), we have $\text{DL-spec}(\Gamma^*_{\text{nccc}}(G)) = \Big\{[0]^1, [\frac{(n+1)z}{2}]^2, [nz]^{\frac{(n-1)z}{2}-1}, [\frac{(n+2)z}{2}]^{z-2}\Big\}$ and $W(\Gamma^*_{\text{nccc}}(G)) = \frac{1}{4}z\left(n^2z-2n+2z-2\right)$. Therefore, $\Delta_D(\Gamma^*_{\text{nccc}}(G)) =$

$\frac{n^2z-2n+2z-2}{n+1}$. We have

$$L_1 := \left|0 - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \frac{n^2z-2n+2z-2}{n+1},$$

$$L_2 := \left|\frac{1}{2}(n+1)z - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = -\frac{-n^2z+2nz+4n-3z+4}{2n+2},$$

$$L_3 := \left|nz - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \frac{n(z+2)-2z+2}{n+1}$$

and

$$L_4 := \left|\frac{1}{2}(n+2)z - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \begin{cases} \frac{-n^2z+3nz+4n-2z+4}{2n+2}, & \text{for } n=4 \ \& \ z=2,3 \\ -\frac{-n^2z+3nz+4n-2z+4}{2n+2}, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} {\rm E_{DL}}(\Gamma^*_{\rm nccc}(G)) &= 1\times L_1 + 2\times L_2 + \left(\frac{(n-1)z}{2}-1\right)\times L_3 + (z-2)\times L_4 \\ &= \begin{cases} \frac{2\left(2n^2z-2n(z+3)+5z-6\right)}{n+1}, & \text{for } n=4 \ \& \ z=2,3 \\ \frac{n^2z^2+2n^2z-3nz^2-2nz-4n+2z^2+2z-4}{n+1}, & \text{otherwise.} \end{cases} \end{aligned}$$

By Theorem 5.1.4(a), we also have DQ-spec$(\Gamma^*_{\rm nccc}(G)) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+2)z}{2}-4\right]^{z-2}$, $\left[\frac{(n+3)z}{2}-4\right]^1, \left[\frac{1}{4}\left(5nz+3z-16-z\sqrt{9n^2-34n+41}\right)\right]^1$,

$\left[\frac{1}{4}\left(5nz+3z-16+z\sqrt{9n^2-34n+41}\right)\right]^1\Big\}$.

We have

$$B_1 := \left|(nz-4) - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \begin{cases} -\frac{nz-2n-2z-2}{n+1}, & \text{for } z=2 \ \& \ n\geq 4; \\ & z=3 \ \& \ n=4,6;\ z=4 \ \& \ n=4 \\ = \frac{nz-2n-2z-2}{n+1}, & \text{otherwise,} \end{cases}$$

$$B_2 := \left|\left(\frac{(n+2)z}{2}-4\right) - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \frac{n^2z-3nz+4n+2z+4}{2n+2},$$

$$B_3 := \left| \left(\frac{(n+3)z}{2} - 4 \right) - \Delta_{\mathrm{D}}(\Gamma^*_{\mathrm{nccc}}(G)) \right| = \frac{n^2 z - 4nz + 4n + z + 4}{2n+2},$$

$$B_4 := \left| \frac{1}{4} \left(-\sqrt{9n^2 - 34n + 41}z + 5nz + 3z - 16 \right) - \Delta_{\mathrm{D}}(\Gamma^*_{\mathrm{nccc}}(G)) \right|$$
$$= \begin{cases} \frac{n^2z - + 8nz - 8n - 5z - 8(nz+z)\sqrt{9n^2-34n+41}}{4(n+1)}, & \text{for } n = 4 \ \& \ z \geq 5 \\ -\frac{n^2z - + 8nz - 8n - 5z - 8(nz+z)\sqrt{9n^2-34n+41}}{4(n+1)}, & \text{otherwise} \end{cases}$$

and

$$B_5 := \left| \frac{1}{4} \left(\sqrt{9n^2 - 34n + 41}z + 5nz + 3z - 16 \right) - \Delta_{\mathrm{D}}(\Gamma^*_{\mathrm{nccc}}(G)) \right|$$
$$= \frac{n^2 z + 8nz - 8n - 5z - 8 + (nz + z)\sqrt{9n^2 - 34n + 41}}{4(n+1)}.$$

Hence,

$$\mathrm{E}_{\mathrm{DQ}}(\Gamma^*_{\mathrm{nccc}}(G)) = \left(\frac{(n-1)z}{2} - 1 \right) \times B_1 + (z-2) \times B_2 + 1 \times B_3 + 1 \times B_4 + 1 \times B_5$$
$$= \begin{cases} \frac{n^2z + n\left(\left(\sqrt{9n^2-34n+41}+8\right)z - 8\right) + \left(\sqrt{9n^2-34n+41}-5\right)z - 8}{2(n+1)}, & \text{for } z = 2 \ \& \ n \geq 4; \\ & z = 3 \ \& \ n = 4, 6; \\ & z = 4 \ \& \ n = 4 \\ \frac{1}{5}\left(6z^2 + 9z - 20\right), & \text{for } n = 4 \ \& \ z \geq 5 \\ \frac{z\left(n^2(2z-3) + n\left(\sqrt{9n^2-26n+33} - 6z + 4\right) + \sqrt{9n^2-26n+33} + 4z + 7\right)}{2(n+1)}, & \text{otherwise.} \end{cases}$$

(b) If n is odd then by Theorem 5.1.4(b), we have $\text{D-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \Big\{ [-2]^{\frac{(n+1)z}{2} - 2},$
$\left[\frac{1}{2}(nz + z - 4 - z\sqrt{n^2 - 4n + 7})\right]^1, \left[\frac{1}{2}(nz + z - 4 + z\sqrt{n^2 - 4n + 7})\right]^1 \Big\}.$

We have

$$A_1' := \left| \frac{1}{2} \left(nz + z - 4 - z\sqrt{n^2 - 4n + 7} \right) \right| = \begin{cases} -\frac{1}{2} \left(nz + z - 4 - z\sqrt{n^2 - 4n + 7} \right), \text{for } z = 1 \\ \frac{1}{2} \left(nz + z - 4 - z\sqrt{n^2 - 4n + 7} \right), \text{for } z \geq 2 \end{cases}$$

and

$$A'_2 := \left|\frac{1}{2}\left(nz + z - 4 + z\sqrt{n^2 - 4n + 7}\right)\right| = \frac{1}{2}\left(nz + z - 4 + z\sqrt{n^2 - 4n + 7}\right).$$

Hence,

$$\begin{aligned} \mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) &= \left(\frac{(n+1)z}{2} - 2\right) \times |-2| + 1 \times A'_1 + 1 \times A'_2 \\ &= \begin{cases} \frac{1}{2}\left(4\sqrt{n^2 - 4n + 7} + n - 3\right), & \text{for } z = 1 \\ \frac{5}{2}(nz + z - 4), & \text{for } z \geq 2. \end{cases} \end{aligned}$$

By Theorem 5.1.4(b), we have DL-spec$(\Gamma^*_{\text{nccc}}(G)) = \Big\{[0]^1, [\frac{(n+1)z}{2}]^1, [nz]^{\frac{(n-1)z}{2}-1}, [\frac{(n+3)z}{2}]^{z-1}\Big\}$ and $W(\Gamma^*_{\text{nccc}}(G)) = \frac{1}{4}z\left(n^2z - 2n + 3z - 2\right)$. Therefore, $\Delta_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G)) = \frac{n^2z-2n+3z-2}{n+1}$. We have

$$L'_1 := \left|0 - \Delta_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G))\right| = \left|-\frac{n^2z - 2n + 3z - 2}{n+1}\right| = \frac{n^2z - 2n + 3z - 2}{n+1},$$

$$L'_2 := \left|\frac{(n+1)z}{2} - \Delta_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G))\right| = \begin{cases} \frac{-n^2z+2nz+4n-5z+4}{2n+2}, & \text{for } n = 3, 5 \ \& \ z = 1 \\ -\frac{-n^2z+2nz+4n-5z+4}{2n+2}, & \text{otherwise,} \end{cases}$$

$$L'_3 := \left|nz - \Delta_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G))\right| = \left|\frac{nz + 2n - 3z + 2}{n+1}\right| = \frac{nz + 2n - 3z + 2}{n+1}$$

and

$$L'_4 = \left|\frac{(n+3)z}{2} - \Delta_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G))\right| = \begin{cases} \frac{-n^2z+4nz+4n-3z+4}{2n+2}, & \text{for } n = 3 \ \& \ z \geq 1; \\ & n{=}5 \ \& \ z{=}1,2; \ n{=}7 \ \& \ z{=}1 \\ -\frac{-n^2z+4nz+4n-3z+4}{2n+2}, & \text{otherwise.} \end{cases}$$

Hence,

$$\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) = 1 \times L'_1 + 1 \times L'_2 + \left(\frac{(n-1)z}{2} - 1\right) L'_3 + (z-1) \times L'_4$$

$$= \begin{cases} \frac{2n^2z-4n+6z-4}{n+1}, & \text{for } n=3,5 \ \& \ z=1 \\ \frac{3n^2z-2nz-8n+11z-8}{n+1}, & \text{for } n=7 \ \& \ z=1;\ n=3 \ \& \ z\geq 2; \\ & n=5 \ \& \ z=2 \\ \frac{n^2z^2+2n^2z-4nz^2-2nz-4n+3z^2+4z-4}{n+1}, & \text{otherwise.} \end{cases}$$

By Theorem 5.1.4(b), we also have DQ-spec$(\Gamma^*_{nccc}(G)) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, [\frac{(n+3)z}{2} - 4]^{z-1}, [\frac{1}{4}(5nz+5z-16-z\sqrt{9n^2-46n+73})]^1, [\frac{1}{4}(5nz+5z-16+z\sqrt{9n^2-46n+73})]^1\Big\}$.

We have

$$B_1' := \left|(nz-4)-\Delta_D(\Gamma^*_{nccc}(G))\right| = \begin{cases} -\frac{nz-2n-3z-2}{n+1}, \text{for } z=1,2 \ \& n\geq 3;\ z=3 \ \& \ n=3,5,7,9; \\ \quad z=4,5 \ \& \ n=3,5; n=3 \ \& \ z\geq 1; \\ \quad n=5 \ \& \ 1\leq z\leq 5;\ n=7 \ \& \ z=1,2,3 \\ \frac{nz-2n-3z-2}{n+1}, \text{otherwise,} \end{cases}$$

$$B_2' := \left|\frac{1}{2}(n+3)z-4-\Delta_D(\Gamma^*_{nccc}(G))\right| = \frac{n^2z-4nz+4n+3z+4}{2n+2},$$

$$B_3' := \left|\frac{1}{4}\left(-\sqrt{9n^2-46n+73}z+5nz+5z-16\right)-\Delta_D(\Gamma^*_{nccc}(G))\right|$$

$$= \begin{cases} \frac{n^2z+10nz-8n-7z-8-(nz+z)\sqrt{9n^2-46n+73}}{4(n+1)}, & \text{for } n=3 \ \& \ z\geq 2; \\ & n=5 \ \& \ z\geq 3;\ n=7 \ \& \ z\geq 56 \\ -\frac{n^2z+10nz-8n-7z-8-(nz+z)\sqrt{9n^2-46n+73}}{4(n+1)}, & \text{otherwise} \end{cases}$$

and

$$B_4' := \left|\frac{1}{4}\left(\sqrt{9n^2-46n+73}z+5nz+5z-16\right)-\Delta_D(\Gamma^*_{nccc}(G))\right|$$

$$= \frac{n^2z+10nz-8n-7z-8+(nz+z)\sqrt{9n^2-46n+73}}{4(n+1)}.$$

Hence,

$$\mathrm{E_{DQ}}(\Gamma^*_{\mathrm{nccc}}(G)) = \left(\frac{(n-1)z}{2} - 1\right) \times B'_1 + (z-1) \times B'_2 + 1 \times B'_3 + 1 \times B'_4$$

$$= \begin{cases} \frac{n^2z+10nz-8n-7z-8}{n+1}, & \text{for } n=3 \ \&\ z \geq 2; \\ & n=5 \ \&\ z=3,4,5 \\ \frac{1}{2}\left(n+\sqrt{n(9n-46)+73}-\frac{16}{n+1}+9\right)z-4, & \text{for } n=3 \ \&\ z=1 \\ \frac{(z-1)(n((n-4)z+4)+3z+4)}{n+1}, & \text{for } n=5 \ \&\ z \geq 6; \\ & n=7 \ \&\ z \geq 56 \\ \frac{1}{2}\left(n+\sqrt{n(9n-46)+73}-\frac{16}{n+1}+9\right)z-4, & \text{for } n=5 \ \&\ z=1,2; \\ & n=7 \ \&\ z=1,2,3 \\ \frac{1}{2}z\left(2\left(\frac{8}{n+1}-5\right)z+n(2z-3)+\sqrt{n(9n-46)+73}+9\right), & \text{otherwise.} \end{cases}$$

□

Since the groups D_{2n} (where $n \geq 3$), $U_{(n,m)}$ (where $m \geq 3$ and $n \geq 2$), Q_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$) and U_{6n} (where $n \geq 2$) belong to the class of groups considered in Theorem 5.2.4, we have the following corollaries.

Corollary 5.2.5. *Let G be the dihedral group D_{2n}, where $n \geq 3$.*

(a) *If n is odd then* $\mathrm{E_D}(\Gamma^*_{\mathrm{nccc}}(G)) = \frac{1}{2}\left(4\sqrt{n^2-4n+7}+n-3\right)$,

$$\mathrm{E_{DL}}(\Gamma^*_{\mathrm{nccc}}(G)) = \begin{cases} \frac{2(n-1)^2}{n+1}, & \text{for } n=3,5 \\ \frac{3n^2-10n+3}{n+1}, & \text{for } n \geq 7 \end{cases}$$

$$\textit{and } \mathrm{E_{DQ}}(\Gamma^*_{\mathrm{nccc}}(G)) = \begin{cases} 2, & \text{for } n=3 \\ \frac{1}{2}\left(\sqrt{9n^2-46n+73}+n-\frac{16}{n+1}+1\right), & \text{for } n=5,\ 7 \\ \frac{1}{2}\left(\sqrt{9n^2-46n+73}-n+\frac{16}{n+1}-1\right), & \text{for } n \geq 9. \end{cases}$$

(b) *If n and $\frac{n}{2}$ are even then* $\mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) = \sqrt{n^2 - 6n + 17} + n - 3$,

$$\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 4, & \text{for } n = 4 \\ \frac{4(n^2-5n+4)}{n+2}, & \text{for } n \geq 8 \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 4, & \text{for } n = 4 \\ \frac{n^2+(\sqrt{9n^2-68n+164}+8)n+2(\sqrt{9n^2-68n+164}-18)}{2(n+2)}, & \text{for } n \geq 8. \end{cases}$$

(c) *If n is even and $\frac{n}{2}$ is odd then* $\mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) = \frac{5(n-2)}{2}$,

$$\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{3n^2-12n+28}{n+2}, & \text{for } n = 6, 10 \\ \frac{4(n^2-6n+8)}{n+2}, & \text{otherwise} \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{n^2+12n-44}{n+2}, & \text{for } n = 6 \\ \frac{1}{2}\left(\sqrt{9n^2 - 92n + 292} + n - \frac{64}{n+2} + 10\right), & \text{for } n = 10,\ 14 \\ \frac{1}{2}\left(\sqrt{9n^2 - 92n + 292} + n + \frac{128}{n+2} - 22\right), & \text{otherwise.} \end{cases}$$

Corollary 5.2.6. *Let G be the group $U_{(n,m)}$, where $m \geq 3$ and $n \geq 2$.*

(a) *If m is odd then* $\mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) = \frac{5}{2}(mn + n - 4)$,

$$\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) =$$
$$\begin{cases} \frac{3m^2n-2mn-8m+11n-8}{m+1}, & \text{for } m = 3 \,\&\, n \geq 2; m = 5 \,\&\, n = 2 \\ \frac{m^2n^2+2m^2n-4mn^2-2mn-4m+3n^2+4n-4}{m+1}, & \text{otherwise} \end{cases}$$

and $\mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G)) =$

$$\begin{cases} \frac{m^2n+10mn-8m-7n-8}{m+1}, & \text{for } m=3\,\&\,n\geq 2; \\ & m=5\,\&\,n=3,4,5 \\ \frac{(n-1)(m((m-4)n+4)+3n+4)}{m+1}, & \text{for } m=5\,\&\,n\geq 6; \\ & m=7\,\&\,n\geq 56 \\ \frac{1}{2}\left(m+\sqrt{m(9m-46)+73}-\frac{16}{m+1}+9\right)n-4, & \text{for } m=5\,\&\,n=2; \\ & m=7\,\&\,n=2,3 \\ \frac{1}{2}n\left(2\left(\frac{8}{m+1}-5\right)n+m(2n-3)+\sqrt{m(9m-46)+73}+9\right), & \text{otherwise.} \end{cases}$$

(b) *If m and $\frac{m}{2}$ are even then*

$$\mathrm{E_D}(\Gamma^*_{\mathrm{nccc}}(G)) = \begin{cases} 12(n-1), & \text{for } m=4 \\ 2(mn+2n-6), & \text{for } m\geq 8, \end{cases}$$

$$\mathrm{E_{DL}}(\Gamma^*_{\mathrm{nccc}}(G)) = \begin{cases} 12(n-1), & \text{for } m=4 \\ \frac{2m^2n(n+1)-4m\left(3n^2+n+1\right)+8\left(2n^2+n-1\right)}{m+2}, & \text{for } m\geq 8 \end{cases}$$

and $\mathrm{E_{DQ}}(\Gamma^*_{\mathrm{nccc}}(G)) =$

$$\begin{cases} 12(n-1), & \text{for } m=4 \\ \frac{136}{5}, & \text{for } m=8\,\&\,n=2 \\ \frac{1}{5}\left(24n^2+18n-20\right), & \text{for } m=8\,\&\,n\geq 3 \\ \frac{n\left(m^2(4n-3)+m\left(\sqrt{9m^2-52m+132}-24n+8\right)+2\left(\sqrt{9m^2-52m+132}+16n+14\right)\right)}{2(m+2)}, & \text{otherwise.} \end{cases}$$

(c) *If m is even and $\frac{m}{2}$ is odd then* $\mathrm{E_D}(\Gamma^*_{\mathrm{nccc}}(G)) = \frac{5}{2}(mn+2n-4)$,

$$\mathrm{E_{DL}}(\Gamma^*_{\mathrm{nccc}}(G)) = \begin{cases} \frac{3m^2n-4m(n+2)+44n-16}{m+2}, & \text{for } m=6\,\&\,n\geq 2; \\ \frac{2\left(m^2n(n+1)-2m\left(4n^2+n+1\right)+12n^2+8n-4\right)}{m+2}, & \text{otherwise} \end{cases}$$

and

$$\mathrm{E}_{\mathrm{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{m^2n+4m(5n-2)-4(7n+4)}{m+2}, & \text{for } m = 6 \,\&\, n \geq 2; \\ & m = 10 \,\&\, n = 2 \\ \frac{(2n-1)\left(m^2n+m(4-8n)+12n+8\right)}{m+2}, & \text{for } m = 10 \,\&\, n \geq 3; \\ & m = 14 \,\&\, n \geq 28 \\ \frac{1}{2}n\left(\sqrt{9m^2-92m+292} - \frac{8(5m-6)n}{m+2} + m(4n-3) + 18\right) & \text{otherwise.} \end{cases}$$

Corollary 5.2.7. *Let G be the group Q_{4n}, where $n \geq 2$.*

(a) *If n is even then* $\mathrm{E}_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G)) = 2n - 3 + \sqrt{4n^2 - 12n + 17}$,

$$\mathrm{E}_{\mathrm{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 4, & \text{for } n = 2 \\ \frac{8n^2-20n+8}{n+1}, & \text{for } n \geq 4 \end{cases}$$

$$\text{and } \mathrm{E}_{\mathrm{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 4, & \text{for } n = 2 \\ \frac{n^2+\left(\sqrt{9n^2-34n+41}+4\right)n+\sqrt{9n^2-34n+41}-9}{n+1}, & \text{for } n \geq 4. \end{cases}$$

(b) *If n is odd then* $\mathrm{E}_{\mathrm{D}}(\Gamma^*_{\text{nccc}}(G)) = 5(n-1)$,

$$\mathrm{E}_{\mathrm{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{6n^2-12n+14}{n+1}, & \text{for } n = 3, 5 \\ \frac{8n^2-24n+16}{n+1}, & \text{otherwise} \end{cases}$$

$$\text{and } \mathrm{E}_{\mathrm{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 8, & \text{for } n = 3 \\ 2\sqrt{17} + \frac{22}{3}, & \text{for } n = 5 \\ 8\sqrt{3} + 10, & \text{for } n = 7 \\ \sqrt{9n^2-46n+73} + n + \frac{32}{n+1} - 11, & \text{otherwise.} \end{cases}$$

Corollary 5.2.8. *Let G be the semidihedral group SD_{8n}, where $n \geq 2$.*

(a) *If n is even then* $\mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) = \sqrt{16n^2 - 24n + 17} + 4n - 3$,

$$\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{56}{5}, & \text{for } n = 2 \\ \frac{32n^2-40n+8}{2n+1}, & \text{otherwise} \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \frac{4n^2 + 2\left(\sqrt{36n^2 - 68n + 41} + 4\right)n + \sqrt{36n^2 - 68n + 41} - 9}{2n+1}.$$

(b) *If n is odd then* $\mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) = 10n$,

$$\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 24, & \text{for } n = 3 \\ \frac{24n^2-76n+60}{n+1}, & \text{otherwise} \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} \frac{4n^2+32n-36}{n+1}, & \text{for } n = 3,\ 5 \\ 2\left(\sqrt{9n^2 - 46n + 73} + 5n + \frac{64}{n+1} - 31\right), & \text{otherwise.} \end{cases}$$

Corollary 5.2.9. *Let G be the group U_{6n}, where $n \geq 2$. Then* $\mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) = 10(n-1)$ *and* $\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) = \mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G)) = 8(n-1)$.

We conclude this section with the following result.

Theorem 5.2.10. *Let G be the group V_{8n}, where $n \geq 2$.*

(a) *If n is even then* $\mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) = 4(2n-1)$,

$$\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 12, & \text{for } n = 2 \\ \frac{12\left(2n^2-5n+3\right)}{n+1}, & \text{for } n \geq 4 \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G)) = \begin{cases} 12, & \text{for } n = 2 \\ \frac{136}{5}, & \text{for } n = 4 \\ \frac{2\left(5n^2-20n+(n+1)\sqrt{n(9n-34)+41}+23\right)}{n+1}, & \text{for } n \geq 6. \end{cases}$$

(b) *If n is odd then* $\mathrm{E_D}(\Gamma^*_{\mathrm{nccc}}(G)) = \sqrt{16n^2 - 24n + 17} + 4n - 3$,

$$\mathrm{E_{DL}}(\Gamma^*_{\mathrm{nccc}}(G)) = \frac{8\left(4n^2 - 5n + 1\right)}{2n+1}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma^*_{\mathrm{nccc}}(G)) = \frac{4n^2 + 2\left(\sqrt{36n^2 - 68n + 41} + 4\right)n + \sqrt{36n^2 - 68n + 41} - 9}{2n+1}.$$

Proof. (a) If n is even then by Theorem 5.1.11(a), we have $\text{D-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[0]^1, [-2]^{2n-1}, \left[2n - 1 - \sqrt{4n^2 - 12n + 17}\right]^1, \left[2n - 1 + \sqrt{4n^2 - 12n + 17}\right]^1\right\}$.

We have

$$A_1 := \left|2n - 1 - \sqrt{4n^2 - 12n + 17}\right| = 2n - 1 - \sqrt{4n^2 - 12n + 17}$$

and

$$A_2 := \left|2n - 1 + \sqrt{4n^2 - 12n + 17}\right| = 2n - 1 + \sqrt{4n^2 - 12n + 17}.$$

Hence,

$$\begin{aligned}\mathrm{E_D}(\Gamma^*_{\mathrm{nccc}}(G)) &= 1 \times |0| + (2n-1) \times |-2| + 1 \times A_1 + 1 \times A_2 \\ &= 4(2n-1).\end{aligned}$$

By Theorem 5.1.11(a), we have $\text{DL-spec}(\Gamma^*_{\mathrm{nccc}}(G)) = \left\{[0]^1, [2n+2]^2, [4n]^{2n-3}, [2n+4]^2\right\}$ and $W(\Gamma^*_{\mathrm{nccc}}(G)) = \frac{1}{2}(8n^2 - 4n + 12)$. Therefore, $\Delta_{\mathrm{D}}(\Gamma^*_{\mathrm{nccc}}(G)) = \frac{4n^2-2n+6}{n+1}$. We have

$$L_1 := \left|0 - \Delta_{\mathrm{D}}(\Gamma^*_{\mathrm{nccc}}(G))\right| = \frac{4n^2 - 2n + 6}{n+1},$$

$$L_2 := \left|(2n+2) - \Delta_{\mathrm{D}}(\Gamma^*_{\mathrm{nccc}}(G))\right| = \frac{2\left(n^2 - 3n + 2\right)}{n+1},$$

$$L_3 := \left|4n - \Delta_{\mathrm{D}}(\Gamma^*_{\mathrm{nccc}}(G))\right| = \frac{6(n-1)}{n+1}$$

and

$$L_4 := \left|(2n+4) - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \begin{cases} 2, & \text{for } n=2 \\ \frac{2\left(n^2-4n+1\right)}{n+1}, & \text{for } n\geq 4. \end{cases}$$

Hence,

$$\begin{aligned} {\rm E}_{\rm DL}(\Gamma^*_{\rm nccc}(G)) &= 1\times L_1 + 2\times L_2 + (2n-3)\times L_3 + 2\times L_4 \\ &= \begin{cases} 12, & \text{for } n=2 \\ \frac{12\left(2n^2-5n+3\right)}{n+1}, & \text{for } n\geq 4. \end{cases}. \end{aligned}$$

By Theorem 5.1.11(a), we also have DQ-spec$(\Gamma^*_{\rm nccc}(G))$=$\{[4n-4]^{2n-3}, [2n]^2, [2n+2]^1,$ $\left[-\sqrt{9n^2-34n+41}+5n-1\right]^1, \left[\sqrt{9n^2-34n+41}+5n-1\right]^1\}$.

We have

$$B_1 := \left|(4n-4) - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \begin{cases} -\frac{2(n-5)}{n+1}, & \text{for } n=2,4 \\ \frac{2(n-5)}{n+1}, & \text{for } n\geq 6, \end{cases}$$

$$B_2 := \left|2n - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \frac{2\left(n^2-2n+3\right)}{n+1},$$

$$B_3 := \left|(2n+2) - \Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \frac{2\left(n^2-3n+2\right)}{n+1},$$

$$B_4 := \left|5n-1-\sqrt{9n^2-34n+41}-\Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = -\frac{n^2+6n-7-(n+1)\sqrt{9n^2-34n+41}}{n+1}$$

and

$$B_5 := \left|\sqrt{9n^2-34n+41}+5n-1-\Delta_{\rm D}(\Gamma^*_{\rm nccc}(G))\right| = \frac{n^2+6n-7+(n+1)\sqrt{9n^2-34n+41}}{n+1}.$$

Hence,

$${\rm E}_{\rm DQ}(\Gamma^*_{\rm nccc}(G)) = (2n-3)\times B_1 + 2\times B_2 + 1\times B_3 + 1\times B_4 + 1\times B_5$$

$$= \begin{cases} 12, & \text{for } n = 2 \\ \frac{136}{5}, & \text{for } n = 4 \\ \frac{2\left(5n^2-20n+(n+1)\sqrt{n(9n-34)+41}+23\right)}{n+1}, & \text{for } n \geq 6. \end{cases}$$

(b) If n is odd then by Result 1.2.24, we have $\Gamma^*_{\text{nccc}}(G) = K_{2n-1,1,1} = \Gamma^*_{\text{nccc}}(D_{2\times 4n})$. Hence, the result follows from Corollary 5.1.5. □

5.3 Comparing different distance energies

Motivated by Problem 1.1.7 – Problem 1.1.13, in this section, we compare the distance energy, distance Laplacian energy and distance signless Laplacian energy of $\Gamma^*_{\text{nccc}}(G)$ for the finite non-abelian groups discussed in the previous sections. We choose graphical method to compare various distance energies of $\Gamma^*_{\text{nccc}}(G)$. The following figures describe the comparison among $\mathrm{E_D}(\Gamma^*_{\text{nccc}}(G))$, $\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G))$ and $\mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G))$ for the groups $G = D_{2n}, Q_{4n}, SD_{8n}, U_{6n}$ and V_{8n}.

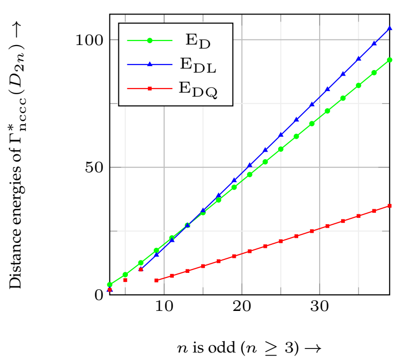

Figure 5.1: Distance energies of $\Gamma^*_{\text{nccc}}(D_{2n})$, n is odd

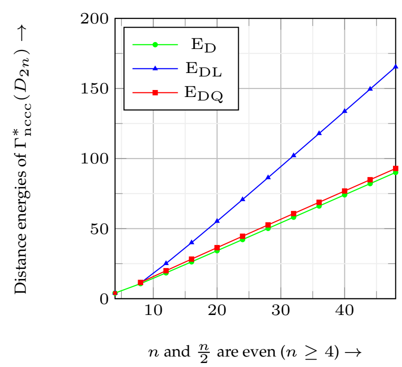

Figure 5.2: Distance energies of $\Gamma^*_{\text{nccc}}(D_{2n})$, n and $\frac{n}{2}$ are even

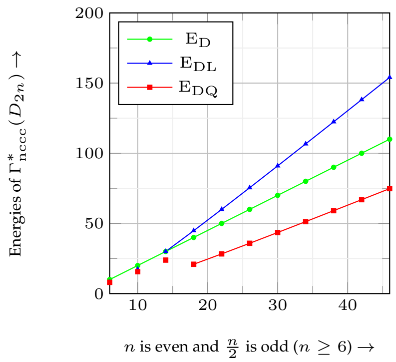

Figure 5.3: Distance energies of $\Gamma^*_{\mathrm{nccc}}(D_{2n})$, n is even and $\frac{n}{2}$ is odd

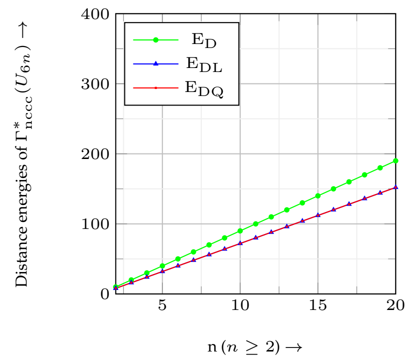

Figure 5.4: Distance energies of $\Gamma^*_{\mathrm{nccc}}(U_{6n})$

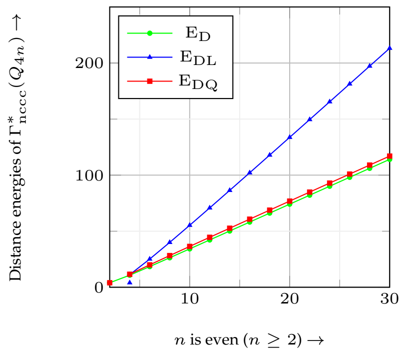

Figure 5.5: Distance energies of $\Gamma^*_{\mathrm{nccc}}(Q_{4n})$, n is even

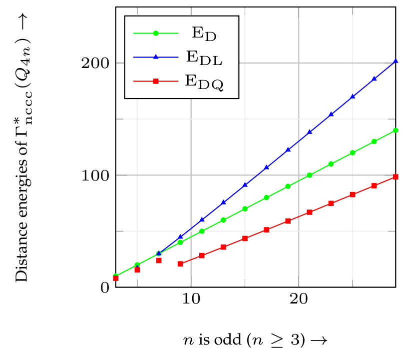

Figure 5.6: Distance energies of $\Gamma^*_{\mathrm{nccc}}(Q_{4n})$, n is odd

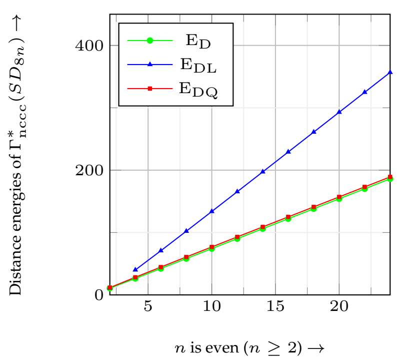

Figure 5.7: Distance energies of $\Gamma^*_{\mathrm{nccc}}(SD_{8n})$, n is even

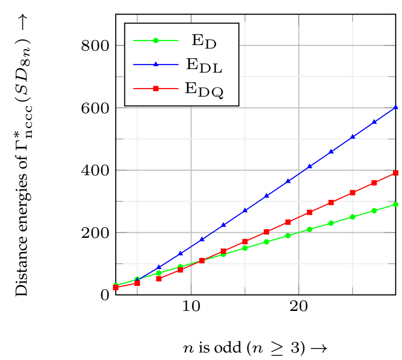

Figure 5.8: Distance energies of $\Gamma^*_{\mathrm{nccc}}(SD_{8n})$, n is odd

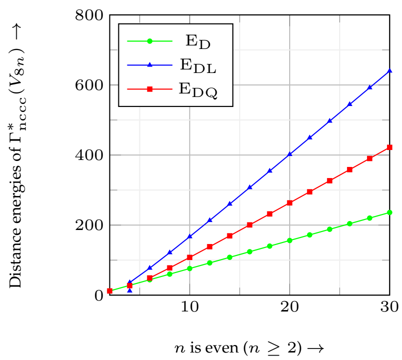

Figure 5.9: Distance energies of $\Gamma^*_{\rm nccc}(V_{8n})$, n is even

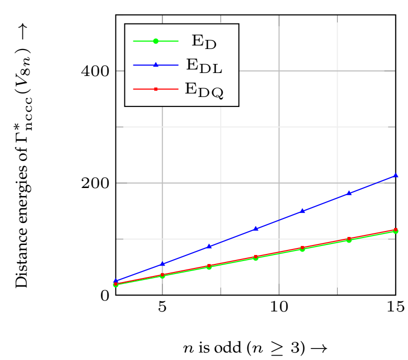

Figure 5.10: Distance energies of $\Gamma^*_{\rm nccc}(V_{8n})$, n is odd

By observing the Figures 5.1 – 5.10 we get the following result.

Theorem 5.3.1. *Let $G = D_{2n}$ (where $n \geq 3$), Q_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$), V_{8n} (where $n \geq 2$) and U_{6n} (where $n \geq 2$). Then*

(a) $E_D(\Gamma^*_{\rm nccc}(G)) = E_{DL}(\Gamma^*_{\rm nccc}(G)) = E_{DQ}(\Gamma^*_{\rm nccc}(G))$ *if and only if $G \cong D_8$ or T_8 or V_{16};*

(b) $E_{DL}(\Gamma^*_{\rm nccc}(G)) = E_{DQ}(\Gamma^*_{\rm nccc}(G)) < E_D(\Gamma^*_{\rm nccc}(G))$ *if and only if $G \cong D_6$ or D_{12} or T_{12} or SD_{24} or U_{6n} ($n \geq 2$);*

(c) $E_{DL}(\Gamma^*_{\rm nccc}(G)) < E_{DQ}(\Gamma^*_{\rm nccc}(G)) < E_D(\Gamma^*_{\rm nccc}(G))$ *if and only if $G \cong D_{10}$;*

(d) $E_{DQ}(\Gamma^*_{\rm nccc}(G)) < E_{DL}(\Gamma^*_{\rm nccc}(G)) < E_D(\Gamma^*_{\rm nccc}(G))$ *if and only if $G \cong D_{14}$ or D_{18} or or D_{20} or D_{22} or D_{26} or T_{20} or SD_{40};*

(e) $E_{DQ}(\Gamma^*_{\rm nccc}(G)) < E_D(\Gamma^*_{\rm nccc}(G)) < E_{DL}(\Gamma^*_{\rm nccc}(G))$ *if and only if $G \cong D_{2n}$ (n is odd and $n \geq 15$; n is even, $\frac{n}{2}$ is odd and $n \geq 18$) or Q_{4n} (n is odd and $n \geq 9$) or SD_{56} or SD_{72} or SD_{88} or v_{32};*

(f) $E_D(\Gamma^*_{\rm nccc}(G)) < E_{DL}(\Gamma^*_{\rm nccc}(G)) < E_{DQ}(\Gamma^*_{\rm nccc}(G))$ *if and only if $G \cong D_{16}$ or T_{16} or SD_{16};*

(g) $E_D(\Gamma^*_{\rm nccc}(G)) < E_{DQ}(\Gamma^*_{\rm nccc}(G)) < E_{DL}(\Gamma^*_{\rm nccc}(G))$ *if and only if $G \cong D_{2n}$ (n, $\frac{n}{2}$ are even and $n \geq 12$) or Q_{4n} (n is even and $n \geq 6$) or SD_{8n} (n is even and $n \geq 4$; n is odd and $n \geq 13$) or V_{8n} (n is even and $n \geq 6$; n is odd);*

(h) $\mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G)) < \mathrm{E_D}(\Gamma^*_{\text{nccc}}(G)) = \mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G))$ *if and only if* $G \cong D_{28}$ *or* T_{28}.

We conclude this chapter with the following corollary related to Problem 1.1.12.

Corollary 5.3.2. *Let* $G = D_{2n}$ *(where* $n \geq 3$*),* Q_{4n} *(where* $n \geq 2$*),* SD_{8n} *(where* $n \geq 2$*),* V_{8n} *(where* $n \geq 2$*) and* U_{6n} *(where* $n \geq 2$*). Then* $\mathrm{E_{DL}}(\Gamma^*_{\text{nccc}}(G)) = \mathrm{E_{DQ}}(\Gamma^*_{\text{nccc}}(G))$ *if and only if* $G \cong D_6, D_8,$ D_{12} *or* T_8, T_{12} *or* V_{16}, SD_{24} *or* U_{6n} $(n \geq 2)$.