CHAPTER 6

Mixed type weighted integral inequalities for the Hardy-Steklov integral operators

6.1 Introduction

In this chapter, we characterize the weights ω, ρ, ϕ and ψ for which the integral operator of Hardy-Steklov type satisfies the following weak type mixed modular inequality

$$\mathcal{U}^{-1}\Big(\int\limits_{\{\mathcal{I}f>\gamma\}} \mathcal{U}(\gamma\omega)\rho\Big) \le \mathcal{V}^{-1}\Big(\int \mathcal{V}(Cf\phi)\psi\Big)$$

for some constant $C > 0$. We also prove the following mixed integral inequality of the extra-weak type under appropriate conditions on the weights ω, ϕ and ψ.

$$\omega\Big(\big\{\mathcal{I}f > \gamma\big\}\Big) \le \mathcal{U} \circ \mathcal{V}^{-1}\bigg(\int \mathcal{V}\Big(\frac{Cf\phi}{\gamma}\Big)\psi\bigg).$$

Further, we discuss the above two integral inequalities for the adjoint of the integral operator of Hardy-Steklov type.

We consider the Hardy-Steklov integral operator $\mathcal{I}$, defined for a measurable function with values in $[0, \infty)$ by

$$\mathcal{I}f(t) = h(t)\int_{\zeta(t)}^{\sigma(t)} \mathcal{R}(t,z)f(z)w(z)dz, \tag{6.1}$$

where $\zeta, \sigma : (a, b) \to \mathbb{R}$ are continuous and increasing functions satisfying $\zeta(z) \le \sigma(z)$ for each $z \in (a, b)$ with $-\infty \le a < b \le \infty$, the functions h and w are measurable with values in $(0, \infty)$ and the kernel $\mathcal{R}(t, z)$ defined on $\{(t, z); \zeta(t) \le z \le \sigma(t)\}$ satisfies the following conditions.

This chapter is based on *Mixed type weighted integral inequalities for the Hardy-Steklov integral operators* by R. Haloi and D. Chutia [28], communicated in *Mathematical Inequalities & Applications.*

(a) $\mathcal{R}(t,z)$ is non-negative.

(b) $\mathcal{R}(t_1,z) \leq \mathcal{R}(t_2,z)$ for $t_1 \leq t_2$.

(c) $\mathcal{R}(t,z_1) \leq \mathcal{R}(t,z_2)$ for $z_2 \leq z_1$.

(d) There exists a constant $M \geq 1$ for which
$$\mathcal{R}(x,y) \leq M\Big[\mathcal{R}(x,\sigma(z)) + \mathcal{R}(z,y)\Big] \tag{6.2}$$
holds, where $z \leq x$ and $\zeta(x) \leq y \leq \sigma(z)$.

For $\mathcal{R} \equiv 1$, the operator (6.1) is reduced to the Hardy-Steklov operator defined by
$$\mathcal{S}f(t) = h(t)\int_{\zeta(t)}^{\sigma(t)} f(z)w(z)dz. \tag{6.3}$$

From (6.3), it is observed that the Hardy-Steklov operator extends the notion of the Hardy operator to the dynamic limits. Weighted weak as well as strong type estimates for the operators of Hardy-Steklov type and its integral version have been studied substantially by several authors [5, 6, 10, 24, 59]. In [5], characterization of weights ρ and ψ has been established such that
$$\Big(\int_{\{t\in(a,b):\mathcal{I}f(t)>\gamma\}} \gamma^q \rho(y)dy\Big)^{\frac{1}{q}} \leq C\Big(\int_{\zeta(a)}^{\sigma(b)} f(y)^p\psi(y)dy\Big)^{\frac{1}{p}} \tag{6.4}$$
holds for a suitable constant $C > 0$ and for the exponents $0 < q < p, \quad 1 < p < \infty$ with $w = 1$. Our first goal is to study the inequality (6.4) in the Orlicz space setting for the Hardy-Steklov integral operator and its adjoint, $\tilde{\mathcal{I}}$ defined by
$$\tilde{\mathcal{I}}f(t) = w(t)\int_{\sigma^{-1}(t)}^{\zeta^{-1}(t)} \mathcal{R}(z,t)f(z)h(z)dz. \tag{6.5}$$

Among the various equivalent generalization of the estimate (6.4) in to the Orlicz space setting, we will consider the following form.
$$\mathcal{U}^{-1}\Bigg(\int\limits_{\{t\in(a,b):\mathcal{I}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Bigg) \leq \mathcal{V}^{-1}\Bigg(\int_{\zeta(a)}^{\sigma(b)} \mathcal{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\Bigg), \tag{6.6}$$
where $\gamma > 0; \omega, \rho, \phi$ and ψ are weights and the conditions on $\mathcal{U}$ and $\mathcal{V}$ will be set down later.

The second objective of this study is to obtain the following weaker version of (6.6), that is
$$\int\limits_{\{t\in(a,b):\mathcal{I}f(t)>\gamma\}} \omega(y)dy \leq \mathcal{U}\circ\mathcal{V}^{-1}\Bigg(\int_{\zeta(a)}^{\sigma(b)} \mathcal{V}\Big(\frac{Cf(y)\phi(y)}{\gamma}\Big)\psi(y)dy\Bigg). \tag{6.7}$$

We arrange the chapter as follows. In Section 6.2, we state a lemma which will be used later. Weak type results are discussed in Section 6.3. Extra-weak type inequalities for the operators (6.1) and (6.5) are considered in Section 6.4.

6.2 Preliminaries

In this section, we state the following lemma [6, pp. 278], which has significant contribution in the remaining sections.

Lemma 6.2.1. *Let* $\Gamma = \{z \in (a,b) : \zeta(z) < \sigma(z)\}$. *Then there exists a countable collection of open intervals* $\{(a_m, b_m)\}$ *such that* $\Gamma = \cup_m (a_m, b_m)$ *and*

(a) $\big(\zeta(a_k), \sigma(b_k)\big) \cap \big(\zeta(a_m), \sigma(b_m)\big) = \phi$ *for* $m \neq k$,

(b) *for each* m, *there exists a sequence of real numbers* $\{\xi_k^m\}$ *satisfying*

  (i) $(a_m, b_m) = \cup_k (\xi_k^m, \xi_{k+1}^m)$ *a.e. for all* m,

  (ii) $a_m \leq \xi_k^m < \xi_{k+1}^m \leq b_m$ *for each* m *and* k,

  (iii) $\zeta(\xi_{k+1}^m) \leq \sigma(\xi_k^m)$ *for each* m, k *and also,* $\zeta(\xi_{k+1}^m) = \sigma(\xi_k^m)$, *if* $a_m < \xi_k^m < \xi_{k+1}^m < b_m$.

6.3 Weak type results

Weak type inequalities for the integral operator of Hardy-Stkelov type and its adjoint are discussed in this section. Following theorem is about the weak type result for the integral operator of Hardy-Steklov type.

Theorem 6.3.1. *Let* $\tilde{\mathcal{V}}$ *be the complementary function corresponding to an* N*-function* $\mathcal{V}$. *Suppose that* $\mathcal{V} \circ \mathcal{U}^{-1}$ *is countably sub-additive, where* $\mathcal{U}$ *is strictly increasing and positive with* $\mathcal{U}(0) = 0$. *Let the function* h *be monotone on* $\mathbb{R}$ *and let the function* $h(\cdot)\mathcal{R}(\cdot, y)$ *satisfies that*

$$\inf_{x\in\Omega} h(x)\mathcal{R}(x,y) = \inf_{x\in(\inf\Omega, \sup\Omega)} h(x)\mathcal{R}(x,y)$$

for all bounded set Ω *and all* y. *Then we have the following equivalent conditions.*

(i) *There exists a positive constant* C *such that*

$$\mathcal{U}^{-1}\left(\int\limits_{\{t\in(a,b):\mathcal{I}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right) \leq \mathcal{V}^{-1}\left(\int_{\zeta(a)}^{\sigma(b)} \mathcal{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\right) \quad (6.8)$$

holds for all functions $f \geq 0$ *and each* $\gamma > 0$.

(ii) There exists $C > 0$ such that

$$\int_{\zeta(\tau)}^{\sigma(t)} \tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\tau)} h)\mathcal{R}(t,s)w(s)\pi(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leq \pi(\gamma;t,\tau) \tag{6.9}$$

and

$$\int_{\zeta(\tau)}^{\sigma(z)} \tilde{\mathcal{V}}\left[\frac{\big(\inf_{(t,\tau)} h(y)\mathcal{R}(y,\sigma(z))\big)w(s)\pi(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leq \pi(\gamma;t,\tau) \tag{6.10}$$

hold, where $a < z \leq t < \tau < b$ satisfying $\zeta(\tau) \leq \sigma(z) \leq \sigma(t)$ and

$$\pi(\gamma;t,\tau) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\tau \mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big).$$

Proof. $(ii) \implies (i)$.

Letting $\{\xi_k^m\}$ be the sequence given by Lemma 6.2.1, we then have

$$\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(a,b):\mathcal{I}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) \leq \sum_{m,k}\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big). \tag{6.11}$$

Now for $t \in (\xi_k^m, \xi_{k+1}^m)$, we use Lemma 6.2.1 to break the integral (6.1) as

$$\begin{aligned}\mathcal{I}f(t) &= h(t)\Big[\int_{\zeta(t)}^{\zeta(\xi_{k+1}^m)} + \int_{\zeta(\xi_{k+1}^m)}^{\sigma(\xi_k^m)} + \int_{\sigma(\xi_k^m)}^{\sigma(t)}\Big]\mathcal{R}(t,y)f(y)w(y)dy &(6.12)\\ &= \mathcal{I}_1 f(t) + \mathcal{I}_2 f(t) + \mathcal{I}_3 f(t). &(6.13)\end{aligned}$$

Thus from (6.13), we observe

$$\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) \leq \sum_{i=1}^{3}\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_i f(t)>\frac{\gamma}{3}\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big). \tag{6.14}$$

We will first estimate $\mathcal{I}_1 f$. Applying the inequality (6.2), we now break the kernel $\mathcal{R}$ as

$$\begin{aligned}\mathcal{I}_1 f(t) &= h(t)\int_{\zeta(t)}^{\zeta(\xi_{k+1}^m)} \mathcal{R}(t,y)f(y)w(y)dy\\ &\leq M\left[h(t)\Big\{\mathcal{R}(t,\sigma(\xi_k^m))\int_{\zeta(t)}^{\zeta(\xi_{k+1}^m)} + \int_{\zeta(t)}^{\zeta(\xi_{k+1}^m)}\mathcal{R}(\xi_k^m,y)\Big\}f(y)w(y)dy\right]\\ &= M\Big[\mathcal{I}_{1,1}f(t) + \mathcal{I}_{1,2}f(t)\Big].\end{aligned}$$

Thus

$$\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_1f(t)>\frac{\gamma}{3}\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\le\sum_{i=1}^{2}\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_{1,i}f(t)>\frac{\gamma}{6M}\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big). \tag{6.15}$$

To estimate $\mathcal{I}_{1,1}f$, we define a sequence $\{x_j\}$ as $x_0=\xi_k^m$ and for each x_{j-1} let x_j be the number given by $\int_{\zeta(x_j)}^{\zeta(\xi_{k+1}^m)}fw=\int_{\zeta(x_{j-1})}^{\zeta(x_j)}fw$. The sequence $\{x_j\}$ increases and satisfies $\int_{\zeta(x_j)}^{\zeta(\xi_{k+1}^m)}fw=4\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}fw$. Let us consider the set

$$\Omega_{1,1}^j=\Big\{t\in(x_j,x_{j+1}):\mathcal{I}_{1,1}f(t)>\frac{\gamma}{6M}\Big\}.$$

We define $\delta_{1,1}^j=\inf\Omega_{1,1}^j$ and $\epsilon_{1,1}^j=\sup\Omega_{1,1}^j$. If $x\in\Omega_{1,1}^j$, then

$$\frac{\gamma}{6M}<4h(x)\mathcal{R}(x,\sigma(\xi_k^m))\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}f(z)w(z)dz. \tag{6.16}$$

As the estimate (6.16) holds for each $x\in\Omega_{1,1}^j$, thus

$$\gamma\le24M\big(\inf_{(\delta_{1,1}^j,\epsilon_{1,1}^j)}h(x)\mathcal{R}(x,\sigma(\xi_k^m))\big)\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}f(z)w(z)dz. \tag{6.17}$$

Let us denote $\pi(\gamma;\delta_{1,1}^j,\epsilon_{1,1}^j)=\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{1,1}^j}^{\epsilon_{1,1}^j}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$. Using (5.9) and (6.17), we obtain

$$\begin{aligned}2\pi(\gamma;\delta_{1,1}^j,\epsilon_{1,1}^j)&\le\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\Big[48MCf(z)\phi(z)\Big]\left[\frac{\big(\inf h(x)\mathcal{R}(x,\sigma(\xi_k^m))\big)w(z)\pi}{C\gamma\phi(z)\psi(z)}\right]\psi(z)dz\\&\le\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz\\&+\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\tilde{\mathcal{V}}\left(\frac{\big(\inf h(x)\mathcal{R}(x,\sigma(\xi_k^m))\big)w(z)\pi}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz.\end{aligned} \tag{6.18}$$

As $\zeta(\epsilon_{1,1}^j)\le\zeta(x_{j+1})\le\zeta(x_{j+2})\le\zeta(\xi_{k+1}^m)\le\sigma(\xi_k^m)\le\sigma(x_j)\le\sigma(\delta_{1,1}^j)$, from (6.10), we have

$$\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\tilde{\mathcal{V}}\left(\frac{\big(\inf h(x)\mathcal{R}(x,\sigma(\xi_k^m))\big)w(z)\pi}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz\le\pi(\gamma;\delta_{1,1}^j,\epsilon_{1,1}^j). \tag{6.19}$$

Combining (6.18) and (6.19), we obtain

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{1,1}^j}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)\le\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz. \tag{6.20}$$

Summing up over j and applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we get

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_{1,1}f(t)>\frac{\gamma}{6M}\}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big)\le\int_{\zeta(\xi_k^m)}^{\zeta(\xi_{k+1}^m)}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz. \tag{6.21}$$

To estimate $\mathcal{I}_{1,2}$, we define a sequence $\{y_j\}$ as $y_0=\xi_k^m$ and for each y_{j-1} let y_j be the number given by $\int_{\zeta(y_j)}^{\zeta(\xi_{k+1}^m)}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz=\int_{\zeta(y_{j-1})}^{\zeta(y_j)}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz$. Then $\{y_j\}$ increases and satisfies $\int_{\zeta(y_j)}^{\zeta(\xi_{k+1}^m)}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz=4\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz$. As in the previous case, we define $\Omega_{1,2}^j=\Big\{t\in(y_j,y_{j+1}):\mathcal{I}_{1,2}f(t)>\frac{\gamma}{6M}\Big\}$ with $\delta_{1,2}^j=\inf\Omega_{1,2}^j$ and $\epsilon_{1,2}^j=\sup\Omega_{1,2}^j$. For $x\in\Omega_{1,2}^j$, we have

$$\frac{\gamma}{6M}<4h(x)\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz. \tag{6.22}$$

As the estimate (6.22) holds for each $x\in\Omega_{1,2}^j$, thus

$$\gamma\le 24M\big(\inf_{(\delta_{1,2}^j,\epsilon_{1,2}^j)}h(x)\big)\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz. \tag{6.23}$$

We denote $\pi(\gamma;\delta_{1,2}^j,\epsilon_{1,2}^j)=\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{1,2}^j}^{\epsilon_{1,2}^j}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$. Applying (5.9) and (6.23), we get

$$\begin{aligned}2\pi(\gamma;\delta_{1,2}^j,\epsilon_{1,2}^j)\le&\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz\\&+\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\tilde{\mathcal{V}}\left(\frac{(\inf h)\mathcal{R}(\xi_k^m,z)h(z)\pi}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz.\end{aligned} \tag{6.24}$$

As $\zeta(\epsilon_{1,2}^j)\le\zeta(y_{j+1})\le\zeta(y_{j+2})\le\sigma(\delta_{1,2}^j)$, thus (6.9) gives

$$\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\tilde{\mathcal{V}}\left(\frac{(\inf h)\mathcal{R}(\xi_k^m,z)h(z)\pi}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz\le\pi(\gamma;\delta_{1,2}^j,\epsilon_{1,2}^j). \tag{6.25}$$

Combining (6.24) and (6.25), we have

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{1,2}^j}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big)\le\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz.$$

Summing up in j and applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, hence we obtain

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_{1,2}f(t)>\frac{\gamma}{6M}\}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big)\le\int_{\zeta(\xi_k^m)}^{\zeta(\xi_{k+1}^m)}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz. \tag{6.26}$$

Arguing similarly as in the previous case, we have

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_2f(t)>\frac{\gamma}{3}\}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\right)\leq\int_{\zeta(\xi_{k+1}^m)}^{\sigma(\xi_k^m)}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz. \tag{6.27}$$

For $\mathcal{I}_3f$, we consider $z_0=\xi_{k+1}^m$ and define a decreasing sequence $\{z_j\}$ as

$$\mathcal{L}(z_j)=\int_{\sigma(\xi_k^m)}^{\sigma(z_j)}\mathcal{R}(z_j,\tau)f(\tau)w(\tau)d\tau=(M+1)^{-j}\mathcal{L}(z_0).$$

Now, we have

$$\begin{aligned}
\mathcal{L}(z_j)&=(M+1)^2\mathcal{L}(z_{j+2})\\
&=(M+1)^2\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},\tau)f(\tau)w(\tau)d\tau \qquad (6.28)\\
&=(M+1)^2\Big[\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}+\int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\Big]\mathcal{R}(z_{j+2},\tau)f(\tau)w(\tau)d\tau\\
&\leq(M+1)^2\left[\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}M\Big\{\mathcal{R}(z_{j+2},\sigma(z_{j+3}))+\mathcal{R}(z_{j+3},\tau)\Big\}+\int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},\tau)\right]f(\tau)w(\tau)d\tau\\
&\leq(M+1)^3\left\{\mathcal{R}(z_{j+2},\sigma(z_{j+3}))\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}+\int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},\tau)\right\}f(\tau)w(\tau)d\tau\\
&\qquad+M(M+1)^2\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}\mathcal{R}(z_{j+3},\tau)f(\tau)w(\tau)d\tau. \qquad (6.29)
\end{aligned}$$

From the construction of the sequence $\{z_j\}$, we have

$$\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}\mathcal{R}(z_{j+3},\tau)f(\tau)w(\tau)d\tau=\mathcal{L}(z_{j+3})=(M+1)^{-(j+3)}\mathcal{L}(z_0)=(M+1)^{-3}\mathcal{L}(z_j).$$

Thus (6.29) implies

$$\mathcal{L}(z_j)\leq(M+1)^4\left[\left\{\mathcal{R}(z_{j+2},\sigma(z_{j+3}))\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}+\int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},y)\right\}f(y)w(y)dy\right].$$

Next, we define $\delta_{3,l}^j=\inf\Omega_{3,l}^j$ and $\epsilon_{3,l}^j=\sup\Omega_{3,l}^j$ for $l=1,2$, where

$$\Omega_{3,1}^j=\left\{y\in(z_{j+1},z_j):h(y)\mathcal{R}(z_{j+2},\sigma(z_{j+3}))\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}f(\tau)w(\tau)d\tau>\frac{\gamma}{6(M+1)^4}\right\},$$

$$\Omega_{3,2}^j=\left\{z\in(z_{j+1},z_j):h(z)\int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},\tau)f(\tau)w(\tau)d\tau>\frac{\gamma}{6(M+1)^4}\right\}.$$

Thus

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t:\mathcal{I}_3 f(t)>\frac{\gamma}{3}\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\le\sum_{j\ge 0}\Big\{\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega^j_{3,1}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)$$
$$+\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega^j_{3,2}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\Big\}. \tag{6.30}$$

For the first part, we define a decreasing sequence $\{d'_i\}$ in (ξ^m_k,ξ^m_{k+1}) with the iteration $d'_0=\xi^m_{k+1}$ and

$$\int_{\sigma(\xi^m_k)}^{\sigma(d'_i)} f(z)w(z)dz = 2^{-i}\int_{\sigma(\xi^m_k)}^{\sigma(\xi^m_{k+1})} f(z)w(z)dz.$$

We define $d_0=d'_0$ and if $d'_i>z_j\ge d'_{i+1}$ then $d_{n+1}=d'_{i+1}$, otherwise we delete the term d'_{i+1} and continue the process. Thus, we get a subsequence $\{d_n\}$ of $\{d'_i\}$. Let $\tilde{\delta}^n_{3,1}=\inf\tilde{\Omega}^n_{3,1}$ and $\tilde{\epsilon}^n_{3,1}=\sup\tilde{\Omega}^n_{3,1}$, where $\tilde{\Omega}^n_{3,1}=\cup_{\{j:d_n>z_{j+3}\ge d_{n+1}\}}\Omega^j_{3,1}$. Now, if $d'_{i+1}=d_{n+1}\le z_{j+3}<d_n$, then $z_{j+3}\le d'_i$ and $d_{n+2}\le d'_{i+2}$. We have

$$\int_{\sigma(\xi^m_k)}^{\sigma(z_{j+3})} f(z)w(z)dz\le\int_{\sigma(\xi^m_k)}^{\sigma(d'_i)} f(z)w(z)dz=4\int_{\sigma(d'_{i+2})}^{\sigma(d'_{i+1})} f(z)w(z)dz\le 4\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} f(z)w(z)dz. \tag{6.31}$$

Now for $x\in\tilde{\Omega}^n_{3,1}$, we obtain

$$\frac{\gamma}{6(M+1)^4}<4h(x)\mathcal{R}(x,\sigma(z_{j+3}))\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} f(\tau)w(\tau)d\tau. \tag{6.32}$$

As (6.32) holds for each $x\in\tilde{\Omega}^n_{3,1}$, thus

$$\gamma\le 24(M+1)^4\inf_{(\tilde{\delta}^n_{3,1},\tilde{\epsilon}^n_{3,1})}\Big(h(x)\mathcal{R}(x,\sigma(z_{j+3}))\Big)\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} f(\tau)w(\tau)d\tau. \tag{6.33}$$

We denote $\pi(\gamma;\tilde{\delta}^n_{3,1},\tilde{\epsilon}^n_{3,1})=\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}^n_{3,1}}^{\tilde{\epsilon}^n_{3,1}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$. From (5.9) and (6.10) we obtain

$$\begin{aligned}2\pi(\gamma;\tilde{\delta}^n_{3,1},\tilde{\epsilon}^n_{3,1})&\le\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau\\&\quad+\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})}\tilde{\mathcal{V}}\left[\frac{\inf\big(h(x)\mathcal{R}(x,\sigma(z_{j+3}))\big)w(\tau)\pi}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau\\&\le\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau+\pi(\gamma;\tilde{\delta}^n_{3,1},\tilde{\epsilon}^n_{3,1}).\end{aligned} \tag{6.34}$$

Thus

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\tilde{\delta}_{3,1}^{n}}^{\tilde{\epsilon}_{3,1}^{n}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\right)\leq\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau.$$

That is

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\tilde{\Omega}_{3,1}^{n}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\right)\leq\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau.$$

Summing up over n and then applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we obtain

$$\sum_{n\geq0}\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\tilde{\Omega}_{3,1}^{n}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\right)\leq\int_{\sigma(\xi_k^m)}^{\sigma(\xi_{k+1}^m)}\mathcal{V}\Big(48(M+1)^4Cf(z)\phi(z)\Big)\psi(z)dz.$$

This implies

$$\sum_{j\geq0}\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\Omega_{3,1}^{j}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\right)\leq\int_{\sigma(\xi_k^m)}^{\sigma(\xi_{k+1}^m)}\mathcal{V}\Big(48(M+1)^4Cf(z)\phi(z)\Big)\psi(z)dz. \tag{6.35}$$

Next, for $\Omega_{3,2}^{j}$ working as similar to the previous cases and thus we have

$$\sum_{j\geq0}\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\Omega_{3,2}^{j}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\right)\leq\int_{\sigma(\xi_k^m)}^{\sigma(\xi_{k+1}^m)}\mathcal{V}\Big(12(M+1)^4Cf(z)\phi(z)\Big)\psi(z)dz. \tag{6.36}$$

Combining (6.14), (6.15), (6.21), (6.26), (6.27), (6.30), (6.35) and (6.36) we obtain

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}f(t)>\gamma\}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\right)\leq\int_{\zeta(\xi_k^m)}^{\sigma(\xi_{k+1}^m)}\mathcal{V}\Big(96(M+1)^4Cf(z)\phi(z)\Big)\psi(z)dz. \tag{6.37}$$

Summing up (6.37) over m and k we obtain the estimate (6.8) with constant $96(M+1)^4C$.

$(i)\implies(ii)$. Conversely, let us assume $t<\tau$ so that $\zeta(\tau)<\sigma(t)$. Corresponding to each $N\in\mathbb{N}$ we consider the set $E_N=\Big\{\zeta(\tau)<s<\sigma(t):\frac{1}{N}\leq\mathcal{R}(t,s),w(s)\leq N\Big\}$ has finite measure. We have

$$\int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy\leq lN^2|E_N|\big(\inf h\big)\tilde{v}(\lambda lkN^2\inf h)<\infty$$

for each $l,k\in\mathbb{N}$ and $\lambda>0$. Given $\mu>0$ we choose λ suitably such that

$$\int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy=(1+\mu)C\gamma,$$

where C is the constant in (6.8). We consider

$$f(y)=\frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}\chi_{E_N}(y).$$

For $t<x<\tau$, we have

$$\begin{aligned}\mathcal{I}f(x)&=h(x)\int_{E_N}\mathcal{R}(x,z)\frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,z)w(z)}{(\phi(z)+1/l)(\psi(z)+1/k)}\right)\frac{\psi(z)+1/k}{\lambda\big(\inf h\big)\mathcal{R}(t,z)w(z)}w(z)dz\\&\geq\int_{E_N}\frac{1}{C\lambda}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,z)w(z)}{(\phi(z)+1/l)(\psi(z)+1/k)}\right)\Big(\psi(z)+1/k\Big)dz\\&=(1+\mu)\gamma>\gamma.\end{aligned}$$

This implies

$$(t,\tau)\subset\{x:\mathcal{I}f(x)>\gamma\}.$$

Thus using (5.10) and (6.8) we obtain

$$\begin{aligned}\pi(\gamma;t,\tau)&=\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_t^\tau\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right)\\&\leq\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_{\{\mathcal{I}f>\gamma\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right)\\&\leq\int_{E_N}\mathcal{V}\left(\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}\phi(y)\right)\psi(y)dy\\&\leq\int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy\\&\leq(1+\mu)C\lambda\gamma.\end{aligned}$$

Since $\tilde{\mathcal{V}}(r)/r$ increases as r increases, we obtain

$$\begin{aligned}&\int_{E_N}\tilde{\mathcal{V}}\left(\frac{\big(\inf h\big)\mathcal{R}(t,y)w(y)\pi(\gamma:t,\tau)}{(1+\mu)C\gamma(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\pi(\gamma:t,\tau)}dy\\&\quad\leq\int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda\gamma}dy=1.\end{aligned}$$

By the Monotone Convergence Theorem, we have

$$\int_{E_N}\tilde{\mathcal{V}}\left(\frac{\big(\inf h\big)\mathcal{R}(t,y)w(y)\pi(\gamma;t,\tau)}{(1+\mu)C\gamma(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\pi(\gamma;t,\tau)}dy\leq 1.$$

Letting $l,N\to\infty$ and $\mu\to 0^+$, we obtain

$$\int_{\zeta(\tau)}^{\sigma(t)}\tilde{\mathcal{V}}\left(\frac{\big(\inf h\big)\mathcal{R}(t,y)w(y)\pi\big(\gamma;t,\tau\big)}{C\gamma\phi(y)\psi(y)}\right)\psi(y)dy\leq\pi\big(\gamma;t,\tau\big).$$

Next, we prove the inequality (6.10). Let $a < z \leq t < \tau < b$ satisfying $\zeta(\tau) \leq \sigma(z)$. For each $m \in \mathbb{N}$ we consider the set $E_m = \big\{\zeta(\tau) < s < \sigma(z) : \frac{1}{m} \leq w(s) \leq m\big\}$ has finite measure. We have

$$\int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\Big(\frac{\psi(s)+1/k}{\lambda}\Big)ds < \infty$$

for each $l, k \in \mathbb{N}$ and $\lambda > 0$. Given $\mu > 0$ we choose λ suitably such that

$$\int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\Big(\frac{\psi(s)+1/k}{\lambda}\Big)ds = (1+\mu)C\gamma,$$

where C is the constant in (6.8). We define

$$f(s) = \frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\frac{\psi(s)+1/k}{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}\chi_{E_m}(s).$$

If $s \in E_m$ and $t < x < \tau$, then $\mathcal{R}(x,s) \geq \mathcal{R}(x,\sigma(z))$. Thus for $t < x < \tau$, we obtain

$$\begin{aligned}
\mathcal{I}f(x) &= h(x)\int_{E_m} \mathcal{R}(x,s)\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\frac{\psi(s)+1/k}{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}w(s)ds \\
&\geq \int_{E_m} \frac{1}{C\lambda}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\big(\psi(s)+1/k\big)ds \\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

This implies

$$(t,\tau) \subset \{x : \mathcal{I}f(x) > \gamma\}.$$

The rest of the proof proceeds similarly. Hence the proof is complete. □

Remark 6.3.2. *For $\mathcal{R} \equiv 1$, estimates (6.9) and (6.10) are equivalent and reduced to the following form*

$$\int_{\zeta(\tau)}^{\sigma(t)} \tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\tau)} h)w(s)\pi(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \leq \pi(\gamma;t,\tau). \tag{6.38}$$

Thus (6.38) characterizes the estimate (6.8) for the operators $\mathcal{S}f(t) = h(t)\int_{\zeta(t)}^{\sigma(t)} f(z)w(z)dz$.

Theorem 6.3.3. *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}$ and $\mathcal{V}\circ\mathcal{U}^{-1}$ satisfy all the conditions stated in Theorem 6.3.1. Suppose that w is monotone on $\mathbb{R}$ and and let the function $w(\cdot)\mathcal{R}(y,\cdot)$ satisfies that*

$$\inf_{x\in\Omega} w(x)\mathcal{R}(y,x) = \inf_{x\in(\inf\Omega,\sup\Omega)} w(x)\mathcal{R}(y,x)$$

for all bounded set Ω and all y. Then we obtain the following equivalent statements.

(i) *There exists a positive constant C such that*

$$\mathcal{U}^{-1}\Big(\int_{\{t\in(a,b):\tilde{\mathcal{I}}f(t)>\gamma\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\leq\mathcal{V}^{-1}\Big(\int_{\sigma^{-1}(a)}^{\zeta^{-1}(b)}\mathcal{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\Big) \tag{6.39}$$

holds for all functions $f\geq 0$ and each $\gamma>0$.

(ii) *There exists $C>0$ for which*

$$\int_{\sigma^{-1}(\tau)}^{\zeta^{-1}(t)}\tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\tau)}w)\mathcal{R}(s,\tau)h(s)\pi(\gamma:t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds\leq\pi(\gamma;t,\tau), \tag{6.40}$$

and

$$\int_{\sigma^{-1}(z)}^{\zeta^{-1}(t)}\tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(t,\tau)}w(y)\mathcal{R}(\sigma^{-1}(z),y))h(s)\pi(\gamma;t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds\leq\pi(\gamma;t,\tau) \tag{6.41}$$

hold for each $a<t<\tau\leq z<b$ satisfying $\sigma^{-1}(\tau)\leq\sigma^{-1}(z)\leq\zeta^{-1}(t)$, where

$$\pi(\gamma;t,\tau)=\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\tau\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big).$$

Proof. (ii) $\implies$ (i). Let $\{\xi_k^m\}$(after renaming) be the sequence given by Lemma 6.2.1 for the limits $\zeta^{-1}(t)$ and $\sigma^{-1}(t)$, and thus (6.11) holds for the adjoint operator $\tilde{\mathcal{I}}f$. We write $\tilde{\mathcal{I}}f$ for $t\in(\xi_k^m,\xi_{k+1}^m)$ as

$$\begin{aligned}\tilde{\mathcal{I}}f(t)&=w(t)\int_{\sigma^{-1}(t)}^{\zeta^{-1}(t)}\mathcal{R}(s,t)f(s)h(s)ds\\&=w(t)\left[\int_{\sigma^{-1}(t)}^{\sigma^{-1}(\xi_{k+1}^m)}+\int_{\sigma^{-1}(\xi_{k+1}^m)}^{\zeta^{-1}(\xi_k^m)}+\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(t)}\right]\mathcal{R}(s,t)f(s)h(s)ds \qquad (6.42)\\&=\tilde{\mathcal{I}}_1f(t)+\tilde{\mathcal{I}}_2f(t)+\tilde{\mathcal{I}}_3f(t). \qquad (6.43)\end{aligned}$$

Thus from (6.43) we obtain

$$\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}f(t)>\gamma\}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big)\leq\sum_{i=1}^3\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_if(t)>\frac{\gamma}{3}\}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big). \tag{6.44}$$

We will first estimate the integral $\tilde{\mathcal{I}}_3f$. Applying (6.2) we obtain

$$\tilde{\mathcal{I}}_3f(t)\leq M\left[w(t)\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(t)}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds+w(t)\mathcal{R}(\sigma^{-1}(\xi_{k+1}^m),t)\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(t)}f(s)h(s)ds\right]$$

$$= M\Big[\tilde{\mathcal{I}}_{3,1}f(t) + \tilde{\mathcal{I}}_{3,2}f(t)\Big].$$

Thus

$$\begin{aligned}
&\mathcal{V}\circ\mathcal{U}^{-1}\bigg(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_3 f(t)>\frac{\gamma}{3}\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\bigg)\\
&\leq \sum_{i=1}^{2}\mathcal{V}\circ\mathcal{U}^{-1}\bigg(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_{3,i} f(t)>\frac{\gamma}{6M}\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\bigg). \qquad (6.45)
\end{aligned}$$

We define a sequence $\{x_j\}$ as $x_0 = \xi_{k+1}^m$ and for each x_{j-1} let x_j be the number given by $\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(x_j)}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds = \int_{\zeta^{-1}(x_j)}^{\zeta^{-1}(x_{j-1})}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds$. Then the sequence $\{x_j\}$ decreases and satisfies

$$\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(x_j)}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds = 4\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds.$$

We define $\delta_{3,1}^j = \inf\Omega_{3,1}^j$ and $\epsilon_{3,1}^j = \sup\Omega_{3,1}^j$, where

$$\Omega_{3,1}^j = \Big\{t\in(x_{j+1},x_j) : \tilde{\mathcal{I}}_{3,1}f(t) > \frac{\gamma}{6M}\Big\}.$$

For $x\in\Omega_{3,1}^j$, we have

$$\frac{\gamma}{6M} < 4w(x)\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{R}(z,\xi_{k+1}^m)f(z)h(z)dz. \qquad (6.46)$$

As the estimate (6.46) holds for each $x\in\Omega_{3,1}^j$, thus

$$\gamma \leq 24M\big(\inf_{(\delta_{3,1}^j,\epsilon_{3,1}^j)} w(x)\big)\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{R}(z,\xi_{k+1}^m)f(z)h(z)dz. \qquad (6.47)$$

Let us denote $\pi(\gamma;\delta_{3,1}^j,\epsilon_{3,1}^j) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{3,1}^j}^{\epsilon_{3,1}^j}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big)$. Using (5.9) and (6.47) we obtain

$$\begin{aligned}
2\pi(\gamma;\delta_{3,1}^j,\epsilon_{3,1}^j) \leq &\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz\\
&+\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\tilde{\mathcal{V}}\left(\frac{(\inf w)\mathcal{R}(z,\xi_{k+1}^m)h(z)\pi}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz. \qquad (6.48)
\end{aligned}$$

As $\sigma^{-1}(\epsilon_{3,1}^j)\leq\sigma^{-1}(\xi_{k+1}^m)\leq\zeta^{-1}(\xi_k^m)\leq\zeta^{-1}(x_j)\leq\zeta^{-1}(\delta_{3,1}^j)$, thus (6.40) gives

$$\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\tilde{\mathcal{V}}\left(\frac{(\inf w)\mathcal{R}(z,\xi_{k+1}^m)h(z)\pi}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz \leq \pi(\gamma;\delta_{3,1}^j,\epsilon_{3,1}^j). \qquad (6.49)$$

Combining (6.48) and (6.49) we have

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\Omega_{3,1}^j}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\bigg)\leq\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz. \qquad (6.50)$$

Summing up in j and applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we get

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_{3,1}f(t)>\frac{\gamma}{6M}\}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\bigg)\leq\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(\xi_{k+1}^m)}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz. \qquad (6.51)$$

To estimate $\tilde{\mathcal{I}}_{3,2}$, we consider a decreasing sequence $\{y_j\}$ defined as $y_0=\xi_{k+1}^m$ and for each y_{j-1} let y_j be the number given by $\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(y_j)}f(s)h(s)ds=\int_{\zeta^{-1}(y_j)}^{\zeta^{-1}(y_{j-1})}f(s)h(s)ds$. From the definition of $\{y_j\}$ we obtain $\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(y_j)}f(s)h(s)ds=4\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}f(s)h(s)ds$. We define $\delta_{3,2}^j=\inf\Omega_{3,2}^j$ and $\epsilon_{3,2}^j=\sup\Omega_{3,2}^j$, where

$$\Omega_{3,2}^j=\Big\{t\in(y_{j+1},y_j):\tilde{\mathcal{I}}_{3,2}f(t)>\frac{\gamma}{6M}\Big\}.$$

For $x\in\Omega_{3,2}^j$, we have

$$\frac{\gamma}{6M}<4w(x)\mathcal{R}(\sigma^{-1}(\xi_{k+1}^m),x)\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}f(z)h(z)dz. \qquad (6.52)$$

As the estimate (6.52) holds for each $x\in\Omega_{3,2}^j$, we have

$$\gamma\leq 24M\big(\inf_{(\delta_{3,2}^j,\epsilon_{3,2}^j)}w(x)\mathcal{R}(\sigma^{-1}(\xi_{k+1}^m),x)\big)\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}f(z)h(z)dz. \qquad (6.53)$$

Considering $\pi(\gamma;\delta_{3,2}^j,\epsilon_{3,2}^j)=\big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\Big(\int_{\delta_{3,2}^j}^{\epsilon_{3,2}^j}\mathcal{U}\big(\gamma\omega(\tau)\big)\rho(\tau)d\tau\Big)$, (5.9) and (6.53) we have

$$\begin{aligned}2\pi(\gamma;\delta_{3,2}^j,\epsilon_{3,2}^j)\leq&\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz\\&+\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}\tilde{\mathcal{V}}\left(\frac{\big(\inf w(x)\mathcal{R}(\sigma^{-1}(\xi_{k+1}^m),x)\big)h(z)\pi}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz. \qquad (6.54)\end{aligned}$$

As $\sigma^{-1}(\epsilon_{3,2}^j)\leq\sigma^{-1}(\xi_{k+1}^m)\leq\zeta^{-1}(\xi_k^m)\leq\zeta^{-1}(y_j)\leq\zeta^{-1}(\delta_{3,2}^j)$, thus (6.41) gives

$$\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}\tilde{\mathcal{V}}\left(\frac{\big(\inf w(x)\mathcal{R}(\sigma^{-1}(\xi_{k+1}^m),x)\big)h(z)\pi}{C\gamma\phi(z)\psi(z)}\right)\psi(z)dz\leq\pi(\gamma;\delta_{3,2}^j,\epsilon_{3,2}^j). \qquad (6.55)$$

Combining (6.54) and (6.55) we have

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\Omega_{3,2}^j}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\bigg)\leq\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}\mathcal{V}\Big(48MCf(z)\phi(z)\Big)\psi(z)dz. \qquad (6.56)$$

Summing up in j and applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we get

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_{3,2}f(t)>\frac{\gamma}{6M}\}}\mathcal{U}\left(\gamma\omega(z)\right)\rho(z)dz\right)\leq\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(\xi_{k+1}^m)}\mathcal{V}\left(48MCf(z)\phi(z)\right)\psi(z)dz. \tag{6.57}$$

Arguing similarly we obtain

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_{2}f(t)>\frac{\gamma}{3}\}}\mathcal{U}\left(\gamma\omega(z)\right)\rho(z)dz\right)\leq\int_{\sigma^{-1}(\xi_{k+1}^m)}^{\zeta^{-1}(\xi_k^m)}\mathcal{V}\left(48MCf(z)\phi(z)\right)\psi(z)dz. \tag{6.58}$$

For $\tilde{\mathcal{I}}_1$, we consider $z_0=\xi_k^m$ and we define an increasing sequence $\{z_j\}$ as

$$\mathcal{L}(z_j)=\int_{\sigma^{-1}(z_j)}^{\sigma^{-1}(\xi_{k+1}^m)}\mathcal{R}(\tau,z_j)f(\tau)h(\tau)d\tau=(M+1)^{-j}\mathcal{L}(z_0).$$

We use the condition (6.2) to estimate the following.

$$\mathcal{L}(z_j)=(M+1)^2\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(\xi_{k+1}^m)}\mathcal{R}(\tau,z_{j+2})f(\tau)h(\tau)d\tau \tag{6.59}$$

$$\begin{aligned}
&=(M+1)^2\Big[\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}+\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\Big]\mathcal{R}(\tau,z_{j+2})f(\tau)h(\tau)d\tau\\
&\leq(M+1)^2\left[\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}\mathcal{R}(\tau,z_{j+2})+\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}M\Big\{\mathcal{R}(\tau,z_{j+3})+\mathcal{R}(\sigma^{-1}(z_{j+3}),z_{j+2})\Big\}\right]f(\tau)h(\tau)d\tau\\
&\leq(M+1)^3\left\{\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}\mathcal{R}(\tau,z_{j+2})+\mathcal{R}(\sigma^{-1}(z_{j+3}),z_{j+2})\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\right\}f(\tau)h(\tau)d\tau\\
&\qquad+M(M+1)^2\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\mathcal{R}(\tau,z_{j+3})f(\tau)h(\tau)d\tau.
\end{aligned} \tag{6.60}$$

From the construction of $\{z_j\}$, we have

$$\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\mathcal{R}(\tau,z_{j+3})f(\tau)h(\tau)d\tau=\mathcal{L}(z_{j+3})=(M+1)^{-(j+3)}\mathcal{L}(z_0)=(M+1)^{-3}\mathcal{L}(z_j).$$

Thus (6.60) implies that

$$\mathcal{L}(z_j)\leq(M+1)^4\left[\left\{\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}\mathcal{R}(\tau,z_{j+2})+\mathcal{R}(\sigma^{-1}(z_{j+3}),z_{j+2})\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\right\}f(y)h(y)dy\right].$$

Next we consider $\delta_{1,l}^j=\inf\Omega_{1,l}^j$ and $\epsilon_{1,l}^j=\sup\Omega_{1,l}^j$ for $l=1,2$, where

$$\Omega_{1,1}^j=\left\{y\in(z_j,z_{j+1}):w(y)\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}\mathcal{R}(\tau,z_{j+2})f(\tau)h(\tau)d\tau>\frac{\gamma}{6(M+1)^4}\right\},$$

$$\Omega^j_{1,2} = \left\{ z \in (z_j, z_{j+1}) : w(z)\mathcal{R}(\sigma^{-1}(z_{j+3}), z_{j+2}) \int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi^m_{k+1})} f(\tau)h(\tau)d\tau > \frac{\gamma}{6(M+1)^4} \right\}.$$

Thus the estimate for $\tilde{\mathcal{I}}_1 f$ is reduced to the following form.

$$\begin{aligned}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t:\mathcal{I}_1 f(t)>\frac{\gamma}{3}\}} \mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big) &\le \sum_{j\ge0}\Big\{\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega^j_{1,1}} \mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big) \\ &+ \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega^j_{1,2}} \mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big)\Big\}.\end{aligned} \tag{6.61}$$

Arguing similarly as in the previous cases, we observe that

$$\sum_{j\ge0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega^j_{1,1}} \mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big) \le \int_{\sigma^{-1}(\xi^m_k)}^{\sigma^{-1}(\xi^m_{k+1})} \mathcal{V}\Big(12(M+1)^4 Cf(z)\phi(z)\Big)\psi(z)dz. \tag{6.62}$$

For the second part, we define an increasing sequence $\{d'_i\}$ in (ξ^m_k, ξ^m_{k+1}) with the iteration $d'_0 = \xi^m_k$ and

$$\int_{\sigma^{-1}(d'_i)}^{\sigma^{-1}(\xi^m_{k+1})} f(z)h(z)dz = 2\int_{\sigma^{-1}(d'_{i+1})}^{\sigma^{-1}(\xi^m_{k+1})} f(z)h(z)dz.$$

Next we define a subsequence $\{d_n\}$ as $d_0 = d'_0$ and if $d'_i < z_j \le d'_{i+1}$ then $d_{n+1} = d'_{i+1}$, otherwise we delete the term d'_{i+1} and continue the process. Thus, we get a subsequence $\{d_n\}$ of $\{d'_i\}$. Let $\tilde{\delta}^n_{1,2} = \inf \tilde{\Omega}^n_{1,2}$ and $\tilde{\epsilon}^n_{1,2} = \sup \tilde{\Omega}^n_{1,2}$, where $\tilde{\Omega}^n_{1,2} = \cup_{\{j:d_n<z_{j+3}\le d_{n+1}\}}\Omega^j_{1,2}$. Now, if $d'_{i+1} = d_{n+1} \ge z_{j+3} > d_n$ then by construction $z_{j+3} \ge d'_i$ and $d_{n+2} \ge d'_{i+2}$. From the construction of $\{d'_i\}$ and $\{d_n\}$, we have

$$\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi^m_{k+1})} \le \int_{\sigma^{-1}(d'_i)}^{\sigma^{-1}(\xi^m_{k+1})} = 4\int_{\sigma^{-1}(d'_{i+1})}^{\sigma^{-1}(d'_{i+2})} \le 4\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}. \tag{6.63}$$

Now for $x \in \tilde{\Omega}^n_{1,2}$, we have

$$\frac{\gamma}{6(M+1)^4} < 4w(x)\mathcal{R}(\sigma^{-1}(z_{j+3}), x)\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})} f(\tau)h(\tau)d\tau. \tag{6.64}$$

As (6.64) holds for each $x \in \tilde{\Omega}^n_{1,2}$, we obtain

$$\gamma \le 24(M+1)^4 \inf_{(\tilde{\delta}^n_{1,2},\tilde{\epsilon}^n_{1,2})} \Big(w(x)\mathcal{R}(\sigma^{-1}(z_{j+3}), x)\Big)\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})} f(\tau)h(\tau)d\tau. \tag{6.65}$$

Let us denote $\pi(\gamma;\tilde{\delta}^n_{1,2},\tilde{\epsilon}^n_{1,2}) = \big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\Big(\int_{\tilde{\delta}^n_{1,2}}^{\tilde{\epsilon}^n_{1,2}}\mathcal{U}\big(\gamma\omega(\tau)\big)\rho(\tau)d\tau\Big)$. Now applying (5.9) and (6.64) we obtain

$$\begin{aligned}
2\pi(\gamma;\tilde{\delta}^n_{1,2},\tilde{\epsilon}^n_{1,2}) &\le \int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau\\
&\quad+\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}\tilde{\mathcal{V}}\left[\frac{\inf\big(w(x)\mathcal{R}(\sigma^{-1}(z_{j+3}),x)\big)h(\tau)\pi}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau\\
&\le \int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau+\pi(\gamma;\tilde{\delta}^n_{1,2},\tilde{\epsilon}^n_{1,2}).
\end{aligned}$$

Thus using the condition (6.41), we have

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\tilde{\delta}^n_{1,2}}^{\tilde{\epsilon}^n_{1,2}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\bigg)\le\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau.$$

Summing up over n and then applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we obtain

$$\sum_{n\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\tilde{\Omega}^n_{1,2}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\bigg)\le\int_{\sigma^{-1}(\xi^m_k)}^{\sigma^{-1}(\xi^m_{k+1})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \tag{6.66}$$

Thus

$$\sum_{j\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\Omega^j_{1,2}}\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\bigg)\le\int_{\sigma^{-1}(\xi^m_k)}^{\sigma^{-1}(\xi^m_{k+1})}\mathcal{V}\Big(48(M+1)^4Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \tag{6.67}$$

Combining (6.44), (6.45), (6.51), (6.57), (6.58), (6.61), (6.62) and (6.67) we obtain

$$\begin{aligned}
&\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\{t\in(\xi^m_k,\xi^m_{k+1}):\tilde{\mathcal{I}}f(t)>\gamma\}}\mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\bigg)\\
&\qquad\le\int_{\sigma^{-1}(\xi^m_k)}^{\zeta^{-1}(\xi^m_{k+1})}\mathcal{V}\Big(96(M+1)^4Cf(z)\phi(z)\Big)\psi(z)dz.
\end{aligned}\tag{6.68}$$

Summing up (6.68) over m and k we obtain the estimate (6.39) with constant $96(M+1)^4C$.

$(i)\implies(ii)$.Conversely let us assume $t<\tau$ such that $\sigma^{-1}(\tau)<\zeta^{-1}(t)$. For each $N\in\mathbb{N}$ we consider the set, $E_N=\Big\{\sigma^{-1}(\tau)<s<\zeta^{-1}(t):\frac{1}{N}\le\mathcal{R}(s,\tau),h(s)\le N\Big\}$ has finite measure. As the integral

$$\int_{E_N}\tilde{\mathcal{V}}\bigg(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\bigg)\bigg(\frac{\psi(y)+1/k}{\lambda}\bigg)dy\le lN^2|E_N|\big(\inf w\big)\tilde{v}(\lambda lkN^2\inf w)<\infty$$

for each $l, k \in \mathbb{N}$ and $\lambda > 0$. Given $\mu > 0$ we choose λ suitably so that

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy = (1+\mu)C\gamma,$$

where C is the constant in (6.39). We consider

$$f(y) = \frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}\chi_{E_N}(y).$$

For $t < x < \tau$, we have

$$\begin{aligned}
\tilde{\mathcal{I}}f(x) &= w(x)\int_{E_N} \mathcal{R}(y,x)\frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}h(y)dy \\
&\geq \int_{E_N} \frac{1}{C\lambda}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\Big(\psi(y)+1/k\Big)dy \\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

This implies that

$$(t,\tau) \subset \{x : \tilde{\mathcal{I}}f(x) > \gamma\}.$$

Applying the condition (5.10) and the assumption (6.39), we get

$$\begin{aligned}
\pi(\gamma; t, \tau) &= \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_t^\tau \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right) \\
&\leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_{\{\tilde{\mathcal{I}}f>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right) \\
&\leq \int_{E_N} \mathcal{V}\left(\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}\phi(y)\right)\psi(y)dy \\
&\leq \int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy \\
&\leq (1+\mu)C\lambda\gamma.
\end{aligned}$$

As $\tilde{\mathcal{V}}(r)/r$ increases as r increases, we obtain

$$\begin{aligned}
&\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)\mathcal{R}(y,\tau)h(y)\pi(\gamma;t,\tau)}{(1+\mu)C\gamma(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\pi(\gamma;t,\tau)}dy \\
&\quad\leq \int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda\gamma}dy = 1.
\end{aligned}$$

By the Monotone Convergence Theorem

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)\mathcal{R}(y,\tau)h(y)\pi(\gamma;t,\tau)}{(1+\mu)C\gamma(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\pi(\gamma:t,\tau)}dy \leq 1.$$

Letting $l, N \to \infty$ and $\mu \to 0^+$, we obtain

$$\int_{\sigma^{-1}(\tau)}^{\zeta^{-1}(t)} \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)\mathcal{R}(y,\tau)h(y)\pi\big(\gamma;t,\tau\big)}{C\gamma\phi(y)\psi(y)}\right)\psi(y)dy \le \pi\big(\gamma;t,\tau\big).$$

Next we prove the estimate (6.41). Let $a < t < \tau \le z < b$ such that $\sigma^{-1}(\tau) \le \sigma^{-1}(z) \le \zeta^{-1}(t)$. For each $m \in \mathbb{N}$ we assume the set $E_m = \left\{\sigma^{-1}(z) < s < \zeta^{-1}(t) : \frac{1}{m} \le h(s) \le m\right\}$ has finite measure. Now for each $l, k \in \mathbb{N}$ and $\lambda > 0$ we have

$$\int_{E_m} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\left(\frac{\psi(s)+1/k}{\lambda}\right)ds < \infty,$$

where the infimum is considered over the interval (t,τ). Given $\mu > 0$ we choose λ suitably so that

$$\int_{E_m} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\left(\frac{\psi(s)+1/k}{\lambda}\right)ds = (1+\mu)C\gamma,$$

where C is the constant in (6.39). We define

$$f(s) = \frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\frac{\psi(s)+1/k}{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)}\chi_{E_m}(s).$$

If $s \in E_m$ and $t < x < \tau$, then $\mathcal{R}(s,x) \ge \mathcal{R}(\sigma^{-1}(z),x)$. Thus for $t < x < \tau$, we have

$$\begin{aligned}
\tilde{\mathcal{I}}f(x) &= w(x)\int_{E_m} \mathcal{R}(s,x)\frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\frac{\psi(s)+1/k}{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)}h(s)ds\\
&\ge \int_{E_m} \frac{1}{C\lambda}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\left(\psi(s)+1/k\right)ds\\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

This implies that

$$(t,\tau) \subset \{x : \tilde{\mathcal{I}}f(x) > \gamma\}.$$

The rest of the proof proceeds similarly. Hence the proof is complete. □

Remark 6.3.4. *In the case of the adjoint if we consider* $\mathcal{R} \equiv 1$, *then estimates* (6.40) *and* (6.41) *are equivalent and reduced to the following form*

$$\int_{\sigma^{-1}(\tau)}^{\zeta^{-1}(t)} \tilde{\mathcal{V}}\left[\frac{(\inf w)h(s)\pi(\gamma : t,\tau)}{C\gamma\phi(s)\psi(s)}\right]\psi(s)ds \le \pi(\gamma : t,\tau). \tag{6.69}$$

Thus (6.69) characterizes the estimate (6.39) for the adjoint of Hardy-Steklov operators $\tilde{\mathcal{S}}f(t) = w(t)\int_{\sigma^{-1}(t)}^{\zeta^{-1}(t)} f(z)h(z)dz$.

6.4 Extra-weak type results

We also prove extra-weak type integral inequalities and the results are as follows.

Theorem 6.4.1. *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}, \mathcal{V}\circ\mathcal{U}^{-1}, h$ and the function $h(\cdot)\mathcal{R}(\cdot, y)$ satisfy all the conditions stated in Theorem 6.3.1. Then the following assertions are equivalent.*

(i) There exists a positive constant C such that

$$\int_{\{t\in(a,b):\mathcal{I}f(t)>\gamma\}} \omega(y)dy \leq \mathcal{U}\circ\mathcal{V}^{-1}\Big(\int_{\zeta(a)}^{\sigma(b)} \mathcal{V}\Big(\frac{Cf(y)\phi(y)}{\gamma}\Big)\psi(y)dy\Big) \tag{6.70}$$

holds for all functions $f \geq 0$ and each $\gamma > 0$.

(ii) There exists $C > 0$ such that

$$\int_{\zeta(\tau)}^{\sigma(t)} \tilde{\mathcal{V}}\Bigg[\frac{(\inf_{(t,\tau)} h)\mathcal{R}(t,s)w(s)\kappa(t,\tau)}{C\phi(s)\psi(s)}\Bigg]\psi(s)ds \leq \kappa(t,\tau) \tag{6.71}$$

and

$$\int_{\zeta(\tau)}^{\sigma(z)} \tilde{\mathcal{V}}\Bigg[\frac{\big(\inf_{(t,\tau)} h(y)\mathcal{R}(y,\sigma(z)\big)w(s)\kappa(t,\tau)}{C\phi(s)\psi(s)}\Bigg]\psi(s)ds \leq \kappa(t,\tau) \tag{6.72}$$

hold, where $a < z \leq t < \tau < b$ satisfying $\zeta(\tau) \leq \sigma(z) \leq \sigma(t)$ and

$$\kappa(t,\tau) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\tau \omega(z)dz\Big).$$

Proof. $(ii) \implies (i)$. Let $\{\xi_k^m\}$ be the sequence given by Lemma 6.2.1, then

$$\omega\Big(\big\{t\in(a,b):\mathcal{I}f(t)>\gamma\big\}\Big) = \sum_{m,k}\omega\Big(\big\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}f(t)>\gamma\big\}\Big). \tag{6.73}$$

Now for $t \in (\xi_k^m, \xi_{k+1}^m)$, we use Lemma 6.2.1 to break the integral (6.1) as

$$\mathcal{I}f(t) = h(t)\Bigg\{\int_{\zeta(t)}^{\zeta(\xi_{k+1}^m)} + \int_{\zeta(\xi_{k+1}^m)}^{\sigma(\xi_k^m)} + \int_{\sigma(\xi_k^m)}^{\sigma(t)}\Bigg\}\mathcal{R}(t,z)f(z)w(z)dz \tag{6.74}$$

$$= \mathcal{I}_1f(t) + \mathcal{I}_2f(t) + \mathcal{I}_3f(t). \tag{6.75}$$

Thus from (6.75), we have

$$\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}f(t)>\gamma\}} \omega(y)dy\Big) \leq \sum_{i=1}^{3}\mathcal{V}\circ\mathcal{U}^{-1}\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_if(t)>\frac{\gamma}{3}\}} \omega(y)dy\Big). \tag{6.76}$$

We will first estimate $\mathcal{I}_1 f$. Applying inequlity (6.2), we now break the kernel $\mathcal{R}$ as

$$\begin{aligned}\mathcal{I}_1 f(t) &= h(t)\int_{\zeta(t)}^{\zeta(\xi_{k+1}^m)} \mathcal{R}(t,z)f(z)w(z)dz \\ &\leq M\left[h(t)\left\{\mathcal{R}(t,\sigma(\xi_k^m))\int_{\zeta(t)}^{\zeta(\xi_{k+1}^m)} + \int_{\zeta(t)}^{\zeta(\xi_{k+1}^m)} \mathcal{R}(\xi_k^m, z)\right\} f(z)w(z)dz\right] \\ &= M\Big[\mathcal{I}_{1,1}f(t) + \mathcal{I}_{1,2}f(t)\Big].\end{aligned}$$

Thus

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Bigg(\int\limits_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_1 f(t)>\frac{\gamma}{3}\}} \omega(y)dy\Bigg) \leq \sum_{i=1}^{2}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Bigg(\int\limits_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_{1,i} f(t)>\frac{\gamma}{6M}\}} \omega(y)dy\Bigg). \tag{6.77}$$

To estimate $\mathcal{I}_{1,1}f$, we define a sequence $\{x_j\}$ as $x_0 = \xi_k^m$ and for each x_{j-1} let x_j be the number given by $\int_{\zeta(x_j)}^{\zeta(\xi_{k+1}^m)} fw = \int_{\zeta(x_{j-1})}^{\zeta(x_j)} fw$. The sequence $\{x_j\}$ increases and satisfies $\int_{\zeta(x_j)}^{\zeta(\xi_{k+1}^m)} fw = 4\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})} fw$. Let us consider the set

$$\Omega_{1,1}^j = \Big\{t\in(x_j,x_{j+1}) : \mathcal{I}_{1,1}f(t) > \frac{\gamma}{6M}\Big\}.$$

We define $\delta_{1,1}^j = \inf\Omega_{1,1}^j$ and $\epsilon_{1,1}^j = \sup\Omega_{1,1}^j$. If $x\in\Omega_{1,1}^j$, then

$$\frac{\gamma}{6M} < 4h(x)\mathcal{R}(x,\sigma(\xi_k^m))\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})} f(z)w(z)dz. \tag{6.78}$$

As the estimate (6.78) holds for each $x\in\Omega_{1,1}^j$, thus

$$\gamma \leq 24M\Big(\inf_{(\delta_{1,1}^j,\epsilon_{1,1}^j)} h(x)\mathcal{R}(x,\sigma(\xi_k^m))\Big)\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})} f(z)w(z)dz. \tag{6.79}$$

Let us denote $\kappa(\delta_{1,1}^j,\epsilon_{1,1}^j) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{1,1}^j}^{\epsilon_{1,1}^j}\omega(z)dz\Big)$. Using (5.9) and (6.79), we obtain

$$\begin{aligned} 2\kappa(\delta_{1,1}^j,\epsilon_{1,1}^j) &\leq \int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\left[\frac{48MCf(z)\phi(z)}{\gamma}\right]\left[\frac{\big(\inf h(x)\mathcal{R}(x,\sigma(\xi_k^m))\big)w(z)\kappa}{C\phi(z)\psi(z)}\right]\psi(z)dz \\ &\leq \int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\mathcal{V}\left(\frac{48MCf(z)\phi(z)}{\gamma}\right)\psi(z)dz \\ &+ \int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\tilde{\mathcal{V}}\left(\frac{\big(\inf h(x)\mathcal{R}(x,\sigma(\xi_k^m))\big)w(z)\kappa}{C\phi(z)\psi(z)}\right)\psi(z)dz. \end{aligned} \tag{6.80}$$

As $\zeta(\epsilon_{1,1}^j) \le \zeta(x_{j+1}) \le \zeta(x_{j+2}) \le \zeta(\xi_{k+1}^m) \le \sigma(\xi_k^m) \le \sigma(x_j) \le \sigma(\delta_{1,1}^j)$, thus from (6.72), we have

$$\int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})} \tilde{\mathcal{V}}\left(\frac{\big(\inf h(x)\mathcal{R}(x,\sigma(\xi_k^m))\big)w(z)\kappa}{C\phi(z)\psi(z)}\right)\psi(z)dz \le \kappa(\delta_{1,1}^j,\epsilon_{1,1}^j). \tag{6.81}$$

Combining (6.80) and (6.81), we obtain

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{1,1}^j}\omega(z)dz\Big) \le \int_{\zeta(x_{j+1})}^{\zeta(x_{j+2})}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \tag{6.82}$$

Summing up over j and applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we get

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_{1,1}f(t)>\frac{\gamma}{6M}\}}\omega(z)dz\Big) \le \int_{\zeta(\xi_k^m)}^{\zeta(\xi_{k+1}^m)}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \tag{6.83}$$

To estimate $\mathcal{I}_{1,2}$, we define a sequence $\{y_j\}$ as $y_0 = \xi_k^m$ and for each y_{j-1} let y_j be the number given by $\int_{\zeta(y_j)}^{\zeta(\xi_{k+1}^m)}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz = \int_{\zeta(y_{j-1})}^{\zeta(y_j)}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz$. Then $\{y_j\}$ increases and satisfies $\int_{\zeta(y_j)}^{\zeta(\xi_{k+1}^m)}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz = 4\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz$. As in the previous case, we define $\Omega_{1,2}^j = \Big\{t\in(y_j,y_{j+1}) : \mathcal{I}_{1,2}f(t) > \frac{\gamma}{6M}\Big\}$ with $\delta_{1,2}^j = \inf\Omega_{1,2}^j$ and $\epsilon_{1,2}^j = \sup\Omega_{1,2}^j$. For $x\in\Omega_{1,2}^j$, we have

$$\frac{\gamma}{6M} < 4h(x)\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz. \tag{6.84}$$

As the estimate (6.84) holds for each $x\in\Omega_{1,2}^j$, thus

$$\gamma \le 24M\big(\inf_{(\delta_{1,2}^j,\epsilon_{1,2}^j)} h(x)\big)\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{R}(\xi_k^m,z)f(z)w(z)dz. \tag{6.85}$$

We denote $\kappa(\delta_{1,2}^j,\epsilon_{1,2}^j) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{1,2}^j}^{\epsilon_{1,2}^j}\omega(z)dz\Big)$. Applying (5.9) and (6.85), we get

$$\begin{aligned} 2\kappa(\delta_{1,2}^j,\epsilon_{1,2}^j) \le & \int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz \\ & + \int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\tilde{\mathcal{V}}\left(\frac{\big(\inf h\big)\mathcal{R}(\xi_k^m,z)h(z)\kappa}{C\phi(z)\psi(z)}\right)\psi(z)dz. \end{aligned} \tag{6.86}$$

As $\zeta(\epsilon_{1,2}^j) \le \zeta(y_{j+1}) \le \zeta(y_{j+2}) \le \sigma(\delta_{1,2}^j)$, from (6.71) we obtain

$$\int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\tilde{\mathcal{V}}\left(\frac{\big(\inf h\big)\mathcal{R}(\xi_k^m,z)h(z)\kappa}{C\phi(z)\psi(z)}\right)\psi(z)dz \le \kappa(\delta_{1,2}^j,\epsilon_{1,2}^j). \tag{6.87}$$

Combining (6.86) and (6.87), we have

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\Omega^j_{1,2}}\omega(z)dz\right) \le \int_{\zeta(y_{j+1})}^{\zeta(y_{j+2})}\mathcal{V}\left(\frac{48MCf(z)\phi(z)}{\gamma}\right)\psi(z)dz.$$

Summing up in j and applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, hence we obtain

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_{1,2}f(t)>\frac{\gamma}{6M}\}}\omega(z)dz\right) \le \int_{\zeta(\xi_k^m)}^{\zeta(\xi_{k+1}^m)}\mathcal{V}\left(\frac{48MCf(z)\phi(z)}{\gamma}\right)\psi(z)dz. \quad (6.88)$$

Arguing similarly as in the previous case, we have

$$\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}_2f(t)>\frac{\gamma}{3}\}}\omega(z)dz\right) \le \int_{\zeta(\xi_{k+1}^m)}^{\sigma(\xi_k^m)}\mathcal{V}\left(\frac{48MCf(z)\phi(z)}{\gamma}\right)\psi(z)dz. \quad (6.89)$$

For $\mathcal{I}_3 f$, we consider $z_0 = \xi_{k+1}^m$ and define a decreasing sequence $\{z_j\}$ as

$$\mathcal{L}(z_j) = \int_{\sigma(\xi_k^m)}^{\sigma(z_j)}\mathcal{R}(z_j,\tau)f(\tau)w(\tau)d\tau = (M+1)^{-j}\mathcal{L}(z_0).$$

Now, we have

$$\begin{aligned}
\mathcal{L}(z_j) &= (M+1)^2\mathcal{L}(z_{j+2}) \\
&= (M+1)^2\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},\tau)f(\tau)w(\tau)d\tau \\
&= (M+1)^2\Big[\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})} + \int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\Big]\mathcal{R}(z_{j+2},\tau)f(\tau)w(\tau)d\tau \\
&\le (M+1)^2\left[\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})} M\Big\{\mathcal{R}(z_{j+2},\sigma(z_{j+3})) + \mathcal{R}(z_{j+3},\tau)\Big\} + \int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},\tau)\right]f(\tau)w(\tau)d\tau \\
&\le (M+1)^3\left\{\mathcal{R}(z_{j+2},\sigma(z_{j+3}))\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})} + \int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},\tau)\right\}f(\tau)w(\tau)d\tau \\
&\quad + M(M+1)^2\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}\mathcal{R}(z_{j+3},\tau)f(\tau)w(\tau)d\tau. \qquad (6.90)
\end{aligned}$$

From the construction of the sequence $\{z_j\}$,

$$\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})}\mathcal{R}(z_{j+3},\tau)f(\tau)w(\tau)d\tau = \mathcal{L}(z_{j+3}) = (M+1)^{-(j+3)}\mathcal{L}(z_0) = (M+1)^{-3}\mathcal{L}(z_j).$$

Thus (6.90) implies

$$\mathcal{L}(z_j) \le (M+1)^4\left[\left\{\mathcal{R}(z_{j+2},\sigma(z_{j+3}))\int_{\sigma(\xi_k^m)}^{\sigma(z_{j+3})} + \int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})}\mathcal{R}(z_{j+2},\tau)\right\}f(\tau)w(\tau)d\tau\right].$$

Next, we define $\delta^j_{3,l} = \inf \Omega^j_{3,l}$ and $\epsilon^j_{3,l} = \sup \Omega^j_{3,l}$ for $l = 1, 2$, where

$$\Omega^j_{3,1} = \left\{ y \in (z_{j+1}, z_j) : h(y)\mathcal{R}(z_{j+2}, \sigma(z_{j+3})) \int_{\sigma(\xi^m_k)}^{\sigma(z_{j+3})} f(\tau)w(\tau)d\tau > \frac{\gamma}{6(M+1)^4} \right\},$$

$$\Omega^j_{3,2} = \left\{ z \in (z_{j+1}, z_j) : h(z) \int_{\sigma(z_{j+3})}^{\sigma(z_{j+2})} \mathcal{R}(z_{j+2}, \tau)f(\tau)w(\tau)d\tau > \frac{\gamma}{6(M+1)^4} \right\}.$$

Thus

$$\begin{aligned}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t:\mathcal{I}_3 f(t)>\frac{\gamma}{3}\}} \omega(y)dy\Big) \leq \sum_{j\geq 0}\Big\{&\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega^j_{3,1}} \omega(y)dy\Big) \\ &+ \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega^j_{3,2}} \omega(y)dy\Big)\Big\}. \qquad (6.91)\end{aligned}$$

For the first part, we define a decreasing sequence $\{d'_i\}$ in (ξ^m_k, ξ^m_{k+1}) with the iteration $d'_0 = \xi^m_{k+1}$ and

$$\int_{\sigma(\xi^m_k)}^{\sigma(d'_i)} f(z)w(z)dz = 2^{-i} \int_{\sigma(\xi^m_k)}^{\sigma(\xi^m_{k+1})} f(z)w(z)dz.$$

We define $d_0 = d'_0$ and if $d'_i > z_j \geq d'_{i+1}$ then $d_{n+1} = d'_{i+1}$, otherwise we delete the term d'_{i+1} and continue the process. Thus, we get a subsequence $\{d_n\}$ of $\{d'_i\}$. Let $\tilde{\delta}^n_{3,1} = \inf \tilde{\Omega}^n_{3,1}$ and $\tilde{\epsilon}^n_{3,1} = \sup \tilde{\Omega}^n_{3,1}$, where $\tilde{\Omega}^n_{3,1} = \cup_{\{j:d_n>z_{j+3}\geq d_{n+1}\}} \Omega^j_{3,1}$. Now, if $d'_{i+1} = d_{n+1} \leq z_{j+3} < d_n$, then $z_{j+3} \leq d'_i$ and $d_{n+2} \leq d'_{i+2}$. We have

$$\int_{\sigma(\xi^m_k)}^{\sigma(z_{j+3})} \leq \int_{\sigma(\xi^m_k)}^{\sigma(d'_i)} = 4 \int_{\sigma(d'_{i+2})}^{\sigma(d'_{i+1})} \leq 4 \int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})}. \qquad (6.92)$$

Now for $x \in \tilde{\Omega}^n_{3,1}$, we obtain

$$\frac{\gamma}{6(M+1)^4} < 4h(x)\mathcal{R}(x, \sigma(z_{j+3})) \int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} f(\tau)w(\tau)d\tau. \qquad (6.93)$$

As (6.93) holds for each $x \in \tilde{\Omega}^n_{3,1}$, thus

$$\gamma \leq 24(M+1)^4 \inf_{(\tilde{\delta}^n_{3,1}, \tilde{\epsilon}^n_{3,1})} \Big(h(x)\mathcal{R}(x, \sigma(z_{j+3}))\Big) \int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} f(\tau)w(\tau)d\tau. \qquad (6.94)$$

We denote $\kappa(\tilde{\delta}^n_{3,1}, \tilde{\epsilon}^n_{3,1}) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}^n_{3,1}}^{\tilde{\epsilon}^n_{3,1}} \omega(\tau)d\tau\Big)$. From (5.9) and (6.72) we obtain

$$2\kappa(\tilde{\delta}^n_{3,1}, \tilde{\epsilon}^n_{3,1}) \leq \int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} \mathcal{V}\Big(\frac{48(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau$$

$$
\begin{aligned}
&+\int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} \tilde{\mathcal{V}}\left[\frac{\inf\big(h(x)\mathcal{R}(x,\sigma(z_{j+3}))\big)w(\tau)\kappa}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \\
&\leq \int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} \mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau + \kappa(\tilde{\delta}_{3,1}^n, \tilde{\epsilon}_{3,1}^n). \qquad (6.95)
\end{aligned}
$$

Thus

$$
\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}_{3,1}^n}^{\tilde{\epsilon}_{3,1}^n} \omega(\tau)d\tau\Big) \leq \int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} \mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau.
$$

That is

$$
\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}_{3,1}^n} \omega(\tau)d\tau\Big) \leq \int_{\sigma(d_{n+2})}^{\sigma(d_{n+1})} \mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau.
$$

Summing up over n and then applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we obtain

$$
\sum_{n\geq 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}_{3,1}^n} \omega(\tau)d\tau\Big) \leq \int_{\sigma(\xi_k^m)}^{\sigma(\xi_{k+1}^m)} \mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau.
$$

This implies

$$
\sum_{j\geq 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{3,1}^j} \omega(\tau)d\tau\Big) \leq \int_{\sigma(\xi_k^m)}^{\sigma(\xi_{k+1}^m)} \mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau. \qquad (6.96)
$$

Next, for $\Omega_{3,2}^j$ we may proceed similarly to the previous cases, obtaining

$$
\sum_{j\geq 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega_{3,2}^j} \omega(z)dz\Big) \leq \int_{\sigma(\xi_k^m)}^{\sigma(\xi_{k+1}^m)} \mathcal{V}\Big(\frac{12(M+1)^4Cf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \qquad (6.97)
$$

Combining (6.76), (6.77), (6.83), (6.88), (6.89), (6.91), (6.96) and (6.97) we obtain

$$
\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\mathcal{I}f(t)>\gamma\}} \omega(z)dz\Big) \leq \int_{\zeta(\xi_k^m)}^{\sigma(\xi_{k+1}^m)} \mathcal{V}\Big(\frac{96(M+1)^4Cf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \qquad (6.98)
$$

Summing up (6.98) over m and k we obtain the estimate (6.70) with constant $96(M+1)^4C$.

$(i) \implies (ii)$. Conversely, let us assume $t<\tau$ for which $\zeta(\tau)<\sigma(t)$. Corresponding to each $N\in\mathbb{N}$ we consider the set $E_N=\big\{\zeta(\tau)<s<\sigma(t) : \frac{1}{N}\leq\mathcal{R}(t,s), w(s)\leq N\big\}$ has finite measure. We have

$$
\int_{E_N}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy \leq lN^2|E_N|\big(\inf h\big)\tilde{v}(\lambda lkN^2\inf h) < \infty
$$

for each $l, k \in \mathbb{N}$ and $\lambda > 0$. Thus for each $\mu > 0$ we choose λ such that

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy = (1+\mu)C,$$

where C is the constant in (6.70). For each $\gamma > 0$ we consider

$$f_\gamma(y) = \frac{\gamma}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}\chi_{E_N}(y).$$

For $t < x < \tau$, we have

$$\begin{aligned}
\mathcal{I}f_\gamma(x) &= h(x)\int_{E_N}\mathcal{R}(x,y)\frac{\gamma}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}w(y)dy\\
&\geq \int_{E_N}\frac{\gamma}{C\lambda}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\Big(\psi(y)+1/k\Big)dy\\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

This implies

$$(t,\tau) \subset \{x : \mathcal{I}f_\gamma(x) > \gamma\}.$$

Thus using (5.10) and (6.70) we obtain

$$\begin{aligned}
\kappa(t,\tau) &= \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_t^\tau \omega(y)dy\right)\\
&\leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_{\{\mathcal{I}f_\gamma>\gamma\}} \omega(y)dy\right)\\
&\leq \int_{E_N}\mathcal{V}\left(\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}\phi(y)\right)\psi(y)dy\\
&\leq \int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy\\
&\leq (1+\mu)C\lambda.
\end{aligned}$$

Since $\tilde{\mathcal{V}}(r)/r$ increases as r increases, we have

$$\begin{aligned}
&\int_{E_N}\tilde{\mathcal{V}}\left(\frac{\big(\inf h\big)\mathcal{R}(t,y)w(y)\kappa(t,\tau)}{(1+\mu)C(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\kappa(t,\tau)}dy\\
&\quad\leq \int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h\big)\mathcal{R}(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda}dy = 1.
\end{aligned}$$

By the Monotone Convergence Theorem, we obtain

$$\int_{E_N}\tilde{\mathcal{V}}\left(\frac{\big(\inf h\big)\mathcal{R}(t,y)w(y)\kappa(t,\tau)}{(1+\mu)C(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\kappa(t,\tau)}dy \leq 1.$$

Letting $l, N \to \infty$ and $\mu \to 0^+$, we obtain

$$\int_{\zeta(\tau)}^{\sigma(t)} \tilde{\mathcal{V}}\left(\frac{\big(\inf h\big)\mathcal{R}(t,y)w(y)\kappa\big(t,\tau\big)}{C\phi(y)\psi(y)}\right)\psi(y)dy \le \kappa\big(t,\tau\big).$$

Next, we prove the condition (6.72). Let $a < z \le t < \tau < b$ satisfying $\zeta(\tau) \le \sigma(z)$. For each $m \in \mathbb{N}$ we consider the set $E_m = \big\{\zeta(\tau) < s < \sigma(z) : \frac{1}{m} \le w(s) \le m\big\}$ has finite measure. For each $l, k \in \mathbb{N}$ and $\lambda > 0$ we have

$$\int_{E_m} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\left(\frac{\psi(s)+1/k}{\lambda}\right)ds < \infty,$$

where the infimum is considered over the interval (t, τ). Given $\mu > 0$ we choose λ suitably such that

$$\int_{E_m} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\left(\frac{\psi(s)+1/k}{\lambda}\right)ds = (1+\mu)C,$$

where C is the constant in (6.70). For $\gamma > 0$ we define

$$f_\gamma(s) = \frac{\gamma}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\frac{\psi(s)+1/k}{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}\chi_{E_m}(s).$$

If $s \in E_m$ and $t < x < \tau$, then $\mathcal{R}(x,s) \ge \mathcal{R}(x,\sigma(z))$. Thus for $t < x < \tau$ we have

$$\begin{aligned}
\mathcal{I}f_\gamma(x) &= h(x)\int_{E_m} \mathcal{R}(x,s)\frac{\gamma}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\frac{\psi(s)+1/k}{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}w(s)ds \\
&\ge \int_{E_m} \frac{\gamma}{C\lambda}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf h(y)\mathcal{R}(y,\sigma(z))\big)w(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\big(\psi(s)+1/k\big)ds \\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

This implies

$$(t,\tau) \subset \{x : \mathcal{I}f_\gamma(x) > \gamma\}.$$

The rest of the proof proceeds similarly. Hence the proof is complete. □

Theorem 6.4.2. *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}$ and $\mathcal{V} \circ \mathcal{U}^{-1}$ satisfy all the conditions stated in Theorem 6.3.1. Let the function w be monotone on $\mathbb{R}$ and and let the function $w(\cdot)\mathcal{R}(y,\cdot)$ satisfies that*

$$\inf_{x\in\Omega} w(x)\mathcal{R}(y,x) = \inf_{x\in(\inf\Omega,\sup\Omega)} w(x)\mathcal{R}(y,x)$$

for all bounded set Ω and all y. Then the following assertions are equivalent.

(i) *For all functions* $f \geq 0$ *and each* $\gamma > 0$, *the estimate*

$$\omega\Big(\big\{t \in (a,b) : \tilde{\mathcal{I}}f(t) > \gamma\big\}\Big) \leq \mathcal{U} \circ \mathcal{V}^{-1}\Big(\int_{\sigma^{-1}(a)}^{\zeta^{-1}(b)} \mathcal{V}\Big(\frac{Cf(y)\phi(y)}{\gamma}\Big)\psi(y)dy\Big), \tag{6.99}$$

holds for an arbitrary constant $C > 0$.

(ii) *There exists* $C > 0$ *such that*

$$\int_{\sigma^{-1}(\tau)}^{\zeta^{-1}(t)} \tilde{\mathcal{V}}\left[\frac{(\inf w)\mathcal{R}(s,\tau)h(s)\kappa(t,\tau)}{C\phi(s)\psi(s)}\right]\psi(s)ds \leq \kappa(t,\tau), \tag{6.100}$$

and

$$\int_{\sigma^{-1}(z)}^{\zeta^{-1}(t)} \tilde{\mathcal{V}}\left[\frac{\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)\kappa(t,\tau)}{C\phi(s)\psi(s)}\right]\psi(s)ds \leq \kappa(t,\tau) \tag{6.101}$$

hold for each $a < t < \tau \leq z < b$ *satisfying* $\sigma^{-1}(\tau) \leq \sigma^{-1}(z) \leq \zeta^{-1}(t)$, *where*

$$\kappa(t,\tau) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_t^\tau \omega(z)dz\Big).$$

Proof. (ii) $\implies$ (i). Let $\{\xi_k^m\}$(after renaming) be the sequence given by Lemma 6.2.1 for the limits $\zeta^{-1}(t)$ and $\sigma^{-1}(t)$, and thus (6.11) holds for the adjoint operator $\tilde{\mathcal{I}}f$. We write $\tilde{\mathcal{I}}f$ for $t \in (\xi_k^m, \xi_{k+1}^m)$ as

$$\begin{aligned}
\tilde{\mathcal{I}}f(t) &= w(t)\int_{\sigma^{-1}(t)}^{\zeta^{-1}(t)} \mathcal{R}(s,t)f(s)h(s)ds \\
&= w(t)\Big\{\int_{\sigma^{-1}(t)}^{\sigma^{-1}(\xi_{k+1}^m)} + \int_{\sigma^{-1}(\xi_{k+1}^m)}^{\zeta^{-1}(\xi_k^m)} + \int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(t)}\Big\}\mathcal{R}(s,t)f(s)h(s)ds \\
&= \tilde{\mathcal{I}}_1f(t) + \tilde{\mathcal{I}}_2f(t) + \tilde{\mathcal{I}}_3f(t).
\end{aligned} \tag{6.102}$$

Thus from (6.102), we have

$$\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\{t \in (\xi_k^m, \xi_{k+1}^m) : \tilde{\mathcal{I}}f(t) > \gamma\}} \omega(y)dy\Big) \leq \sum_{i=1}^{3}\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\{t \in (\xi_k^m, \xi_{k+1}^m) : \tilde{\mathcal{I}}_if(t) > \frac{\gamma}{3}\}} \omega(y)dy\Big). \tag{6.103}$$

We will first estimate the integral $\tilde{\mathcal{I}}_3f$. Applying (6.2) we obtain

$$\tilde{\mathcal{I}}_3f(t) \leq M\left[w(t)\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(t)} \mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds + w(t)\mathcal{R}(\sigma^{-1}(\xi_{k+1}^m),t)\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(t)} f(s)h(s)ds\right]$$

$$= M\Big[\tilde{\mathcal{I}}_{3,1}f(t) + \tilde{\mathcal{I}}_{3,2}f(t)\Big].$$

Thus

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_3 f(t)>\frac{\gamma}{3}\}}\omega(y)dy\Big) \le \sum_{i=1}^{2}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_{3,i} f(t)>\frac{\gamma}{6M}\}}\omega(y)dy\Big). \tag{6.104}$$

We define a sequence $\{x_j\}$ as $x_0 = \xi_{k+1}^m$ and for each x_{j-1} let x_j be the number given by $\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(x_j)}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds = \int_{\zeta^{-1}(x_j)}^{\zeta^{-1}(x_{j-1})}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds$. Then the sequence $\{x_j\}$ decreases and satisfies

$$\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(x_j)}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds = 4\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{R}(s,\xi_{k+1}^m)f(s)h(s)ds.$$

We define $\delta_{3,1}^j = \inf\Omega_{3,1}^j$ and $\epsilon_{3,1}^j = \sup\Omega_{3,1}^j$, where

$$\Omega_{3,1}^j = \Big\{t\in(x_{j+1},x_j) : \tilde{\mathcal{I}}_{3,1}f(t) > \frac{\gamma}{6M}\Big\}.$$

For $x\in\Omega_{3,1}^j$, we have

$$\frac{\gamma}{6M} < 4w(x)\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{R}(z,\xi_{k+1}^m)f(z)h(z)dz. \tag{6.105}$$

As the estimate (6.105) holds for each $x\in\Omega_{3,1}^j$, we have

$$\gamma \le 24M\big(\inf_{(\delta_{3,1}^j,\epsilon_{3,1}^j)} w(x)\big)\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{R}(z,\xi_{k+1}^m)f(z)h(z)dz. \tag{6.106}$$

Let us denote $\kappa(\delta_{3,1}^j,\epsilon_{3,1}^j) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{3,1}^j}^{\epsilon_{3,1}^j}\omega(\tau)d\tau\Big)$, then using (5.9) and (6.106) we obtain

$$\begin{aligned}2\kappa(\delta_{3,1}^j,\epsilon_{3,1}^j) \le &\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz\\ &+\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)\mathcal{R}(z,\xi_{k+1}^m)h(z)\kappa}{C\phi(z)\psi(z)}\right)\psi(z)dz.\end{aligned} \tag{6.107}$$

As $\sigma^{-1}(\epsilon_{3,1}^j) \le \sigma^{-1}(\xi_{k+1}^m) \le \zeta^{-1}(\xi_k^m) \le \zeta^{-1}(x_j) \le \zeta^{-1}(\delta_{3,1}^j)$, thus (6.100) gives

$$\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)\mathcal{R}(z,\xi_{k+1}^m)h(z)\kappa}{C\phi(z)\psi(z)}\right)\psi(z)dz \le \kappa(\delta_{3,1}^j,\epsilon_{3,1}^j). \tag{6.108}$$

Combining (6.107) and (6.108) we have

$$\big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\Big(\int_{\Omega^j_{3,1}}\omega(\tau)d\tau\Big)\leq\int_{\zeta^{-1}(x_{j+2})}^{\zeta^{-1}(x_{j+1})}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \tag{6.109}$$

Summing up in j and applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we get

$$\big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\Big(\int_{\{t\in(\xi^m_k,\xi^m_{k+1}):\tilde{\mathcal{I}}_{3,1}f(t)>\frac{\gamma}{6M}\}}\omega(z)dz\Big)\leq\int_{\zeta^{-1}(\xi^m_k)}^{\zeta^{-1}(\xi^m_{k+1})}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \tag{6.110}$$

To estimate $\tilde{\mathcal{I}}_{3,2}$, we consider a decreasing sequence $\{y_j\}$ defined as $y_0=\xi^m_{k+1}$ and for each y_{j-1} let y_j be the number given by $\int_{\zeta^{-1}(\xi^m_k)}^{\zeta^{-1}(y_j)}f(s)h(s)ds=\int_{\zeta^{-1}(y_j)}^{\zeta^{-1}(y_{j-1})}f(s)h(s)ds$. From the definition of $\{y_j\}$ we obtain $\int_{\zeta^{-1}(\xi^m_k)}^{\zeta^{-1}(y_j)}f(s)h(s)ds=4\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}f(s)h(s)ds$. We define $\delta^j_{3,2}=\inf\Omega^j_{3,2}$ and $\epsilon^j_{3,2}=\sup\Omega^j_{3,2}$, where

$$\Omega^j_{3,2}=\Big\{t\in(y_{j+1},y_j):\tilde{\mathcal{I}}_{3,2}f(t)>\frac{\gamma}{6M}\Big\}.$$

For $x\in\Omega^j_{3,2}$, we have

$$\frac{\gamma}{6M}<4w(x)\mathcal{R}(\sigma^{-1}(\xi^m_{k+1}),x)\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}f(z)h(z)dz. \tag{6.111}$$

As the estimate (6.111) holds for each $x\in\Omega^j_{3,2}$, thus

$$\gamma\leq 24M\big(\inf_{(\delta^j_{3,2},\epsilon^j_{3,2})}w(x)\mathcal{R}(\sigma^{-1}(\xi^m_{k+1}),x)\big)\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}f(z)h(z)dz. \tag{6.112}$$

Considering $\kappa(\delta^j_{3,2},\epsilon^j_{3,2})=\big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\Big(\int_{\delta^j_{3,2}}^{\epsilon^j_{3,2}}\omega(\tau)d\tau\Big)$ and from (5.9) and (6.112) we have

$$\begin{aligned}2\kappa(\delta^j_{3,2},\epsilon^j_{3,2})\leq&\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz\\&+\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}\tilde{\mathcal{V}}\left(\frac{\big(\inf w(x)\mathcal{R}(\sigma^{-1}(\xi^m_{k+1}),x)\big)h(z)\kappa}{C\phi(z)\psi(z)}\right)\psi(z)dz.\end{aligned} \tag{6.113}$$

As $\sigma^{-1}(\epsilon^j_{3,2})\leq\sigma^{-1}(\xi^m_{k+1})\leq\zeta^{-1}(\xi^m_k)\leq\zeta^{-1}(y_j)\leq\zeta^{-1}(\delta^j_{3,2})$, (6.101) gives

$$\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}\tilde{\mathcal{V}}\left(\frac{\big(\inf w(x)\mathcal{R}(\sigma^{-1}(\xi^m_{k+1}),x)\big)h(z)\kappa}{C\phi(z)\psi(z)}\right)\psi(z)dz\leq\kappa(\delta^j_{3,2},\epsilon^j_{3,2}). \tag{6.114}$$

Combining (6.113) and (6.114) we have

$$\big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\Big(\int_{\Omega^j_{3,2}}\omega(\tau)d\tau\Big)\leq\int_{\zeta^{-1}(y_{j+2})}^{\zeta^{-1}(y_{j+1})}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \tag{6.115}$$

Summing up in j and applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we get

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_{3,2}f(t)>\frac{\gamma}{6M}\}}\omega(z)dz\Big)\leq\int_{\zeta^{-1}(\xi_k^m)}^{\zeta^{-1}(\xi_{k+1}^m)}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \tag{6.116}$$

Arguing similarly we obtain

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi_k^m,\xi_{k+1}^m):\tilde{\mathcal{I}}_2f(t)>\frac{\gamma}{3}\}}\omega(z)dz\Big)\leq\int_{\sigma^{-1}(\xi_{k+1}^m)}^{\zeta^{-1}(\xi_k^m)}\mathcal{V}\Big(\frac{48MCf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \tag{6.117}$$

For $\tilde{\mathcal{I}}_1$, we consider $z_0=\xi_k^m$ and we define an increasing sequence $\{z_j\}$ as

$$\mathcal{L}(z_j)=\int_{\sigma^{-1}(z_j)}^{\sigma^{-1}(\xi_{k+1}^m)}\mathcal{R}(\tau,z_j)f(\tau)h(\tau)d\tau=(M+1)^{-j}\mathcal{L}(z_0).$$

We use the condition (6.2) to estimate the following.

$$\begin{aligned}
\mathcal{L}(z_j)&=(M+1)^2\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(\xi_{k+1}^m)}\mathcal{R}(\tau,z_{j+2})f(\tau)h(\tau)d\tau \qquad (6.118)\\
&=(M+1)^2\Big[\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}+\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\Big]\mathcal{R}(\tau,z_{j+2})f(\tau)h(\tau)d\tau\\
&\leq(M+1)^2\Bigg[\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}\mathcal{R}(\tau,z_{j+2})+\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}M\Big\{\mathcal{R}(\tau,z_{j+3})+\mathcal{R}(\sigma^{-1}(z_{j+3}),z_{j+2})\Big\}\Bigg]f(\tau)h(\tau)d\tau\\
&\leq(M+1)^3\Bigg\{\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}\mathcal{R}(\tau,z_{j+2})+\mathcal{R}(\sigma^{-1}(z_{j+3}),z_{j+2})\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\Bigg\}f(\tau)h(\tau)d\tau\\
&\quad+M(M+1)^2\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\mathcal{R}(\tau,z_{j+3})f(\tau)h(\tau)d\tau. \qquad (6.119)
\end{aligned}$$

From the construction of $\{z_j\}$, we have

$$\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}\mathcal{R}(\tau,z_{j+3})f(\tau)h(\tau)d\tau=\mathcal{L}(z_{j+3})=(M+1)^{-(j+3)}\mathcal{L}(z_0)=(M+1)^{-3}\mathcal{L}(z_j).$$

Thus (6.119) implies that

$$\mathcal{L}(z_j)\leq(M+1)^4\Bigg[\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}\mathcal{R}(\tau,z_{j+2})f(\tau)h(\tau)d\tau+\mathcal{R}(\sigma^{-1}(z_{j+3}),z_{j+2})\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)}f(\tau)h(\tau)d\tau\Bigg].$$

Next we consider $\delta_{1,l}^j=\inf\Omega_{1,l}^j$ and $\epsilon_{1,l}^j=\sup\Omega_{1,l}^j$ for $l=1,2$, where

$$\Omega_{1,1}^j=\Bigg\{y\in(z_j,z_{j+1}):w(y)\int_{\sigma^{-1}(z_{j+2})}^{\sigma^{-1}(z_{j+3})}\mathcal{R}(\tau,z_{j+2})f(\tau)h(\tau)d\tau>\frac{\gamma}{6(M+1)^4}\Bigg\},$$

$$\Omega_{1,2}^j = \left\{ z \in (z_j, z_{j+1}) : w(z)\mathcal{R}(\sigma^{-1}(z_{j+3}), z_{j+2}) \int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)} f(\tau)h(\tau)d\tau > \frac{\gamma}{6(M+1)^4} \right\}.$$

Thus the estimate for $\tilde{\mathcal{I}}_1 f$ is reduced to the following form.

$$\begin{aligned}\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t:\mathcal{I}_1 f(t)>\frac{\gamma}{3}\}} \omega(z)dz\right) \leq \sum_{j\geq 0}\Bigg\{ &\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\Omega_{1,1}^j} \omega(z)dz\right) \\ &+ \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\Omega_{1,2}^j} \omega(z)dz\right)\Bigg\}. \end{aligned} \tag{6.120}$$

Arguing similarly as in the previous cases, we observe that

$$\sum_{j\geq 0}\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\Omega_{1,1}^j} \omega(z)dz\right) \leq \int_{\sigma^{-1}(\xi_k^m)}^{\sigma^{-1}(\xi_{k+1}^m)} \mathcal{V}\Big(\frac{12(M+1)^4 Cf(z)\phi(z)}{\gamma}\Big)\psi(z)dz. \tag{6.121}$$

For the second part, we define an increasing sequence $\{d_i'\}$ in (ξ_k^m, ξ_{k+1}^m) with the iteration $d_0' = \xi_k^m$ and

$$\int_{\sigma^{-1}(d_i')}^{\sigma^{-1}(\xi_{k+1}^m)} f(z)h(z)dz = 2\int_{\sigma^{-1}(d_{i+1}')}^{\sigma^{-1}(\xi_{k+1}^m)} f(z)h(z)dz.$$

Next we define a subsequence $\{d_n\}$ as $d_0 = d_0'$ and if $d_i' < z_j \leq d_{i+1}'$ then $d_{n+1} = d_{i+1}'$, otherwise we delete the term d_{i+1}' and continue the process. Thus, we get a subsequence $\{d_n\}$ of $\{d_i'\}$. Let $\tilde{\delta}_{1,2}^n = \inf \tilde{\Omega}_{1,2}^n$ and $\tilde{\epsilon}_{1,2}^n = \sup \tilde{\Omega}_{1,2}^n$, where $\tilde{\Omega}_{1,2}^n = \cup_{\{j:d_n<z_{j+3}\leq d_{n+1}\}}\Omega_{1,2}^j$. Now, if $d_{i+1}' = d_{n+1} \geq z_{j+3} > d_n$ then by construction $z_{j+3} \geq d_i'$ and $d_{n+2} \geq d_{i+2}'$. From the construction of $\{d_i'\}$ and $\{d_n\}$, we have

$$\int_{\sigma^{-1}(z_{j+3})}^{\sigma^{-1}(\xi_{k+1}^m)} \leq \int_{\sigma^{-1}(d_i')}^{\sigma^{-1}(\xi_{k+1}^m)} = 4\int_{\sigma^{-1}(d_{i+1}')}^{\sigma^{-1}(d_{i+2}')} \leq 4\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}. \tag{6.122}$$

Now for $x \in \tilde{\Omega}_{1,2}^n$, we have

$$\frac{\gamma}{6(M+1)^4} < 4w(x)\mathcal{R}(\sigma^{-1}(z_{j+3}), x)\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})} f(\tau)h(\tau)d\tau. \tag{6.123}$$

As (6.123) holds for each $x \in \tilde{\Omega}_{1,2}^n$, thus

$$\gamma \leq 24(M+1)^4 \inf_{(\tilde{\delta}_{1,2}^n, \tilde{\epsilon}_{1,2}^n)} \left(w(x)\mathcal{R}(\sigma^{-1}(z_{j+3}), x)\right)\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})} f(\tau)h(\tau)d\tau. \tag{6.124}$$

Let us denote $\kappa(\tilde{\delta}_{1,2}^n, \tilde{\epsilon}_{1,2}^n) = \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\tilde{\delta}_{1,2}^n}^{\tilde{\epsilon}_{1,2}^n} \omega(\tau)d\tau\right)$. Now applying (5.9) and (6.124) we obtain

$$2\kappa(\tilde{\delta}_{1,2}^n, \tilde{\epsilon}_{1,2}^n) \leq \int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})} \mathcal{V}\Big(\frac{48(M+1)^4 Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau$$

$$+\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}\tilde{\mathcal{V}}\left[\frac{\inf\left(w(x)\mathcal{R}(\sigma^{-1}(z_{j+3}),x)\right)h(\tau)\kappa}{C\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau$$
$$\leq \int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}\mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau+\kappa(\tilde{\delta}^n_{1,2},\tilde{\epsilon}^n_{1,2}).$$

Thus using the condition (6.101), we have

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}^n_{1,2}}^{\tilde{\epsilon}^n_{1,2}}\omega(\tau)d\tau\Big)\leq\int_{\sigma^{-1}(d_{n+1})}^{\sigma^{-1}(d_{n+2})}\mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau.$$

Summing up over n and then applying the sub-additivity of $\mathcal{V}\circ\mathcal{U}^{-1}$, we obtain

$$\sum_{n\geq 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\Omega}^n_{1,2}}\omega(\tau)d\tau\Big)\leq\int_{\sigma^{-1}(\xi^m_k)}^{\sigma^{-1}(\xi^m_{k+1})}\mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau.\qquad(6.125)$$

Thus

$$\sum_{j\geq 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\Omega^j_{1,2}}\omega(\tau)d\tau\Big)\leq\int_{\sigma^{-1}(\xi^m_k)}^{\sigma^{-1}(\xi^m_{k+1})}\mathcal{V}\Big(\frac{48(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau.\qquad(6.126)$$

Combining (6.103), (6.104), (6.110), (6.116), (6.117), (6.120), (6.121) and (6.126) we obtain

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{t\in(\xi^m_k,\xi^m_{k+1}):\tilde{\mathcal{I}}f(t)>\gamma\}}\omega(z)dz\Big)$$
$$\leq\int_{\sigma^{-1}(\xi^m_k)}^{\zeta^{-1}(\xi^m_{k+1})}\mathcal{V}\Big(\frac{96(M+1)^4Cf(z)\phi(z)}{\gamma}\Big)\psi(z)dz.\qquad(6.127)$$

Summing up (6.127) over m and k we obtain the estimate (6.39) with constant $96(M+1)^4C$.

$(i)\implies(ii)$. Conversely let us assume $t<\tau$ such that $\sigma^{-1}(\tau)<\zeta^{-1}(t)$. For each $N\in\mathbb{N}$ we consider the set, $E_N=\Big\{\sigma^{-1}(\tau)<s<\zeta^{-1}(t):\frac{1}{N}\leq\mathcal{R}(s,\tau),h(s)\leq N\Big\}$ has finite measure. As the integral

$$\int_{E_N}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy\leq lN^2|E_N|\big(\inf w\big)\tilde{v}(\lambda lkN^2\inf w)<\infty$$

for each $l,k\in\mathbb{N}$ and $\lambda>0$. Given $\mu>0$ we choose λ suitably so that

$$\int_{E_N}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy=(1+\mu)C,$$

where C is the constant in (6.99). For each $\gamma>0$ we consider

$$f_\gamma(y)=\frac{\gamma}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}\chi_{E_N}(y).$$

If $t < x < \tau$, then

$$\begin{aligned}
\tilde{\mathcal{I}} f_\gamma(x) &= w(x) \int_{E_N} \mathcal{R}(y,x) \frac{\gamma}{C} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)} \right) \frac{\psi(y)+1/k}{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)} h(y)dy \\
&\geq \int_{E_N} \frac{\gamma}{C\lambda} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)} \right) \Big(\psi(y)+1/k\Big) dy \\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

This implies that

$$(t,\tau) \subset \{x : \tilde{\mathcal{I}} f_\gamma(x) > \gamma\}.$$

Applying the condition (5.10) and the assumption (6.99) we get

$$\begin{aligned}
\kappa(t,\tau) &= \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_t^\tau \omega(y)dy\right) \\
&\leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_{\{\tilde{\mathcal{I}} f_\gamma > \gamma\}} \omega(y)dy\right) \\
&\leq \int_{E_N} \mathcal{V}\left(\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)} \right) \frac{\psi(y)+1/k}{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)} \phi(y) \right) \psi(y)dy \\
&\leq \int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)} \right) \psi(y)dy \\
&\leq (1+\mu)C\lambda.
\end{aligned}$$

As $\tilde{\mathcal{V}}(r)/r$ increases as r increases, we have

$$\begin{aligned}
&\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)\mathcal{R}(y,\tau)h(y)\kappa(t,\tau)}{(1+\mu)C(\phi(y)+1/l)(\psi(y)+1/k)} \right) \frac{\psi(y)+1/k}{\kappa(t,\tau)} dy \\
&\quad \leq \int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)\mathcal{R}(y,\tau)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)} \right) \frac{\psi(y)+1/k}{(1+\mu)C\lambda} dy = 1.
\end{aligned}$$

By the Monotone Convergence Theorem, we get

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)\mathcal{R}(y,\tau)h(y)\kappa(t,\tau)}{(1+\mu)C(\phi(y)+1/l)\psi(y)} \right) \frac{\psi(y)}{\kappa(t,\tau)} dy \leq 1.$$

Letting $l, N \to \infty$ and $\mu \to 0^+$, we obtain

$$\int_{\sigma^{-1}(\tau)}^{\zeta^{-1}(t)} \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)\mathcal{R}(y,\tau)h(y)\kappa\big(t,\tau\big)}{C\phi(y)\psi(y)} \right) \psi(y)dy \leq \kappa\big(t,\tau\big).$$

Next we prove the estimate (6.101). Let $a < t < \tau \leq z < b$ such that $\sigma^{-1}(\tau) \leq \sigma^{-1}(z) \leq \zeta^{-1}(t)$. For each $m \in \mathbb{N}$ we consider the set, $E_m = \Big\{\sigma^{-1}(z) < s < \zeta^{-1}(t) : \frac{1}{m} \leq h(s) \leq m\Big\}$ has finite measure. Now for each $l, k \in \mathbb{N}$ and $\lambda > 0$ we have

$$\int_{E_m} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z),y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)} \right) \left(\frac{\psi(s)+1/k}{\lambda} \right) ds < \infty,$$

where the infimum is considered over the interval (t, τ). Given $\mu > 0$ we choose λ suitably so that

$$\int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z), y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\Big(\frac{\psi(s)+1/k}{\lambda}\Big)ds = (1+\mu)C,$$

where C is the constant in (6.99). We define

$$f_\gamma(s) = \frac{\gamma}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z), y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\frac{\psi(s)+1/k}{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z), y)\big)h(s)}\chi_{E_m}(s).$$

If $s \in E_m$ and $t < x < \tau$, then $\mathcal{R}(s, x) \geq \mathcal{R}(\sigma^{-1}(z), x)$. Thus for $t < x < \tau$ we have

$$\begin{aligned}
\tilde{\mathcal{I}} f_\gamma(x) &= w(x)\int_{E_m} \mathcal{R}(s, x)\frac{\gamma}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z), y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\frac{\psi(s)+1/k}{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z), y)\big)h(s)}h(s)ds \\
&\geq \int_{E_m} \frac{\gamma}{C\lambda}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w(y)\mathcal{R}(\sigma^{-1}(z), y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\Big)\Big(\psi(s)+1/k\Big)ds \\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

This implies that

$$(t, \tau) \subset \{x : \tilde{\mathcal{I}} f_\gamma(x) > \gamma\}.$$

The rest of the proof proceeds similarly. Hence the proof is complete. $\square$

Similarly, the extra-weak type integral estimates for the operator of Hardy-Steklov type and its adjoint follow directly from the above two theorems by considering $\mathcal{R} \equiv 1$.

Future Plan

(1) The integral operator of Hardy type is defined as

$$K(f)(x) = \int_{-\infty}^{x} k(x,y)f(y)dy, \tag{6.128}$$

where $k(\geq 0)$ is defined on the set $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : y < x\}$. For $k = 1$, the integral operator is known as the Hardy operator and it is defined as

$$H(f)(x) = \int_{-\infty}^{x} f(y)dy.$$

We refer to [1,29] for the weighted estimates for the integral operator (6.128). In [9,33], the authors defined a bilinear version of Hardy operator as

$$H(f,g)(x) = \int_{-\infty}^{x}\int_{-\infty}^{x} f(s)g(t)dsdt$$

and they characterized the positive measurable functions w, w_1 and w_2 such that the inequality

$$\left(\int_{-\infty}^{\infty} H(f,g)(x)^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{-\infty}^{\infty} f(x)^{p_1}w_1(x)dx\right)^{\frac{1}{p_1}}\left(\int_{-\infty}^{\infty} g(x)^{p_2}w_2(x)dx\right)^{\frac{1}{p_2}} \tag{6.129}$$

holds for some positive constant C and for any pair of nonnegative measurable function (f,g) with $q, p_1, p_2 > 1$.

We plan to consider a bilinear integral operator of Hardy type, and want to characterize the weights w, w_1, w_2 such that the inequality (6.129) holds for the bilinear integral operator.

(2) In [20, 25, 32], the autors considered Hardy type operators involving suprema, and established weighted estimates for such operators. We plan to study weighted inequalities for the Hardy-Steklov type operators involving suprema.

(3) We also plan to study weighted estimates for maximal operators and integral operators in Sobolev spaces.