CHAPTER 1

Introduction

The theory of weights is a well-established area in Fourier analysis. The area has got a new appearance in the 1970s after the introduction of $\mathcal{A}_p$ weights by Benjamin Muckenhoupt and then received remarkable attention. It has considerable significance in the field of partial differential equations(PDE). Some applications of weighted inequalities include the study of boundary value problems for PDE, regularity of PDE. The significant purpose is to obtain certain norm inequalities for maximal functions and singular integrals.

In this thesis, we emphasize weighted norm inequalities for certain maximal functions and integral operators of Hardy type. It is noteworthy to mention that the Hardy-Littlewood maximal function has a significant role in obtaining convergence of some integral operators and also plays a crucial role in proving the Lebesgue differentiation theorem. An independent proof for the Sobolev inequality is also feasible by using the concept of maximal functions. Also, the weighted inequalities of maximal functions are used to study the regularity of certain PDE. The one-sided version of the maximal function is a particular case of ergodic maximal functions and, hence the study of one-sided maximal functions has wide applications in ergodic theory. The concept of the minimal function comes naturally from the maximal function. The class of weights for the minimal function are used to characterize the class of functions which satisfy reverse Hölder inequality. Apart from it, the minimal function is also useful in obtaining convergence of some integral operators. Weighted inequalities for the integral operators of Hardy type are applied to the spectral problem, fractional regularity, in the construction of energy estimates for PDE, especially in hyperbolic PDE, etc. In general, Hardy type inequalities can be visualized as the weighted version of the Poincaré inequality with some extra conditions. The Hardy-Steklov operator extends the notion of the Hardy operator to dynamic limits. Many fruitful applications of this operator mainly include the study of delay differential equations and analyzing the future value of a stock in the financial market.

1.1 Preliminaries

By a weight, we mean a non-negative and locally integrable function which takes values in $(0,\infty)$ almost everywhere. Corresponding to a measurable set $\mathcal{E}$ in $\mathbb{R}$ and a weight σ, we consider $\sigma(\mathcal{E}) = \int_{\mathcal{E}} \sigma(y)dy$. Given a weight σ and $1 \le p < \infty$, we define the weighted Lebesgue spaces, $L^p_\sigma(\mathbb{R}^n)$ as

$$L^p_\sigma(\mathbb{R}^n) = \Big\{ f : \mathbb{R}^n \to \mathbb{R} \ : \int_{\mathbb{R}^n} |f(x)|^p \sigma(x)dx < \infty \Big\}.$$

For $f \in L^p_\sigma(\mathbb{R}^n)$, we consider the norm of f as

$$\|f\|_{L^p_\sigma} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x)dx \Big)^{\frac{1}{p}}.$$

$L^\infty_\sigma(\mathbb{R}^n)$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\sigma\Big(\{x \in \mathbb{R}^n : |f(x)| > C\} \Big) = 0$$

for some $C > 0$ and the norm of $f \in L^\infty_\sigma(\mathbb{R}^n)$ is defined as

$$\|f\|_{L^\infty_\sigma} = \inf \Big\{ C > 0 : \sigma\big(\{x \in \mathbb{R}^n : |f(x)| > C\}\big) = 0 \Big\}.$$

Let us define the weak $L^p_\sigma(\mathbb{R}^n), 1 \le p < \infty$ space denoted by $L^{p,\infty}_\sigma(\mathbb{R}^n)$ as the collection of all measurable functions f in $\mathbb{R}^n$ such that

$$\sigma\Big(\{x \in \mathbb{R}^n : |f(x)| > \gamma\} \Big) \le \frac{C^p}{\gamma^p}$$

holds for all $\gamma > 0$ and for a suitable constant $C > 0$. For $f \in L^{p,\infty}_\sigma(\mathbb{R}^n)$ we define

$$\|f\|_{L^{p,\infty}_\sigma} = \inf \Big\{ C > 0 : \sigma\Big(\{x \in \mathbb{R}^n : |f(x)| > \gamma\} \Big) \le \frac{C^p}{\gamma^p} \text{ for each } \gamma > 0 \Big\}.$$

For $1 \le p < \infty$, $L^{p,\infty}_\sigma(\mathbb{R}^n)$ with $\|\cdot\|_{L^{p,\infty}_\sigma}$ is a quasi Banach spaces. For $p = \infty$, the weak $L^p_\sigma(\mathbb{R}^n)$ and $L^p_\sigma(\mathbb{R}^n)$ are the same.

Let (X, ζ) be a measure space and $\mathcal{T}$ be an operator defined from $L^p_\sigma(\mathbb{R}^n)$ into $\mathcal{F}$, where $\mathcal{F}$ is the collection of all measurable functions from X to $\mathbb{R}$. For $p, q \in [1,\infty)$, we say the operator $\mathcal{T}$ is weak (p,q) if there exists a constant $C > 0$ such that

$$\zeta\Big(\Big\{ x \in X : |\mathcal{T}f(x)| > \gamma \Big\} \Big) \le \Big(\frac{C}{\gamma} \Big(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x)dx \Big)^{\frac{1}{p}} \Big)^q.$$

We say $\mathcal{T}$ is of weak (p,∞) if it is bounded from $L^p_\sigma(\mathbb{R}^n)$ to $L^\infty(X,\zeta)$. $\mathcal{T}$ is said to satisfy strong (p,q) if it is bounded from $L^p_\sigma(\mathbb{R}^n)$ to $L^q(X,\zeta)$. By definition, the weak (p,q) follows from strong (p, q). In general, the converse is not valid, but it is achievable from two endpoint estimates and the result is the following [58].

Marcinkiewicz Interpolation Theorem: Let $\mathcal{T}$ be a sublinear operator such that it satisfies weak (q, q) and weak (r, r), where $1 \leq q < r \leq \infty$. Then $\mathcal{T}$ is of strong (p, p) for $q < p < r$.

1.2 Hardy-Littlewood maximal function

The classical Hardy-Littlewood maximal function, $\mathcal{M}(f)$ for a locally integrable function, f on $\mathbb{R}^n$ is defined as

$$\mathcal{M}(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{1.1}$$

where the supremum is considered over all those cubes Q having x and whose sides are parallel to the coordinate axes. The expression $|Q|$ denotes the Lebesgue measure of Q in $\mathbb{R}^n$. The maximal function defined in (1.1) is also corresponds to the non-centered maximal function. In the case of centered maximal function at a point x, the supremum is considered over all those cubes having center at x. The centered and non-centered maximal functions are point-wise equivalent to each other and hence their results can be interchangeable [27].

The maximal function satisfies the following two estimates.

$$\int_{\mathbb{R}^n} \mathcal{M}(f)(x)^p dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p dx, \quad 1 < p < \infty, \tag{1.2}$$

where C_p is a positive constant, and for each $\gamma > 0$ there exists a constant $C > 0$ such that

$$|\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > \gamma\}| \leq \frac{C}{\gamma} \int_{\mathbb{R}^n} |f(x)| dx. \tag{1.3}$$

Naturally we ask, whether we can extend the inequalities (1.2) and (1.3) in the weighted Lebesgue spaces. It is due to Muckenhoupt who gave a new direction to the weighted inequalities for the classical Hardy-Littlewood maximal function. Muckenhoupt [47] obtained a beautiful class, known as $\mathcal{A}_p$ weights in the context of generalization of the inequalities (1.2) and (1.3) with weights.

The fundamental problem in weighted theory is to characterize the class of weights σ and ϕ for which the maximal operator is of weak type $(p, p), 1 \leq p < \infty$, that is

$$\sigma\Big(\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > \gamma\}\Big) \leq \frac{C}{\gamma^p} \int_{\mathbb{R}^n} |f(x)|^p \phi(x) dx \tag{1.4}$$

and of strong type $(p,p), 1 < p < \infty$, that is

$$\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^p \sigma(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \phi(x)dx \tag{1.5}$$

for some constant $C > 0$.

In order to obtain the estimate (1.4), Muckenhoupt [47] characterized the class of $\mathcal{A}_p$ weights as follows.

Definition 1.2.1. *Suppose that (σ, ϕ) is a couple of weights on $\mathbb{R}^n$.*

(i) The pair (σ, ϕ) is said to satisfy the $\mathcal{A}_p$ condition for $1 < p < \infty$ if

$$\left(\frac{1}{|Q|}\int_Q \sigma dy\right)\left(\frac{1}{|Q|}\int_Q \phi^{-\frac{1}{p-1}} dy\right)^{p-1} \leq C$$

holds for a constant $C > 0$,

(ii) We say the pair (σ, ϕ) satisfies $\mathcal{A}_1$ if there exists $C > 0$ such that

$$\mathcal{M}(\sigma)(x) \leq C\phi(x) \quad a.e.\ x \in \mathbb{R}^n.$$

It was proved that the condition $(\sigma, \phi) \in \mathcal{A}_p$ is necessary and sufficient for the inequality (1.4). Muckenhoupt [47] proved the inequality (1.5) for equal weights, that is for $\sigma = \phi$ using the condition $\mathcal{A}_p$, and also established that the condition $\mathcal{A}_p$ is not sufficient for the strong type (p,p) estimate for the maximal operator with two different weights. Later, Muckenhoupt and Wheeden [48] showed that the inequality (1.5) implies

$$\left(\frac{1}{|Q|}\int_Q \mathcal{M}(\chi_Q)(x)^p \sigma(x)dx\right)\left(\frac{1}{|Q|}\int_Q \phi(x)^{-\frac{1}{p-1}} dx\right)^{p-1} \leq C \tag{1.6}$$

for some constant $C > 0$ and conjectured that (1.6) is sufficient for (1.5). Sawyer [55] proved that the conjecture is not true and further defined a testing type condition, $\mathcal{S}_p$ as follows.

Definition 1.2.2. *Let $1 < p < \infty$ and suppose that (σ, ϕ) be a couple of weights on $\mathbb{R}^n$. The pair $(\sigma, \phi) \in \mathcal{S}_p$ if*

$$\int_Q \mathcal{M}(\phi^{1-p'}\chi_Q)(x)^p \sigma(x)dx \leq C \int_Q \phi(x)^{1-p'} dx < \infty$$

holds for a positive constant $C > 0$ and for all cubes Q in $\mathbb{R}^n$.

Sawyer found that the condition $(\sigma, \phi) \in \mathcal{S}_p$ is necessary and sufficient for the estimate (1.5). Fefferman and Stein [21] proved a weighted inequality for Hardy-Littlewood maximal function similar to that of Muckenhoupt [47] but with some new weight $\mathcal{M}(\sigma)$ on the right hand side.

Lemma 1.2.3. *[21] Let $q > 1$ and f, σ be two non-negative locally integrable functions on $\mathbb{R}^n$. Then there exists a positive constant C_q such that*

$$\int_{\mathbb{R}^n} \mathcal{M}(f)(x)^q \sigma(x) dx \leq C_q \int_{\mathbb{R}^n} f(x)^q \mathcal{M}(\sigma)(x) dx. \tag{1.7}$$

The inequality (1.7), also known as Fefferman-Stein's weighted inequality, can be seen as a consequence of the condition $(\sigma, \mathcal{M}(\sigma)) \in \mathcal{A}_1$ for any arbitrary weight σ and Marcinkiewicz interpolation theorem. It has many applications in the theory of $\mathcal{A}_p$ weights. Using Lemma 1.2.3, Fefferman and Stein [21] proved the L^p boundedness of the operator $\mathcal{M}$ in the vector-valued setting. It also plays a vital role in the work of Anderson and John [2] where they established the following result.

For $1 \leq q < \infty$ we will now define the l_q norm of a sequence of functions $f = \{f_j\}_{j \in \mathbb{N}}$ at a point as

$$\|f(z)\|_q = \left(\sum_j |f_j(z)|^q \right)^{\frac{1}{q}}.$$

Similarly, for the sequence $f = \{f_j\}$, we define $\mathcal{M}(f) = \{\mathcal{M}(f_j)\}$ and the l_q norm of $\mathcal{M}(f)$ as

$$\|\mathcal{M}(f)(z)\|_q = \left(\sum_j |\mathcal{M}(f_j)(z)|^q \right)^{\frac{1}{q}}.$$

Theorem 1.2.4. *[2] Let $f = \{f_j\}_j$ be a sequence of locally integrable functions on $\mathbb{R}^n$ and $1 < q < \infty$. Then*

(i) for $1 < p < \infty$, the following

$$\left(\int_{\mathbb{R}^n} \|\mathcal{M}f(x)\|_q^p \sigma(x) dx \right)^{\frac{1}{p}} \leq C_{p,q} \left(\int_{\mathbb{R}^n} \|f(x)\|_q^p \sigma(x) dx \right)^{\frac{1}{p}}$$

holds for some constant $C_{p,q} > 0$ if and only if $\sigma \in \mathcal{A}_p$,

(ii) for $p = 1$, the following

$$\sigma\left(\left\{ x : \|\mathcal{M}f(x)\|_q > \gamma \right\} \right) \leq \frac{C_q}{\gamma} \left(\int_{\mathbb{R}^n} \|f(x)\|_q \sigma(x) dx \right)$$

holds for some constant $C_q > 0$ if and only if $\sigma \in \mathcal{A}_1$.

There are many authors who used different approaches to prove the vector-valued weighted inequality for the maximal operator. Garcia-Cuerva and Rubio de Francia [23] used the idea of a vector-valued singular integral operator with operator valued kernel and Rubio de Francia et al. [53] used the technique of extrapolation theory of $\mathcal{A}_p$ weights to prove the result. Shrivastava [57] obtained the vector-valued weighted inequality for a one-sided maximal function on $\mathbb{R}$. For a nice introduction on the maximal function and related results we refer to [19, 23, 27]. In Chapter 2, we consider a generalized one-sided maximal function, $\mathcal{M}_\psi^+$ and prove the weighted norm inequality for the maximal function in vector-valued setup.

Let ψ be a positive and locally integrable function on $\mathbb{R}$. For $f \in L^1_{loc,\psi}(\mathbb{R})$, we define the general one-sided maximal function as

$$\mathcal{M}_\psi^+(f)(x) = \sup_{\xi>0} \left(\frac{1}{\int_x^{x+\xi} \psi dy} \right) \left(\int_x^{x+\xi} |f|\psi dy \right).$$

Similarly, in the interval $[x-\xi, x)$, we denote the one-sided maximal function as $\mathcal{M}_\psi^-$ and it is defined as

$$\mathcal{M}_\psi^-(f)(x) = \sup_{\xi>0} \left(\frac{1}{\int_{x-\xi}^{x} \psi dy} \right) \left(\int_{x-\xi}^{x} |f|\psi dy \right).$$

By writing $f \in L^1_{loc,\psi}(\mathbb{R})$, we want to express that $f\psi$ is locally integrable. For $\psi = 1$, the operators $\mathcal{M}_\psi^+$ and $\mathcal{M}_\psi^-$ are reduced to the classical one-sided Hardy-Littlewood maximal operators $\mathcal{M}^+$ and $\mathcal{M}^-$ respectively. The forward maximal operator $\mathcal{M}^+$ is defined as

$$\mathcal{M}^+(f)(x) = \sup_{\xi>0} \frac{1}{\xi} \int_x^{x+\xi} |f| dy.$$

Similarly, the left maximal operator or the backward maximal operator $\mathcal{M}^-$ is defined as

$$\mathcal{M}^-(f)(x) = \sup_{\xi>0} \frac{1}{\xi} \int_{x-\xi}^{x} |f| dy.$$

In the context of weighted norm inequality for the one-sided maximal function, Sawyer [56] introduced two new conditions $\mathcal{A}_p^+$ and $\mathcal{A}_p^-$. The $\mathcal{A}_p^+$ condition is used to obtain boundedness for $\mathcal{M}^+$ and similarly $\mathcal{A}_p^-$ is used for $\mathcal{M}^-$.

Definition 1.2.5. *Let σ and ϕ be two weights on $\mathbb{R}$. Suppose that $\alpha \in \mathbb{R}$ and $\xi > 0$.*

(i) The pair $(\sigma, \phi) \in \mathcal{A}_p^+, 1 < p < \infty$ if

$$\left(\frac{1}{\xi} \int_{\alpha-\xi}^{\alpha} \sigma dy \right) \left(\frac{1}{\xi} \int_{\alpha}^{\alpha+\xi} \phi^{-\frac{1}{p-1}} dy \right)^{p-1} \leq C.$$

And the pair $(\sigma, \phi) \in \mathcal{A}_1^+$ if

$$\mathcal{M}^-(\sigma)(x) \leq C\phi(x) \quad a.e.$$

(ii) The pair $(\sigma, \phi) \in \mathcal{A}_p^-, 1 < p < \infty$ *if*

$$\left(\frac{1}{\xi}\int_{\alpha}^{\alpha+\xi} \sigma dy\right)\left(\frac{1}{\xi}\int_{\alpha-\xi}^{\alpha} \phi^{-\frac{1}{p-1}} dy\right)^{p-1} \leq C.$$

And the pair $(\sigma, \phi) \in \mathcal{A}_1^-$ *if*

$$\mathcal{M}^+(\sigma)(x) \leq C\phi(x) \quad a.e.$$

Later we will use the notation $\sigma \in \mathcal{A}_p^+$ to denote $(\sigma, \sigma) \in \mathcal{A}_p^+$. Using $\mathcal{A}_p^+$ condition, Sawyer proved the following result for the maximal operator $\mathcal{M}^+$.

Theorem 1.2.6. *[56] Suppose that* $p \in (1, \infty)$, *then the following weighted estimate*

$$\left(\int_{\mathbb{R}} |\mathcal{M}^+(f)(x)|^p \sigma(x) dx\right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{R}} |f(x)|^p \sigma(x) dx\right)^{\frac{1}{p}}$$

holds for a constant $C_p > 0$ *if and only if* $\sigma \in \mathcal{A}_p^+$.

Also, one-sided Muckenhoupt weight, that is $\mathcal{A}_p^+$ condition is not sufficient for the strong type (p, p) inequality for the operator $\mathcal{M}^+$ with two different weights. Sawyer [56] defined a testing type condition for $\mathcal{M}^+$ as follows.

Definition 1.2.7. *Let* $1 < p < \infty$ *and suppose that* σ *and* ϕ *be two weights on* $\mathbb{R}$. *Let* $\Gamma = (\alpha, \beta)$ *be any interval with* $\int_{-\infty}^{\alpha} \sigma(x) dx > 0$. *The pair of weight* $(\sigma, \phi) \in \mathcal{S}_p^+$ *if there exists a constant* $C > 0$ *such that*

$$\int_{\Gamma} \mathcal{M}^+(\phi^{1-p'}\chi_\Gamma)(x)^p \sigma(x) dx \leq C \int_{\Gamma} \phi(x)^{1-p'} dx < \infty. \tag{1.8}$$

Sawyer proved that the strong type (p, p), $1 < p < \infty$ inequality for $\mathcal{M}^+$

$$\int_{\mathbb{R}} \mathcal{M}^+(f)(x)^p \sigma(x) dx \leq C_p \int_{\mathbb{R}} |f(x)|^p \phi(x) dx \tag{1.9}$$

holds for some constant $C_p > 0$ if and only if the pair $(\sigma, \phi) \in \mathcal{S}_p^+$. Sawyer also established that the weak type $(p, p), 1 \leq p < \infty$ estimate for $\mathcal{M}^+$ with respect to the pair (σ, ϕ) holds if and only if $(\sigma, \phi) \in \mathcal{A}_p^+$. Shrivastava [57] extended the Theorem 1.2.6 to the vector-valued setting by using techniques from [2] and the main result is the following.

Theorem 1.2.8. *[57] Let* $f = \{f_j\}_j$ *be a sequence of measurable functions and* $1 < q < \infty$. *Then the following two weighted estimates hold.*

(i) If $\sigma \in \mathcal{A}_p^+, 1 < p < \infty$ *then we have*

$$\left(\int_{\mathbb{R}} \|\mathcal{M}^+(f)(x)\|_q^p \sigma(x) dx \right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}} \|f(x)\|_q^p \sigma(x) dx \right)^{\frac{1}{p}}.$$

(ii) If $\sigma \in \mathcal{A}_1^+$, *then the following weak type* $(1,1)$ *holds, that is*

$$\sigma\left(\left\{ x : \|\mathcal{M}^+(f)(x)\|_q > \gamma \right\} \right) \leq \frac{C_q}{\gamma} \left(\int_{\mathbb{R}} \|f(x)\|_q \sigma(x) dx \right).$$

Martín-Reyes [44] introduced two new classes of weights $\mathcal{A}_p^+(\psi)$ and $\mathcal{S}_p^+(\psi)$ for the weak type (p,p) and strong type (p,p) estimates for the operator $\mathcal{M}_\psi^+$ and also established an equivalence relation between these two family in one weight case. In Chapter 2, we prove weighted norm inequality for the one-sided maximal operator $\mathcal{M}_\psi^+$ in the vector-valued setup. This will extend the work of Shrivastava [57] and the existing results for the one-sided maximal functions in the scalar setting. The results of [57] can be deduced from our result by taking ψ as a constant function.

Note that, though a lot of progress has been made in the context of the maximal functions, in the case of the one-sided maximal functions there are still some open problems which are yet to be addressed. It is little more difficult to visualize the one-sided maximal function as some techniques do not work in one-sided setting. Sawyer proved the weighted inequalities for one-sided maximal function in $\mathbb{R}$. Forzani et al. [22] established the weak type $(p,p), 1 \leq p < \infty$ estimates for a two dimensional one-sided maximal function. As far as we are concerned, the weighted weak and strong type estimates for one-sided maximal function are yet to be established in $\mathbb{R}^n$ for $n \geq 3$. We refer to [2, 23, 37, 41, 42, 44] for more detail about the one-sided maximal function and its applications.

Ombrosi [49] overcame this problem and established a significant result for a new one-sided maximal function in $\mathbb{R}^n$, defined via dyadic cubes.

Definition 1.2.9. *A dyadic interval in* $\mathbb{R}$ *is of the following form,*

$$[m2^{-k}, (m+1)2^{-k}),$$

where $m, k \in \mathbb{Z}$. *Similarly a dyadic cube in* $\mathbb{R}^n$ *is of the following form,*

$$\prod_{j=1}^{n} \left[m_j 2^{-k}, (m_j + 1) 2^{-k} \right),$$

where $m_1, m_2, \ldots, m_n, k \in \mathbb{Z}$.

Ombrosi [49] defined the one-sided dyadic maximal function $\mathcal{M}^{+,d}$ as

$$\mathcal{M}^{+,d}(f)(x) = \sup_{x\in Q,\ Q \text{ dyadic}} \frac{1}{|Q|}\int_{Q^+}|f|dy,$$

where the supremum is taken over all those dyadic cubes Q containing x and the sides are parallel to the coordiante axes. The cube Q^+ related to Q is defined as $Q^+ = [\beta_1, 2\beta_1 - \alpha_1) \times [\beta_2, 2\beta_2 - \alpha_2) \times \ldots \times [\beta_n, 2\beta_n - \alpha_n)$, where $Q = [\alpha_1, \beta_1) \times [\alpha_2, \beta_2) \times \ldots \times [\alpha_n, \beta_n)$ is any dyadic cube in $\mathbb{R}^n$ with vertices $\alpha_i, \beta_i \in \mathbb{R}, 1 \le i \le n$.

Ombrosi [49] proved the weak type (p, p) estimate in the range $1 \le p < \infty$. Martín-Reyes and de la Torre [42] studied the two weighted strong type (p, q) estimate for a one-sided dyadic maximal function on $\mathbb{R}$. Lorente and Martín-Reyes [40] considered a different version of dyadic one-sided maximal function on $\mathbb{R}$ and established weak and strong (p, p) estimates. In Chapter 4, we consider the fractional version of the maximal function $\mathcal{M}^{+,d}$ and prove weak and strong type weighted inequalities. The results extend the work of Ombrosi [49].

1.3 Minimal function

It is quite natural to verify how the weighted estimate will behave if we replace supremum by infimum in the construction of maximal function. In this regard, Cruz-Uribe and Neugebauer [15] introduced a new concept called minimal function and later it motivated them to characterize the class of functions for which the reverse Hölder inequality makes sense. In [15], the authors defined the minimal function as

$$m(f)(x) = \inf_{x\in Q} \frac{1}{|Q|}\int_{Q}|f|dy,$$

where the infimum is considered over the cubes as similar to the case for maximal function.

From the definition of the minimal function the locally integrability condition on f is unnecessary, it is enough to assume that f is a measurable function on $\mathbb{R}^n$ and it turns out to be the main difference between the maximal and minimal function. Cruz-Uribe et al. [17] proved the weak and strong type $(p, p), p > 0$ estimates for the minimal function on $\mathbb{R}$. To establish the weak type (p, p), $p > 0$ estimate for the minimal function, the authors [17] defined a certain weight class, $\mathcal{W}_p$ which is similar to the Muckenhoupt weight class, $\mathcal{A}_p$ but with p replaced by $-p$. Precisely the weight class $\mathcal{W}_p$ is as follows.

Definition 1.3.1. *Let (U, V) be a pair of weight on $\mathbb{R}$. Given $p > 0$, we say $(U, V) \in \mathcal{W}_p$ if there exists $C > 0$ such that*

$$\frac{1}{|\Gamma|}\int_\Gamma U \le C\left(\frac{1}{|\Gamma|}\int_\Gamma V^{\frac{1}{p+1}}\right)^{p+1}$$

holds for every interval Γ *in* $\mathbb{R}$.

With the help of the $\mathcal{W}_p$ condition, the following result regarding the weak type estimate has been established.

Theorem 1.3.2. *[17] Let* (U,V) *be a pair of weight on* $\mathbb{R}$. *For* $p,\gamma > 0$ *the following statements are equivalent.*

(i) The pair $(U,V) \in \mathcal{W}_p$.

(ii) The weak (p,p) *holds for the minimal function, that is*

$$U\left(\left\{x \in \mathbb{R} : m(f)(x) < \frac{1}{\gamma}\right\}\right) \leq \frac{C}{\gamma^p}\int_{\mathbb{R}} \frac{V}{|f|^p}$$

holds for some constant $C > 0$.

As similar to the Sawyer's testing type condition in the case of maximal function, we need an another weight class to show the two weight strong type inequality for the minimal function and the condition was defined in [17] as follows.

Definition 1.3.3. *Let* (U,V) *be a pair of weight on* $\mathbb{R}$. *Given* $p > 0$, *we say* $(U,V) \in \mathcal{W}_p^*$ *if there exists* $C > 0$ *such that*

$$\int_\Gamma \frac{U}{m(\omega/\chi_\Gamma)^p} \leq C\int_\Gamma \omega$$

holds for every interval Γ *in* $\mathbb{R}$, *where* $\omega = V^{\frac{1}{p+1}}$.

The strong type estimate (p,p), $p > 0$ for the minimal function follows from the weight class $\mathcal{W}_p^*$ and this result has been proved in [17].

Theorem 1.3.4. *[17] Let* (U,V) *be a pair of weights on* $\mathbb{R}$. *For* $p > 0$ *the following statements are equivalent.*

(i) The pair $(U,V) \in \mathcal{W}_p^*$.

(ii) The strong (p,p) *holds for the minimal function, that is*

$$\int_{\mathbb{R}} \frac{U}{(m(f))^p} \leq C\int_{\mathbb{R}} \frac{V}{|f|^p}$$

holds for some constant $C > 0$.

Cruz-Uribe et al. [17] proved an interesting result which establishes the equivalence between the weak and strong type norm inequalities which is not true in the case of maximal function, and also showed that unlike maximal functions, centered and non-centered minimal functions are not equivalent to each other. More interestingly, the natural domain for minimal operator becomes the class of all measurable functions f for which $1/f \in L^p$ for $p > 0$.

The one-sided version of the minimal function was studied by Cruz-Uribe et al. [16] and they defined the one-sided minimal function, m^+ for a measurable function f as

$$m^+(f)(x) = \inf_{\xi>0} \frac{1}{\xi} \int_x^{x+\xi} |f| dy.$$

A backward version of the minimal function, m^- is defined as

$$m^-(f)(x) = \inf_{\xi>0} \frac{1}{\xi} \int_{x-\xi}^{x} |f| dy.$$

Cruz-Uribe et al. [16] proved the two weight weak and strong type estimates for the one-sided minimal function on $\mathbb{R}$ with the help of two different weight classes $\mathcal{W}_p^+$ and $(\mathcal{W}_p^+)^*$ rspectively and use these for the study of differentiability of the integral. Further they established an equivalence relationship between these two weight classes. They also characterized the class of functions which satisfy one-sided reverse Hölder inequality in weak sense.

The weight class, $\mathcal{W}_p^+$ corresponding to the weak weighted inequality for m^+ is defined as follows.

Definition 1.3.5. *Given $p > 0$, we say a pair of weights $(U, V) \in \mathcal{W}_p^+$ if there exists a constant $C > 0$ such that*

$$\frac{1}{|\Gamma^-|} \int_{\Gamma^-} U \leq C \left(\frac{1}{|\Gamma|} \int_{\Gamma} V^{\frac{1}{p+1}} \right)^{p+1}$$

holds for each interval $\Gamma = [\alpha, \beta]$ in $\mathbb{R}$ with $2|\Gamma^-| = |\Gamma|$ and $\Gamma^- = [\alpha, \gamma]$.

Cruz-Uribe et al. [16] established the two weight weak (p, p) for m^+ with the help of $\mathcal{W}_p^+$ weight and their main result is the following.

Theorem 1.3.6. *[16] Let $p, \gamma > 0$. Then the pair $(U, V) \in \mathcal{W}_p^+$ if and only if the weak (p, p) inequality*

$$U\left(\left\{ x \in \mathbb{R} : m^+(f)(x) < \frac{1}{\gamma} \right\} \right) \leq \frac{C}{\gamma^p} \int_{\mathbb{R}} \frac{V}{|f|^p}$$

holds for some constant $C > 0$.

An inequality similar to the one-sided Sawyer's testing type condition is used to prove the two weighted strong (p, p) for m^+. The condition is denoted by $(\mathcal{W}_p^+)^*$ and it is defined as follows.

Definition 1.3.7. *Given $p > 0$, we say the pair $(U, V) \in (\mathcal{W}_p^+)^*$ if*

$$\int_\Gamma \frac{U}{m^+(\omega/\chi_\Gamma)^p} \leq C \int_\Gamma \omega$$

holds for some constant $C > 0$ and for any interval Γ in $\mathbb{R}$.

The two weighted strong (p, p) for one-sided minimal function is obtained using the $(\mathcal{W}_p^+)^*$ condition and the main result is the following.

Theorem 1.3.8. *[16] Given $p > 0$ and for a pair of weights (U, V) on $\mathbb{R}$, the following statements are equivalent.*

(i) The pair $(U, V) \in (\mathcal{W}_p^+)^*$.

(ii) The strong (p, p) holds for the one-sided minimal function. i.e.

$$\int_\mathbb{R} \frac{U}{(m^+(f))^p} \leq C \int_\mathbb{R} \frac{V}{|f|^p}$$

holds for some constant $C > 0$.

Motivated by the work of Cruz-Uribe et al. [16], we consider the fractional version of one-sided minimal function on $\mathbb{R}$ in Chapter 3 of our thesis. This will extend the existing results for one-sided minimal function. For this, we define the fractional version of one-sided minimal function as follows.

Let f be a measurable function on $\mathbb{R}$. For $0 \leq \mu < \infty$, we define a one-sided fractional minimal function of order μ as

$$m_\mu^+(f)(x) = \inf_{\xi>0} \frac{1}{\xi^{1+\mu}} \int_x^{x+\xi} |f(t)|dt.$$

If we consider the integral from $x - \xi$ to x, then we get an another version of one-sided minimal function and precisely it is given as

$$m_\mu^+(f)(x) = \inf_{\xi>0} \frac{1}{\xi^{1+\mu}} \int_{x-\xi}^{\xi} |f(t)|dt.$$

If $\mu = 0$, then we get the one-sided minimal function m^+, defined in [16]. Thus m^+ is a particular case of m_μ^+. In Chapter 3, we prove weighted weak and strong type estimates for the minimal operator m_μ^+.

1.4 Integral operators of Hardy type

We define the modified integral Hardy operators, $\mathcal{I}$ for a non-negative measurable function f on the real line as

$$\mathcal{I}f(t) = h(t)\int_0^t K(t,\tau)f(\tau)w(\tau)d\tau, \tag{1.10}$$

where h and w are two positive measurable function on $\mathbb{R}$. Suppose, the kernel K defined on $\{(t,\tau) : 0 \leq \tau \leq t\}$ is non-negative, non-decreasing in the first variable, non-increasing in the second variable and satisfies

$$K(t,\tau) \leq M[K(t,z) + K(z,\tau)] \tag{1.11}$$

for $0 \leq \tau \leq z \leq t$ and some constant $M \geq 1$.

The kernel satisfying condition (1.11) is also known as Oinarov's kernel. Particular examples of this type of kernel are $(t-\tau)^\mu$ and $\log^\mu\left(\frac{t}{\tau}\right)$, where $\mu \geq 0, t > \tau > 0$. For $K = 1$, the integral operator (1.10) is reduced to the modified Hardy operators defined by

$$\mathcal{H}_h f(t) = h(t)\int_0^t f(y)w(y)dy. \tag{1.12}$$

For $h = 1 = w$, the operators (1.10) and (1.12) are usually termed as the integral Hardy and Hardy operators respectively. Weighted inequalities for the operators (1.10) and (1.12) have been studied substantially [8, 34, 39, 43, 45, 46] due to their considerable influence in PDE. Martín-Reyes and Salvador [43] characterized the pair of weights (ρ, ψ) such that

$$\left(\int_{\{t\in(0,\infty):\mathcal{H}_h f(t)>\gamma\}} \gamma^q \rho(t)dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(t)^p\psi(t)dt\right)^{\frac{1}{p}} \tag{1.13}$$

holds for the range $1 \leq p \leq q < \infty$.

In Chapter 5, we examine the estimate (1.13) in the Orlicz space setting for the operators (1.10) and (1.12) and also for their conjugates $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{H}}_h$ respectively defined by

$$\tilde{\mathcal{I}}f(t) = w(t)\int_t^\infty K(\tau,t)f(\tau)h(\tau)d\tau, \tag{1.14}$$

$$\tilde{\mathcal{H}}_h f(t) = w(t)\int_t^\infty f(\tau)h(\tau)d\tau. \tag{1.15}$$

Among the various equivalent generalization of the estimate (1.13), we will consider the following integral inequality

$$\mathcal{U}^{-1}\left(\int_{\{t\in(0,\infty);\mathcal{T}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(t)\Big)\rho(t)dt\right) \leq \mathcal{V}^{-1}\left(\int_0^\infty \mathcal{V}\Big(Cf(t)\phi(t)\Big)\psi(t)dt\right), \tag{1.16}$$

where $\mathcal{U}$ and $\mathcal{V}$ are N-functions; ω, ρ, ϕ and ψ are positive weights on $\mathbb{R}$; $\mathcal{T} = \mathcal{I}, \tilde{\mathcal{I}}, \mathcal{H}_h, \tilde{\mathcal{H}}_h$.

Briefly, an N- function is a continuous and convex function with some extra conditions. Given an N-function $\mathcal{U}$, and a postive locally integrable function σ on $\mathbb{R}^n$, the Orlicz space $L_{\mathcal{U}}(\sigma)$ consists of all measurable functions f in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \mathcal{U}\Big(\frac{|f(x)|}{\gamma}\Big)\sigma(x)dx \leq 1 \tag{1.17}$$

for some $\gamma > 0$. Orlicz space norm for $f \in L_{\mathcal{U}}(\sigma)$ is defined as the infimum over all such $\gamma > 0$ for which the inequality (1.17) is true. In particular, if we consider $\mathcal{V}(t) = t^p$ and $\mathcal{U}(t) = t^q$ for $1 \leq p \leq q < \infty$, then the inequality (1.16) extends the estimate (1.13). Bloom and Kerman [7] addressed the estimate (1.16) for the Hardy operators and its integral version, where the authors used the monotonic condition of the operators. Bloom and Kerman also proved the following stronger version of (1.16) in the case $h = 1 = w$.

$$\mathcal{U}^{-1}\bigg(\int_0^\infty \mathcal{U}\Big(\mathcal{H}_h f(t)\omega(t)\Big)\rho(t)dt\bigg) \leq \mathcal{V}^{-1}\bigg(\int_0^\infty \mathcal{V}\Big(Cf(t)\phi(t)\Big)\psi(t)dt\bigg). \tag{1.18}$$

It is again due to the monotonic property of the Hardy operators that the estimates (1.16) and (1.18) are equivalent for the Hardy operators [7] but yet to be verified for the modified Hardy operators. The techniques of Bloom and Kerman [7] do not work for the operators (1.10) and (1.12) as they do not possess monotone property for non-increasing h. Salvador [50] analyzed (1.16) with two weights that is by considering $\rho = 1 = \phi$.

The main objective of Chapter 5 is to investigate conditions on the four weights ω, ρ, ϕ and ψ such that the estimate (1.16) holds for a suitable constant $C > 0$. Also, we will consider a weaker version of (1.16) of the form

$$\omega\bigg(\{t \in (0,\infty) : \mathcal{T}f(t) > \gamma\}\bigg) \leq \mathcal{U} \circ \mathcal{V}^{-1}\bigg(\int_0^\infty \mathcal{V}\Big(\frac{Cf(t)\phi(t)}{\gamma}\Big)\psi(t)dt\bigg), \tag{1.19}$$

where $\mathcal{T} = \mathcal{I}, \tilde{\mathcal{I}}, \mathcal{H}_h, \tilde{\mathcal{H}}_h$, defined in (1.10), (1.12), (1.14) and (1.15). The inequality (1.19) is called as the extra-weak type integral inequality as it follows from (1.16) but does not follow contrarily. One of the prime importance of the extra-weak type mixed integral inequalities is to obtain exquisite bounds for the strong type integral estimates [3, 4]. We refer to [26, 35, 36, 38, 51] for a detailed investigation on weighted integral inequalities for the integral operators and maximal functions.

1.5 Hardy-Steklov integral operators

We consider the Hardy-Steklov integral operators $\mathcal{I}$, for a non-negative measurable function f on $-\infty \leq a < b \leq \infty$, defined as

$$\mathcal{I}f(t) = h(t) \int_{\zeta(t)}^{\sigma(t)} K(t,z) f(z) w(z) dz, \tag{1.20}$$

where $\zeta, \sigma : (a,b) \to \mathbb{R}$ are continuous and increasing functions satisfying $\zeta(z) \leq \sigma(z)$ for each $z \in (a,b)$, h and w are positive measurable functions, and the kernel $K(t,z)$ defined on $\{(t,z) : \zeta(t) \leq z \leq \sigma(t)\}$ satisfies the following conditions.

(a) $K(t,z) \geq 0$.

(b) $K(t,z)$ is non-decreasing in t and non-increasing in z.

(c) Suppose that $M \geq 1$ is a constant independent of t, z and τ such that

$$K(t,z) \leq M\Big[K(t,\sigma(\tau)) + K(\tau,z)\Big], \tag{1.21}$$

where $\tau \leq t$ and $\zeta(t) \leq z \leq \sigma(\tau)$.

For $K \equiv 1$, the operator (1.20) is reduced to the Hardy-Steklov operator defined as

$$\mathcal{S}f(t) = h(t) \int_{\zeta(t)}^{\sigma(t)} f(z) w(z) dz. \tag{1.22}$$

From (1.22), it is observed that the Hardy-Steklov operator extends the notion of the Hardy operator to the dynamic limits. Riemann-Liouville integral operators of the form $\int_{\zeta(t)}^{\sigma(t)} (t-z)^{\mu} f(z) dz,\ \ \mu > 0$; Steklov operators $\int_{t-\gamma}^{t+\gamma} f(z)dz, \gamma > 0$ are some particular examples of the Hardy-Steklov operators.

Weighted weak and strong type estimates for the Hardy-Steklov type operators have been studied substantially by several authors [5, 6, 10, 24, 30, 54, 59]. Beranardis et al. [5] characterized the pair of weights ρ and ψ such that

$$\Big(\int_{\{t \in (a,b) : \mathcal{I}f(t) > \gamma\}} \gamma^q \rho(y) dy\Big)^{\frac{1}{q}} \leq C\Big(\int_{\zeta(a)}^{\sigma(b)} f(y)^p \psi(y) dy\Big)^{\frac{1}{p}} \tag{1.23}$$

holds for a suitable constant $C > 0$ and in the range $0 < q < p,\ \ 1 < p < \infty$ with $w = 1$. Heinig and Sinnamon [30] obtained strong type (p,q) estimate for the operator (1.22)(with $h = 1 = w$) assuming monotonically increasing and differentiability conditions on ζ and σ in the range $0 < q < p,\ 1 < p < \infty$. In the case $p \leq q$, Gogatishvili and Lang [24] obtained the weak and strong type (p, q) estimates in Banach function spaces for the operator (1.20) assuming only monotone conditions on the limit functions. Stepanov and Ushakova [59] proved $L_p - L_q$ boundedness of the operator (1.20) considering h and w as weight functions.

Among the various equivalent generalization of the estimate (1.23) in to the Orlicz space setting, we will consider the following form.

$$\mathcal{U}^{-1}\left(\int_{\{t\in(a,b):\mathcal{I}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\right) \le \mathcal{V}^{-1}\left(\int_{\zeta(a)}^{\sigma(b)} \mathcal{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\right), \tag{1.24}$$

where $\gamma > 0; \omega, \rho, \phi$ and ψ are weights and the conditions on $\mathcal{U}$ and $\mathcal{V}$ will be set down later.

Salvador and Torreblanca [52] established the inequality (1.24) with $\omega = 1 = \phi$ for the Hardy-Steklov operators.

In Chapter 6, we address the inequality (1.24) for the Hardy-Steklov integral operator and its adjoint $\tilde{\mathcal{I}}$ defined by

$$\tilde{\mathcal{I}}f(t) = w(t)\int_{\sigma^{-1}(t)}^{\zeta^{-1}(t)} K(z,t)f(z)h(z)dz. \tag{1.25}$$

The second objective of Chapter 6 is to prove the extra-weak type integral inequalities for the Hardy-Steklov integral operator and its adjoint.