CHAPTER 2

Weighted estimates for a one-sided maximal function in the vector-valued setting

2.1 Introduction

In this chapter, we consider a one-sided generalized version, $\mathcal{M}_\psi^+$ of the classical maximal function on $\mathbb{R}$. We establish the one weighted weak $(1,1)$ and strong (p,p), $p \in (1,\infty)$ estimates of $\mathcal{M}_\psi^+$ in the vector-valued setup. We provide a necessary as well as sufficient condition for the weighted inequalities of $\mathcal{M}_\psi^+$ in the vector-valued setting. We also establish an inequality for the operator $\mathcal{M}_\psi^+$ in the scalar setting similar to that of Fefferman-Stein's weighted lemma and its analog to the vector-valued setup is also a further purpose of this chapter. It extends the results of the one weighted vector-valued norm inequality for one-sided maximal function proved in [57].

Let the function ψ be locally integrable and take positive values on $\mathbb{R}$. For $f \in L^1_{loc,\psi}(\mathbb{R})$ let us consider the maximal function, $\mathcal{M}_\psi^+$ as

$$\mathcal{M}_\psi^+(f)(t) = \sup_{\xi>0} \left(\frac{1}{\int_t^{t+\xi} \psi dy} \right) \left(\int_t^{t+\xi} |f|\psi dy \right).$$

Similarly in the backward interval $[t-\xi, t)$, let us define the maximal function, $\mathcal{M}_\psi^-$ as

$$\mathcal{M}_\psi^-(f)(t) = \sup_{\xi>0} \left(\frac{1}{\int_{t-\xi}^{t} \psi dy} \right) \left(\int_{t-\xi}^{t} |f|\psi dy \right).$$

By writing $f \in L^1_{loc,\psi}(\mathbb{R})$, we want to express that $f\psi$ is locally integrable. If we consider $\psi = 1$, then $\mathcal{M}_\psi^+$ and $\mathcal{M}_\psi^-$ are reduced respectively to the classical maximal operators $\mathcal{M}^+$ and $\mathcal{M}^-$ defined in Chapter 1.

This chapter is based on the published work *Weighted norm inequality for general one-sided vector valued maximal function* by D. Chutia and R. Haloi [11].

As the integrals are meaningful only when f is integrable over the intervals $[t, t+\xi)$ and $[t-\xi, t)$ for each $t \in \mathbb{R}$. Hence the assumption of local integrability on f is essential. The weighted estimates for the operators $\mathcal{M}_{\psi}^{+}$ and $\mathcal{M}_{\psi}^{-}$ are almost identical and can be interchanged with a little modification in the corresponding weight classes.

For $1 \leq q < \infty$ we will now define the l_q norm of a sequence of functions $f = \{f_j\}_{j\in\mathbb{N}}$ at a point as

$$\|f(z)\|_q = \left(\sum_j |f_j(z)|^q\right)^{\frac{1}{q}}.$$

Similarly, for the sequence $f = \{f_j\}$ we define $\mathcal{M}_{\psi}^{+}(f) = \{\mathcal{M}_{\psi}^{+}(f_j)\}$ and the l_q norm of $\mathcal{M}_{\psi}^{+}(f)$ as

$$\left\|\mathcal{M}_{\psi}^{+}(f)(z)\right\|_q = \left(\sum_j |\mathcal{M}_{\psi}^{+}(f_j)(z)|^q\right)^{\frac{1}{q}}.$$

Given $1 < p < \infty$, let us consider p' satisfying $p^{-1} + (p')^{-1} = 1$. The letter C refers to an arbitrary constant not necessarily same in all cases.

We organize the Chapter as follows. Section 2.2 is devoted to stating some known and existing results. The statement and proof of the main results are given in Section 2.3.

2.2 Preliminaries

Now, we define the $\mathcal{A}_p^{+}(\psi)$ condition and briefly state some existing results for the maximal function $\mathcal{M}_{\psi}^{+}(f)$ that are used to prove the main results.

First, we discuss a generalized one-sided $\mathcal{A}_p$ condition, i.e. the weights corresponding to the maximal operator $\mathcal{M}_{\psi}^{+}$. We have used the term generalized because the results of $\mathcal{M}_{\psi}^{+}$ are reduced to the results of $\mathcal{M}^{+}$ by considering $\psi = 1$. Martín-Reyes et al. [44] introduced two new classes of weights $\mathcal{A}_p^{+}(\psi)$ and $\mathcal{S}_p^{+}(\psi)$ for the operator $\mathcal{M}_{\psi}^{+}$ and also established an equivalence relation between these two weights. The authors [44] defined the class of weight $\mathcal{A}_p^{+}(\psi)$ as similar to the $\mathcal{A}_p^{+}$ condition as follows.

Definition 2.2.1. *Suppose that (σ, ϕ) are a couple of weight on $\mathbb{R}$. Then*

(i) the pair (σ, ϕ) is said to satisfy the $\mathcal{A}_p^{+}(\psi)$ condition for $1 < p < \infty$ if

$$\left(\int_{\alpha}^{\beta} \sigma dx\right)\left(\int_{\beta}^{\delta} \psi^{p'} \rho dx\right)^{p-1} \leq C\left(\int_{\alpha}^{\delta} \psi dx\right)^{p},$$

for each triple $\alpha \leq \beta \leq \delta$, where $\rho = \phi^{-\frac{1}{p-1}}$, *(ii) we say (σ, ϕ) satisfies the $\mathcal{A}_1^+(\psi)$ condition if*

$$\mathcal{M}_\psi^-\left(\frac{\sigma}{\psi}\right)(x) \leq C\left(\frac{\phi}{\psi}\right)(x) \quad a.e.$$

The weighted weak and strong type estimates were proved by Martín-Reyes et al. [44]. They obtained that the class of weight $\mathcal{A}_p^+(\psi)$ is necessary as well as sufficient to prove the weak (p,p) of the function $\mathcal{M}_\psi^+$, but not sufficient to obtain the strong (p,p) of $\mathcal{M}_\psi^+$ with two different weights. Hence to establish the strong (p,p) estimate with two different weights, a testing type condition, $\mathcal{S}_p^+(\psi)$ was developed in [44], which provides the necessary and sufficient condition. Now we state some results for the operator $\mathcal{M}_\psi^+$.

Theorem 2.2.2. *[44] Suppose that $p \in [1,\infty)$ and let σ and ϕ be two weights on $\mathbb{R}$. Then $\mathcal{M}_\psi^+$ holds the weak (p,p) estimate with respect to ϕ and σ if and only if the pair (σ,ϕ) satisfies the $\mathcal{A}_p^+(\psi)$ condition.*

Theorem 2.2.3. *[44] Suppose that $p \in (1,\infty)$. Then $\mathcal{M}_\psi^+$ satisfies the strong (p,p) estimate with respect to the weight ϕ if and only if $\phi \in \mathcal{A}_p^+(\psi)$.*

Thus we observe that the condition $\mathcal{A}_p^+(\psi)$ is necessary as well as sufficient to obtain the strong type estimate for $\mathcal{M}_\psi^+$ in the scalar setting with one weight. Thus we can conclude that the class of weight $\mathcal{A}_p^+(\psi)$ is necessary in the vector valued setup for the one weight problem of $\mathcal{M}_\psi^+$.

In the requirement of proving the main results, we will now state some elementary properties shared by $\mathcal{A}_p^+(\psi)$ condition. The properties are somewhat similar with the properties satisfied by $\mathcal{A}_p$ and $\mathcal{A}_p^+$ weights.

Theorem 2.2.4. *[44] If $\phi \in \mathcal{A}_1^+(\psi)$, then for $\alpha < \beta$ there exists $r > 0$ for which the estimate*

$$\int_\alpha^\beta \phi^{1+r}\psi^{-r}dx \leq C_r\Big(\int_\alpha^\beta \phi dx\Big)\Big(\phi\psi^{-1}\Big)^r(\beta)$$

holds a.e. for a suitable constant $C_r > 0$, and for this particular r, $\dfrac{\phi^{1+r}}{\psi^r} \in \mathcal{A}_1^+(\psi)$.

Corollary 2.2.5. *[44] The followings are true.*

(i) $\phi \in \mathcal{A}_p^+(\psi)$ if and only if $\psi^{p'}\rho \in \mathcal{A}_{p'}^-(\psi)$.

(ii) Let $\phi \in \mathcal{A}_p^+(\psi)$ for $p > 1$. Then there exists $r > 0$ for which $\phi \in \mathcal{A}_{p-r}^+(\psi)$ with $p-r > 1$.

2.3 Main results

We first establish the one weighted vector-valued norm inequality for $\mathcal{M}_\psi^+$ and conclude the chapter by proving Fefferman-Stein's weighted lemma in the vector-valued setup. We begin with the following elementary relation satisfied by the generalized one-sided weights.

Lemma 2.3.1. *Let ϕ be a weight such that $\phi \in \mathcal{A}_p^+(\psi)$. Then for $r > p$, we have $\phi \in \mathcal{A}_r^+(\psi)$.*

Proof. Let $\phi \in \mathcal{A}_p^+(\psi)$. Thus, for any α, β, δ with $\alpha < \beta < \delta$ we have,

$$\left(\int_\alpha^\beta \phi dx\right)\left(\int_\beta^\delta \psi^{p'}\phi^{-\frac{1}{p-1}}dx\right)^{p-1} < \left(\int_\alpha^\delta \psi dx\right)^p < \infty.$$

Using the condition $\phi \in \mathcal{A}_p^+(\psi)$ and applying Hölder's inequality for $\frac{r-1}{p-1}$, we obtain

$$\begin{aligned}
\left(\int_\alpha^\beta \phi dx\right)\left(\int_\beta^\delta \psi^{r'}\phi^{-\frac{1}{r-1}}dx\right)^{r-1} &\leq \left(\int_\alpha^\beta \phi dx\right)\left(\int_\beta^\delta \psi^{p'}\phi^{-\frac{1}{p-1}}dx\right)^{p-1}\left(\int_\beta^\delta \psi dx\right)^{r-p} \\
&\leq \left(\int_\alpha^\delta \psi dx\right)^{p}\left(\int_\alpha^\delta \psi dx\right)^{r-p} \\
&= \left(\int_\alpha^\delta \psi dx\right)^{r} < \infty.
\end{aligned}$$

□

Next, we prove an inequality in the scalar setting similar to that of Fefferman-Stein's lemma based on the techniques used in [37, 57].

Lemma 2.3.2. *Suppose that $p \in (1, \infty)$ and let $C_p > 0$ be a constant for which the inequality*

$$\int_{\mathbb{R}} \mathcal{M}_\psi^+(f)^p g dt \leq C_p \int_{\mathbb{R}} f^p \mathcal{M}_\psi^-\left(\frac{g}{\psi}\right)\psi dt \tag{2.1}$$

holds for functions $f, g \geq 0$.

Proof. We observe that the validity of the estimate (2.1) is similar with the strong (p,p) of the operator $f \mapsto \mathcal{M}_\psi^+(f)$ with respect to the weights $\mathcal{M}_\psi^-(\frac{g}{\psi})\psi$ and g. Thus applying the Marcinkiewicz interpolation theorem, it is enough for us to prove the weak $(1,1)$ and weak (∞,∞) of the operator $f \mapsto \mathcal{M}_\psi^+(f)$.

Let the function f be bounded with compact support in $\mathbb{R}$. For $\gamma > 0$ let us consider the set $\Lambda_\gamma = \{x \in \mathbb{R} : \mathcal{M}_\psi^+(f)(x) > \gamma\}$. Then there exists a sequence of bounded intervals $\{(\alpha_j, \beta_j)\}$ having pairwise disjoint property [44] such that $\Lambda_\gamma = \cup(\alpha_j, \beta_j)$ and

$$\gamma \leq \frac{1}{\psi(x, \beta_j)}\int_x^{\beta_j} f\psi dy \quad \text{for every } \alpha_j \leq x < \beta_j.$$

As the sequence (α_j, β_j) is pairwise disjoint, hence it is sufficient to prove the following inequality

$$\int_{\alpha_j}^{\beta_j} gdt \leq \frac{C}{\gamma} \int_{\alpha_j}^{\beta_j} f\mathcal{M}_\psi^-(\frac{g}{\psi})\psi dt.$$

For a fixed j, we break the interval (α_j, β_j) as follows. We put $\gamma_j^0 = \alpha_j$ and for each $k \in \mathbb{N}$, we define γ_j^k as

$$\int_{\gamma_j^k}^{\beta_j} f\psi dy = \frac{1}{2^k} \int_{\alpha_j}^{\beta_j} f\psi dy.$$

Thus, we get an increasing sequence $\{\gamma_j^k\}$ with $(\alpha_j, \beta_j) = \bigcup\limits_{k=1}^{\infty} (\gamma_j^{k-1}, \gamma_j^k]$ and

$$\gamma \leq \frac{4}{\psi(\gamma_j^k, \gamma_j^{k+2})} \int_{\gamma_j^{k+1}}^{\gamma_j^{k+2}} f\psi dy.$$

Now,

$$\begin{aligned}
\int_{\gamma_j^k}^{\gamma_j^{k+1}} f\psi dy &= \int_{\gamma_j^k}^{\beta_j} f\psi dy - \int_{\gamma_j^{k+1}}^{\beta_j} f\psi dy \\
&= \frac{1}{2^k} \int_{\alpha_j}^{\beta_j} f\psi dy - \frac{1}{2^{k+1}} \int_{\alpha_j}^{\beta_j} f\psi dy \\
&= \frac{1}{2^{k+1}} \int_{\alpha_j}^{\beta_j} f\psi dy \\
&= \frac{1}{4} \int_{\gamma_j^{k-1}}^{\beta_j} f\psi dy.
\end{aligned}$$

Thus we obtain by using the change of the order of integration,

$$\begin{aligned}
\gamma \int_{\gamma_j^{k-1}}^{\gamma_j^k} gdy &\leq \left(\frac{1}{\psi(\gamma_j^{k-1}, \beta_j)} \int_{\gamma_j^{k-1}}^{\beta_j} f\psi dt \right) \left(\int_{\gamma_j^{k-1}}^{\gamma_j^k} gdy \right) \\
&= \left(\frac{4}{\psi(\gamma_j^{k-1}, \beta_j)} \int_{\gamma_j^k}^{\gamma_j^{k+1}} f\psi dt \right) \left(\int_{\gamma_j^{k-1}}^{\gamma_j^k} gdy \right) \\
&= 4 \int_{\gamma_j^k}^{\gamma_j^{k+1}} f \left(\frac{1}{\psi(\gamma_j^{k-1}, \beta_j)} \int_{\gamma_j^{k-1}}^{x} gdy \right) \psi dt \\
&= 4 \int_{\gamma_j^k}^{\gamma_j^{k+1}} f \left(\frac{\psi(\gamma_j^{k-1}, x)}{\psi(\gamma_j^{k-1}, \beta_j)} \frac{1}{\psi(\gamma_j^{k-1}, x)} \int_{\gamma_j^{k-1}}^{x} \left(\frac{g}{\psi} \right) \psi dy \right) \psi dt \\
&\leq 4 \int_{\gamma_j^k}^{\gamma_j^{k+1}} f\mathcal{M}_\psi^- \left(\frac{g}{\psi} \right) \psi dt.
\end{aligned}$$

Summing the integral over k, we get

$$\int_{\alpha_j}^{\beta_j} g dt \leq \frac{C}{\gamma} \int_{\alpha_j}^{\beta_j} f \mathcal{M}_{\psi}^{-}\Big(\frac{g}{\psi}\Big)\psi dt.$$

Thus

$$\int_{\Lambda_\gamma} g dt \leq \frac{C}{\gamma} \int_{\alpha_j}^{\beta_j} f \mathcal{M}_{\psi}^{-}(\frac{g}{\psi})\psi dt.$$

This proves the weak $(1,1)$ condition. Hence the proof is complete. □

Theorem 2.3.3. *Let $f = \{f_j\}$ be a sequence of functions such that $f_j \in L^1_{loc,\psi}(\mathbb{R})$ for each j. Then we have the following statements for $1 < q < \infty$.*

(i) Let $\sigma \in \mathcal{A}_p^+(\psi), 1 < p < \infty$, then the estimate

$$\bigg(\int_{\mathbb{R}} \|\mathcal{M}_{\psi}^{+}(f)(t)\|_q^p \sigma dt\bigg)^{\frac{1}{p}} \leq C\bigg(\int_{\mathbb{R}} \|f(t)\|_q^p \sigma dt\bigg)^{\frac{1}{p}}$$

holds for some constant $C > 0$.

(ii) Let $\sigma \in \mathcal{A}_1^+(\psi)$ and for $\gamma > 0$, the weak $(1,1)$ estimate

$$\sigma\bigg(\Big\{t \in \mathbb{R} : \|\mathcal{M}_{\psi}^{+}(f)(t)\|_q > \gamma\Big\}\bigg) \leq \frac{C}{\gamma}\int_{\mathbb{R}} \|f(t)\|_q \sigma dt$$

holds for some constant $C > 0$.

Proof. Let us break the proof into three different parts depending on the exponents p and q. Let the weight $\sigma \in \mathcal{A}_p^+(\psi)$.
Case I : When $p = q$. We have

$$\begin{aligned}
\bigg(\int_{\mathbb{R}} \|\mathcal{M}_{\psi}^{+}(f)(t)\|_q^p \sigma dt\bigg)^{\frac{1}{p}} &= \bigg(\int_{\mathbb{R}} \|\mathcal{M}_{\psi}^{+}(f)(t)\|_p^p \sigma dt\bigg)^{\frac{1}{p}} \\
&= \bigg(\int_{\mathbb{R}} \Big(\sum_j \mathcal{M}_{\psi}^{+}(f_j)(t)^p\Big)\sigma dt\bigg)^{\frac{1}{p}} \\
&= \bigg(\sum_j \int_{\mathbb{R}} \mathcal{M}_{\psi}^{+}(f_j)(t)^p \sigma dt\bigg)^{\frac{1}{p}} \\
&\leq C\bigg(\sum_j \int_{\mathbb{R}} f_j(t)^p \sigma dt\bigg)^{\frac{1}{p}}, \quad [\text{as } \sigma \in \mathcal{A}_p^+(\psi)]
\end{aligned}$$

$$= \left(\int_{\mathbb{R}} \Big(\sum_j |f_j(t)|^p \Big) \sigma dt \right)^{\frac{1}{p}}$$
$$= \left(\int_{\mathbb{R}} \|f(t)\|^p \sigma dt \right)^{\frac{1}{p}}.$$

Thus we obtain the result for $p = q$.

Case II : When $p > q$. Since $\sigma \in \mathcal{A}_p^+(\psi)$, thus from the Corollary 2.2.5, $\exists\ \pi \in (0, p-1)$ for which $\sigma \in \mathcal{A}_{p-\pi}^+(\psi)$. This is same as saying that there exists $r_0, 1 < r_0 < p$ such that $\sigma \in \mathcal{A}_{s_0}^+(\psi)$ with $s_0 = \frac{p}{r_0}$. Thus, applying Corollary 2.2.5 we get $\sigma \in \mathcal{A}_s^+(\psi)$ and $\psi^{s'}\sigma^{-\frac{1}{s-1}} \in \mathcal{A}_{s'}^-(\psi)$, for all $s \geq s_0$.

For $V \in L_\sigma^{s'}(\mathbb{R})$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \|\mathcal{M}_\psi^+(f)(t)\|_q^q V\sigma dt \\
&= \int_{\mathbb{R}} \Big(\sum_j \mathcal{M}_\psi^+(f_j)(x)^q \Big) V\sigma dt \\
&= \sum_j \int_{\mathbb{R}} \mathcal{M}_\psi^+(f_j)(t)^q V\sigma dt \\
&\leq C \sum_j \int_{\mathbb{R}} |f_j(t)|^q \mathcal{M}_\psi^- \Big(\frac{V\sigma}{\psi} \Big) \psi dt \\
&= C \int_{\mathbb{R}} \sum_j |f_j(t)|^q \mathcal{M}_\psi^- \Big(\frac{V\sigma}{\psi} \Big) \psi dt \\
&= C \int_{\mathbb{R}} \|f(t)\|_q^q \mathcal{M}_\psi^- \Big(\frac{V\sigma}{\psi} \Big) \psi dt \\
&\leq C \left(\int_{\mathbb{R}} \|f(t)\|_q^{sq} \sigma dt \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}} \left(\mathcal{M}_\psi^- \Big(\frac{V\sigma}{\psi} \Big) \psi \right)^{s'} \sigma^{-\frac{s'}{s}} dt \right)^{\frac{1}{s'}} \\
&= C \left(\int_{\mathbb{R}} \|f(t)\|_q^{sq} \sigma dt \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}} \left(\mathcal{M}_\psi^- \Big(\frac{V\sigma}{\psi} \Big) \right)^{s'} \psi^{s'} \sigma^{-\frac{1}{s-1}} dt \right)^{\frac{1}{s'}} \\
&\leq C \left(\int_{\mathbb{R}} \|f(t)\|_q^{sq} \sigma dt \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}} \left(M_\psi^- \Big(\frac{V\sigma}{\psi} \Big) \right)^{s'} \psi^{s'} \sigma^{-\frac{1}{s-1}} dt \right)^{\frac{1}{s'}} \\
&= C \left(\int_{\mathbb{R}} \|f(t)\|_q^{sq} \sigma dt \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}} V^{s'} \sigma dt \right)^{\frac{1}{s'}}.
\end{aligned}$$

In the first inequality we use Lemma 2.3.2 with exponent q. We choose $s = \frac{p}{q}$ and then taking supremum over all V in the set $B = \{V \geq 0 : \int_{\mathbb{R}} V^{s'} \sigma dt = 1\}$, we get

$$\left(\int_{\mathbb{R}} \|\mathcal{M}_\psi^+(f)(t)\|_q^p \sigma dt \right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}} \|f(t)\|_q^p \sigma dt \right)^{\frac{1}{p}}, \quad 1 < q \leq r_0 < p. \tag{2.2}$$

From the Case I, we have got the inequality (2.2) for $q = p$. Thus by Marcinkiewicz interpolation theorem, the inequality (2.2) holds for all q with $r_0 < q < p$. Hence, we conclude that

$$\left(\int_{\mathbb{R}} \|\mathcal{M}_\psi^+(f)(t)\|_q^p \sigma dt \right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}} \|f(t)\|_q^p \sigma dt \right)^{\frac{1}{p}}, \quad 1 < q < p.$$

Case III: When $p < q$. We have already established the result for $p = q$ and $p > q$. Now we plan to establish the weak (p, p) version of the result. Thus by using the result for $p = q$, weak (p, p) and interpolation theorem we get the main result for $p < q$. Hence it is sufficient for us to prove the following estimate

$$\sigma\Big(\{t \in \mathbb{R} : \|\mathcal{M}_\psi^+(f)(t)\|_q > \gamma\} \Big) \leq \frac{C}{\gamma^p} \int_{\mathbb{R}} \|f(t)\|_q^p \sigma dt, \quad \text{where } \gamma > 0.$$

We prove this estimate via breaking f as $f = u + w$ and then proving the same estimate for u and w and finally we use sub-linear property of the maximal operator $\mathcal{M}_\psi^+$. The details of the proof are given below.

For $\gamma > 0$, we consider the set $\Lambda = \{t \in \mathbb{R} : \mathcal{M}_\psi^+(\|f(t)\|_q) > \gamma\}$. Thus we get a sequence of bounded intervals $\{(\alpha_j, \beta_j)\}_j$ having disjoint property and satisfying the following three conditions

(i) $\Lambda = \bigcup_j (\alpha_j, \beta_j)$,

(ii) $\gamma = \frac{1}{\psi(\alpha_j, \beta_j)} \int_{\alpha_j}^{\beta_j} \|f(t)\|_q \psi dt$, for all j,

(iii) For $t \in (\alpha_j, \beta_j)$, we have $\gamma \leq \frac{1}{\psi(t, \beta_j)} \int_t^{\beta_j} \|f(y)\|_q \psi dy$, for all j.

Thus $\|f(t)\|_q \leq \gamma$, for a.e. $t \in \Lambda^c$.

Now, we consider two sequences of measurable functions $u = \{u_k\}$ and $w = \{w_k\}$ defined by $u_k = f_k \chi_{\Lambda^c}$ and $w_k = f_k - u_k$ for each k. Using the sublinearity of $\mathcal{M}_\psi^+$ we have

$$\|\mathcal{M}_\psi^+(f)(t)\|_q \leq \|\mathcal{M}_\psi^+(u)(t)\|_q + \|\mathcal{M}_\psi^+(w)(t)\|_q. \tag{2.3}$$

Next, we establish the following estimates to complete the proof.

$$\sigma\left(\left\{ t \in \mathbb{R} : \|\mathcal{M}_\psi^+(u)(t)\|_q > \gamma \right\} \right) \leq \frac{C}{\gamma^p} \int_{\mathbb{R}} \|f(t)\|_q^p \sigma dt, \tag{2.4}$$

$$\sigma\left(\left\{t \in \mathbb{R} : \|\mathcal{M}_\psi^+(w)(t)\|_q > \gamma\right\}\right) \le \frac{C}{\gamma^p}\int_{\mathbb{R}} \|f(t)\|_q^p \sigma dt. \tag{2.5}$$

First we prove the estimate (2.4).

As the weight $\sigma \in \mathcal{A}_p^+(\psi)$, thus $\sigma \in \mathcal{A}_q^+(\psi)$. From the first case, we get

$$\begin{aligned}
\sigma\left(\left\{t \in \mathbb{R} : \|\mathcal{M}_\psi^+(u)(t)\|_q > \gamma\right\}\right) &\le \frac{C}{\gamma^q}\int_{\mathbb{R}} \|u(t)\|_q^q \sigma dt \\
&= \frac{C}{\gamma^q}\int_{\Lambda^c} \|u(t)\|_q^q \sigma dt \\
&= \frac{C}{\gamma^q}\int_{\Lambda^c} \|u(t)\|_q^{q+p-p} \sigma dt \\
&\le \frac{C}{\gamma^p}\int_{\Lambda^c} \|f(t)\|_q^p \sigma dt \\
&\le \frac{C}{\gamma^p}\int_{\mathbb{R}} \|f(t)\|_q^p \sigma dt,
\end{aligned}$$

where the first inequality comes from the Case I and for the second inequality we use that $\|u(t)\|_q \le \|f(t)\|_q \le \gamma$ for a.e. $t \in \Lambda^c$. This proves the estimate (2.4).

Next we prove the estimate (2.5). We construct a new function $f' = \{f'_k\}$ as

$$f'_k(t) = \begin{cases} \frac{1}{\psi(\alpha_j,\beta_j)}\int_{\alpha_j}^{\beta_j} f_k(y)\psi(y)dy, & \text{if } t \in (\alpha_j, \beta_j) \\ 0, & \text{otherwise.} \end{cases}$$

For an interval $J = (\alpha, \beta)$, set $J^- = (2\alpha - \beta, \alpha)$ and define $\Lambda^* = \Lambda \cup \Lambda^-$, where $\Lambda^- = \bigcup_j (2\alpha_j - \beta_j, \alpha_j)$. Let us first assume that the following three inequalities hold.

$$\|\mathcal{M}_\psi^+(w)(t)\|_q \le C\|\mathcal{M}_\psi^+(f')(t)\|_q, \quad \text{for a.e. } t \notin \Lambda^*, \tag{2.6}$$

$$\sigma\big(2\alpha_j - \beta_j, \alpha_j\big) \le C\sigma\big(\alpha_j, \beta_j\big), \tag{2.7}$$

$$\sigma\big(\alpha_j, \beta_j\big) \le \frac{C}{\gamma^p}\int_{\alpha_j}^{\beta_j} \|f(t)\|_q^p \sigma dt. \tag{2.8}$$

Using the estimates (2.6), (2.7) and (2.8) we get

$$\begin{aligned}
&\sigma\left(\left\{t \in \mathbb{R} : \|\mathcal{M}_\psi^+(w)(t)\|_q > \gamma\right\}\right) \\
&= \sigma\left(\left\{t \in \Lambda^* \cup \Lambda^{*^c} : \|\mathcal{M}_\psi^+(w)(t)\|_q > \gamma\right\}\right) \\
&= \sigma(\Lambda^*) + \sigma\left(\left\{t \notin \Lambda^* : \|\mathcal{M}_\psi^+(w)(t)\|_q > \gamma\right\}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sigma(\Lambda) + \sigma\Bigg(\Big\{t \notin \Lambda^* : \|\mathcal{M}_\psi^+(f')(t)\|_q > \gamma\Big\}\Bigg) \\
&\leq C\sigma(\Lambda) + \frac{C}{\gamma^q}\int_{\mathbb{R}} \|f'(t)\|_q^q \sigma dt \\
&\leq C\Big(\sigma(\Lambda) + \frac{1}{\gamma^q}\int_{\Lambda} \|f'(t)\|_q^q \sigma dt\Big) \\
&= C\Big(\sigma(\Lambda) + \frac{1}{\gamma^q}\int_{\Lambda} \sum_k |f_k'(t)|^q \sigma dt\Big) \\
&\leq C\Big(\sigma(\Lambda) + \frac{1}{\gamma^q}\int_{\Lambda} \sum_k \Big\{\frac{1}{\psi(\alpha_j,\beta_j)}\int_{\alpha_j}^{\beta_j} |f_k|\psi dy\Big\}^q \sigma dt\Big) \\
&\leq C\Bigg(\sigma(\Lambda) + \frac{1}{\gamma^q}\int_{\Lambda} \sum_k \Big\{\frac{1}{\psi(\alpha_j,\beta_j)}\Big(\int_{\alpha_j}^{\beta_j} |f_k|^q\psi dy\Big)^{\frac{1}{q}} \\
&\qquad\qquad \Big(\int_{\alpha_j}^{\beta_j} \psi^{(1-\frac{1}{q})q'} dy\Big)^{\frac{1}{q'}}\Big\}^q \sigma dt\Bigg) \\
&\leq C\Bigg(\sigma(\Lambda) + \frac{1}{\gamma^q}\int_{\Lambda} \frac{1}{\psi(\alpha_j,\beta_j)}\Big(\int_{\alpha_j}^{\beta_j} \sum_k |f_k|^q\psi dy\Big)\sigma dt\Bigg) \\
&\leq C\Bigg(\sigma(\Lambda) + \frac{1}{\gamma^q}\int_{\Lambda} \Bigg(\frac{1}{\psi(\alpha_j,\beta_j)}\int_{\alpha_j}^{\beta_j} \gamma^q\psi dy\Bigg)\sigma dt\Bigg) \\
&\leq C(\sigma(\Lambda) + \sigma(\Lambda)) \\
&= C\sigma(\Lambda) \\
&\leq \frac{C}{\gamma^p}\int_{\mathbb{R}} \|f(t)\|_q^p \sigma dt.
\end{aligned}
$$

We apply results from Case I to obtain the second inequality and the Hölder inequality in obtaining the fifth inequality. In the seventh inequality, we use the fact that $\|f(t)\|_q \leq \gamma$.

Next we shall prove the three assumptions. We begin with the proof of the inequality (2.6). Let $t \notin \Lambda^*$, thus $t \notin \Lambda$ and $t \notin \Lambda^-$. For each $\zeta > 0$ consider the set $Y_\zeta = \{y : t < y < t + \zeta\}$. We have,

$$\frac{1}{\psi(Y_\zeta)}\int_{Y_\zeta} w_k\psi dy = \frac{1}{\psi(Y_\zeta)}\sum_{j\in J}\int_{Y_\zeta\cap(\alpha_j,\beta_j)} w_k\psi dy$$

where $J = \{j : Y_\zeta \cap (\alpha_j,\beta_j) \neq \phi\}$. Clearly $t \notin (\alpha_j,\beta_j)$ and $t \notin (2\alpha_j - \beta_j, \alpha_j)$. Now if $j \in J$, then $(\alpha_j,\beta_j) \subset Y_{2\zeta}$. Thus

$$
\begin{aligned}
\frac{1}{\psi(Y_\zeta)}\int_{Y_\zeta} w_k\psi dy &= \frac{1}{\psi(Y_\zeta)}\sum_{j\in J}\int_{Y_\zeta\cap(\alpha_j,\beta_j)} w_k\psi dy \\
&\leq \frac{1}{\psi(Y_\zeta)}\sum_{j\in J}\int_{\alpha_j}^{\beta_j} w_k\psi dy
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\psi(Y_\zeta)} \sum_{j\in J} \int_{\alpha_j}^{\beta_j} (f_k - u_k)\psi dy \\
&= \frac{1}{\psi(Y_\zeta)} \sum_{j\in J} \int_{\alpha_j}^{\beta_j} f_k' \psi dy \\
&\le \frac{\psi(Y_{2\zeta})}{\psi(Y_\zeta)} \frac{1}{\psi(Y_{2\zeta})} \int_{Y_{2\zeta}} f_k'\psi dy \\
&\le C\mathcal{M}_\psi^+(f_k')(t).
\end{aligned}
$$

This proves the estimate (2.6).

Next we proof the the inequality (2.7) Letting $J = (\alpha, \beta)$ and $t \in J^- = (2\alpha - \beta, \alpha)$, we obtain

$$
\begin{aligned}
1 &= \frac{1}{\psi(\alpha,\beta)} \int_\alpha^\beta \psi dy \\
&= \frac{\psi(t,\beta)}{\psi(\alpha,\beta)} \frac{1}{\psi(t,\beta)} \int_t^\beta \chi_J \psi dy \\
&\le \frac{\psi(2\alpha-\beta,\beta)}{\psi(\alpha,\beta)} \frac{1}{\psi(t,\beta)} \int_t^\beta \chi_J\psi dy \\
&\le C\mathcal{M}_\psi^+(\chi_J)(t).
\end{aligned}
$$

Thus $t \in J^-$ implies that $\mathcal{M}_\psi^+(\chi_J)(t) \ge \frac{1}{C}$ i.e. $J^- \subset \{t : \mathcal{M}_\psi^+(\chi_J)(t) \ge \frac{1}{C}\}$. Now

$$
\begin{aligned}
\sigma(J^-) &\le \sigma\Big\{t : \mathcal{M}_\psi^+(\chi_J)(t) \ge \frac{1}{C}\Big\} \\
&\le C^p \int_{\mathbb{R}} (\chi_J)^p \sigma dt \\
&= C\int_J \sigma dt = C\sigma(J).
\end{aligned}
$$

This establishes that $\sigma\big(2\alpha_j - \beta_j, \alpha_j\big) \le C\sigma\big(\alpha_j, \beta_j\big)$ for each j.

To prove the third inequality (2.8), we use the earlier technique to break the interval (α_j, β_j) into a disjoint increasing sequence $\{\gamma_k\}_{k=0}^\infty$. Set $\gamma_j^0 = \alpha_j$ and for each $k \in \mathbb{N}$, we define γ_j^k as

$$
\int_{\gamma_j^k}^{\beta_j} \|f(t)\|_q \psi dt = \frac{1}{2^k} \int_{\alpha_j}^{\beta_j} \|f(t)\|_q \psi dt.
$$

Thus, we get an increasing sequence $\{\gamma_j^k\}$ with $(\alpha_j, \beta_j) = \bigcup_{k=1}^{\infty} (\gamma_j^{k-1}, \gamma_j^k]$. Now, we have

$$
\sigma\Big(\gamma_j^{k-1}, \gamma_j^k\Big)
$$

$$
\begin{aligned}
&\leq \Big(\frac{1}{\gamma\psi(\gamma_j^{k-1},\beta_j)}\int_{\gamma_j^{k-1}}^{\beta_j}\|f(t)\|_q\psi dt\Big)^p\Big(\int_{\gamma_j^{k-1}}^{\gamma_j^k}\sigma dt\Big)\\
&=\Big(\frac{4}{\gamma\psi(\gamma_j^{k-1},\beta_j)}\int_{\gamma_j^{k-1}}^{\gamma_j^{k+1}}\|f(t)\|_q\sigma^{\frac{1}{p}-\frac{1}{p}}\psi dt\Big)^p\Big(\int_{\gamma_j^{k-1}}^{\gamma_j^k}\sigma dt\Big)\\
&\leq \Big(\frac{4}{\gamma\psi(\gamma_j^{k-1},\beta_j)}\Big(\int_{\gamma_j^k}^{\gamma_j^{k+1}}\|f(t)\|_q^p\sigma dt\Big)^{\frac{1}{p}}\Big(\int_{\gamma_j^k}^{\gamma_j^{k+1}}\psi^{p'}\sigma^{-\frac{p'}{p}}dt\Big)^{\frac{1}{p'}}\Big)^p\Big(\int_{\gamma_j^{k-1}}^{\gamma_j^k}\sigma dt\Big)\\
&\leq \frac{C}{\gamma^p}\Big(\int_{\gamma_j^k}^{\gamma_j^{k+1}}\|f(t)\|_q^p\sigma dt\Big)\frac{1}{\psi(\gamma_j^{k-1},\gamma_j^{k+1})^p}\Big(\int_{\gamma_j^{k-1}}^{\gamma_j^k}\sigma dt\Big)\Big(\int_{\gamma_j^k}^{\gamma_j^{k+1}}\psi^{p'}\sigma^{-\frac{1}{p-1}}dt\Big)^{p-1}\\
&\leq \frac{C}{\gamma^p}\Big(\int_{\gamma_j^k}^{\gamma_j^{k+1}}\|f(t)\|_q^p\sigma dt\Big).
\end{aligned}
$$

Summing over all k, we get

$$\sigma(\alpha_j,\beta_j)\leq \frac{C}{\gamma^p}\Big(\int_{\alpha_j}^{\beta_j}\|f(t)\|_q^p\sigma dt\Big).$$

Thus we obtain the estimate (2.8). Hence Case III follows. The weak $(1,1)$ version for the operator $\mathcal{M}_\psi^+$ in the vector valued setup is immediate from the Case III. Hence the proof is complete. □

Corollary 2.3.4. *Let $1<p,q<\infty$ and $f=\{f_j\}_j$ be a sequence of measurable functions such that $f_j\in L^1_{loc,\psi}(\mathbb{R})$ for each j. If $g\in\mathcal{A}_1^+(\psi)$, then*

$$\int_{\mathbb{R}}\|\mathcal{M}_\psi^+(f)(t)\|_q^p g dt\leq C\int_{\mathbb{R}}\|f(t)\|_q^p\mathcal{M}_\psi^-\Big(\frac{g}{\psi}\Big)\psi dt$$

holds for some constant $C>0$.

Proof. We first show that $\mathcal{M}_\psi^-\Big(\frac{g}{\psi}\Big)\psi\in\mathcal{A}_1^+(\psi)$. For this we need to establish that

$$\mathcal{M}_\psi^-\left(\psi^{-1}\mathcal{M}_\psi^-\Big(\frac{g}{\psi}\Big)\psi\right)(t)\leq C\left(\psi^{-1}\mathcal{M}_\psi^-\Big(\frac{g}{\psi}\Big)\psi\right)(t),\quad\text{for a.e. } t.$$

We have

$$
\begin{aligned}
\mathcal{M}_\psi^-\Big(\psi^{-1}\mathcal{M}_\psi^-(\frac{g}{\psi})\psi\Big)(t)&=\sup_{\xi>0}\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^t\mathcal{M}_\psi^-(\frac{g}{\psi})\psi dy\\
&\geq\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^t\mathcal{M}_\psi^-(\frac{g}{\psi})\psi dy.
\end{aligned}
$$

Consider $\Gamma=(t-\xi,t)$ and $\Gamma^-=(t-2\xi,t)$. Then

$$\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^t\mathcal{M}_\psi^-(\frac{g}{\psi})\psi dy=\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^t\mathcal{M}_\psi^-(\frac{g}{\psi})\chi_{\Gamma^-\cup(\mathbb{R}\setminus\Gamma^-)}\psi dy$$

$$\begin{aligned}&\leq \frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^{t}\mathcal{M}_{\psi}^{-}(\frac{g}{\psi})\chi_{\Gamma^{-}}\psi dy\\&+\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^{t}\mathcal{M}_{\psi}^{-}(\frac{g}{\psi})\chi_{(\mathbb{R}\setminus\Gamma^{-})}\psi dy.\end{aligned}\tag{2.9}$$

First we approximate the second integrand as

$$\mathcal{M}_{\psi}^{-}\Big(\frac{g}{\psi}\chi_{(\mathbb{R}\setminus\Gamma^{-})}\Big)(y)\leq \mathcal{M}_{\psi}^{-}(\frac{g}{\psi})(t)\quad \text{a.e. } y\in\Gamma.\tag{2.10}$$

The inequality (2.10) has the following equivalent form.

$$\frac{1}{\psi(y-\zeta,y)}\int_{y-\zeta}^{y}\Big(\frac{g}{\psi}\chi_{(\mathbb{R}\setminus\Gamma^{-})}\Big)\psi dz\leq \mathcal{M}_{\psi}^{-}(\frac{g}{\psi})(t)\quad\text{for } \zeta>0.$$

Now, we have either $(y-\zeta,y)\cap(\mathbb{R}\setminus\Gamma^{-})=\phi$ or $(y-\zeta,y)\cap(\mathbb{R}\setminus\Gamma^{-})\neq\phi$. When $(y-\zeta,y)\cap(\mathbb{R}\setminus\Gamma^{-})=\phi$, then the integral on the left side becomes zero and hence the result follows. If $(y-\zeta,y)\cap(\mathbb{R}\setminus\Gamma^{-})\neq\phi$, then $\zeta>2\xi$ and we obtain

$$\begin{aligned}\frac{1}{\psi(y-\zeta,y)}\int_{y-\zeta}^{y}\Big(\frac{g}{\psi}\chi_{(\mathbb{R}\setminus\Gamma^{-})}\Big)\psi dz&\leq\frac{1}{\psi(t-\zeta,t)}\int_{t-\zeta}^{t}\Big(\frac{g}{\psi}\Big)\psi dz\\&\leq\mathcal{M}_{\psi}^{-}\Big(\frac{g}{\psi}\Big)(t).\end{aligned}$$

For the first part of (2.9), we have

$$\begin{aligned}&\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^{t}\mathcal{M}_{\psi}^{-}\Big(\frac{g}{\psi}\chi_{J^{-}}\Big)\psi dy\\&=\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^{t}\mathcal{M}_{\psi}^{-}\Big(\frac{g}{\psi}\chi_{J^{-}}\Big)\psi^{\frac{1}{r}}\psi^{\frac{r-1}{r}}dy\\&\leq\frac{1}{\psi(t-\xi,t)}\left(\int_{t-\xi}^{t}\mathcal{M}_{\psi}^{-}\Big(\frac{g}{\psi}\chi_{J^{-}}\Big)^{r}\psi dy\right)^{\frac{1}{r}}\left(\int_{t-\xi}^{t}\psi^{\frac{r-1}{r}r'}dy\right)^{\frac{1}{r'}}\\&=\left(\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^{t}\mathcal{M}_{\psi}^{-}\Big(\frac{g}{\psi}\chi_{J^{-}}\Big)^{r}\psi dy\right)^{\frac{1}{r}}\\&\leq C\left(\frac{1}{\psi(t-\xi,t)}\int_{t-\xi}^{t}\Big(\frac{g}{\psi}\chi_{J^{-}}\Big)^{r}\psi dy\right)^{\frac{1}{r}}\\&\leq C\mathcal{M}_{\psi}^{-}\left(\Big(\frac{g}{\psi}\chi_{J^{-}}\Big)(t)^{r}\right)^{\frac{1}{r}}\\&\leq C\mathcal{M}_{\psi}^{-}\left(\Big(\frac{g}{\psi}\chi_{J^{-}}\Big)(t)\right)\end{aligned}$$

$$\leq C\mathcal{M}_\psi^-\left(\left(\frac{g}{\psi}\right)(t)\right).$$

[In the fourth inequality, we use the weak reverse Hölder inequality, a consequence of Theorem 2.2.4]

This implies that

$$\mathcal{M}_\psi^-\left(\frac{g}{\psi}\right)\psi \in \mathcal{A}_1^+(\psi).$$

It follows that

$$\mathcal{M}_\psi^-(\frac{g}{\psi})\psi \in \mathcal{A}_p^+(\psi), \quad \text{for } p > 1. \tag{2.11}$$

Also by Lebesgue differentiation theorem there exists $\delta > 0$ such that

$$g \leq \left(\mathcal{M}_\psi^-\left((\psi^{-1}g)^{1+\delta}\right)\right)^{1/1+\delta}\psi. \tag{2.12}$$

Applying (2.11), (2.12), the weak reverse Hölder inequality, and the weighted vector valued inequality of $\mathcal{M}_\psi^+$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \|\mathcal{M}_\psi^+(f)(t)\|_q^p g dt &\leq \int_{\mathbb{R}} \|\mathcal{M}_\psi^+(f)(t)\|_q^p \left(\mathcal{M}_\psi^-(\frac{g}{\psi})^{1+\delta}\right)^{1/1+\delta}\psi dt \\
&\leq C\int_{\mathbb{R}} \|\mathcal{M}_\psi^+(f)(t)\|_q^p \mathcal{M}_\psi^-\left(\frac{g}{\psi}\right)\psi dt \\
&\leq C\int_{\mathbb{R}} \|f(t)\|_q^p \mathcal{M}_\psi^-\left(\frac{g}{\psi}\right)\psi dt.
\end{aligned}$$

Hence the proof is complete. $\square$