CHAPTER 3

Weighted estimates for a one-sided fractional minimal function

3.1 Introduction

In this chapter, we define a fractional version of one-sided minimal function, m_μ^+ for $0 \leq \mu < \infty$ on $\mathbb{R}$. We prove weighted weak and strong type norm estimates for m_μ^+ by defining two new weight classes. We also establish an equivalence relation between the two weight classes.

Let us consider a measurable function f on $\mathbb{R}$. We denote the one-sided fractional minimal function for f of order $0 \leq \mu < \infty$ by m_μ^+ and is defined as

$$m_\mu^+(f)(x) = \inf_{\xi>0} \frac{1}{\xi^{1+\mu}} \int_x^{x+\xi} |f(t)|dt.$$

Considering the limits $x-\xi$ to x, we obtain an another version of one-sided minimal function as follows

$$m_\mu^-(f)(x) = \inf_{\xi>0} \frac{1}{\xi^{1+\mu}} \int_{x-\xi}^{x} |f(t)|dt.$$

For $\mu = 0$, we get the usual one-sided minimal function m^+ defined in [16]. Thus m_μ^+ is a generalization of m^+. Some authors use forward minimal function for m_μ^+ and backward minimal function for m_μ^-. As like maximal functions, the results of forward and backward minimal function can be interchangeable with a little modification in the corresponding weight classes.

Corresponding to a non-negative locally integrable function w and a measurable subset Γ of $\mathbb{R}$, we consider the w measure of Γ as $w(\Gamma) = \int_\Gamma w dy$. Given $1 < p < \infty$, we consider

This chapter is based on the published work *Weighted inequalities for one-sided fractional minimal function* by D. Chutia and R. Haloi [12].

p' satisfying $p^{-1} + (p')^{-1} = 1$. Throughout this chapter U and V will denote two weights on $\mathbb{R}$ and we define $\omega = V^{1/p+1}$ for $1 < p < \infty$. The letter C refers to an arbitrary positive constant not necessarily same in all cases.

The structure of the chapter is of the following form. The weight classes $\mathcal{W}^+_{p,q}$ and $(\mathcal{W}^+_{p,q})^*$ are defined in Section 3.2. We prove the weak and strong type results in Section 3.3 and their equivalence relation is given in Section 3.4.

3.2 Preliminaries

We state a basic lemma required in proving the main results. For the proof we refer [18].

Lemma 3.2.1. *Let Γ be an interval in $\mathbb{R}$. Let $\{P_a\}_a$ be a collection of subintervals of Γ such that given a function w and for $0 \leq \mu < \infty$ the relation $\int_{P_a} w dx \leq C|P_a|^{1+\mu}$ holds for each a. If $\Lambda = \cup_a P_a$, then $\int_\Lambda w dx \leq C(2|\Lambda|)^{1+\mu}$.*

We define the weight class $\mathcal{W}^+_{p,q}$ corresponding to the weak (p,q) estimate for m^+_μ.

Definition 3.2.2. *Let $p, q \in (0,\infty)$ with $q \geq p$ and $\mu \in [0,\infty)$. Then the pair of weights $(U,V) \in \mathcal{W}^+_{p,q}$ if*

$$\frac{1}{|\Gamma^-|}\int_{\Gamma^-} U \leq \frac{C}{|\Gamma|^{1+(\mu-\frac{1}{p})q}}\left(\frac{1}{|\Gamma|}\int_\Gamma V^{\frac{1}{p+1}}\right)^{(1+\frac{1}{p})q}$$

holds for each interval $\Gamma = [\alpha,\beta]$ in $\mathbb{R}$ with $2|\Gamma^-| = |\Gamma|$ and $\Gamma^- = [\alpha,\lambda]$.

For $\mu = 0$ and $p = q$, $\mathcal{W}^+_{p,q}$ coincides with $\mathcal{W}^+_p$, defined by Cruz-Uribe et al. [16]. Thus $\mathcal{W}^+_{p,q}$ is a generalized form of $\mathcal{W}^+_p$. In this definition, we consider two intervals Γ^- and Γ with common left end point such that $2|\Gamma^-| = |\Gamma|$. We generalize the Definition 3.2.2 to improve the condition $2|\Gamma^-| = |\Gamma|$. To do this, we consider any subinterval Γ^- of Γ with common left end point such that $0 < \frac{|\Gamma^-|}{|\Gamma|} < 1$.

Definition 3.2.3. *Let $\Gamma^- = [\alpha,\lambda]$ be any subinterval of $\Gamma = [\alpha,\beta]$. We set $\pi = \dfrac{|\Gamma^-|}{|\Gamma|}$ such that $0 < \pi < 1$. For $0 \leq \mu < \infty$, the pair $(U,V) \in \mathcal{W}^+_{p,q,\pi}$ if*

$$\frac{1}{|\Gamma^-|}\int_{\Gamma^-} U \leq \frac{C}{\pi(1-\pi)^{(1+\mu)q}|\Gamma|^{1+(\mu-\frac{1}{p})q}}\left(\frac{1}{|\Gamma|}\int_\Gamma V^{\frac{1}{p+1}}\right)^{(1+\frac{1}{p})q}$$

holds for some constant $C > 0$ and $0 < p \leq q < \infty$.

We need a decomposition of a finite interval in $\mathbb{R}$ to relate the weight classes $\mathcal{W}^+_{p,q}$ and $\mathcal{W}^+_{p,q,\pi}$ which is known as plus-minus decomposition, introduced by Cruz-Uribe et al. [16].

Definition 3.2.4. *Let us consider the finite interval* $\Gamma = [\alpha, \beta]$ *in* $\mathbb{R}$. *We define a sequence* $\{\kappa_m\}_{m\geq 0}$ *recursively from the interval* Γ *as, set* $\kappa_0 = \alpha$ *and for each* $m \geq 1$ *we define*

$$\kappa_m = \frac{\beta + \kappa_{m-1}}{2}.$$

For $m \geq 1$, *we construct three subintervals of* Γ *from the sequence* $\{\kappa_m\}$ *as* $\Lambda_m^- = [\kappa_{m-1}, \kappa_m]$, $\Lambda_m^+ = [\kappa_m, \kappa_{m+1}]$ *and* $\Lambda_m = [\kappa_{m-1}, \kappa_{m+1}]$. *From the construction itself* $\Gamma = \cup_{m\geq 0}\Lambda_m^-$.

Next, we define a weight class similar to that of Sawyer's testing type condition and use this condition to obtain the strong (p, q) estimate for the minimal function, m_μ^+.

Definition 3.2.5. *Let* $0 < p \leq q < \infty$. *If*

$$\int_\Gamma \frac{U}{m_\mu^+(\omega/\chi_\Gamma)^q} \leq C\left(\int_\Gamma \omega\right)^{\frac{q}{p}}$$

holds for each interval Γ *in* $\mathbb{R}$ *and for some constant* $C > 0$, *then* $(U, V) \in (\mathcal{W}_{p,q}^+)^*$.

3.3 Weak and strong type results

In this section, we prove the weak as well as strong type weighted estimates for m_μ^+. The following lemma will be used to prove the weak and strong type results.

Lemma 3.3.1. *Suppose* $q \geq p$ *with* $p, q \in (0, \infty)$ *and* $\pi \in (0, 1)$. *Then* (U, V) *belongs to* $\mathcal{W}_{p,q}^+$ *if and only if* (U, V) *lies in* $\mathcal{W}_{p,q,\pi}^+$.

Proof. Let $(U, V) \in \mathcal{W}_{p,q,\pi}^+, 0 < \pi < 1$. If we choose $\pi = \frac{1}{2}$, then it follows from the definition that $(U, V) \in \mathcal{W}_{p,q}^+$.

For the converse part, we break the proof into two parts depending on π.

Case I. The case when $0 < \pi = \frac{|\Gamma^-|}{|\Gamma|} < \frac{1}{2}$, with $\Gamma = [\alpha, \beta]$ and $\Gamma^- = [\alpha, \delta]$. Suppose $\Lambda = [\alpha, \lambda]$ be a subinterval of Γ with $2|\Lambda| = |\Gamma|$. By the definition of $\mathcal{W}_{p,q}^+$, we have

$$\int_\Lambda U \leq \frac{C|\Lambda|}{|\Gamma|^{1+(\mu-\frac{1}{p})q}}\left(\frac{1}{|\Gamma|}\int_\Gamma V^{\frac{1}{p+1}}\right)^{\frac{(p+1)q}{p}}. \tag{3.1}$$

Since $2|\Lambda| = |\Gamma| > 2|\Gamma^-|$. Thus $\Gamma^- \subset \Lambda$. From the inequality (3.1), we have

$$\frac{1}{|\Gamma^-|}\int_{\Gamma^-} U \leq \frac{1}{|\Gamma^-|}\int_\Lambda U \leq \frac{C|\Lambda|}{|\Gamma^-||\Gamma|^{1+(\mu-\frac{1}{p})q}}\left(\frac{1}{|\Gamma|}\int_\Gamma \omega\right)^{(1+1/p)q}$$

$$\leq \frac{C}{\pi(1-\pi)^{(1+\mu)q}|\Gamma|^{1+(\mu-\frac{1}{p})q}}\left(\frac{1}{|\Gamma|}\int_{\Gamma}\omega\right)^{(1+1/p)q}.$$

Thus $(U,V)\in W^{+}_{p,q,\pi},\ 0<\pi\leq\frac{1}{2}$.

Case II. The case when $\frac{1}{2}<\pi=\frac{|\Gamma^{-}|}{|\Gamma|}<1$. We choose the smallest $M_0\in\mathbb{N}$ such that $\pi\leq 1-\frac{1}{2^{M_0}}$. As M_0 is the smallest, so

$$1-\pi<\frac{1}{2^{M_0-1}}. \tag{3.2}$$

Let $\{\Lambda^{-}_{m}\}_m$ be a collection of subintervals of Γ formed by the plus-minus decomposition of the interval Γ. We have,

$$\Gamma^{-}\subset\cup_{m=1}^{M_0}\Lambda^{-}_{m}. \tag{3.3}$$

From the definition of $\mathcal{W}^{+}_{p,q}$, we have

$$\begin{aligned}\int_{\Lambda^{-}_{m}}U &\leq \frac{C|\Lambda^{-}_{m}|}{(2|\Lambda^{-}_{m}|)^{1+(\mu-\frac{1}{p})q}}\left(\frac{1}{2|\Lambda^{-}_{m}|}\int_{\Gamma}V^{\frac{1}{p+1}}\right)^{\frac{(p+1)q}{p}}\\ &\leq C|\Gamma^{-}|\left(\frac{\Gamma}{2|\Lambda^{-}_{m}|}\right)^{(1+\mu)q}\left[\frac{1}{|\Gamma|^{1+(\mu-\frac{1}{p})q}}\left(\frac{1}{|\Gamma|}\int_{\Gamma}V^{\frac{1}{p+1}}\right)^{\frac{(p+1)q}{p}}\right].\end{aligned} \tag{3.4}$$

For $m\geq 1$, we have

$$\frac{|\Gamma|}{2|\Lambda^{-}_{m}|}\leq 2^{m-1}. \tag{3.5}$$

Using the estimates (3.2), (3.3), (3.4) and (3.5), we obtain

$$\frac{1}{|\Gamma^{-}|}\int_{\Gamma^{-}}U\leq\frac{1}{|\Gamma^{-}|}\sum_{m=1}^{M_0}\int_{\Lambda^{-}_{m}}U\leq\frac{CM_0}{2}\frac{1}{\pi(1-\pi)^{(1+\mu)q}}\left[\frac{1}{|\Gamma|^{1+(\mu-\frac{1}{p})q}}\left(\frac{1}{|\Gamma|}\int_{\Gamma}V^{\frac{1}{p+1}}\right)^{\frac{(p+1)q}{p}}\right].$$

□

We prove the two weighted weak type estimate for m^{+}_{μ} in the following theorem.

Theorem 3.3.2. *Suppose $q\geq p$ with $p,q\in(0,\infty)$ and $\mu\in[0,\infty)$. Then the following statements are equivalent.*

(i) The pair of weights $(U,V)\in\mathcal{W}^{+}_{p,q}$.

(ii) For $\gamma>0$ the following estimate

$$U\left(\left\{t\in\mathbb{R}:m^{+}_{\mu}(f)(t)<\frac{1}{\gamma}\right\}\right)\leq\frac{C}{\gamma^{q}}\left(\int_{\mathbb{R}}\frac{V}{|f|^{p}}\right)^{\frac{q}{p}} \tag{3.6}$$

holds for a suitable positive constant C.

Proof. (*i*) $\implies$ (*ii*). We prove the statement for the function f with $\frac{1}{f}$ having compact support and the result for any measurable f follows due to the method developed in [16].

We assume that $(U,V) \in \mathcal{W}^+_{p,q}, 0 < p \leq q < \infty$. For each $\gamma > 0$, we denote the set $O_\gamma = \{t \in \mathbb{R} : m^+_\mu(f)(t) < \frac{1}{\gamma}\}$. Since the minimal function is upper semi-continuous and $\frac{1}{f}$ has compact support, so we get a disjoint sequence of bounded and open interval $\{\Gamma_k\}_{k\geq 1}$ such that $O_\gamma = \cup_{k\geq 1}\Gamma_k$. Also by the plus-minus decomposition of Γ_k, we have

$$\frac{1}{|\Lambda^+_m|^{1+\mu}} \int_{\Lambda_m} |f| \leq \frac{8^{1+\mu}}{\gamma}. \tag{3.7}$$

Using $(U,V) \in \mathcal{W}^+_{p,q}$ and the inequality (3.7), we get

$$\begin{aligned}
\int_{\Gamma_k} U = \sum_l \int_{\Lambda^-_m} U &\leq \frac{3^{(1+\mu)q}C}{\gamma^q} \sum_m \frac{1}{|\Lambda_m|^{(1+\mu)q}} \left(\int_{\Lambda_m} V^{\frac{1}{p+1}} \right)^{\frac{(p+1)q}{p}} \left(\frac{(8|\Lambda^+_m|)^{(1+\mu)q}}{\int_{\Lambda_m} |f|} \right)^q \\
&\leq \frac{C}{\gamma^q} \sum_m \left(\int_{\Lambda_m} \frac{V}{|f|^p} \right)^{\frac{q}{p}} \\
&\leq \frac{C}{\gamma^q} \left(\sum_m \int_{\Lambda_m} \frac{V}{|f|^p} \right)^{\frac{q}{p}} \quad \text{(since } p \leq q) \\
&\leq \frac{C}{\gamma^q} \left(\int_{\Gamma_k} \frac{V}{|f|^p} \right)^{\frac{q}{p}} \quad \text{for each } k.
\end{aligned}$$

Thus

$$U(O_\gamma) = U(\cup_k \Gamma_k) = \sum_k U(\Gamma_k) \leq \sum_k \frac{C}{\gamma^q} \left(\int_{\Gamma_k} \frac{V}{|f|^p} \right)^{\frac{q}{p}} \leq \frac{C}{\gamma^q} \left(\int_{\mathbb{R}} \frac{V}{|f|^p} \right)^{\frac{q}{p}}.$$

(*ii*) $\implies$ (*i*). Let $\Gamma = [\alpha, \beta]$ be a finite interval in $\mathbb{R}$. Let $\Gamma^- = [\alpha, \delta]$ and $\Gamma^+ = [\delta, \beta]$ be two subintervals of Γ with $|\Gamma^-| = |\Gamma^+|$. For $f = V^{\frac{1}{p+1}}/\chi_\Gamma$ and $t \in \Gamma^-$, we obtain

$$m^+_\mu(f)(t) \leq \frac{1}{|\Gamma^+|^{1+\mu}} \int_\Gamma V^{\frac{1}{p+1}}. \tag{3.8}$$

We choose $\gamma > 0$ such that

$$\frac{1}{\gamma} = \frac{1}{|\Gamma^+|^{1+\mu}} \int_\Gamma V^{\frac{1}{p+1}}. \tag{3.9}$$

Incorporating the estimates (3.8) and (3.9) in the inequality (3.6), we get

$$\begin{aligned}
U(\Gamma^-) \leq U\left(\left\{ t \in \mathbb{R} : m^+_\mu(f)(t) < \frac{1}{\gamma} \right\} \right) &\leq \frac{C}{\gamma^q} \left(\int_\Gamma \frac{V}{V^{\frac{p}{p+1}}} \right)^{\frac{q}{p}} \\
&= C \frac{|\Gamma^-|}{|\Gamma|^{1+(\mu-\frac{1}{p})q}} \left(\int_\Gamma V^{\frac{1}{p+1}} \right)^{\frac{(p+1)q}{p}}.
\end{aligned}$$

Thus $(U,V) \in \mathcal{W}^+_{p,q}$ and completes the proof. $\square$

We obtain the two weighted strong type (p,q) estimate using $(\mathcal{W}^+_{p,q})^*$ condition as follows.

Theorem 3.3.3. *Let us assume $q \geq p$ with $p,q \in (0,\infty)$. Then we have the following equivalent statements.*

(i) The pair of weights $(U,V) \in (\mathcal{W}^+_{p,q})^*$.

(ii) The strong (p,q) holds for the fractional one-sided minimal function, that is

$$\int_{\mathbb{R}} \frac{U}{(m^+_\mu(f))^q} \leq C\bigg(\int_{\mathbb{R}} \frac{V}{|f|^p}\bigg)^{\frac{q}{p}}.$$

Proof. $(i) \implies (ii)$. For each $\kappa \in \mathbb{Z}$, we define

$$\Omega_\kappa = \Big\{t \in \mathbb{R} : m^+_\mu(f)(t) < \frac{1}{2^\kappa}\Big\}.$$

Thus we get a disjoint sequence of bounded open intervals $\{\Gamma_{i,\kappa}\}_i$ with $\Omega_\kappa = \cup_i \Gamma_{i,\kappa}$ and for each $t \in \Gamma_{i,\kappa} = (\delta_{i,\kappa}, \epsilon_{i,\kappa})$,

$$\int_t^{\epsilon_{i,\kappa}} |f| \leq 2^{1+\mu-\kappa}|\epsilon_{i,\kappa} - t|^{1+\mu}. \tag{3.10}$$

We construct another pairwise disjoint set $\Omega_{i,\kappa}$ for each integer i and κ as

$$\Omega_{i,\kappa} = \Big\{t \in \Gamma_{i,\kappa} : m^+_\mu(f)(t) \geq \frac{1}{2^{\kappa+1}}\Big\}.$$

Then

$$\begin{aligned}\int_{\mathbb{R}} \frac{U}{(m^+_\mu(f))^q} = \sum_{i,\kappa} \int_{\Omega_{i,\kappa}} \frac{U}{m^+_\mu(f)^q} &\leq \sum_{i,\kappa} \int_{\Omega_{i,\kappa}} 2^{(\kappa+1)q} U \\ &\leq 2^{q(2+\mu)} \sum_{i,\kappa} \int_{\Omega_{i,\kappa}} \bigg(\frac{1}{|\epsilon_{i,\kappa} - t|^{1+\mu}} \int_t^{\epsilon_{i,\kappa}} |f|\bigg)^{-q} U.\end{aligned}$$

Let η stand for usual Lebesgue measure and ζ represent the counting measure defined respectively in $\mathbb{R}$ and $\mathbb{Z}$. Using the argument in [55], we consider a measure ν on $\mathcal{Z} = \mathbb{Z}\times\mathbb{Z}\times\mathbb{R}$ as $\zeta \times \zeta \times \eta$. Let ρ be a weight on $\mathcal{Z}$ defined by

$$\rho(i,\kappa,t) = \bigg(\frac{1}{|\epsilon_{i,\kappa} - t|^{1+\mu}} \int_t^{\epsilon_{i,\kappa}} \omega dy\bigg)^{-q} \chi_{\Omega_{i,\kappa}}(t) U(t).$$

We define two operators $\mathcal{L}$ and $\mathcal{G}$ by

$$\mathcal{L}(g)(i,\kappa,t) = \chi_{\Omega_{i,\kappa}}(t) \frac{1}{\int_t^{\epsilon_{i,\kappa}} g\omega dy} \int_t^{\epsilon_{i,\kappa}} \omega dy, \;\; \mathcal{G}(g)(i,\kappa,t) = \chi_{\Omega_{i,\kappa}}(t) \frac{1}{\int_t^{\epsilon_{i,\kappa}} \omega dy} \int_t^{\epsilon_{i,\kappa}} g\omega dy.$$

Using Hölder's inequality with exponent s and s' for $s>1$, we have

$$\frac{1}{\int_t^{\epsilon_{i,\kappa}} g\omega dy}\int_t^{\epsilon_{i,\kappa}}\omega dy \le \frac{1}{\int_t^{\epsilon_{i,\kappa}} g\omega dy}\left(\int_t^{\epsilon_{i,\kappa}} g\omega dy\right)\left(\frac{1}{\int_t^{\epsilon_{i,\kappa}}\omega dy}\left(\int_t^{\epsilon_{i,\kappa}} g^{1-s'}\omega dy\right)\right)^{s-1}$$
$$= \left(\frac{1}{\int_t^{\epsilon_{i,\kappa}}\omega dy}\left(\int_t^{\epsilon_{i,\kappa}} g^{1-s'}\omega dy\right)\right)^{s-1}.$$

This implies that

$$\mathcal{L}(g)(i,\kappa,t) \le \left(\mathcal{G}(g^{1-s'})(i,\kappa,t)\right)^{s-1}. \tag{3.11}$$

We first assume the strong $\left(\frac{q}{p},\frac{q^2}{p^2}\right)$ inequality of the operator $\mathcal{G}$ with respect to the weights ω and $\rho d\nu$ in $\mathbb{R}$ and $\mathcal{Z}$, respectively. Considering $s=1+\frac{q}{p^2}$ and the estimate (3.11), we have

$$2^{q(2+\mu)}\int_{\mathcal{Z}}\mathcal{L}\left(\frac{|f|}{\omega}\right)^q\rho d\nu \le 2^{q(2+\mu)}\left(\int_{\mathbb{R}}\left(\left(\frac{\omega}{|f|}\right)^{s'-1}\right)^{\frac{q}{p}}\omega\right)^{\frac{p}{q}\times\frac{q^2}{p^2}} \le C\left(\int_{\mathbb{R}}\frac{V}{|f|^p}\right)^{\frac{q}{p}}.$$

Hence the result follows. Now, it is left to establish the boundedness of $\mathcal{G}$. As the operator $\mathcal{G}$ satisfies the strong type (∞,∞), thus applying Marcinkiewicz interpolation theorem, it is sufficient for us to establish the weak type $(1,\frac{q}{p})$ of $\mathcal{G}$. That is we need to show that

$$\int_{\{|\mathcal{G}(g)|>\gamma\}}\phi d\tau \le C\left(\frac{1}{\gamma}\int g\omega dx\right)^{\frac{q}{p}}.$$

Let $\theta_{i\kappa}(\gamma)=\inf\mathcal{E}_{i,\kappa}(\gamma)$, where $\mathcal{E}_{i,\kappa}(\gamma)=\{t\in\Omega_{i,\kappa}:\mathcal{G}(g)(i,\kappa,t)>\gamma\}$ and $\Lambda_{i\kappa}=\Lambda_{i\kappa}(\gamma)=[\theta_{i\kappa}(\gamma),\epsilon_{i\kappa})$. Then

$$\frac{1}{\int_{\Lambda_{i\kappa}}\omega}\int_{\Lambda_{i\kappa}} g\omega \ge \gamma$$

and either $\Lambda_{i_1\kappa_1}\cap\Lambda_{i_1\kappa_2}=\phi$ or one is contained in the other.
Let $\{\Lambda_j\}$ be the maximal elements of the family $\{\Lambda_{i\kappa}\}$ so that Λ_j's are disjoint. Thus

$$\int_{\{|\mathcal{G}(g)|>\gamma\}}\phi d\tau = \sum_{\kappa,i}\int_{\mathcal{E}_{i,\kappa}(\gamma)}\left(\frac{1}{|\epsilon_{i,\kappa}-t|^{1+\mu}}\right)^{-q}U dt$$
$$=\sum_j\sum_{\{(\kappa,i):\Lambda_{i\kappa}\subset\Lambda_j\}}\int_{\mathcal{E}_{i\kappa}(\gamma)}\left(\frac{1}{|\epsilon_{i,\kappa}-t|^{1+\mu}}\right)^{-q}U dt$$
$$\le\sum_j\int_{\Lambda_j}\frac{U}{m_\mu^+(\omega/\chi_{\Lambda_j})^q}dt$$
$$\le C\sum_j\left(\int_{\Lambda_j}\omega dt\right)^{\frac{q}{p}} \le C\sum_j\left(\frac{1}{\gamma}\int_{\Lambda_j}g\omega dt\right)^{\frac{q}{p}} \le C\left(\frac{1}{\gamma}\int_{\mathbb{R}}g\omega dt\right)^{\frac{q}{p}}.$$

This proves that H is weak $(1, \frac{q}{p})$ and hence it concludes the proof.

$(ii) \implies (i)$. We take, $f = \omega/\chi_\Gamma, \omega = V^{\frac{1}{p+1}}$ and hence it follows immediately.

□

3.4 Relation between weak and strong type estimates

The following result is quite interesting as it gives the equivalence between the two weight classes $\mathcal{W}^+_{p,q}$ and $(\mathcal{W}^+_{p,q})^*$.

Theorem 3.4.1. *Suppose that $q \geq p$ with $p, q \in (0, \infty)$. Then the weight classes $\mathcal{W}^+_{p,q}$ and $(\mathcal{W}^+_{p,q})^*$ are equivalent, that is a pair (U, V) satisfies the $\mathcal{W}^+_{p,q}$ condition if and only if it satisfies the condition $(\mathcal{W}^+_{p,q})^*$.*

We prove Theorem 3.4.1 using the following two lemmas. Next, we state these two lemmas and the equivalence between $\mathcal{W}^+_{p,q}$ and $(\mathcal{W}^+_{p,q})^*$ follows immediately.

Lemma 3.4.2. *Let the pair $(U, V) \in \mathcal{W}^+_{p,q}, 0 < p \leq q < \infty$. We consider any finite interval $\Gamma = [\alpha, \beta]$ in $\mathbb{R}$. Then*

$$\int_{\Gamma^-} \frac{U}{m^+_\mu(\omega/\chi_\Gamma)^q} \leq C \left(\int_{\Gamma^- \cup \Gamma^+} \omega \right)^{\frac{q}{p}}$$

holds for Γ^- and Γ^+, where $\Gamma^- = [\alpha, \delta]$ and $\Gamma^+ = [\delta, \lambda]$ with $2|\Gamma^-| = |\Gamma| = 4|\Gamma^+|$.

Proof. Let $\gamma > 0$ and consider $\Omega_\gamma = \left\{ x \in \Gamma^- : m^+_\mu(\omega/\chi_\Gamma)(x) < \frac{1}{\gamma} \right\}$. Thus we obtain a sequence of intervals $\{\Gamma_j\}$ that are disjoint and that satisfy $\Omega_\gamma = \cup_j \Gamma_j$. Also, we have

$$\int_{\Gamma^-} \frac{U}{m^+_\mu(\omega/\chi_\Gamma)^q} dx = q \int_0^\infty \gamma^{q-1} U(\Omega_\gamma) d\gamma = I_1 + I_2,$$

where $I_1 = q \int_0^\epsilon \gamma^{q-1} U(\Omega_\gamma) d\gamma$ and $I_2 = q \int_\epsilon^\infty \gamma^{q-1} U(\Omega_\gamma) d\gamma$, for some $\epsilon > 0$ and the value of ϵ to be specified later. Now, we obtain

$$I_1 = q \int_0^\epsilon \gamma^{q-1} U(\Omega_\gamma) d\gamma \leq U(\Gamma^-) q \int_0^\epsilon \gamma^{q-1} = U(\Gamma^-) \epsilon^q. \tag{3.12}$$

For a fixed interval $\{\Gamma_j\}$, we construct two sequences of intervals $\{\Lambda^-_m\}$ and $\{\Lambda^+_m\}$ as the plus-minus decomposition of the interval Γ_j. Then

$$U(\Gamma_j) = \sum_{m \geq 1} U(\Lambda^-_m) \leq 3^{(1+\mu)q} C \sum_{m \geq 1} \frac{3|\Lambda^-_m|}{2|\Lambda_m|^{1+(1+\mu)q}} \left(\int_{\Lambda_m} \omega \right)^{(1+1/p)q} \tag{3.13}$$

$$\leq C\sum_{m\geq 1}\left(\frac{1}{\gamma}\right)^{(1+1/p)q}|\Lambda_m|^{(1+\mu)\frac{q}{p}} \leq C\left(\frac{1}{\gamma}\right)^{(1+1/p)q}|\Gamma_j|^{(1+\mu)\frac{q}{p}}. \tag{3.14}$$

Using the disjoint decomposition of Ω_γ and the inequality (3.14), we thus obtain

$$\begin{aligned} I_2 = q\int_\epsilon^\infty \gamma^{q-1}U(\Omega_\gamma)d\gamma &\leq q\int_\epsilon^\infty \gamma^{q-1}\left(\frac{1}{\gamma}\right)^{\frac{(p+1)q}{p}}|\Gamma^-|^{(1+\mu)\frac{q}{p}} \\ &\leq C\left(\frac{1}{\epsilon}\right)^{\frac{q}{p}}|\Gamma^-|^{(1+\mu)\frac{q}{p}}. \end{aligned} \tag{3.15}$$

We choose ϵ such that

$$\epsilon^q = \frac{(\omega(\Gamma^-\cup\Gamma^+))^{\frac{q}{p}}}{U(\Gamma^-)}. \tag{3.16}$$

As the pair $(U, V) \in \mathcal{W}^+_{p,q}$, we have

$$\begin{aligned} U(\Gamma^-) &\leq C\frac{3^{1+(1+\mu)q}|\Gamma^-|}{2|\Gamma^-\cup\Gamma^+|^{1+(1+\mu)q}}\left(\omega(\Gamma^-\cup\Gamma^+)\right)^{\frac{(p+1)q}{p}} \\ &\leq \frac{C}{|\Gamma^-|^{(1+\mu)q}}\left(\omega(\Gamma^-\cup\Gamma^+)\right)^{\frac{(p+1)q}{p}}. \end{aligned} \tag{3.17}$$

From the inequality (3.16) and (3.17), we obtain

$$\left(\frac{1}{\epsilon}\right)^{\frac{q}{p}} \leq \frac{C}{|\Gamma^-|^{(1+\mu)\frac{q}{p}}}\left(\omega(\Gamma^-\cup\Gamma^+)\right)^{\frac{q}{p}}. \tag{3.18}$$

Using the inequality (3.12), (3.15), and (3.18), we conclude that

$$\Gamma_1, \Gamma_2 \leq (\omega(\Gamma^-\cup\Gamma^+))^{\frac{q}{p}}.$$

Hence the proof is complete. □

Lemma 3.4.3. *Let $\{\mathcal{K}_l\}_{l\geq 0}$ be a decreasing sequence of nested intervals with the property that $|\mathcal{K}_l| \to 0$ as $l \to \infty$. Then for the pair $(U, V) \in \mathcal{W}^+_{p,q}$, $0 < p \leq q < \infty$, we have*

$$\lim_{l\to\infty}\int_{\mathcal{K}_l}\frac{U}{m^+_\mu(\omega/\chi_{\mathcal{K}_l})^q} = 0.$$

Proof. As the sequence $\{\mathcal{K}_l\}$ is decreasing, thus for $0 \leq \mu < \infty$, m^+_μ satisfies

$$m^+_\mu(\omega/\chi_{\mathcal{K}_l}) \leq m^+_\mu(\omega/\chi_{\mathcal{K}_{l+1}}),\ k \geq 0.$$

Also, letting $l \to \infty$, we have $1/m^+_\mu(\omega/\chi_{\mathcal{K}_l}) \to 0$ *a.e.* on $\mathcal{K}_0$.

We set $\Gamma^- = \mathcal{K}_0$ and by the Lemma 3.4.2, we obtain

$$\int_{\mathcal{K}_0} \frac{U}{m_\mu^+(\omega/\chi_{\mathcal{K}_0})} < \infty.$$

Using the property $\mathcal{K}_{l+1} \subset \mathcal{K}_l$ and from the dominated convergence theorem, we have

$$\lim_{l\to\infty} \int_{\mathcal{K}_l} \frac{U}{m_\mu^+(\omega/\chi_{\mathcal{K}_l})} \le \lim_{l\to\infty} \int_{\mathcal{K}_0} \frac{U}{m_\mu^+(\omega/\chi_{\mathcal{K}_l})} = 0.$$

Hence the result holds. □

Proof of Theorem 3.4.1. Clearly $(\mathcal{W}_{p,q}^+)^*$ implies $\mathcal{W}_{p,q}^+$. We only need to establish the other part.

Let $(U, V) \in \mathcal{W}_{p,q}^+$. For a interval $\Gamma = [\alpha, \beta]$ in $\mathbb{R}$, we establish the inequality

$$\int_\Gamma \frac{U}{m_\mu^+(\omega/\chi_\Gamma)^q} \le C\omega(\Gamma)^{\frac{q}{p}}$$

for any arbitrary $C > 0$. Suppose $\{\Lambda'_m\}$ be a sequence of intervals defined by $\Lambda'_m = [\kappa_m, \beta]$ for each $m \ge 1$, where $\{\kappa_m\}$ is the sequence constructed to form the plus-minus decomposition, Λ_m^- and Λ_m^+, of the interval Γ. Then $\Gamma = \Lambda_1^- \cup \Lambda'_1$ and thus we have

$$\int_\Gamma \frac{U}{m_\mu^+(\omega/\chi_\Gamma)^q} \le \int_{\Lambda_1^-} + \int_{\Lambda'_1}.$$

From the Lemma 3.4.2

$$\int_{\Lambda_1^-} \le C\Big(\omega(\Lambda_1)\Big)^{\frac{q}{p}}.$$

For the second integral, we write $\Lambda'_1 = \Lambda_2^- \cup \Lambda'_2$ and we repeat the process. Continuing this process up to l many times, we get

$$\int_\Gamma \frac{U}{m_\mu^+(\omega/\chi_\Gamma)^q} \le C \sum_{m=1}^{l} \Big(\omega(\Lambda_m)\Big)^{\frac{q}{p}} + \int_{\Lambda'_l} \frac{U}{m_\mu^+(\omega/\chi_{\Lambda'_l})^q}.$$

Using Lemma 3.4.3, we obtain

$$\int_{\Lambda'_l} \frac{U}{m_\mu^+(\omega/\chi_{\Lambda'_l})^q} \to 0 \text{ as } l \to \infty.$$

Now, letting $l \to \infty$ and by the relation $\frac{q}{p} \ge 1$, we obtain

$$\int_\Gamma \frac{U}{m_\mu^+(\omega/\chi_\Gamma)^q} \le C\Big(\sum_{m=1}^{\infty} \omega(\Lambda_m)\Big)^{\frac{q}{p}} \le C\Big(\omega(\Gamma)\Big)^{\frac{q}{p}}.$$

Hence the proof is complete.