CHAPTER 4

Weighted estimates for the one-sided dyadic fractional maximal function

4.1 Introduction

We devote this chapter to study a dyadic version of one-sided fractional maximal function, $\mathcal{M}_{\mu}^{+,d}(f)$ for $0 \leq \mu < 1$ on $\mathbb{R}^n$. We introduce dyadic Muckenhoupt type weight and Sawyer's testing type condition and applying the weight classes we obtain two weighted weak and strong type inequalities respectively for $\mathcal{M}_{\mu}^{+,d}$. We also prove one weighted strong type inequalities using Muckenhoupt type condition.

Let Γ denotes a dyadic cube with all the sides parallel to the coordinate axes. The cubes Γ^+ and Γ^-, related to Γ are defined as $\Gamma^+ = [\beta_1, 2\beta_1 - \alpha_1) \times [\beta_2, 2\beta_2 - \alpha_2) \times \ldots \times [\beta_n, 2\beta_n - \alpha_n)$ and $\Gamma^- = [2\alpha_1 - \beta_1, \alpha_1) \times [2\alpha_2 - \beta_2, \alpha_2) \times \ldots \times [2\alpha_n - \beta_n, \alpha_n)$, where $\Gamma = [\alpha_1, \beta_1) \times [\alpha_2, \beta_2) \times \ldots \times [\alpha_n, \beta_n)$ is any dyadic cube in $\mathbb{R}^n$ with vertices $\alpha_i, \beta_i \in \mathbb{R}, 1 \leq i \leq n$.

We assume the function f to be locally integrable in $\mathbb{R}^n$. For $0 \leq \mu < 1$, let us define the dyadic one-sided fractional maximal function, $\mathcal{M}_{\mu}^{+,d}(f)$ at any point $t \in \mathbb{R}^n$ as

$$\mathcal{M}_{\mu}^{+,d}(f)(t) = \sup_{t \in \Gamma,\ \Gamma \text{ dyadic}} \frac{1}{|\Gamma|^{1-\mu}} \int_{\Gamma^+} |f| dy,$$

where the supremum is considered over each dyadic cube $t \in \Gamma$ with all the sides of Γ are parallel with the respective coordinate axes. Similarly the backward maximal function is defined as

$$\mathcal{M}_{\mu}^{-,d}(f)(t) = \sup_{t \in \Gamma} \frac{1}{|\Gamma|^{1-\mu}} \int_{\Gamma^-} |f| dy,$$

where the supremum is considered similarly as in the earlier case. The results for forward

This chapter is based on the published work *A note on one-sided dyadic fractional maximal function on* $\mathbb{R}^n$ by D. Chutia and S. Pal [13].

and backward maximal functions are similar and can be obtained from the other with a slight modification in the corresponding weight class.

For $\epsilon > 0$, we construct two more cubes $\Gamma^{\epsilon,+}$ and $\Gamma^{\epsilon,-}$ associated with Γ as $\Gamma^{\epsilon,+} = [\alpha_1, \alpha_1 + \epsilon(\beta_1 - \alpha_1)) \times [\alpha_2, \alpha_2 + \epsilon(\beta_2 - \alpha_2)) \times \ldots \times [\alpha_n, \alpha_n + \epsilon(\beta_n - \alpha_n))$ and $\Gamma^{\epsilon,-} = [\beta_1 - \epsilon(\beta_1 - \alpha_1), \beta_1) \times [\beta_2 - \epsilon(\beta_2 - \alpha_2), \beta_2) \times \ldots \times [\beta_n - \epsilon(\beta_n - \alpha_n), \beta_n)$, where $\Gamma = [\alpha_1, \beta_1) \times [\alpha_2, \beta_2) \times \ldots \times [\alpha_n, \beta_n)$ is any dyadic cube similar to the previous case. From the above constructions, we have the following remarks.

(i) $|\Gamma^-| = |\Gamma| = |\Gamma^+| = \epsilon^{-n}|\Gamma^{\epsilon,+}| = \epsilon^{-n}|\Gamma^{\epsilon,-}|$.

(ii) If Γ_1 and Γ_2 are any two cubes with $\Gamma_1^+ \subset \Gamma_2^+$, then $\Gamma_1 \subset \Gamma_2^{2,+}$.

Next, we introduce two classes of weight $\mathcal{A}_{p,q}^{+,d}$ and $\mathcal{S}_{p,q}^{+,d}$ to prove the main results. The conditions $\mathcal{A}_{p,q}^{+,d}$ and $\mathcal{S}_{p,q}^{+,d}$ are also called as the dyadic version of one-sided Muckenhoupt weight and Sawyer's testing type condition respectively.

Definition 4.1.1. *Suppose $q \geq p$ with $p, q \in (1, \infty)$ and we define $\mu = \frac{1}{p} - \frac{1}{q}$. Then the pair of weights (σ, ϕ) verifies the $\mathcal{A}_{p,q}^{+,d}$ condition if for each dyadic cube Γ the following*

$$\Big(\frac{1}{|\Gamma|}\int_{\Gamma} \sigma\Big)^{\frac{1}{q}} \Big(\frac{1}{|\Gamma|}\int_{\Gamma^+} \phi^{1-p'}\Big)^{\frac{p-1}{p}} \leq C$$

holds for a positive constant C.

Definition 4.1.2. *Suppose $q \geq p$ with $p, q \in (1, \infty)$. Then the pair (σ, ϕ) satisfies the $\mathcal{S}_{p,q}^{+,d}$ condition if*

$$\int_{\Gamma \cup \Gamma^+} \psi < \infty \qquad \text{and} \qquad \int_{\Gamma \cup \Gamma^+} (\mathcal{M}_{\mu}^{+,d}(\psi \chi_{\Gamma^+}))^q \sigma \leq C\Big(\int_{\Gamma^+} \psi\Big)^{\frac{q}{p}},$$

hold for each dyadic cube Γ with $\int_{(\Gamma^-)^- \cup \Gamma^-} \sigma < \infty$, where $\psi = \phi^{-\frac{1}{p-1}}$.

The chapter is arranged as follows. In Section 4.2, we discuss the weak type result. In Section 4.3, we establish the two weighted strong type result using Sawyer's condition and also establish the one weighted strong type result using the $\mathcal{A}_{p,q}^{+,d}$ condition.

4.2 Weak type result

We now state the two weighted weak type estimate.

Theorem 4.2.1. *Suppose $q \geq p$ with $p, q \in (1, \infty)$ and we define $\mu = \frac{1}{p} - \frac{1}{q}$. Then for $\gamma > 0$ we have the following equivalent statements.*

(a) *The pair of weights* (σ, ϕ) *verifies the* $\mathcal{A}^{+,d}_{p,q}$ *condition.*

(b) $\mathcal{M}^{+,d}_{\mu}$ *satisfies the weak* (p,q) *inequality. More precisely, we have*

$$\sigma\Big(\{t \in \mathbb{R}^n : \mathcal{M}^{+,d}_{\mu}(f)(t) > \gamma\}\Big) \le C\left(\frac{4}{\gamma}\right)^q 2^{n+2}\Big(\int_{\mathbb{R}^n} |f|^p \phi\Big)^{\frac{q}{p}}$$

for some suitable constant $C > 0$.

It is sufficient to consider the function f to be non-negative and locally integrable on $\mathbb{R}^n$. For each $\lambda > 0$ we consider the set $\mathcal{O}_\lambda$ as

$$\mathcal{O}_\lambda = \Big\{t \in \mathbb{R}^n : \mathcal{M}^{+,d}_{\mu}(f)(t) > \lambda\Big\}.$$

Thus we get a family of dyadic cubes $\{\Gamma_k\}$ having pairwise disjoint property and satisfying

$$\mathcal{O}_\lambda = \cup_{k\ge 1}\Gamma_k \ \text{ and } \ \lambda < f_{k,\mu} = \frac{1}{|\Gamma_k|^{1-\mu}}\int_{\Gamma_k^+} f, \ \text{ for k} \in \mathbb{N}.$$

We rearrange(rename) the collection of cubes $\{\Gamma_k\}_{k\ge 1}$ satisfying the following order

$$L_k \le L_{k+1}, \ \text{ for all } k \ge 1, \ \text{ where } \ L_k = |\Gamma_k|.$$

We construct $\mathcal{F}_\lambda$, a collection of indices k, for which $\lambda < f_{k,\mu} \le 2\lambda$ is hold.

Based on the ideas from [49], we now state a lemma which is the key ingredient for proving Theorem 4.2.1.

Lemma 4.2.2. *Let the pair* $(\sigma, \phi) \in \mathcal{A}^{+,d}_{p,q}, 1 < p \le q < \infty$. *If for each* $\lambda > 0$, $\{\Gamma_k\}_{k\in\mathcal{F}_\lambda}$ *is a family of disjoint dyadic cubes, then*

$$\int_{\cup_{k\in\mathcal{F}_\lambda}\Gamma_k} \sigma \le C\left(\frac{2}{\lambda}\right)^q 2^{n+2}\Big(\int_{\cup_{k\in\mathcal{F}_\lambda}\Gamma_k^+} |f|^p\phi\Big)^{\frac{q}{p}}$$

holds for a suitable constant $C > 0$.

Proof. For $l \in \mathbb{N} \cup \{0\}$, let us define a subcollection j_l of $\mathcal{F}_\lambda$ as

$$j_l = \{k \in \mathcal{F}_\lambda : \exists \text{ exactly } \ l \ \text{ cubes } \Gamma_r^+ \text{ such that } \Gamma_k^+ \subset \Gamma_r^+ \text{ with } \ r \in \mathcal{F}_\lambda\ \}$$

and we define

$$\rho_l = \cup_{k\in j_l}\Gamma_k^+.$$

Then the following are valid.

(i) Clearly, $\mathcal{F}_\lambda = \cup_{l\ge 0} j_l$. Also $\{\Gamma_i\}_{i\in j_l}$ are mutually disjoint.

(ii) For any integer $0 \leq m \leq l$ with $i \in j_{l+1} (l \geq 0)$, we get an unique $i_m \in j_m$ satisfying $\Gamma_i^+ \subset \Gamma_{i_m}^+$. This follows that $\rho_{l+1} \subset \rho_l$.

(iii) For a fixed l_0, we choose $k_0 \in j_{l_0}$. Now if $\Gamma_k^+ \subset \Gamma_{k_0}^+$, then $k \in j_l$ for all $l > l_0$. Also $\Gamma_k \subset (\Gamma_{k_0})^{2,+}$.

As the collection $\{\Gamma_k\}$ is disjoint, thus we obtain

$$\cup_{l>l_0} \cup_{\{k \in j_l : \Gamma_k^+ \subset \Gamma_{k_0}^+\}} \Gamma_k \subset (\Gamma_{k_0})^{2,+}.$$

Thus, we obtain

$$\sum_{l>l_0} \sum_{\{k \in j_l : \Gamma_k^+ \subset \Gamma_{k_0}^+\}} |\Gamma_k| \leq 2^n |\Gamma_{k_0}|.$$

Now, we have

$$\begin{aligned}
\sum_{l>l_0} \sum_{\{k \in j_l : \Gamma_k^+ \subset \Gamma_{k_0}^+\}} |\Gamma_k|^{1-\mu} &= \sum_{l>l_0} \sum_{\{k \in j_l : \Gamma_k^+ \subset \Gamma_{k_0}^+\}} \frac{|\Gamma_k|}{|\Gamma_k|^\mu} \\
&\leq \left(\frac{1}{L_1}\right)^\mu \sum_{l>l_0} \sum_{\{k \in j_l : \Gamma_k^+ \subset \Gamma_{k_0}^+\}} |\Gamma_k| \\
&\leq \tilde{C} 2^n |\Gamma_{k_0}|^{1-\mu},
\end{aligned}$$

where $\tilde{C} = \left(\frac{L_{k_0}}{L_1}\right)^\mu$.

We have

$$\begin{aligned}
\sum_{l>l_0} \int_{\rho_l \cap \Gamma_{k_0}^+} f &= \sum_{l>l_0} \sum_{\{k \in j_l : \Gamma_k^+ \subset \Gamma_{k_0}^+\}} \int_{\Gamma_k^+} f \\
&\leq 2\lambda \sum_{l>l_0} \sum_{\{k \in j_l : \Gamma_k^+ \subset \Gamma_{k_0}^+\}} |\Gamma_k|^{1-\mu} \\
&\leq 2\lambda \tilde{C} 2^n |\Gamma_{k_0}|^{1-\mu} \\
&< \tilde{C} 2^{n+1} \int_{\Gamma_{k_0}^+} f.
\end{aligned}$$

Thus

$$\sum_{l=l_0+1}^{l_0+2^{n+1}} \int_{\rho_l \cap \Gamma_{k_0}^+} f < \tilde{C} 2^{n+1} \int_{\Gamma_{k_0}^+} f.$$

From this relation we get an integer $l, l_0 + 1 \leq l \leq l_0 + 2^{n+2}$, such that

$$\int_{\rho_l \cap \Gamma_{k_0}^+} f < \frac{\tilde{C}}{2} \int_{\Gamma_{k_0}^+} f.$$

Since $\rho_{l+1} \subset \rho_l$, thus

$$\int_{\rho_{l+1}\cap\Gamma_{k_0}^+} f \leq \int_{\rho_l\cap\Gamma_{k_0}^+} f.$$

For $k_0 \in j_{l_0}$, we have

$$\begin{aligned} &\frac{2}{\tilde{C}} \int_{\Gamma_{k_0}^+\cap\rho_{l_0+2^{n+2}}} f < \int_{\Gamma_{k_0}^+} f \\ \Longrightarrow\ &\frac{2}{\tilde{C}} \int_{\Gamma_{k_0}^+\setminus\rho_{l_0+2^{n+2}}} > \int_{\Gamma_{k_0}^+} f. \end{aligned}$$

Thus from the last inequality we obtain

$$\lambda < \frac{2}{\tilde{C}|\Gamma_{k_0}|^{1-\mu}} \int_{\Gamma_{k_0}^+\setminus\rho_{l_0+2^{n+2}}} f. \tag{4.1}$$

Using the assumption $(\sigma, \phi) \in \mathcal{A}_{p,q}^{+,d}$, estimate (4.1) and Hölder inequality we obtain

$$\begin{aligned} \sum_{k\in\mathcal{F}_\lambda} \sigma(\Gamma_k) &= \sum_{l=0}^{\infty}\sum_{k\in j_l} \sigma(\Gamma_k) \\ &\leq \left(\frac{2}{\tilde{C}\lambda}\right)^q \sum_{l=0}^{\infty}\sum_{k\in j_l} \left(\int_{\Gamma_k}\sigma\right)\left(\frac{1}{|\Gamma_k|^{1-\mu}}\int_{\Gamma_k^+\setminus\rho_{l+2^{n+2}}} f\right)^q \\ &= \left(\frac{2}{\tilde{C}\lambda}\right)^q \sum_{l=0}^{\infty}\sum_{k\in j_l} \left(\int_{\Gamma_k}\sigma\right)\left(\frac{1}{|\Gamma_k|^{1-\mu}}\int_{\Gamma_k^+\setminus\rho_{l+2^{n+2}}} f\phi^{\frac{1}{p}}\phi^{-\frac{1}{p}}\right)^q \\ &\leq \left(\frac{2}{\tilde{C}\lambda}\right)^q \sum_{l=0}^{\infty}\sum_{k\in j_l} \left(\frac{1}{|\Gamma_k|^{1-\mu}}\right)^q \left(\int_{\Gamma_k}\sigma\right)\left(\int_{\Gamma_k^+\setminus\rho_{l+2^{n+2}}} \phi^{-\frac{1}{p-1}}\right)^{\frac{q(p-1)}{p}} \left(\int_{\Gamma_k^+\setminus\rho_{l+2^{n+2}}} f^p\phi\right)^{\frac{q}{p}} \\ &\leq \left(\frac{2}{\tilde{C}\lambda}\right)^q \sum_{l=0}^{\infty}\sum_{k\in j_l} \left(\frac{1}{|\Gamma_k|^{1-\mu}}\right)^q \left(\int_{\Gamma_k}\sigma\right)\left(\int_{\Gamma_k^+} \phi^{-\frac{1}{p-1}}\right)^{\frac{q(p-1)}{p}} \left(\int_{\Gamma_k^+\setminus\rho_{l+2^{n+2}}} f^p\phi\right)^{\frac{q}{p}} \\ &\leq C\left(\frac{2}{\lambda}\right)^q \sum_{l=0}^{\infty} \left(\int_{\rho_l\setminus\rho_{l+2^{n+2}}} f^p\phi\right)^{\frac{q}{p}}, \quad [\text{ for some positive constant } C] \\ &= C\left(\frac{2}{\lambda}\right)^q \sum_{m=0}^{2^{n+2}-1}\sum_{l=0}^{\infty} \left(\int_{\rho_{m+l2^{n+2}}\setminus\rho_{m+(l+1)2^{n+2}}} f^p\phi\right)^{\frac{q}{p}} \\ &= C\left(\frac{2}{\lambda}\right)^q \sum_{m=0}^{2^{n+2}-1} \left(\int_{\rho_m} f^p\phi\right)^{\frac{q}{p}} \\ &\leq C\left(\frac{2}{\lambda}\right)^q 2^{n+2}\left(\int_{\rho_0} f^p\phi\right)^{\frac{q}{p}}. \end{aligned}$$

Thus we obtain the result. □

Proof of Theorem 4.2.1.

Suppose the condition (a) holds, i.e. the pair $(\sigma, \phi) \in \mathcal{A}^{+,d}_{p,q}$. For $\gamma > 0$, we define a set

$$\theta_\gamma = \Big\{ t \in \mathbb{R}^n : \mathcal{M}^{+,d}_\mu(f)(t) > \gamma \Big\}.$$

Thus we obtain a family of dyadic cubes $\{\Gamma_k\}_k$ having disjoint property and satisfies

$$\theta_\gamma = \cup_k \Gamma_k$$

and for each integer k, we have $f_{k,\mu} > \gamma$.

Further, we construct a family of indices γ_m for each fixed non-negative integer m as

$$\gamma_m = \Big\{ k : 2^m\gamma < f_{\Gamma_k^+} \leq 2^{m+1}\gamma \Big\}.$$

Now we use Lemma 4.2.2 for the collection $\{\Gamma_k\}_{k \in \gamma_m}$ by taking $\lambda = 2^m\gamma$ and we obtain

$$\sum_{k \in \gamma_m} \sigma(\Gamma_k) \leq C \left(\frac{2}{2^m\gamma} \right)^q 2^{n+2} \left(\int_{\mathbb{R}^n} f^p \phi \right)^{\frac{q}{p}}.$$

Hence

$$\begin{aligned}
\sigma(\theta_\gamma) &= \sum_k \sigma(\Gamma_k) \\
&= \sum_{m=0}^{\infty} \sum_{k \in \gamma_m} \sigma(\Gamma_k) \\
&\leq C \sum_{m=0}^{\infty} \left(\frac{2}{2^m\gamma} \right)^q 2^{n+2} \left(\int_{\mathbb{R}^n} f^p \phi \right)^{\frac{q}{p}} \\
&\leq C \left(\frac{4}{\gamma} \right)^q 2^{n+2} \left(\int_{\mathbb{R}^n} f^p \phi \right)^{\frac{q}{p}}.
\end{aligned}$$

We now prove the remaining part $\Big((b) \implies (a) \Big)$ of the theorem using the method of [23, pp. 390]. For a fixed dyadic cube Γ and for a non-negative function f, we have

$$f_{\Gamma^+} \leq \mathcal{M}^{+,d}_\mu(f\chi_{\Gamma^+})(x), \quad \text{for each } x \in \Gamma$$

and suppose that

$$f_{\Gamma^+} = \frac{1}{|\Gamma|^{1-\mu}} \int_{\Gamma^+} f(t)dt > 0.$$

Thus for each γ with $0 < \gamma < f_{\Gamma^+}$, we have

$$\Gamma \subset \Omega_\gamma = \{x \in \mathbb{R}^n : \mathcal{M}_\mu^{+,d}(f\chi_{\Gamma^+})(x) > \gamma\}. \tag{4.2}$$

From the weak (p,q) estimate of $\mathcal{M}_\mu^{+,d}$ and applying (4.2), we obtain

$$\sigma(\Gamma) \leq \frac{C}{\gamma^q}\Big(\int_{\Gamma^+} f(t)^p \phi(t)dt\Big)^{\frac{q}{p}}$$

for each γ with $0 < \gamma < f_{\Gamma^+}$. Thus

$$\Big(f_{\Gamma^+}\Big)^q \sigma(\Gamma) \leq C\Big(\int_{\Gamma^+} f(t)^p \phi(t)dt\Big)^{\frac{q}{p}}. \tag{4.3}$$

For $m \in \mathbb{N}$, we define $\{\mathcal{B}_m\}_{m\geq 1}$ as $\mathcal{B}_m = \{x \in \Gamma^+ : \phi(x) > \frac{1}{m}\}$. Putting $f = \phi^{-\frac{1}{p-1}}\chi_{\mathcal{B}_m}$ in the estimate (4.3), we get

$$\Big(\frac{1}{|\Gamma|^{1-\mu}}\int_{\mathcal{B}_m} \phi(t)^{-\frac{1}{p-1}}dt\Big)^q \sigma(\Gamma) \leq C\Big(\int_{\mathcal{B}_m} \phi(t)^{-\frac{1}{p-1}}dt\Big)^{\frac{q}{p}}. \tag{4.4}$$

Rewriting the inequality (4.4), we obtain

$$\Big(\frac{1}{|\Gamma|}\int_{\Gamma} \sigma(t)dt\Big)^{\frac{1}{q}}\Big(\frac{1}{|\Gamma|}\int_{\mathcal{B}_m} \phi(t)^{-\frac{1}{p-1}}dt\Big)^{\frac{1}{p'}} \leq C. \tag{4.5}$$

It is possible to write in the above form due to the assumption that $\mu = \frac{1}{p} - \frac{1}{q}$. As the sequence $\{\mathcal{B}_m\}$ is monotonically increasing, thus passing the limit as $m \to \infty$ in (4.5), we get

$$\Big(\frac{1}{|\Gamma|}\int_{\Gamma} \sigma(t)dt\Big)^{\frac{1}{q}}\Big(\frac{1}{|\Gamma|}\int_{\Gamma^+} \phi(t)^{-\frac{1}{p-1}}dt\Big)^{\frac{p-1}{p}} \leq C. \tag{4.6}$$

Hence the proof is complete.

4.3 Strong type results

Next we prove the one and two weighted strong type (p,q) inequalities. We first state the two weighted result with the help of testing type condition $\mathcal{S}_{p,q}^{+,d}$.

Theorem 4.3.1. *Let σ and ϕ be two weights on $\mathbb{R}^n$. Then we obtain the following equivalent conditions.*

(a) $(\sigma,\phi) \in \mathcal{S}_{p,q}^{+,d}$.

(b) *For $q \geq p$ with $p, q \in (1, \infty)$ the following strong (p, q) estimate*

$$\Big(\int_{\mathbb{R}^n} \Big(\mathcal{M}_\mu^{+,d}(f)\Big)^q \sigma dt\Big)^{\frac{1}{q}} \leq C\Big(\int_{\mathbb{R}^n} f^p \phi dt\Big)^{\frac{1}{p}} \tag{4.7}$$

holds for a suitable constant $C > 0$.

Proof. First assume the condition (a) holds. Let us consider the set

$$\Omega_m = \{t \in \mathbb{R}^n : \mathcal{M}_\mu^{+,d}(f)(t) > 2^m, m \in \mathbb{Z}\}.$$

Thus for each $t \in \Omega_m$, there exists a dyadic cube $\Gamma_{t,m}$ such that $|\Gamma_{t,m}^+|^{1-\mu} 2^m < \int_{\Gamma_{t,m}^+} f$. We choose a maximal disjoint collection $\{\Gamma_{l,m}\}_{l\geq 0}$ such that

(i) $\Omega_m = \cup_{l\geq 0}\Gamma_{l,m}$,

(ii) $2^m|\Gamma_{l,m}^+|^{1-\mu} < \int_{\Gamma_{l,m}^+} f$.

Next we define

$$\mathcal{O}_{l,m} = \{t \in \Gamma_{l,m} : 2^m < \mathcal{M}_\mu^{+,d}(f)(t) \leq 2^{m+1}\}.$$

Now,

$$\begin{aligned}
\int_{\mathbb{R}^n} \big(\mathcal{M}_\mu^{+,d}(f)\big)^q \sigma &\leq \sum_m \int_{\Omega_m} \big(\mathcal{M}_\mu^{+,d}(f)\big)^q \sigma \\
&\leq \sum_{l,m} 2^{q(m+1)} \sigma(\mathcal{O}_{l,m}) \\
&\leq 2^q \sum_{l,m} \sigma(\mathcal{O}_{l,m}) \Big(\frac{1}{|\Gamma_{l,m}|^{1-\mu}} \int_{\Gamma_{l,m}^+} f\Big)^q \\
&= 2^q \sum_{l,m} \sigma(\mathcal{O}_{l,m}) \Big(\frac{1}{\psi(\Gamma_{l,m}^+)} \int_{\Gamma_{l,m}^+} f\Big)^q \Big(\frac{1}{|\Gamma_{l,m}|^{1-\mu}} \int_{\Gamma_{l,m}^+} \psi\Big)^q \\
&= 2^q \sum_{l,m} \zeta_{l,m} \mathcal{G}(f\psi^{-1})^q,
\end{aligned} \tag{4.8}$$

where $\zeta_{l,m}$ and the operator $\mathcal{G}$ are defined by

$$\zeta_{l,m} = \sigma(\mathcal{O}_{l,m}) \Big(\frac{1}{|\Gamma_{l,m}|^{1-\mu}} \int_{\Gamma_{l,m}^+} \psi\Big)^q$$

and

$$\mathcal{G}(h) = \frac{1}{\psi(\Gamma_{l,m}^+)} \int_{\Gamma_{l,m}^+} h\psi.$$

First we prove that $\mathcal{G}$ satisfies the weak $(1, \frac{q}{p})$ estimate with respect to the measures $\zeta_{l,m}$ and ψ defined respectively in $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{R}^n$. From the collection of dyadic cubes $\{\Gamma^+_{l,m}\}$, let us extract a maximal collection $\{\Gamma^+_k\}$ with respect to the property $\mathcal{G}(h) > \gamma$. For $t \in \mathcal{O}_{l,m}$,

$$\frac{1}{|\Gamma^+_{l,m}|^{1-\mu}} \int_{\Gamma^+_{l,m}} \psi \leq \mathcal{M}^{+,d}_{\mu}(\psi \chi_{\Gamma^+_{l,m}})(t).$$

Now, we have

$$\begin{aligned}\sum_{l,m} \{\zeta_{l,m}; \mathcal{G}(h) > \gamma\} &\leq \sum_{l,m} \int_{\mathcal{O}_{l,m}} \sigma\big(\mathcal{M}^{+,d}_{\mu}(\psi \chi_{\Gamma^+_{l,m}})\big)^q \\ &\leq \sum_{k} \sum_{\Gamma^+_{l,m} \subset \Gamma^+_k} \int_{\mathcal{O}_{l,m}} \sigma\big(\mathcal{M}^{+,d}_{\mu}(\psi \chi_{\Gamma^+_{l,m}})\big)^q. \end{aligned} \tag{4.9}$$

Again from the relation $\Gamma^+_{l,m} \subset \Gamma^+_k$, we observe that $\mathcal{O}_{l,m} \subset \Gamma_k \cup \Gamma^+_k$. Thus

$$\begin{aligned}\sum_{l,m} \{\zeta_{l,m}; \mathcal{G}(h) > \gamma\} &\leq \sum_{k} \int_{\Gamma_k \cup \Gamma^+_k} \sigma\big(\mathcal{M}^{+,d}_{\mu}(\psi \chi_{\Gamma^+_{l,m}})\big)^q \\ &\leq C \sum_{k} \Big(\int_{\Gamma^+_k} \psi\Big)^{\frac{q}{p}} \\ &\leq C \Big(\sum_{k} \int_{\Gamma^+_k} \psi\Big)^{\frac{q}{p}} \\ &\leq \Big(\frac{1}{\gamma} \int |h| \psi\Big). \end{aligned} \tag{4.10}$$

Thus $\mathcal{G}$ satisfies strong type (p, q) with respect to $\zeta_{l,m}$ and ψ. Hence from (4.8), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \big(\mathcal{M}^{+,d}_{\mu}(f)\big)^q \sigma dt &\leq C\Big(\int_{\mathbb{R}^n} f^p \psi^{1-p} dt\Big)^{\frac{q}{p}}, \quad \text{for some constant } C > 0 \\ &= C\Big(\int_{\mathbb{R}^n} f^p \phi dt\Big)^{\frac{q}{p}}, \text{ as } \psi = \phi^{-1/p-1}. \end{aligned}$$

$(b) \implies (a)$. Suppose the condition (b) holds. If $\int_{\Gamma \cup \Gamma^+} \psi = \infty$, then we choose f suitably such that $\int_{\mathbb{R}^n} f^p \phi < \infty$ and $\int_{\Gamma \cup \Gamma^+} f = \infty$. Thus for $t \in \Gamma^- \cup (\Gamma^-)^-$ we have $\mathcal{M}^{+,d}_{\mu}(f)(t) = \infty$, which contradicts (b). For the second part we consider $f = \psi_{\chi_{\Gamma^+}}$. □

The next result is about the equivalence relation between the conditions $\mathcal{S}^{+,d}_{p,q}$ and $\mathcal{A}^{+,d}_{p,q}$ in the case of equal weight and thus we obtain the one weighted strong type result using $\mathcal{A}^{+,d}_{p,q}$ condition.

Theorem 4.3.2. *Suppose $q \geq p$ with $p, q \in (1, \infty)$ and we consider $\mu = \frac{1}{p} - \frac{1}{q}$. Suppose σ be a weight on $\mathbb{R}^n$ and we define $\xi = \sigma^{-p'}$. Then we obtain the following equivalent statements.*

(a) The pair of weight $(\sigma^q, \sigma^p) \in \mathcal{A}^{+,d}_{p,q}$.

(b) $\mathcal{M}^{+,d}_{\mu}$ *satisfies the following strong* (p, q) *estimate*

$$\Big(\int\limits_{\mathbb{R}^n} \Big(\mathcal{M}^{+,d}_{\mu}(f)(t)\sigma(t)\Big)^q dt\Big)^{\frac{1}{q}} \leq C\Big(\int_{\mathbb{R}^n} \big(f(t)\sigma(t)\big)^p dt\Big)^{\frac{1}{p}} \tag{4.11}$$

for a constant $C > 0$.

Proof. $(a) \implies (b)$. We first assume that the pair of weight (σ^q, σ^p) satisfies $\mathcal{A}^{+,d}_{p,q}$ condition. Because of Theorem 4.3.1, it is sufficient to prove that the pair $(\sigma^q, \sigma^p) \in \mathcal{S}^{+,d}_{p,q}$.

Let us denote $\xi = \sigma^{-p'}$. Now, if $\int_{\Gamma\cup(\Gamma^-)^-} \sigma^q > 0$ then from the assumption (a), we obtain that $\int_{\Gamma\cup\Gamma^+} \xi < \infty$. For $t \in \Gamma$, we have

$$\mathcal{M}^{+,d}_{\mu}(\xi\chi_{\Gamma^+})(t) = \sup_{t\in R} \frac{1}{|R|^{1-\mu}} \int_{R^+} \xi\chi_{\Gamma^+}.$$

As the two cubes have to satisfy $R^+ \cap \Gamma^+ \neq \phi$, thus we have $R = \Gamma$. Thus we obtain

$$\mathcal{M}^{+,d}_{\mu}(\xi\chi_{\Gamma^+})(t) = \frac{1}{|\Gamma|^{1-\mu}} \int_{\Gamma^+} \xi.$$

Thus

$$\begin{aligned}
\int\limits_{\Gamma} \mathcal{M}^{+,d}_{\mu}(\xi\chi_{\Gamma^+})^q\sigma^q &\leq \Big(\int_{\Gamma} \sigma^q\Big)\Big(\frac{1}{|\Gamma|^{1-\mu}} \int_{\Gamma^+} \xi\Big)^q \\
&\leq C|\Gamma|^{q(1-\mu)}\Big(\int_{\Gamma^+} \xi\Big)^{-\frac{q}{p'}}\Big(\int_{\Gamma^+} \xi\Big)^q \frac{1}{|\Gamma|^{q(1-\mu)}} \\
&= C\Big(\int_{\Gamma^+} \xi\Big)^{\frac{q}{p}}.
\end{aligned} \tag{4.12}$$

If $t \in \Gamma^+$, then

$$\mathcal{M}^{+,d}_{\mu}(\xi\chi_{\Gamma^+})(t) = \sup_{t\in R} \frac{1}{|R|^{1-\mu}} \int_{R^+} \xi\chi_{\Gamma^+}.$$

Again we have $R \cap \Gamma^+ \neq \phi$ and $R^+ \cap \Gamma^+ \neq \phi$. This implies $R \cup R^+ \subset \Gamma^+$. Thus

$$\frac{1}{|R|^{1-\mu}} \int_{R^+} \xi\chi_{\Gamma^+} = \frac{1}{|R|^{1-\mu}} \int_{R^+} \xi.$$

Now using the condition $(\sigma^q, \sigma^p) \in \mathcal{A}^{+,d}_{p,q}$, we have

$$\Big(\frac{1}{|R|^{1-\mu}} \int_{R^+} \xi\Big)^{\frac{1}{p'}} \leq C|R|^{(1-\mu)(1-\frac{1}{p'})}\Big(\int_{R} \sigma^q\Big)^{-\frac{1}{q}}. \tag{4.13}$$

Thus we get

$$\Big(\frac{1}{|R|^{1-\mu}} \int_{R^+} \xi\Big)^q \leq C|R|^{qp'(1-\mu)(1-\frac{1}{p'})}\Big(\int_{R} \sigma^q\Big)^{-p'}$$

$$
\begin{aligned}
&= C|R|^{p'+\mu q}\Big(\int_R \sigma^q\Big)^{-p'}\\
&= C\left\{\frac{1}{\big(\int_R \sigma^q\big)^{\frac{p'}{p'+\mu q}}}\int_R \sigma^{-q}\chi_{\Gamma^+}\sigma^q\right\}^{p'+\mu q}. \qquad (4.14)
\end{aligned}
$$

Thus from (4.14), we obtain

$$
\mathcal{M}_\mu^{+,d}(\xi\chi_{\Gamma^+})(t)^q\sigma^q(t) \le C M_{\eta,\sigma^q}(\sigma^{-q}\chi_{\Gamma^+})(t)^r\sigma^q(t), \qquad (4.15)
$$

where $r = p' + \mu q, \eta = 1 - \frac{p'}{p'+\mu q}$ and

$$
M_{\eta,\sigma^q}(f)(t) = \sup_{t\in\Gamma}\frac{1}{\big(\int_\Gamma \sigma^q\big)^{1-\eta}}\int_\Gamma |f|\sigma^q dy.
$$

Integrating (4.15) over Γ^+ and using the boundedness of M_{η,σ^q} from $L^s(\sigma^q)$ to $L^r(\sigma^q)$ with $\eta = \frac{1}{s} - \frac{1}{r}$, we obtain

$$
\begin{aligned}
\int_{\Gamma^+}\mathcal{M}_\mu^{+,d}(\xi\chi_{\Gamma^+})(t)^q\sigma^q(t)dt &\le C\int_{\Gamma^+} M_{\eta,\sigma^q}(\sigma^{-q}\chi_{\Gamma^+})(t)^r\sigma^q(t)dt\\
&\le C\left(\int_{\Gamma^+}\sigma^{-qs}(t)\sigma^q(t)dt\right)^{\frac{r}{s}}\\
&= C\left(\int_{\Gamma^+}\sigma^{q(1-s)}(t)dt\right)^{\frac{q}{p}}\\
&= C\left(\int_{\Gamma^+}\xi(t)dt\right)^{\frac{q}{p}}. \qquad (4.16)
\end{aligned}
$$

Thus from (4.12) and (4.16), we obtain

$$
\int_{\Gamma\cup\Gamma^+}\mathcal{M}_\mu^{+,d}(\xi\chi_{\Gamma^+})(t)^q\sigma^q(t)dt \le C\left(\int_{\Gamma^+}\xi(t)dt\right)^{\frac{q}{p}}.
$$

This implies that the pair $(\sigma^q,\sigma^p)\in\mathcal{S}_{p,q}^{+,d}$ and hence using Theorem 4.3.1, we obtain (b).

Conversely, suppose the condition (b) holds. By Theorem 4.3.1, the pair $(\sigma^q,\sigma^p)\in\mathcal{S}_{p,q}^{+,d}$, thus

$$
\int_\Gamma \mathcal{M}_\mu^{+,d}(\xi\chi_{\Gamma^+})^q\sigma^q \le C\Big(\int_{\Gamma^+}\xi\Big)^{\frac{q}{p}}. \qquad (4.17)
$$

If $t\in\Gamma$, then

$$
\mathcal{M}_\mu^{+,d}(\xi\chi_{\Gamma^+})(t) = \frac{1}{|\Gamma|^{1-\mu}}\int_{\Gamma^+}\xi. \qquad (4.18)
$$

Thus combining the inequalities (4.17) and (4.18), we get

$$\Big(\int_{\Gamma}\sigma^q\Big)\Big(\frac{1}{|\Gamma|^{1-\mu}}\int_{\Gamma^+}\xi\Big)^q \le C\Big(\int_{\Gamma^+}\xi\Big)^{\frac{q}{p}}.$$

That is

$$\Big(\int_{\Gamma}\sigma^q\Big)^{\frac{1}{q}}\Big(\int_{\Gamma^+}\xi\Big)^{\frac{1}{p'}} \le C|\Gamma|^{1-\mu}.$$

This implies $(\sigma^q, \sigma^p) \in \mathcal{A}^{+,d}_{p,q}$. Hence the proof is complete. $\square$