CHAPTER 5

On weighted estimates of mixed type for the modified integral Hardy operators

5.1 Introduction

We present weighted integral inequalities of mixed type for the modified integral Hardy operators. We provide a suitable relation on the weights ω, ρ, ϕ and ψ to hold the following weak type modular estimate.

$$\mathcal{U}^{-1}\Big(\int_{\{|\mathcal{I}f|>\gamma\}} \mathcal{U}\Big(\gamma\omega\Big)\rho\Big) \leq \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(C|f|\phi\Big)\psi\Big),$$

where $\mathcal{I}$ is the modified integral Hardy operators defined below. We also establish the following extra-weak type integral inequality.

$$\omega\Big(\Big\{|\mathcal{I}f| > \gamma\Big\}\Big) \leq \mathcal{U}\circ\mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(\frac{C|f|\phi}{\gamma}\Big)\psi\Big).$$

Further, we discuss the above two inequalities for the conjugate of the modified integral Hardy operators. It will extend the existing results for the Hardy operators and its integral version.

We define the modified integral Hardy operators, $\mathcal{I}$ for a measurable function, $f \geq 0$ as

$$\mathcal{I}f(t) = h(t)\int_0^t K(t,z)f(z)w(z)dz, \tag{5.1}$$

where both the functions h and w are measurable and take positive values on $\mathbb{R}$. Let us assume the following conditions for the kernel K defined on $\{(t,z) : 0 \leq z \leq t\}$.

This chapter is based on *Weighted integral inequalities for modified integral Hardy operators* by D. Chutia and R. Haloi [14], to appear in *Bulletin of the Korean Mathematical Society.*

(i) $K(t,z)$ is non-negative.

(ii) $K(t_1,z) \leq K(t_2,z)$ for $0 \leq z \leq t_1 \leq t_2$.

(iii) $K(t,z_1) \leq K(t,z_2)$ for $0 \leq z_2 \leq z_1 \leq t$.

(iv) For $0 \leq x \leq z \leq t$, we have

$$K(t,x) \leq M[K(t,z) + K(z,x)] \text{ for some } M > 1. \tag{5.2}$$

For $K = 1$, the integral operator (5.1) is reduced to the modified Hardy operators of the follwoing form,

$$\mathcal{H}_h f(t) = h(t) \int_0^t f(z) w(z) dz. \tag{5.3}$$

Martín-Reyes and Salvador [43] characterized the pair of weights (ρ, ψ) for the following estimate.

$$\left(\int_{\{t \in (0,\infty): \mathcal{H}_h f(t) > \gamma\}} \gamma^q \rho(t) dt \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f(t)^p \psi(t) dt \right)^{\frac{1}{p}}, \quad 1 \leq p \leq q < \infty. \tag{5.4}$$

We study the inequality (5.4) in the Orlicz space setting for the operator (5.3) and its conjugate $\tilde{\mathcal{H}}_h$, defined by

$$\tilde{\mathcal{H}}_h f(t) = w(t) \int_t^\infty f(z) h(z) dz. \tag{5.5}$$

Among the various equivalent generalization of the estimate (5.4) we will consider the following integral inequality

$$\mathcal{U}^{-1}\Bigg(\int\limits_{\{t \in (0,\infty): \mathcal{H}_h f(t) > \gamma\}} \mathcal{U}\Big(\gamma \omega(t)\Big) \rho(t) dt \Bigg) \leq \mathcal{V}^{-1}\Bigg(\int_0^\infty \mathcal{V}\Big(C f(t) \phi(t)\Big) \psi(t) dt \Bigg), \tag{5.6}$$

where $\mathcal{U}$ and $\mathcal{V}$ are N-functions and ω, ρ, ϕ and ψ are positive weights on $\mathbb{R}$. The techniques of Bloom and Kerman [7], used for the Hardy operator do not work for the modified Hardy operator as it does not possess monotone property for non-increasing h. Using the methods of Salvador [50], we establish the estimate (5.6) for the integral operator (5.1) and its conjugate operator $\tilde{\mathcal{I}}$, defined by

$$\tilde{\mathcal{I}} f(t) = w(t) \int_t^\infty K(z,t) f(z) h(z) dz. \tag{5.7}$$

Also, our plan is to discuss a weaker version of (5.6), that is of the form

$$\omega\Big(\{t \in (0,\infty) : \mathcal{H}_h f(t) > \gamma\} \Big) \leq \mathcal{U} \circ \mathcal{V}^{-1}\Bigg(\int_0^\infty \mathcal{V}\Big(\frac{C f(t) \phi(t)}{\gamma} \Big) \psi(t) dt \Bigg). \tag{5.8}$$

We also discuss (5.8) for the operators (5.1), (5.5) and (5.7). The inequality (5.8) is known as the extra weak type mixed integral inequality as it follows from (5.6) but not contrariwise.

In Section 5.2, we discuss some basic definitions and results associated with N-functions. The weak and extra weak type results are proved in Section 5.3 and Section 5.4 respectively.

5.2 Preliminaries

In the context of our discussion, let us now briefly state the definition of N-function and some elementary properties shared by it [31]. A convex continuous function $\mathcal{U}$ defined on $[0,\infty)$ is said to be an N-function provided $\mathcal{U}(0)=0$ and $\frac{\mathcal{U}(t)}{t}\to 0$ (and ∞) when $t\to 0$ (and ∞). It is always possible to write an N- function $\mathcal{U}$ in the integral form as, $\mathcal{U}(t)=\int_0^t u(y)dy$, where $u(>0)$ is increasing and right continuous at each point and satisfies $u(0)=0$, $u(s)>0$ for $s>0$ and $u(s)$ increases to infinity as s increases to infinity. The complementary function $\tilde{\mathcal{U}}$ corresponding to a given N-function $\mathcal{U}$ is defined by $\tilde{\mathcal{U}}(t)=\sup_{\tau\geq 0}(t\tau-\mathcal{U}(\tau))$ also verifies properties of N-functions. For $\alpha,\beta>0$, the pair $(\mathcal{U},\tilde{\mathcal{U}})$ satisfies the following relations [7].

$$\alpha\beta\leq\mathcal{U}(\alpha)+\tilde{\mathcal{U}}(\beta), \tag{5.9}$$

$$\mathcal{U}\Big(\frac{\tilde{\mathcal{U}}(\alpha)}{\alpha}\Big)\leq\tilde{\mathcal{U}}(\alpha), \tag{5.10}$$

$$\mathcal{U}(\alpha)\leq\alpha u(\alpha)\leq\mathcal{U}(2\alpha). \tag{5.11}$$

Next, we state a result, which was used by Martín-Reyes and Salvador [43]. Let the function h be monotone on $\mathbb{R}$, then

$$\inf_{x\in\Omega}h(x)=\inf_{x\in(a,b)}h(x), \tag{5.12}$$

where Ω is a bounded set in $\mathbb{R}$ and $a=\inf\Omega, b=\sup\Omega$. In general, the equality (5.12) is not true. We assume that the function $h(\cdot)K(\cdot,y)$ satisfies that

$$\inf_{x\in\Omega}h(x)K(x,y)=\inf_{x\in(\inf\Omega,\sup\Omega)}h(x)K(x,y) \tag{5.13}$$

and the function $w(\cdot)K(y,\cdot)$ satisfies that

$$\inf_{x\in\Omega}w(x)K(y,x)=\inf_{x\in(\inf\Omega,\sup\Omega)}w(x)K(y,x) \tag{5.14}$$

for all bounded set Ω and all y.

5.3 Weak type results

We now state the results of weak type for the operators (5.1), (5.3), (5.5) and (5.7).

5.3.1 Statement of the results

First we state the result for the modified integral Hardy operators, $\mathcal{I}$.

Theorem 5.3.1. *Let $\tilde{\mathcal{V}}$ stands for the complementary N- function of an N- function $\mathcal{V}$. We assume that the function $\mathcal{U}$ is positive and strictly increasing such that $\mathcal{V}\circ\mathcal{U}^{-1}$ is countably sub-additive. Let the function h be monotone on $\mathbb{R}$ and the function $h(\cdot)K(\cdot,y)$ satisfies the condition (5.13). Suppose that the weights ω,ρ,ϕ and ψ are positive and locally integrable on $(0,\infty)$. Then we have the following equivalent results.*

(a) Suppose that $\gamma>0$ and $C>0$ be a constant such that

$$\mathcal{U}^{-1}\Big(\int_{\{t\in(0,\infty):\mathcal{I}f(t)>\gamma\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\leq\mathcal{V}^{-1}\Big(\int_0^\infty\mathcal{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\Big) \quad (5.15)$$

holds for all non-negative functions f.

(b) For $0<t<\alpha$ and $\gamma>0$, we have

$$\int_0^t\tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\alpha)}h)K(t,y)w(y)\eta(\gamma;t,\alpha)}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy\leq\eta(\gamma;t,\alpha) \quad (5.16)$$

for some constant $C>0$. Also for $0<\xi\leq t<\alpha$, we have

$$\int_0^\xi\tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(t,\alpha)}h(y)K(y,\xi))w(\tau)\eta(\gamma;t,\alpha)}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau\leq\eta(\gamma;t,\alpha) \quad (5.17)$$

for some constant $C>0$, where

$$\eta(\gamma;t,\alpha)=\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\alpha\mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

If we consider $K\equiv 1$, then the conditions (5.16) and (5.17) are the same and reduced to the following.

$$\int_0^t\tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\alpha)}h)w(y)\eta(\gamma;t,\alpha)}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy\leq\eta(\gamma;t,\alpha).$$

We now state the weak type estimate for the modified Hardy operator.

Corollary 5.3.2. *Let $\mathcal{U},\mathcal{V},\tilde{\mathcal{V}},\mathcal{V}\circ\mathcal{U}^{-1}$ and h satisfy all the assumptions stated in Theorem 5.3.1. Then we have the following equivalent statements.*

(a) If $\gamma>0$, then

$$\mathcal{U}^{-1}\Big(\int_{\{t\in(0,\infty):\mathcal{H}_hf(t)>\gamma\}}\mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\leq\mathcal{V}^{-1}\Big(\int_0^\infty\mathcal{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\Big) \quad (5.18)$$

holds for all non-negative functions f and for some constant $C>0$.

(b) *For each* $0 < t < \alpha$ *and* $\gamma > 0$ *the following inequality*

$$\int_0^t \tilde{\mathcal{V}}\left[\frac{(\inf h)w(y)\eta(\gamma;t,\alpha)}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy \leq \eta(\gamma;t,\alpha) \tag{5.19}$$

holds for some constant $C > 0$, *where*

$$\eta(\gamma;t,\alpha) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big).$$

Next, we state the weak type result for the conjugate of the modified integral operator.

Theorem 5.3.3. *Let* $\mathcal{U},\mathcal{V},\tilde{\mathcal{V}}$ *and* $\mathcal{V}\circ\mathcal{U}^{-1}$ *satisfy all the assumptions stated in Theorem 5.3.1. Suppose that the function* w *be monotone on* $\mathbb{R}$ *and the function* $w(\cdot)K(y,\cdot)$ *satisfies the condition (5.14). Then we obtain the following equivalent conditions.*

(a) *Suppose that* $\gamma > 0$ *and* $C > 0$ *be a constant such that*

$$\mathcal{U}^{-1}\Big(\int_{\{t\in(0,\infty):\tilde{\mathcal{I}}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) \leq \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\Big) \tag{5.20}$$

holds for all non-negative functions f.

(b) *Let* $0 \leq \epsilon < t < \alpha$ *and* $\gamma > 0$ *such that*

$$\int_t^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{(\epsilon,t)} w)K(y,t)h(y)\eta(\gamma;\epsilon,t)}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy \leq \eta(\gamma;\epsilon,t) \tag{5.21}$$

and for $0 \leq \epsilon < t \leq \xi < \alpha$ *the following inequality*

$$\int_\xi^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(\epsilon,t)} w(y)K(\xi,y))h(\tau)\eta(\gamma;\epsilon,t)}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \leq \eta(\gamma;\epsilon,t) \tag{5.22}$$

hold for some $C > 0$, *where*

$$\eta(\gamma;\epsilon,t) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

Again the result for the conjugate of the modified Hardy operators can be obtained from the Theorem 5.3.3 by considering $K \equiv 1$.

Corollary 5.3.4. *Let* $\mathcal{U},\mathcal{V},\tilde{\mathcal{V}}$ *and* $\mathcal{V}\circ\mathcal{U}^{-1}$ *satisfy all the assumptions stated in Theorem 5.3.1. Let the function* w *be monotone in* $\mathbb{R}$. *Then we have the following equivalent conditions.*

(a) *Suppose that* $\gamma > 0$ *and* $C > 0$ *be a constant such that*

$$\mathcal{U}^{-1}\Big(\int\limits_{\{t\in(0,\infty):\tilde{\mathcal{H}}_h f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) \leq \mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(Cf(y)\phi(y)\Big)\psi(y)dy\Big) \quad (5.23)$$

holds for all non-negative functions f.

(b) *Let* $0 \leq \epsilon < t < \alpha$ *and* $\gamma > 0$ *such that*

$$\int_t^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{(\epsilon,t)} w)h(y)\eta(\gamma;\epsilon,t)}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy \leq \eta(\gamma;\epsilon,t) \quad (5.24)$$

holds for a suitable constant $C > 0$.

Next we state the proof of the results.

5.3.2 Proof of Theorem 5.3.1

Proof. $(b) \implies (a)$. Following the argument from [42], we assume that f is non-negative with support $(0, L)$ for $L > 0$. We define a decreasing sequence $\{\zeta_m\}_{m\geq 0}$ with the iteration $\zeta_0 = L$ and given ζ_m, we consider

$$P(\zeta_m) = \int_0^{\zeta_m} K(\zeta_m, z)f(z)w(z)dz = (M+1)^{-m}P(\zeta_0). \quad (5.25)$$

Now, using the inequality (5.2) and (5.25) we have

$$\begin{aligned}
P(\zeta_m) &= (M+1)^2 P(\zeta_{m+2}) \\
&= (M+1)^2 \int_0^{\zeta_{m+2}} K(\zeta_{m+2}, z)f(z)w(z)dz \\
&= (M+1)^2 \Big[\int_0^{\zeta_{m+3}} + \int_{\zeta_{m+3}}^{\zeta_{m+2}}\Big] K(\zeta_{m+2}, z)f(z)w(z)dz \\
&\leq (M+1)^2 \left[\int_0^{\zeta_{m+3}} M\Big\{K(\zeta_{m+2}, \zeta_{m+3}) + K(\zeta_{m+3}, z)\Big\} + \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2}, z)\right] f(z)w(z)dz \\
&\leq (M+1)^3 \left[K(\zeta_{m+2}, \zeta_{m+3}) \int_0^{\zeta_{m+3}} f(z)w(z)dz + \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2}, z)f(z)w(z)dz\right] \\
&\quad + M(M+1)^2 \int_0^{\zeta_{m+3}} K(\zeta_{m+3}, z)f(z)w(z)dz. \quad (5.26)
\end{aligned}$$

From the construction of $P(\zeta_m)$, we have

$$\int_0^{\zeta_{m+3}} K(\zeta_{m+3}, z)f(z)w(z)dz = P(\zeta_{m+3}) = (M+1)^{-(m+3)}P(\zeta_0) = (M+1)^{-3}P(\zeta_m).$$

Thus (5.26) implies that

$$P(\zeta_m) \le (M+1)^4 \left[K(\zeta_{m+2},\zeta_{m+3}) \int_0^{\zeta_{m+3}} f(z)w(z)dz + \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2},z)f(z)w(z)dz \right]. \tag{5.27}$$

For $k=1,2$, we define $\delta_{k,m} = \inf \mathcal{O}_{k,m}$ and $\beta_{k,m} = \sup \mathcal{O}_{k,m}$, where

$$\mathcal{O}_{1,m} = \left\{ x \in (\zeta_{m+1},\zeta_m) : h(x)K(\zeta_{m+2},\zeta_{m+3}) \int_0^{\zeta_{m+3}} f(z)w(z)dz > \frac{\gamma}{2(M+1)^4} \right\},$$

$$\mathcal{O}_{2,m} = \left\{ y \in (\zeta_{m+1},\zeta_m) : h(y) \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2},z)f(z)w(z)dz > \frac{\gamma}{2(M+1)^4} \right\}.$$

Thus inequalities (5.25) and (5.27) give

$$\begin{aligned}\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t:\mathcal{I}f(t)>\gamma\}} \mathcal{U}\left(\gamma\omega(y)\right)\rho(y)dy\right) \le \sum_{m\ge 0} \Bigg\{ &\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\mathcal{O}_{1,m}} \mathcal{U}\left(\gamma\omega(y)\right)\rho(y)dy\right) \\ &+ \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\mathcal{O}_{2,m}} \mathcal{U}\left(\gamma\omega(y)\right)\rho(y)dy\right)\Bigg\}.\end{aligned} \tag{5.28}$$

Next, we use the idea developed in [38, Lemma 1, pp. 654] to estimate the first sum of (5.28). For this, we define a cover sequence $\{t_k\}$ for the interval $(0,L)$ with the iteration $t_0 = L$ and

$$\int_0^{t_k} fw = 2\int_0^{t_{k+1}} fw.$$

Then $\{t_k\}$ is decreasing and satisfies

$$\int_0^{t_k} fw = 4\int_{t_{k+2}}^{t_{k+1}} fw.$$

From the sequence $\{t_k\}$, we construct a subsequence $\{t'_n\}$ with the iteration $t'_0 = t_0$ and if $t_{k+1} \le \zeta_m < t_k$, then $t'_{n+1} = t_{k+1}$, otherwise we delete the term t_{k+1} and continue the process. Thus, we get a subsequence $\{t'_n\}$ of $\{t_k\}$. Let $\tilde{\delta}_{1,n} = \inf \tilde{\mathcal{O}}_{1,n}$ and $\tilde{\beta}_{1,n} = \sup \tilde{\mathcal{O}}_{1,n}$, where $\tilde{\mathcal{O}}_{1,n} = \cup_{\{m:t'_{n+1}\le\zeta_{m+3}<t'_n\}}\mathcal{O}_{1,m}$. Now, if $t_{k+1} = t'_{n+1} \le \zeta_{m+3} < t'_n$, then by construction $\zeta_{m+3} \le t_k$ and $t'_{n+2} \le t_{k+2}$. Suppose that $\tilde{\mathcal{O}}_{1,n} \ne \phi$, then for $t \in \tilde{\mathcal{O}}_{1,n}$, we have

$$\frac{\gamma}{2(M+1)^4} < h(t)K(\zeta_{m+2},\zeta_{m+3}) \int_0^{\zeta_{m+3}} f(z)w(z)dz \le h(t)K(t,\zeta_{m+3}) \int_0^{t_k} f(z)w(z)dz. \tag{5.29}$$

As the estimate (5.29) holds for each $t \in \tilde{\mathcal{O}}_{1,n}$, thus

$$\gamma \leq 8(M+1)^4 \inf h(t) K(t, \zeta_{m+3}) \int_{t'_{n+2}}^{t'_{n+1}} f(z)w(z)d(z), \tag{5.30}$$

where the infimum is considered over the interval $(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})$. Let us denote

$$\eta(\gamma; \tilde{\delta}_{1,n}, \tilde{\beta}_{1,n}) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}_{1,n}}^{\tilde{\beta}_{1,n}} \mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big).$$

We use (5.9) and (5.30) to obtain

$$\begin{aligned} 2\eta(\gamma; \tilde{\delta}_{1,n}, \tilde{\beta}_{1,n}) &\leq \int_{t'_{n+2}}^{t'_{n+1}} \Big[16(M+1)^4 C f(y)\phi(y)\Big]\left[\frac{\big(\inf h(t)K(t,\zeta_{m+3})\big)w(y)\eta(\gamma; \tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy \\ &\leq \int_{t'_{n+2}}^{t'_{n+1}} \mathcal{V}\Big(16(M+1)^4 C f(y)\phi(y)\Big)\psi(y)dy \\ &\quad + \int_{t'_{n+2}}^{t'_{n+1}} \tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)K(t,\zeta_{m+3})\big)w(y)\eta(\gamma; \tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy. \end{aligned} \tag{5.31}$$

Since $K(t, y)$ is non-increasing in y, from (5.17) we get

$$\int_{t'_{n+2}}^{t'_{n+1}} \tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)K(t,\zeta_{m+3})\big)w(y)\eta(\gamma; \tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})}{C\gamma\phi(y)\psi(y)}\right]\psi(y)dy \leq \eta(\gamma; \tilde{\delta}_{1,n}, \tilde{\beta}_{1,n}). \tag{5.32}$$

Combining (5.31) and (5.32), we have

$$\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\mathcal{O}}_{1,n}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) \leq \int_{t'_{n+2}}^{t'_{n+1}} \mathcal{V}\Big(16(M+1)^4 C f(y)\phi(y)\Big)\psi(y)dy. \tag{5.33}$$

Applying the sub-additivity of $\mathcal{V} \circ \mathcal{U}^{-1}$, we obtain

$$\begin{aligned} \sum_{m\geq 0}\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\mathcal{O}_{1,m}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) &\leq \sum_{n\geq 0}\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\mathcal{O}}_{1,n}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) \\ &\leq \int_0^\infty \mathcal{V}\Big(16(M+1)^4 C f(y)\phi(y)\Big)\psi(y)dy. \end{aligned} \tag{5.34}$$

We now estimate the second part of (5.28). Let us denote

$$\eta(\gamma; \delta_{2,m}, \beta_{2,m}) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\delta_{2,m}}^{\beta_{2,m}} \mathcal{U}\Big(\gamma\omega(z)\Big)\rho(z)dz\Big).$$

If $t \in \mathcal{O}_{2,m}$, then we have

$$\frac{\gamma}{2(M+1)^4} < h(t) \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2}, z) f(z)w(z)dz. \tag{5.35}$$

As the estimate (5.35) holds for each $t \in \mathcal{O}_{2,m}$, thus

$$\gamma \leq 2(M+1)^4 \inf_{(\delta_{2,m},\beta_{2,m})} h(t) \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2}, z) f(z) w(z) dz. \tag{5.36}$$

Now, applying (5.9) and (5.36) we have

$$\begin{aligned} 2\eta(\gamma;\delta_{2,m},\beta_{2,m}) &\leq \int_{\zeta_{m+3}}^{\zeta_{m+2}} \Big[4(M+1)^4 C f(\tau)\phi(\tau)\Big] \left[\frac{\big(\inf h(t)\big) K(\zeta_{m+2},\tau) w(\tau)\eta(\gamma;\delta_{2,m},\beta_{2,m})}{C\gamma\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau \\ &\leq \int_{\zeta_{m+3}}^{\zeta_{m+2}} \mathcal{V}\Big(4(M+1)^4 C f(\tau)\phi(\tau)\Big)\psi(\tau)d\tau \\ &\quad + \int_{\zeta_{m+3}}^{\zeta_{m+2}} \tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)\big) K(\zeta_{m+3},\tau) w(\tau)\eta(\gamma;\delta_{2,m},\beta_{2,m})}{C\gamma\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau. \end{aligned} \tag{5.37}$$

As $\zeta_{m+2} \leq \delta_{2,m} \leq \beta_{2,m}$, from the condition (5.16) we have

$$\int_{\zeta_{m+3}}^{\zeta_{m+2}} \tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)\big) K(\zeta_{m+2},\tau) w(\tau)\eta(\gamma;\delta_{2,m},\beta_{2,m})}{C\gamma\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau \leq \eta(\gamma;\delta_{2,m},\beta_{2,m}). \tag{5.38}$$

Combining (5.37) and (5.38), and then summing up in m, we obtain

$$\sum_{m\geq 0} \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_{\mathcal{O}_{2,m}} \mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\right) \leq \int_0^\infty \mathcal{V}\Big(4(M+1)^4 C f(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \tag{5.39}$$

Thus from (5.28), (5.34) and (5.39) we obtain (5.15) with constant $32(M+1)^4C$.

$(a) \implies (b)$. Let $0 < t < \alpha$ and for each $N \in \mathbb{N}$ we consider the set

$$E_N = \{0 < s < t : \frac{1}{N} \leq K(t,s), w(s) \leq N\}.$$

Applying $\tilde{\mathcal{V}}(s) \leq s\tilde{v}(s)$, we obtain

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy \leq ltN^2(\inf h)\tilde{v}(\lambda(\inf h)lkN^2) < \infty,$$

where the infimum is considered over (t,α). We choose $\lambda > 0$ suitably such that given $\mu > 0$, we have

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy = (1+\mu)C\gamma.$$

We define the function f as

$$f(y) = \frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\chi_{E_N}(y),$$

where C is the constatnt in the etimate (5.15). Let $t < \beta < \alpha$, then

$$\begin{aligned}\mathcal{I}f(\beta) &= h(\beta)\int_0^\beta K(\beta,y)\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\Big)\chi_{E_N}(y)w(y)dy\\ &\geq \int_{E_N}\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy\\ &= (1+\mu)\gamma > \gamma.\end{aligned}$$

This implies that

$$(t,\alpha) \subset \{y : \mathcal{I}f(y) > \gamma\}.$$

Thus using (5.10) and (5.15) we obtain

$$\begin{aligned}\eta(\gamma;t,\alpha) &= \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\\ &\leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{\mathcal{I}f>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\\ &\leq \int_{E_N}\mathcal{V}\Bigg(\tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\Big)\phi(y)\Bigg)\psi(y)dy\\ &\leq \int_{E_N}\tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\psi(y)dy\\ &= (1+\mu)C\lambda\gamma.\end{aligned}$$

As $\tilde{\mathcal{V}}(s)/s$ increases with s, we have

$$\begin{aligned}&\int_{E_N}\tilde{\mathcal{V}}\Big(\frac{(\inf h)K(t,y)w(y)\eta(\gamma;t,\alpha)}{(1+\mu)C\gamma(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\eta(\gamma;t,\alpha)}dy\\ &\quad\leq \int_{E_N}\tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{(1+\mu)C\lambda\gamma}dy = 1.\end{aligned}$$

By the Monotone Convergence Theorem, we have

$$\int_{E_N}\tilde{\mathcal{V}}\Big(\frac{(\inf h)K(t,y)w(y)\eta(\gamma;t,\alpha)}{(1+\mu)C\gamma(\phi(y)+1/l)\psi(y)}\Big)\frac{\psi(y)}{\eta(\gamma;t,\alpha)}dy \leq 1.$$

Letting $l, N \to \infty$ and $\mu \to 0^+$, we get

$$\int_0^t \tilde{\mathcal{V}}\Bigg(\frac{(\inf h)K(t,y)w(y)\eta(\gamma;t,\alpha)}{C\gamma\phi(y)\psi(y)}\Bigg)\psi(y)dy \leq \eta(\gamma;t,\alpha).$$

Thus we obtain (5.16). Next we prove (5.17). For $0 < \xi \leq t < \alpha$ and $N \in \mathbb{N}$, let us consider the set F_N as

$$F_N = \{0 < s < \xi : \frac{1}{N} \leq w(s) \leq N\}.$$

For $\kappa > 0$ we choose $\pi > 0$ such that

$$\int_{F_N} \tilde{\mathcal{V}}\Big(\frac{\pi\big(\inf h(y)K(y,\xi)\big)w(z)}{(\phi(z)+1/l)(\psi(z)+1/k)}\Big)\Big(\frac{\psi(z)+1/k}{\pi}\Big)dz = (1+\kappa)C\gamma,$$

where C is the constant of (5.15) and $l, k \in \mathbb{N}$. Let us consider the function f as

$$f(z) = \frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\pi(\inf_{y\in(t,\alpha)} h(y)K(y,\xi))w(z)}{(\phi(z)+1/l)(\psi(z)+1/k)}\Big)\frac{\psi(z)+1/k}{\pi(\inf_{y\in(t,\alpha)} h(y)K(y,\xi))w(z)}\chi_{F_N}(z).$$

If $z \in F_N$ and $t < \tau < \alpha$, then $K(\tau, z) \geq K(\tau, \xi)$, which implies

$$h(\tau)K(\tau,z) \geq \inf_{y\in(t,\alpha)} h(y)K(y,\xi).$$

Let $t < \tau < \alpha$, then

$$\begin{aligned}\mathcal{I}f(\tau) &= h(\tau)\int_0^\tau K(\tau,z)f(z)w(z)dz\\ &\geq h(\tau)\int_{F_N} K(\tau,z)\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\pi(\inf h(y)K(y,\xi))w(z)}{(\phi(z)+1/l)(\psi(z)+1/k)}\Big)\frac{\psi(z)+1/k}{\pi(\inf h(y)K(y,\xi))w(z)}w(z)dz\\ &\geq (1+\kappa)\gamma > \gamma. \end{aligned} \tag{5.40}$$

This implies that

$$(t,\alpha) \subset \{z : \mathcal{I}f(z) > \gamma\}.$$

The rest of the proof proceeds similarly. Hence the proof is complete. □

5.3.3 Proof of Theorem 5.3.3

Proof. $(b) \implies (a)$. We consider the function f to be non-negative having support $(0,\infty)$. Let $\{\zeta_m\}_{m\geq 0}$ be a sequence defined by the iteration $\zeta_0 = 0$ and given ζ_m, we consider

$$Q(\zeta_m) = \int_{\zeta_m}^\infty K(\tau,\zeta_m)f(\tau)h(\tau)d\tau = (M+1)^{-m}Q(\zeta_0). \tag{5.41}$$

Using (5.2) and (5.41), we have

$$\begin{aligned} Q(\zeta_m) &= (M+1)^2 Q(\zeta_{m+2})\\ &= (M+1)^2\int_{\zeta_{m+2}}^\infty K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau\\ &= (M+1)^2\Big[\int_{\zeta_{m+2}}^{\zeta_{m+3}} + \int_{\zeta_{m+3}}^\infty\Big]K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau\\ &\leq (M+1)^2\Bigg[\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2}) + \int_{\zeta_{m+3}}^\infty M\Big\{K(\tau,\zeta_{m+3}) + K(\zeta_{m+3},\zeta_{m+2})\Big\}\Bigg]f(\tau)h(\tau)d\tau \end{aligned}$$

$$\leq (M+1)^3\left[\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau + K(\zeta_{m+3},\zeta_{m+2})\int_{\zeta_{m+3}}^{\infty} f(\tau)h(\tau)d\tau\right]$$
$$+ M(M+1)^2\int_{\zeta_{m+3}}^{\infty} K(\tau,\zeta_{m+3})f(\tau)h(\tau)d\tau. \tag{5.42}$$

From the construction of $Q(\zeta_m)$, we have

$$\int_{\zeta_{m+3}}^{\infty} K(\tau,\zeta_{m+3})f(\tau)h(\tau)d\tau = Q(\zeta_{m+3}) = (M+1)^{-(m+3)}Q(\zeta_0) = (M+1)^{-3}Q(\zeta_m).$$

Thus (5.42) implies that

$$Q(\zeta_m) \leq (M+1)^4\left[\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau + K(\zeta_{m+3},\zeta_{m+2})\int_{\zeta_{m+3}}^{\infty} f(\tau)h(\tau)d\tau\right]. \tag{5.43}$$

For $k=1,2$, we define $\delta_{k,m} = \inf \mathcal{O}_{k,m}$ and $\beta_{k,m} = \sup \mathcal{O}_{k,m}$, where

$$\mathcal{O}_{1,m} = \left\{x \in (\zeta_m,\zeta_{m+1}) : w(x)\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(z,\zeta_{m+2})f(z)h(z)dz > \frac{\gamma}{2(M+1)^4}\right\},$$

$$\mathcal{O}_{2,m} = \left\{y \in (\zeta_m,\zeta_{m+1}) : w(y)K(\zeta_{m+3},\zeta_{m+2})\int_{\zeta_{m+3}}^{\infty} f(z)h(z)dz > \frac{\gamma}{2(M+1)^4}\right\}.$$

From the construction of $Q(\zeta_m)$ and (5.43), we have

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\{t:\tilde{\mathcal{I}}f(t)>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\bigg) \leq \sum_{m\geq 0}\bigg\{\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\mathcal{O}_{1,m}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\bigg)$$
$$+\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\bigg(\int_{\mathcal{O}_{2,m}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\bigg)\bigg\}. \tag{5.44}$$

We now estimate the first sum of (5.44). Now, if $t \in \mathcal{O}_{1,m}$, we then have

$$\frac{\gamma}{2(M+1)^4} < w(t)\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau. \tag{5.45}$$

Since the inequality (5.45) is valid for each $t \in \mathcal{O}_{1,m}$, we have

$$\gamma \leq 2(M+1)^4\big(\inf_{(\delta_{1,m},\beta_{1,m})} w(t)\big)\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau. \tag{5.46}$$

Let us denote

$$\eta(\gamma;\delta_{1,m},\beta_{1,m}) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\delta_{1,m}}^{\beta_{1,m}} \mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big).$$

Applying (5.9) and (5.46), we obtain

$$\begin{aligned} 2\eta(\gamma;\delta_{1,m},\beta_{1,m}) &\le \int_{\zeta_{m+2}}^{\zeta_{m+3}} \Big[4(M+1)^4 Cf(\tau)\phi(\tau)\Big]\left[\frac{\big(\inf w(t)\big)K(\tau,\zeta_{m+2})h(\tau)\eta(\gamma;\delta_{1,m},\beta_{1,m})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \\ &\le \int_{\zeta_{m+2}}^{\zeta_{m+3}} \mathcal{V}\Big(4(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau \\ &\quad + \int_{\zeta_{m+2}}^{\zeta_{m+3}} \tilde{\mathcal{V}}\left[\frac{\big(\inf w(t)\big)K(\tau,\zeta_{m+2})h(\tau)\eta(\gamma;\delta_{1,m},\beta_{1,m})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau. \quad (5.47) \end{aligned}$$

As $\zeta_{m+2} \ge \beta_{1,m} \ge \delta_{1,m}$, thus from (5.21) we get

$$\int_{\zeta_{m+2}}^{\zeta_{m+3}} \tilde{\mathcal{V}}\left[\frac{\big(\inf w(t)\big)K(\tau,\zeta_{m+2})h(\tau)\eta(\gamma;\delta_{1,m},\beta_{1,m})}{C\gamma\phi(\tau)\psi(\tau)}\right]\psi(\tau)d\tau \le \eta(\gamma;\delta_{1,m},\beta_{1,m}). \quad (5.48)$$

Combining (5.47), (5.48) and then summing up in m, we get

$$\sum_{m\ge 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\mathcal{O}_{1,m}} \mathcal{U}\Big(\gamma\omega(\tau)\Big)\rho(\tau)d\tau\Big) \le \int_0^\infty \mathcal{V}\Big(4(M+1)^4 Cf(\tau)\phi(\tau)\Big)\psi(\tau)d\tau. \quad (5.49)$$

For the second sum of the inequality (5.44), we define a sequence $\{t_k\}$ on $(0,\infty)$ with the iteration $t_0 = 0$ and given $\{t_k\}$, we define $\{t_{k+1}\}$ as

$$\int_{t_k}^\infty fh = 2\int_{t_{k+1}}^\infty fh.$$

Then $\{t_k\}$ is increasing and satisfies

$$\int_{t_k}^\infty fh = 4\int_{t_{k+1}}^{t_{k+2}} fh.$$

From the sequence $\{t_k\}$ we define a subsequence $\{t'_n\}$ with the iteration as $t'_0 = t_0$ and if $t_k < \zeta_m \le t_{k+1}$ then $t'_{n+1} = t_{k+1}$, otherwise we delete the term t_{k+1} and continue the process. Thus, we get a subsequence $\{t'_n\}$ of $\{t_k\}$. Let $\tilde{\delta}_{2,n} = \inf \tilde{\mathcal{O}}_{2,n}$ and $\tilde{\beta}_{2,n} = \sup \tilde{\mathcal{O}}_{2,n}$, where $\tilde{\mathcal{O}}_{2,n} = \cup_{\{m: t'_n < \zeta_{m+3} \le t'_{n+1}\}}\mathcal{O}_{2,m}$. Now, if $t_{k+1} = t'_{n+1} \ge \zeta_{m+3} > t'_n$, then by construction $\zeta_{m+3} > t_k$ and $t'_{n+2} \ge t_{k+2}$. If $t \in \tilde{\mathcal{O}}_{2,n}$, we then have

$$\frac{\gamma}{2(M+1)^4} < w(t)K(\zeta_{m+3},\zeta_{m+2})\int_{\zeta_{m+3}}^\infty f(y)h(y)dy \le 4w(t)K(\zeta_{m+3},t)\int_{t_k}^\infty f(y)h(y)dy. \quad (5.50)$$

As the estimate (5.50) holds for each $t \in \tilde{\mathcal{O}}_{2,n}$, thus

$$\gamma \leq 8(M+1)^4 \inf_{(\tilde{\delta}_{2,n}, \tilde{\beta}_{2,n})} w(t)K(\zeta_{m+3}, t) \int_{t'_{n+1}}^{t'_{n+2}} f(y)h(y)dy. \tag{5.51}$$

Let us denote

$$\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n}) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}_{2,n}}^{\tilde{\beta}_{2,n}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big).$$

Applying (5.9) and (5.51), we have

$$\begin{aligned} 2\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n}) &\leq \int_{t'_{n+1}}^{t'_{n+2}} \Big[16(M+1)^4 Cf(y)\phi(y)\Big]\Bigg[\frac{\big(\inf w(t)K(\zeta_{m+3}, t)\big)h(y)\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n})}{C\gamma\phi(y)\psi(y)}\Bigg]\psi(y)dy \\ &\leq \int_{t'_{n+1}}^{t'_{n+2}} \mathcal{V}\Big(16(M+1)^4 Cf(y)\phi(y)\Big)\psi(y)dy \\ &\quad + \int_{t'_{n+1}}^{t'_{n+2}} \tilde{\mathcal{V}}\Bigg[\frac{\big(\inf w(t)K(\zeta_{m+3}, t)\big)h(y)\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n})}{C\gamma\phi(y)\psi(y)}\Bigg]\psi(y)dy. \end{aligned} \tag{5.52}$$

Since the kernel K is increasing in the first variable, from (5.22) we have

$$\int_{t'_{n+1}}^{t'_{n+2}} \tilde{\mathcal{V}}\Bigg[\frac{\big(\inf w(t)K(\zeta_{m+3}, t)\big)h(y)\eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n})}{C\gamma\phi(y)\psi(y)}\Bigg]\psi(y)dy \leq \eta(\gamma; \tilde{\delta}_{2,n}, \tilde{\beta}_{2,n}). \tag{5.53}$$

Combining (5.52) and (5.53), we obtain

$$\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\mathcal{O}}_{2,n}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) \leq \int_{t'_{n+1}}^{t'_{n+2}} \mathcal{V}\Big(16(M+1)^4 Cf(y)\phi(y)\Big)\psi(y)dy. \tag{5.54}$$

Thus, we have

$$\begin{aligned} \sum_{m\geq 0} \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\mathcal{O}_{2,m}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) &\leq \sum_{n\geq 0} \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\mathcal{O}}_{2,n}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big) \\ &\leq \int_0^\infty \mathcal{V}\Big(16(M+1)^4 Cf(y)\phi(y)\Big)\psi(y)dy. \end{aligned} \tag{5.55}$$

Thus from (5.44), (5.49) and (5.55) we obtain (5.20) with constant $32(M+1)^4C$.

$(a) \implies (b)$. Let $0 \leq \epsilon < t < \alpha$. For each $m \in \mathbb{N}$, let us define the set

$$E_m = \Big\{y \in (t, \alpha) : \frac{1}{m} \leq K(y, t), h(y) \leq m\Big\}.$$

For $\lambda > 0$ and $l, k \in \mathbb{N}$ we have

$$\int_{E_m} \tilde{\mathcal{V}}\Bigg(\frac{\lambda\big(\inf w\big)K(y, t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Bigg)\Bigg(\frac{\psi(y)+1/k}{\lambda}\Bigg)dy \leq lm^2(\alpha - t)\big(\inf w\big)\tilde{v}(\lambda lkm^2 \inf w) < \infty,$$

where the infimum is considered over (ϵ, t). Given $\mu > 0$ we choose $\lambda > 0$ suitably so that the inequality

$$\int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy = (1+\mu)C\gamma$$

holds for the constant C of (5.20). Let us define the function f by

$$f(y) = \frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\chi_{E_m}(y).$$

If $\epsilon < \beta < t$, then

$$\begin{aligned}\tilde{\mathcal{I}}f(\beta) &= w(\beta)\int_{E_m} K(y,\beta)\Big[\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\Big]h(y)dy\\ &\geq \int_{E_m}\frac{1}{C}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy\\ &= (1+\mu)\gamma > \gamma.\end{aligned}$$

This implies that

$$(\epsilon, t) \subset \{y : \tilde{\mathcal{I}}f(y) > \gamma\}.$$

Thus applying (5.10) and (5.20) we obtain

$$\begin{aligned}\eta(\gamma;\epsilon,t) &= \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\\ &\leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\{\tilde{\mathcal{I}}f>\gamma\}} \mathcal{U}\Big(\gamma\omega(y)\Big)\rho(y)dy\Big)\\ &\leq \int_{E_m}\mathcal{V}\Big(\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\phi(y)\Big)\psi(y)dy\\ &\leq \int_{E_m}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\psi(y)dy\\ &= (1+\mu)C\lambda\gamma.\end{aligned}$$

Since $\tilde{\mathcal{V}}(s)/s$ increases as s increases, we have

$$\begin{aligned}&\int_{E_m}\tilde{\mathcal{V}}\Big(\frac{\big(\inf w\big)K(y,t)h(y)\eta(\gamma;\epsilon,t)}{(1+\mu)C\gamma(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{\eta(\gamma;\epsilon,t)}dy\\ &\qquad\leq \int_{E_m}\tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\frac{\psi(y)+1/k}{(1+\mu)C\lambda\gamma}dy = 1.\end{aligned}$$

Using the Monotone Convergence Theorem, we obtain

$$\int_{E_m}\tilde{\mathcal{V}}\Big(\frac{\big(\inf w\big)K(y,t)h(y)\eta(\gamma;\epsilon,t)}{(1+\mu)C\gamma(\phi(y)+1/l)\psi(y)}\Big)\frac{\psi(y)}{\eta(\gamma;\epsilon,t)}dy \leq 1.$$

Letting $l, m \to \infty$ and $\mu \to 0^+$, we thus obtain

$$\int_t^\alpha \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)K(y,t)h(y)\eta(\gamma;\epsilon,t)}{C\gamma\phi(y)\psi(y)}\right)\psi(y)dy \le \eta(\gamma;\epsilon,t).$$

Thus we obtain (5.21). Next we state the proof of (5.22). Let $0 \le \epsilon < t \le \xi < \alpha$ and for $m \in \mathbb{N}$ we define the set F_m as

$$F_m = \{s \in (\xi,\alpha) : \frac{1}{m} \le h(s) \le m\}.$$

Given $\kappa > 0$ we choose $\pi > 0$ such that

$$\int_{F_m} \tilde{\mathcal{V}}\left(\frac{\pi\big(\inf w(y)K(\xi,y)\big)h(s)}{(\phi(s)+1/l)(\psi(s)+1/k)}\right)\left(\frac{\psi(s)+1/k}{\pi}\right)ds = (1+\kappa)C\gamma, \tag{5.56}$$

where C is the constant of (5.20) and $l, k \in \mathbb{N}$. Let us define the function f by

$$f(z) = \frac{1}{C}\tilde{\mathcal{V}}\left(\frac{\pi\big(\inf_{(\epsilon,t)} w(y)K(\xi,y)\big)h(z)}{(\phi(z)+1/l)(\psi(z)+1/k)}\right)\frac{\psi(z)+1/k}{\pi\big(\inf_{(\epsilon,t)} w(y)K(\xi,y)\big)h(z)}\chi_{F_m}(z).$$

If $z \in F_m$ and $0 \le \epsilon < \beta < t$, then

$$\inf_{y\in(\epsilon,t)} w(y)K(\xi,y) \le w(\beta)K(z,\beta). \tag{5.57}$$

Now, combining (5.56) and (5.57) and for $0 \le \epsilon < \beta < t$, we have

$$\begin{aligned}\tilde{\mathcal{I}}f(\beta) &= w(\beta)\int_\beta^\infty K(z,\beta)f(z)h(z)dz\\ &\ge (1+\kappa)\gamma > \gamma.\end{aligned}$$

This implies that

$$(\epsilon,t) \subset \{x : \tilde{\mathcal{I}}f(x) > \gamma\}.$$

The rest of the proof proceeds similarly. Hence the proof is complete. □

5.4 Extra-weak type results

Next, we discuss extra-weak type results for the modified integral Hardy operators and its conjugate operators. We also state the results for the modified Hardy operator and its conjugate operator.

5.4.1 Statement of the results

We first state the extra-weak type results for the modified integral Hardy operator.

Theorem 5.4.1. *Let $\tilde{\mathcal{V}}$ stands for the complementary N- function of an N- function $\mathcal{V}$. We assume that the function $\mathcal{U}$ is positive and strictly increasing such that $\mathcal{V}\circ\mathcal{U}^{-1}$ is countably sub-additive. Let the function h be monotone on $\mathbb{R}$ and the function $h(\cdot)K(\cdot, y)$ satisfies the condition (5.13). Suppose that the weights ω, ϕ and ψ are positive and locally integrable on $(0,\infty)$. Then we have the following equivalent results.*

(a) For each non-negative function f and $\gamma > 0$ the following

$$\int_{\{t\in(0,\infty):\mathcal{I}f(t)>\gamma\}} \omega(y)dy \leq \Big(\mathcal{U}\circ\mathcal{V}^{-1}\Big)\Big(\int_0^\infty \mathcal{V}\Big(\frac{Cf(y)\phi(y)}{\gamma}\Big)\psi(y)dy\Big) \tag{5.58}$$

holds for some $C > 0$.

(b) Let $0 < t < \alpha$ and $\gamma > 0$ such that

$$\int_0^t \tilde{\mathcal{V}}\left[\frac{(\inf_{(t,\alpha)} h)K(t,y)w(y)\theta(t,\alpha)}{C\phi(y)\psi(y)}\right]\psi(y)dy \leq \theta(t,\alpha) \tag{5.59}$$

and for $0 < \xi \leq t < \alpha$ the following

$$\int_0^\xi \tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(t,\alpha)} h(y)K(y,\xi))w(z)\theta(t,\alpha)}{C\phi(z)\psi(z)}\right]\psi(z)dz \leq \theta(t,\alpha) \tag{5.60}$$

hold for some $C > 0$, where

$$\theta(t,\alpha) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \omega(z)dz\Big).$$

Corollary 5.4.2. *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}, \mathcal{V}\circ\mathcal{U}^{-1}$ and h satisfy all the assumptions stated in Theorem 5.4.1. Then we obtain the following equivalent conditions.*

(a) For each non-negative function f and $\gamma > 0$ the following

$$\int_{\{t\in(0,\infty):\mathcal{H}_h f(t)>\gamma\}} \omega(y)dy \leq \mathcal{U}\circ\mathcal{V}^{-1}\Big(\int_0^\infty \mathcal{V}\Big(\frac{Cf(y)\phi(y)}{\gamma}\Big)\psi(y)dy\Big). \tag{5.61}$$

holds for a suitable constant $C > 0$.

(b) For each $0 < t < \alpha$ the inequality

$$\int_0^t \tilde{\mathcal{V}}\left[\frac{(\inf h)w(y)\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \omega(y)dy\Big)}{C\phi(y)\psi(y)}\right]\psi(y)dy \leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_t^\alpha \omega(y)dy\Big) \tag{5.62}$$

holds for a suitable constant $C > 0$.

Next, we state the extra-weak type result for the conjugate of the modified integral operator.

Theorem 5.4.3. *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}$ and $\mathcal{V} \circ \mathcal{U}^{-1}$ satisfy all the assumptions of Theorem 5.4.1. Suppose that w be monotone on $\mathbb{R}$ the function $w(\cdot)K(y, \cdot)$ satisfies the condition (5.14). Then we obtain the following equivalent statements.*

(a) For each non-negative function f and $\gamma > 0$, the following

$$\int_{\{t\in(0,\infty):\tilde{\mathcal{I}}f(t)>\gamma\}} \omega(y)dy \leq \Big(\mathcal{U} \circ \mathcal{V}^{-1}\Big)\Big(\int_0^\infty \mathcal{V}\Big(\frac{Cf(y)\phi(y)}{\gamma}\Big)\psi(y)dy\Big) \tag{5.63}$$

holds for some $C > 0$.

(b) Let $0 \leq \epsilon < t < \alpha$ and $\gamma > 0$ such that

$$\int_t^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{(\epsilon,t)} w)K(y,t)h(y)\theta(\epsilon,t)}{C\phi(y)\psi(y)}\right]\psi(y)dy \leq \theta(\epsilon,t) \tag{5.64}$$

and for $0 \leq \epsilon < t \leq \xi < \alpha$, the following

$$\int_\xi^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{y\in(\epsilon,t)} w(y)K(\xi,y))h(z)\theta(\epsilon,t)}{C\phi(z)\psi(z)}\right]\psi(z)dz \leq \theta(\epsilon,t) \tag{5.65}$$

hold for some $C > 0$, where

$$\theta(\epsilon,t) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \omega(\tau)d\tau\Big).$$

Similarly, the extra-weak type result for the modified Hardy operator follows from Theorem 5.4.3 by considering $K \equiv 1$.

Corollary 5.4.4. *Let $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{V}}$, and $\mathcal{V} \circ \mathcal{U}^{-1}$ satisfy all the assumptions stated in Theorem 5.4.1. Suppose the function w be monotone on $\mathbb{R}$. Then we obtain the following equivalet results.*

(a) For each non-negative function f and $\gamma > 0$, the following

$$\int_{\{t\in(0,\infty):\tilde{\mathcal{H}}_h f(t)>\gamma\}} \omega(y)dy \leq \Big(\mathcal{U} \circ \mathcal{V}^{-1}\Big)\Big(\int_0^\infty \mathcal{V}\Big(\frac{Cf(y)\phi(y)}{\gamma}\Big)\psi(y)dy\Big) \tag{5.66}$$

holds for some $C > 0$.

(b) For each $0 \leq \epsilon < t < \alpha$, the inequality

$$\int_t^\alpha \tilde{\mathcal{V}}\left[\frac{(\inf_{(\epsilon,t)} w)h(y)\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \omega(y)dy\Big)}{C\phi(y)\psi(y)}\right]\psi(y)dy \leq \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \omega(y)dy\Big) \tag{5.67}$$

holds for a suitable constant $C > 0$.

Next we prove the results.

5.4.2 Proof of Theorem 5.4.1

Proof. $(b) \implies (a)$. We consider the function f is non-negative and measurable function with support $(0, L)$ in $\mathbb{R}$. Next we consider a decreasing sequence $\{\zeta_m\}_{m\geq 0}$ with the iteration $\zeta_0 = L$ and given ζ_m, we define

$$P(\zeta_m) = \int_0^{\zeta_m} K(\zeta_m, z)f(z)w(z)dz = (M+1)^{-m}P(\zeta_0). \tag{5.68}$$

We have already proved that

$$P(\zeta_m) \leq (M+1)^4\left[K(\zeta_{m+2}, \zeta_{m+3})\int_0^{\zeta_{m+3}} f(z)w(z)dz + \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2}, z)f(z)w(z)dz\right]. \tag{5.69}$$

As similarly in the weak type result we consider the set as

$$\mathcal{O}_{1,m} = \left\{x \in (\zeta_{m+1}, \zeta_m) : h(x)K(\zeta_{m+2}, \zeta_{m+3})\int_0^{\zeta_{m+3}} f(z)w(z)dz > \frac{\gamma}{2(M+1)^4}\right\},$$

$$\mathcal{O}_{2,m} = \left\{y \in (\zeta_{m+1}, \zeta_m) : h(y)\int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2}, z)f(z)w(z)dz > \frac{\gamma}{2(M+1)^4}\right\}.$$

Now, from (5.68) and (5.69) we obtain

$$\begin{aligned}\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t:\mathcal{I}f(t)>\gamma\}} \omega(z)dz\right) \leq \sum_{m\geq 0}\Bigg\{&\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\mathcal{O}_{1,m}} \omega(z)dz\right) \\ &+ \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\mathcal{O}_{2,m}} \omega(z)dz\right)\Bigg\}.\end{aligned} \tag{5.70}$$

Next, we consider the first sum of (5.70). For this, we construct a decreasing sequence $\{t_k\}$ on $(0, L)$ with the iteration $t_0 = L$ and for each t_k we define t_{k+1} as

$$\int_0^{t_k} fw = 2\int_0^{t_{k+1}} fw.$$

Thus we obtain a decreasing sequence $\{t_k\}$ with the following property.

$$\int_0^{t_k} fw = 4\int_{t_{k+2}}^{t_{k+1}} fw.$$

Again we consider a subsequence $\{t'_n\}$ from $\{t_k\}$ with the iteration $t'_0 = t_0$ and if $t_{k+1} \leq \zeta_m < t_k$, then $t'_{n+1} = t_{k+1}$, otherwise we delete the term t_{k+1} and continue the process. Thus, we get a subsequence $\{t'_n\}$ of $\{t_k\}$. Let $\tilde{\delta}_{1,n} = \inf \tilde{\mathcal{O}}_{1,n}$ and $\tilde{\beta}_{1,n} = \sup \tilde{\mathcal{O}}_{1,n}$, where

$\tilde{\mathcal{O}}_{1,n} = \cup_{\{m: t'_{n+1} \le \zeta_{m+3} < t'_n\}} \mathcal{O}_{1,m}$. Now, if $t_{k+1} = t'_{n+1} \le \zeta_{m+3} < t'_n$, then by construction $\zeta_{m+3} \le t_k$ and $t'_{n+2} \le t_{k+2}$. If we consider $t \in \tilde{\mathcal{O}}_{1,n}$, then

$$\frac{\gamma}{2(M+1)^4} < h(t)K(\zeta_{m+2}, \zeta_{m+3}) \int_0^{\zeta_{m+3}} f(z)w(z)dz \le h(t)K(t, \zeta_{m+3}) \int_0^{t_k} f(z)w(z)dz. \tag{5.71}$$

Since the inequality (5.71) is valid for each $t \in \tilde{\mathcal{O}}_{1,n}$, we have

$$\gamma \le 8(M+1)^4 \inf_{(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})} h(t)K(t, \zeta_{m+3}) \int_{t'_{n+2}}^{t'_{n+1}} f(\tau)w(\tau)d\tau. \tag{5.72}$$

Let us denote

$$\theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n}) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}_{1,n}}^{\tilde{\beta}_{1,n}} \omega(\tau)d\tau\Big).$$

Using the estimates (5.9) and (5.72) we obtain

$$\begin{aligned} 2\theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n}) &\le \int_{t'_{n+2}}^{t'_{n+1}} \Big[\frac{16(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big] \Bigg[\frac{\big(\inf h(t)K(t, \zeta_{m+3})\big) w(\tau)\theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})}{C\phi(\tau)\psi(\tau)}\Bigg] \psi(\tau)d\tau \\ &\le \int_{t'_{n+2}}^{t'_{n+1}} \mathcal{V}\Big(\frac{16(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau \\ &\quad + \int_{t'_{n+2}}^{t'_{n+1}} \tilde{\mathcal{V}}\Bigg[\frac{\big(\inf h(t)K(t, \zeta_{m+3})\big) w(\tau)\theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})}{C\phi(\tau)\psi(\tau)}\Bigg]\psi(\tau)d\tau. \end{aligned} \tag{5.73}$$

Using non-increasing property of the kernel K with respect to its second variable, from (5.60) we get

$$\int_{t'_{n+2}}^{t'_{n+1}} \tilde{\mathcal{V}}\Bigg[\frac{\big(\inf h(t)K(t, \zeta_{m+3})\big) w(\tau)\theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n})}{C\phi(\tau)\psi(\tau)}\Bigg]\psi(\tau)d\tau \le \theta(\tilde{\delta}_{1,n}, \tilde{\beta}_{1,n}). \tag{5.74}$$

Combining (5.73) and (5.74), we obtain

$$\Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\mathcal{O}}_{1,n}} \omega(\tau)d\tau\Big) \le \int_{t'_{n+2}}^{t'_{n+1}} \mathcal{V}\Big(\frac{16(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau. \tag{5.75}$$

Using the sub-additivity of $\mathcal{V} \circ \mathcal{U}^{-1}$, we obtain

$$\begin{aligned} \sum_{m \ge 0} \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\mathcal{O}_{1,m}} \omega(\tau)d\tau\Big) &\le \sum_{n \ge 0} \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\mathcal{O}}_{1,n}} \omega(\tau)d\tau\Big) \\ &\le \int_0^{\infty} \mathcal{V}\Big(\frac{16(M+1)^4 f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau. \end{aligned} \tag{5.76}$$

Next, we estimate the second part of (5.70). For this, let us define $\delta_{2,m} = \inf \mathcal{O}_{2,m}$ and $\beta_{2,m} = \sup \mathcal{O}_{2,m}$ and denote

$$\theta(\delta_{2,m}, \beta_{2,m}) = \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\delta_{2,m}}^{\beta_{2,m}} \omega(\tau) d\tau\Big).$$

Letting $t \in \mathcal{O}_{2,m}$, we then have

$$\frac{\gamma}{2(M+1)^4} < h(t) \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2}, \tau) f(\tau) w(\tau) d\tau. \tag{5.77}$$

As the estimate (5.77) is valid for each $t \in \mathcal{O}_{2,m}$, we have

$$\gamma \leq 2(M+1)^4 \inf_{(\delta_{2,m}, \beta_{2,m})} h(t) \int_{\zeta_{m+3}}^{\zeta_{m+2}} K(\zeta_{m+2}, \tau) f(\tau) w(\tau) d\tau. \tag{5.78}$$

Now, using the condition (5.9) and from (5.78) we have

$$\begin{aligned} 2\theta(\delta_{2,m}, \beta_{2,m}) &\leq \int_{\zeta_{m+3}}^{\zeta_{m+2}} \Big[\frac{4(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big] \left[\frac{\big(\inf h(t)\big) K(\zeta_{m+2}, \tau) w(\tau) \theta(\delta_{2,m}, \beta_{2,m})}{C\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau \\ &\leq \int_{\zeta_{m+3}}^{\zeta_{m+2}} \mathcal{V}\Big(\frac{4(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big) \psi(\tau) d\tau \\ &\quad + \int_{\zeta_{m+3}}^{\zeta_{m+2}} \tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)\big) K(\zeta_{m+2}, \tau) w(\tau) \theta(\delta_{2,m}, \beta_{2,m})}{C\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau. \end{aligned} \tag{5.79}$$

As $\zeta_{m+2} \leq \delta_{2,m} \leq \beta_{2,m}$, from the condition (5.59), we have

$$\int_{\zeta_{m+3}}^{\zeta_{m+2}} \tilde{\mathcal{V}}\left[\frac{\big(\inf h(t)\big) K(\zeta_{m+2}, \tau) w(\tau) \eta(\gamma; \delta_{2,m}, \beta_{2,m})}{C\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau \leq \theta(\delta_{2,m}, \beta_{2,m}). \tag{5.80}$$

Combining (5.79), (5.80) and then summing up in m, we obtain

$$\sum_{m \geq 0} \Big(\mathcal{V} \circ \mathcal{U}^{-1}\Big)\Big(\int_{\mathcal{O}_{2,m}} \omega(\tau) d\tau\Big) \leq \int_0^\infty \mathcal{V}\Big(\frac{4(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big) \psi(\tau) d\tau. \tag{5.81}$$

Thus from the estimates (5.70), (5.76) and (5.81) we obtain (5.58) with constant $32(M+1)^4 C$.

$(a) \implies (b)$. For each $N \in \mathbb{N}$, we define E_N as

$$E_N = \{0 < s < t : \frac{1}{N} \leq K(t, y), w(y) \leq N\}.$$

We have

$$\int_{E_N} \tilde{\mathcal{V}}\Big(\frac{\lambda(\inf h) K(t, y) w(y)}{(\phi(y) + 1/l)(\psi(y) + 1/k)}\Big)\Big(\frac{\psi(y) + 1/k}{\lambda}\Big) dy \leq ltN^2 (\inf h) \tilde{v}(\lambda(\inf h) lkN^2) < \infty$$

for each $l, k \in \mathbb{N}$ and $\lambda > 0$. We choose $\lambda > 0$ such that

$$\int_{E_N} \tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy = (1+\mu)C,$$

where C is the constant in (5.58). For each $\gamma > 0$ we consider

$$f(y) = \frac{\gamma}{C}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\chi_{E_N}(y).$$

If $t < \beta < \alpha$, then

$$\begin{aligned}
\mathcal{I}f(\beta) &= h(\beta)\int_0^\beta K(\beta,y)\frac{\gamma}{C}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\right)\chi_{E_N}(y)w(y)dy \\
&\geq \int_{E_N}\frac{\gamma}{C\lambda}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\psi(y)+1/k\right)dy \\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

Thus we obtain

$$(t,\alpha) \subset \{y : \mathcal{I}f(y) > \gamma\}.$$

Thus using (5.10) and the assumption (5.58) we obtain

$$\begin{aligned}
\theta(t,\alpha) &\leq \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_t^\alpha \omega(y)dy\right) \\
&\leq \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{\mathcal{I}f>\gamma\}} \omega(y)dy\right) \\
&\leq \int_{E_N}\mathcal{V}\left(\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda(\inf h)K(t,y)w(y)}\right)\phi(y)\right)\psi(y)dy \\
&\leq \int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy \\
&\leq (1+\mu)C\lambda.
\end{aligned}$$

Using the increasing property of $\tilde{\mathcal{V}}(s)/s$ corresponding to s, we get

$$\begin{aligned}
&\int_{E_N}\tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\theta(t,\alpha)}{(1+\mu)C(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\theta(t,\alpha)}dy \\
&\quad\leq \int_{E_N}\tilde{\mathcal{V}}\left(\frac{\lambda(\inf h)K(t,y)w(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda}dy = 1.
\end{aligned}$$

By the Monotone Convergence Theorem

$$\int_{E_N}\tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\theta(t,\alpha)}{(1+\mu)C(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\theta(t,\alpha)}dy \leq 1.$$

Letting $l, N \to \infty$ and $\mu \to 0^+$, we thus obtain

$$\int_0^t \tilde{\mathcal{V}}\left(\frac{(\inf h)K(t,y)w(y)\big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\left(\int_t^\alpha \omega\right)}{C\phi(y)\psi(y)}\right)\psi(y)dy \leq \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_t^\alpha \omega(y)dy\right).$$

We skip the proof of (5.60) as it proceeds similarly. Hence the proof is complete. □

5.4.3 Proof of Theorem 5.4.3

Proof. $(b) \implies (a)$. Let f be non-negative and measurable with support $(0,\infty)$ in $\mathbb{R}$. We define an increasing sequence $\{\zeta_m\}_{m\geq 0}$ with the iteration $\zeta_0 = 0$ and given ζ_m, we consider

$$Q(\zeta_m) = \int_{\zeta_m}^\infty K(\tau,\zeta_m)f(\tau)h(\tau)d\tau = (M+1)^{-m}Q(\zeta_0)$$

We have already established that

$$Q(\zeta_m) \leq (M+1)^4\left[\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau + K(\zeta_{m+3},\zeta_{m+2})\int_{\zeta_{m+3}}^\infty f(\tau)h(\tau)d\tau\right]. \tag{5.82}$$

For $k = 1, 2$, we define $\delta_{k,m} = \inf \mathcal{O}_{k,m}$ and $\beta_{k,m} = \sup \mathcal{O}_{k,m}$, where

$$\mathcal{O}_{1,m} = \left\{y \in (\zeta_m,\zeta_{m+1}) : w(y)\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau > \frac{\gamma}{2(M+1)^4}\right\},$$

$$\mathcal{O}_{2,m} = \left\{z \in (\zeta_m,\zeta_{m+1}) : w(z)K(\zeta_{m+3},\zeta_{m+2})\int_{\zeta_{m+3}}^\infty f(\tau)h(\tau)d\tau > \frac{\gamma}{2(M+1)^4}\right\}.$$

Now from the construction of $Q(\zeta_m)$ and (5.82) we have

$$\begin{aligned}\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\{t:\tilde{\mathcal{I}}f(t)>\gamma\}} \omega(t)dt\right) \leq \sum_{m\geq 0}\Bigg\{&\left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\mathcal{O}_{1,m}} \omega(t)dt\right)\\ &+ \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\mathcal{O}_{2,m}} \omega(t)dt\right)\Bigg\}.\end{aligned} \tag{5.83}$$

Next, we estimate the first part of (5.83). Let us denote

$$\theta(\delta_{1,m},\beta_{1,m}) = \left(\mathcal{V}\circ\mathcal{U}^{-1}\right)\left(\int_{\delta_{1,m}}^{\beta_{1,m}} \omega(\tau)d\tau\right).$$

Now, if $t \in \mathcal{O}_{1,m}$ then

$$\frac{\gamma}{2(M+1)^4} < w(t)\int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2})f(\tau)h(\tau)d\tau. \tag{5.84}$$

Since the inequality (5.84) holds for each $t \in \mathcal{O}_{1,m}$, thus we have

$$\gamma \leq 2(M+1)^4 \big(\inf_{(\delta_{1,m},\beta_{1,m})} w(t) \big) \int_{\zeta_{m+2}}^{\zeta_{m+3}} K(\tau,\zeta_{m+2}) f(\tau) h(\tau) d\tau. \tag{5.85}$$

Applying the condition (5.9) and the inequality(5.85), we obtain

$$\begin{aligned} 2\theta(\delta_{1,m},\beta_{1,m}) &\leq \int_{\zeta_{m+2}}^{\zeta_{m+3}} \Big[\frac{4(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big] \left[\frac{\big(\inf w(t)\big) K(\tau,\zeta_{m+2}) h(\tau)\theta(\delta_{1,m},\beta_{1,m})}{C\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau \\ &\leq \int_{\zeta_{m+2}}^{\zeta_{m+3}} \mathcal{V}\Big(\frac{4(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau) d\tau \\ &\quad + \int_{\zeta_{m+2}}^{\zeta_{m+3}} \tilde{\mathcal{V}}\left[\frac{\big(\inf w(t)\big) K(\tau,\zeta_{m+2}) h(\tau)\theta(\delta_{1,m},\beta_{1,m})}{C\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau. \end{aligned} \tag{5.86}$$

As $\zeta_{m+2} \geq \beta_{1,m} \geq \delta_{1,m}$, thus the assumption (5.64) gives

$$\int_{\zeta_{m+2}}^{\zeta_{m+3}} \tilde{\mathcal{V}}\left[\frac{\big(\inf w(t)\big) K(\tau,\zeta_{m+2}) h(\tau)\theta(\delta_{1,m},\beta_{1,m})}{C\phi(\tau)\psi(\tau)}\right] \psi(\tau) d\tau \leq \theta(\delta_{1,m},\beta_{1,m}). \tag{5.87}$$

Combining (5.86) and (5.87) we get

$$\sum_{m\geq 0} \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\mathcal{O}_{1,m}} \omega(\tau) d\tau\Big) \leq \int_0^\infty \mathcal{V}\Big(\frac{4(M+1)^4 C f(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau) d\tau. \tag{5.88}$$

Let us estimate the second part of (5.83). For this we define an increasing sequence on $(0,\infty)$ with the iteration $t_0 = 0$ and

$$\int_{t_k}^\infty fh = 2\int_{t_{k+1}}^\infty fh.$$

Then $\{t_k\}$ is increasing and satisfies

$$\int_{t_k}^\infty fh = 4\int_{t_{k+1}}^{t_{k+2}} fh.$$

Next, we extract a subsequence $\{t'_n\}$ from the sequence $\{t_k\}$ with the iteration $t'_0 = t_0$ and if $t_k < \zeta_m \leq t_{k+1}$, then $t'_{n+1} = t_{k+1}$, otherwise we delete the term t_{k+1} and continue the process. Thus, we get a subsequence $\{t'_n\}$ of $\{t_k\}$. Let $\tilde{\delta}_{2,n} = \inf \tilde{\mathcal{O}}_{2,n}$ and $\tilde{\beta}_{2,n} = \sup \tilde{\mathcal{O}}_{2,n}$, where $\tilde{\mathcal{O}}_{2,n} = \cup_{\{m: t'_n < \zeta_{m+3} \leq t'_{n+1}\}} \mathcal{O}_{2,m}$. Now, if $t_{k+1} = t'_{n+1} \geq \zeta_{m+3} > t'_n$, then by construction $\zeta_{m+3} > t_k$ and $t'_{n+2} \geq t_{k+2}$. Suppose that $\tilde{\mathcal{O}}_{2,n} \neq \phi$. If $t \in \tilde{\mathcal{O}}_{2,n}$, then

$$\frac{\gamma}{2(M+1)^4} < w(t) K(\zeta_{m+3},\zeta_{m+2}) \int_{\zeta_{m+3}}^\infty f(\tau)h(\tau) d\tau \leq 4 w(t) K(\zeta_{m+3}, t) \int_{t_k}^\infty f(\tau) h(\tau) d\tau. \tag{5.89}$$

As the estimate (5.89) holds for each $t \in \tilde{\mathcal{O}}_{2,n}$, thus

$$\gamma \leq 8(M+1)^4 \inf_{(\tilde{\delta}_{2,n},\tilde{\beta}_{2,n})} w(t)K(\zeta_{m+3},t)\int_{t'_{n+1}}^{t'_{n+2}} f(\tau)h(\tau)d\tau. \tag{5.90}$$

Let us denote

$$\theta(\tilde{\delta}_{2,n},\tilde{\beta}_{2,n}) = \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\delta}_{2,n}}^{\tilde{\beta}_{2,n}} \omega(\tau)d\tau\Big).$$

Now, using (5.9) and (5.90) we obtain

$$\begin{aligned} 2\theta(\tilde{\delta}_{2,n},\tilde{\beta}_{2,n}) &\leq \int_{t'_{n+1}}^{t'_{n+2}} \Big[\frac{16(M+1)^4 Cf(\tau)\phi(\tau)}{\gamma}\Big]\Bigg[\frac{\big(\inf w(t)K(\zeta_{m+3},t)\big)h(\tau)\theta(\tilde{\delta}_{2,n},\tilde{\beta}_{2,n})}{C\phi(\tau)\psi(\tau)}\Bigg]\psi(\tau)d\tau \\ &\leq \int_{t'_{n+1}}^{t'_{n+2}} \mathcal{V}\Big(\frac{16(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau \\ &\quad + \int_{t'_{n+1}}^{t'_{n+2}} \tilde{\mathcal{V}}\Bigg[\frac{\big(\inf w(t)K(\zeta_{m+3},t)\big)h(\tau)\theta(\tilde{\delta}_{2,n},\tilde{\beta}_{2,n})}{C\phi(\tau)\psi(\tau)}\Bigg]\psi(\tau)d\tau. \end{aligned} \tag{5.91}$$

Since the kernel K is non-decreasing in the first variable, thus using (5.65) we obtain

$$\int_{t'_{n+1}}^{t'_{n+2}} \tilde{\mathcal{V}}\Bigg[\frac{\big(\inf w(t)K(\zeta_{m+3},t)\big)h(\tau)\theta(\tilde{\delta}_{2,n},\tilde{\beta}_{2,n})}{C\phi(\tau)\psi(\tau)}\Bigg]\psi(\tau)d\tau \leq \theta(\tilde{\delta}_{2,n},\tilde{\beta}_{2,n}). \tag{5.92}$$

Combining (5.91) and (5.92),

$$\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\mathcal{O}}_{2,n}} \omega(\tau)d\tau\Big) \leq \int_{t'_{n+1}}^{t'_{n+2}} \mathcal{V}\Big(\frac{16(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau. \tag{5.93}$$

Thus

$$\begin{aligned} \sum_{m\geq 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\mathcal{O}_{2,m}} \omega(\tau)d\tau\Big) &\leq \sum_{n\geq 0}\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_{\tilde{\mathcal{O}}_{2,n}} \omega(\tau)d\tau\Big) \\ &\leq \int_0^\infty \mathcal{V}\Big(\frac{16(M+1)^4Cf(\tau)\phi(\tau)}{\gamma}\Big)\psi(\tau)d\tau. \end{aligned} \tag{5.94}$$

Thus we obtain (5.63) with constant $32(M+1)^4C$.

(a) $\Longrightarrow$ *(b)*.

Let $0 \leq \epsilon < t < \alpha$. For each $m \in \mathbb{N}$, we define the collection E_m as

$$E_m = \Big\{z \in (t,\alpha) : \frac{1}{m} \leq K(z,t), h(z) \leq m\Big\}.$$

As,

$$\int_{E_m} \tilde{\mathcal{V}}\Big(\frac{\lambda\big(\inf_{(\epsilon,t)} w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\Big)\Big(\frac{\psi(y)+1/k}{\lambda}\Big)dy \leq lm^2(\alpha-t)\big(\inf w\big)\tilde{v}(\lambda lkm^2\inf w) < \infty$$

for each $l, k \in \mathbb{N}$ and $\lambda > 0$. Thus for each $\mu > 0$ we can choose λ such that

$$\int_{E_m} \tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf_{(\epsilon,t)} w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\left(\frac{\psi(y)+1/k}{\lambda}\right)dy = (1+\mu)C.$$

We consider the function f as

$$f(y) = \frac{\gamma}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\chi_{E_m}(y).$$

For $\epsilon < \beta < t$, we have

$$\begin{aligned}
\tilde{\mathcal{I}}f(\beta) &= w(\beta)\int_{E_m} K(y,\beta)\left[\frac{\gamma}{C}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\right]h(y)dy\\
&\geq \int_{E_m}\frac{\gamma}{C\lambda}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\Big(\psi(y)+1/k\Big)dy\\
&= (1+\mu)\gamma > \gamma.
\end{aligned}$$

This implies that

$$(\epsilon, t) \subset \{y : \tilde{\mathcal{I}}f(y) > \gamma\}.$$

Thus, using (5.10) and the assumption (5.63) we obtain

$$\begin{aligned}
\Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_\epsilon^t \omega(y)dy\right) &\leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\left(\int_{\{\tilde{\mathcal{I}}f>\gamma\}} \omega(y)dy\right)\\
&\leq \int_{E_m}\mathcal{V}\left(\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\lambda\big(\inf w\big)K(y,t)h(y)}\phi(y)\right)\psi(y)dy\\
&\leq \int_{E_m}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\psi(y)dy\\
&\leq (1+\mu)C\lambda\gamma.
\end{aligned}$$

As $\tilde{\mathcal{V}}(s)/s$ increases with s, thus

$$\begin{aligned}
&\int_{E_m}\tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)K(y,t)h(y)\theta(\epsilon,t)}{(1+\mu)C(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{\theta(\epsilon,t)}dy\\
&\quad\leq \int_{E_m}\tilde{\mathcal{V}}\left(\frac{\lambda\big(\inf w\big)K(y,t)h(y)}{(\phi(y)+1/l)(\psi(y)+1/k)}\right)\frac{\psi(y)+1/k}{(1+\mu)C\lambda}dy = 1.
\end{aligned}$$

By the Monotone Convergence Theorem

$$\int_{E_m}\tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)K(y,t)h(y)\theta(\epsilon,t)}{(1+\mu)C(\phi(y)+1/l)\psi(y)}\right)\frac{\psi(y)}{\theta(\epsilon,t)}dy \leq 1.$$

Letting $l, m \to \infty$ and $\mu \to 0^+$, we obtain

$$\int_t^\alpha \tilde{\mathcal{V}}\left(\frac{\big(\inf w\big)K(y,t)h(y)\big(\mathcal{V}\circ\mathcal{U}^{-1}\big)\Big(\int_\epsilon^t \omega\Big)}{C\phi(y)\psi(y)}\right)\psi(y)dy \leq \Big(\mathcal{V}\circ\mathcal{U}^{-1}\Big)\Big(\int_\epsilon^t \omega(y)dy\Big).$$

We skip the proof of (5.65) as it proceeds similarly. Hence the proof is complete.

□