Chapter 8

A Study on Some Properties of Poisson Size-biased Quasi Lindley Distribution

8.1 Introduction

As mentioned in chapter 2 and 3, size-biased distribution is a special case of the more general form known as weighted distribution. These distributions were first introduced by Fisher (1934) to model ascertainment biases which were later formalized by Rao (1965) in a unifying theory. Size-biased distributions arise in practice when observations from a sample are recorded with unequal probabilities having probability proportional to some measure of unit size. About size-biased distribution, we have already discussed in previous chapters 2 and 3.

In this chapter, the Poisson size-biased quasi-Lindley (PSBQL) distribution of which the Poisson size-biased Lindley distribution of Adhikari and Srivastava (2014) [mentioned in chapter 3] is a particular case has been introduced. An attempt has been made to study some properties of the distribution including probability generating function, a general expression for the r^{th} order factorial moments, raw and central moments, coefficient of variation, index of dispersion, coefficients of skewness and kurtosis and problem of parameter estimation etc.

8.2 Poisson size-biased quasi Lindley distribution

The Poisson size-biased quasi Lindley (PSBQL) distribution can be obtained by compounding the Poisson distribution with size-biased quasi-Lindley distribution having probability density function (pdf)

$$f(\lambda;\ \theta,\alpha) = \frac{\theta^2}{\alpha+2}\lambda(\alpha+\lambda\theta)e^{-\theta\lambda}\ ;\ \ \lambda > 0, \theta > 0, \alpha > -2 \tag{8.2.1}$$

The resultant probability mass function (pmf) of the PSBQL distribution may be obtained as

$$P(x;\ \theta,\alpha) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{(\alpha+2)}\lambda(\alpha+\theta\lambda)e^{-\theta\lambda}d\lambda \tag{8.2.2}$$

$$= \frac{\theta^2}{\alpha+2}\frac{1}{x!}\int_0^\infty e^{-(1+\theta)\lambda}(\alpha\lambda^{x+1}+\theta\lambda^{x+2})d\lambda$$

$$= \frac{\theta^2}{\alpha+2}\left[\frac{\alpha(x+1)}{(1+\theta)^{x+2}} + \frac{\theta(x+1)(x+2)}{(1+\theta)^{x+3}}\right]$$

$$= \frac{\theta^2}{(\alpha+2)}\frac{(1+x)(\alpha+\alpha\theta+2\theta+\theta x)}{(1+\theta)^{x+3}}\ ;\ x = 0,1,2,\ldots,\theta > 0, \alpha > -2 \tag{8.2.3}$$

While θ is the scale and α the shape parameters of the distribution. Simply denote it by PSBQL (θ,α). It can be seen that, the Poisson size-biased Lindley distribution of Adhikari and Srivastava (2014) mentioned in chapter 3 is a particular case of the PSBQL distribution at $\alpha = \theta$.

Lemma 8.1 $P(x;\ \theta,\alpha)$ given in (8.2.3) is a well-defined probability mass function.

Since, the ratio

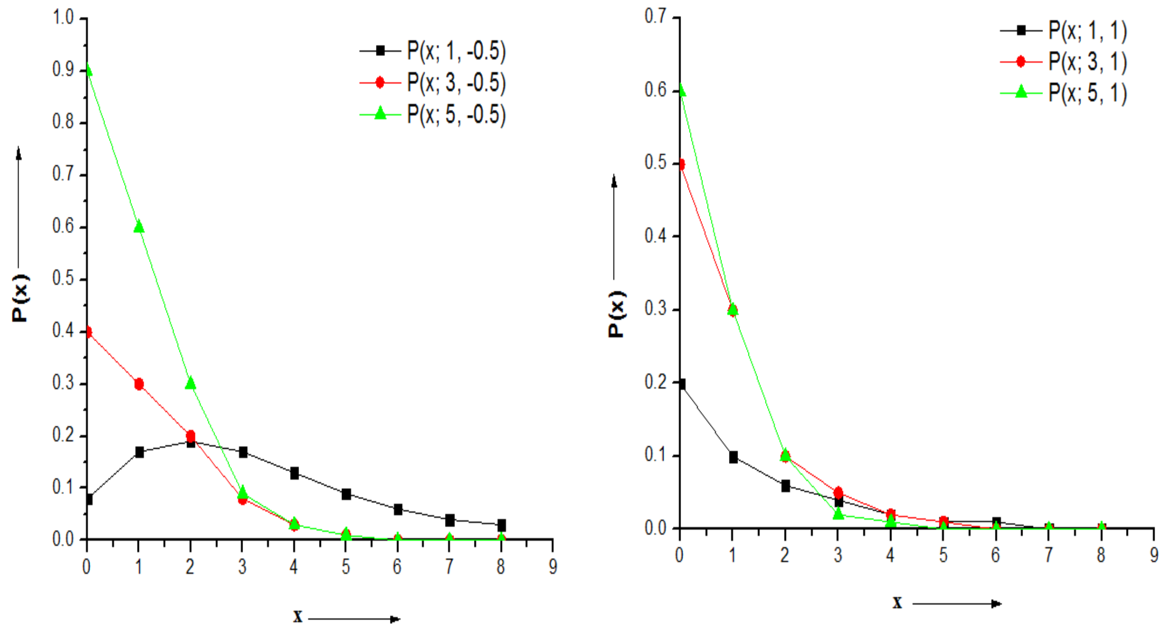
$$\frac{P(x+1;\theta,\alpha)}{P(x;\theta,\alpha)} = \left(1+\frac{1}{x+1}\right)\left(1+\frac{\theta}{\alpha+\alpha\theta+2\theta+\theta x}\right) \tag{8.2.4}$$

is a decreasing function of x, $P(x;\,\theta,\alpha)$ is log-concave. Therefore, the PSBQL distribution is unimodel, has an increasing failure rate. [cf. Johnson et al. (2005)]

8.3 Graphical Representations

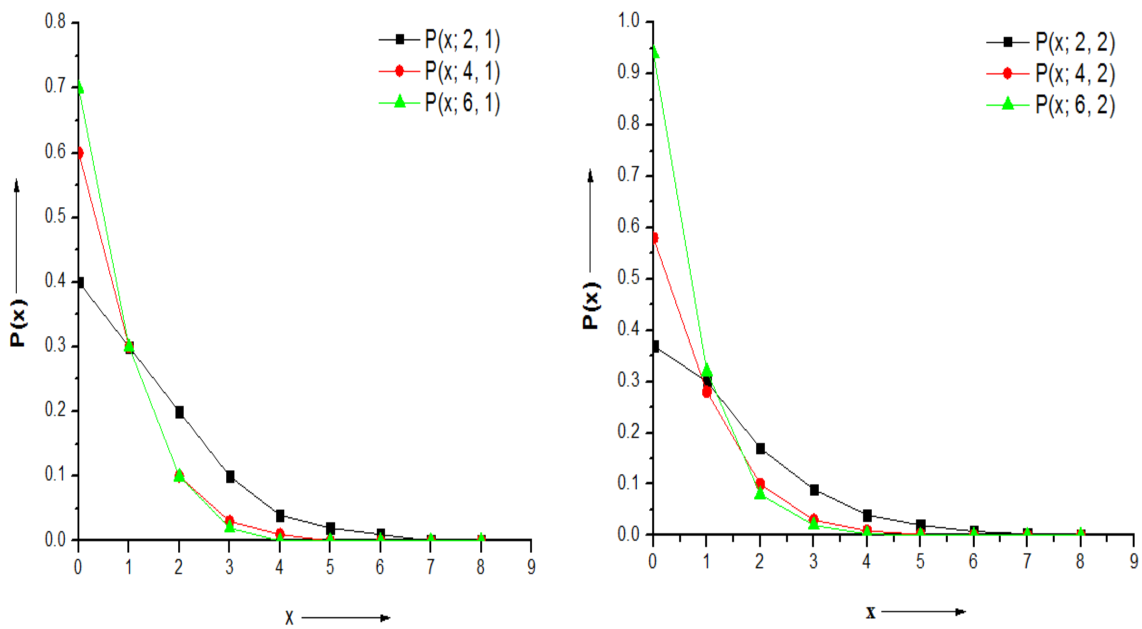
To see the behavior of the PSBQL distribution plots the probabilities of the distribution by taking values of x along the horizontal axis and the probabilities along the vertical axis for different values of parameters θ and α in **Fig. 8.1** and **8.2.**

Fig. 8.1 Plots of probability P(x; θ,α) for (i) $\alpha = -0.5$ and $\theta = 1, 3, 5$, (ii) $\alpha = 1, \theta = 1,3,$ respectively for the PSBQL distribution.



From the above figures it is seen that, when α is fixed and θ increases slowly, the value of P(x; θ,α) is maximum at $x = 0$ and decreases for other increasing values of x .

Fig. 8.2 Plots of probability P(x; θ, α) for (i)$\alpha = 1$ and $\theta = 2,4,6$, (ii) $\alpha = 2, \theta = 2,4$ and 6 respectively for the PSBQL distribution.



From the above figure it is observed that, the value P(x; θ, α) is maximum at $x = 0$ and decreases sharply to a certain point then slowly decreases giving a L-shaped model.

8.4 Statistical Properties

In this section, some properties of the PSBQL distribution have been studied as follows:

8.4.1 Probability generating function

Let X be a random variable with PSBQL (θ, α) distribution. Then the probability generating function (pgf), $g(t)$ of X can be obtained as

$$g(t) = \sum_{x=o}^{\infty} t^x P(x;\ \theta, \alpha)$$

$$= \frac{\theta^2}{(\alpha+2)(1+\theta)^3} \sum_{x=0}^{\infty} t^x \frac{(1+x)(\alpha+\alpha\theta+2\theta+\theta x)}{(1+\theta)^x}$$

$$= \frac{\theta^2}{(\alpha+2)} \frac{(\alpha+\alpha\theta+2\theta-\alpha t)}{(1+\theta-t)^3}, \quad \theta > 0, \alpha > -2 \qquad (8.4.1)$$

The pgf, $g(t) = \frac{\theta^3}{(\theta+2)} \frac{(\theta+3-t)}{(1+\theta-t)^3}$; $\theta > 0$ of the Poisson size-biased Lindley distribution, discussed in chapter 3 may be obtained as a particular case of the PSBQL distribution at $\alpha = \theta$.

The expression for probabilities of the PSBQL distribution obtained from (8.4.1) is given as

$$P_r = \frac{1}{(1+\theta)^3} [3(1+\theta)^2 P_{r-1} - 3(1+\theta) P_{r-2} + P_{r-3}]\ ;\ \ r > 3 \qquad (8.4.2)$$

where, $P_0 = \frac{\theta^2}{(\alpha+2)} \frac{(\alpha+\alpha\theta+2\theta)}{(1+\theta)^3}$, $P_1 = \frac{\theta^2}{(\alpha+2)} \frac{2(\alpha+\alpha\theta+3\theta)}{(1+\theta)^4}$

$P_2 = \frac{\theta^2}{(\alpha+2)} \frac{3(\alpha+\alpha\theta+4\theta)}{(1+\theta)^5}$, $P_3 = \frac{\theta^2}{(\alpha+2)} \frac{4(\alpha+\alpha\theta+4\theta)}{(1+\theta)^6}$ so on.

The probability $P(X = r)$ of the PSBQL distribution may also be written as

$$P_r = \frac{\theta^2}{(\alpha+2)} \frac{(r+1)[\alpha+\alpha\theta+(r+2)\theta]}{(1+\theta)^{r+3}}\ ; r = 0, 1, 2, \dots.;\ \theta > 0, \alpha > -2 \qquad (8.4.3)$$

The higher order probabilities may be obtained very easily by using expression (8.4.3).

8.4.2 Moment generating function

The distribution of a random variable is often characterized in terms of its moment generating function (mgf), a real function whose derivatives at zero are equal to the moments of the random variable. Moment generating functions have great practical relevance not only because they can be used to easily derive moments, but also because a probability distribution is uniquely determined by its moment generating function.

If $X \backsim PSBQL\ (\theta, \alpha)$, then the mgf of X can be derived as

$$M_x(t) = \sum_{x=0}^{\infty} e^{tx}\, P(x;\ \alpha, \theta) = \sum_{x=0}^{\infty} e^{tx}\, \frac{\theta^2}{(\alpha+2)} \frac{(1+x)(\alpha+\alpha\theta+2\theta+\theta x)}{(1+\theta)^{x+3}}$$

$$= \frac{\theta^2}{\alpha+2} \frac{(\alpha+\alpha\theta+2\theta-\alpha e^t)}{(1+\theta-e^t)^3}\ ;\ \theta > 0, \alpha > -2 \qquad (8.4.4)$$

The mgf of the PSBQL distribution is same as that of the Poisson size-biased Lindley distribution of Adhikari and Srivastava (2014) at $\alpha = \theta$.

8.4.3 Factorial Moments

The r^{th} order factorial moment of the PSBQL distribution can be obtained as

$$\mu'_{(r)} = E\left[E\left(X^{(r)}|\lambda\right)\right] \qquad (8.4.5)$$

where, $X^{(r)} = X(X-1)(X-2) \ldots\ldots (X-r+1)$

From equation (8.2.2), we get

$$\mu'_{(r)} = \int_0^{\infty} \left[\sum_{x=o}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}\right] \frac{\theta^2}{(\alpha+2)} \lambda(\alpha+\theta\lambda) e^{-\theta\lambda} d\lambda \qquad (8.4.6)$$

$$= \int_0^{\infty} \left[\lambda^r \sum_{x=r}^{\infty} \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right] \frac{\theta^2}{(\alpha+2)} \lambda(\alpha+\theta\lambda) e^{-\theta\lambda} d\lambda \qquad (8.4.7)$$

Taking $(x+r)$ in place of x, we get

$$\mu'_{(r)} = \int_0^\infty \lambda^r \left[\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}\right] \frac{\theta^2}{(\alpha+2)} \lambda(\alpha+\theta\lambda)e^{-\theta\lambda} d\lambda \qquad (8.4.8)$$

The expression within bracket is equal to one and hence we have

$$\mu'_{(r)} = \frac{\theta^2}{(\alpha+2)} \int_0^\infty \lambda^{r+1}(\alpha+\theta\lambda)e^{-\theta\lambda} d\lambda \qquad (8.4.9)$$

Using gamma integral, we get finally, after a little simplification, a general expression for the r^{th}order factorial moment of the PSBQL distribution as

$$\mu'_{(r)} = \frac{\Gamma(r+2)}{(\alpha+2)\theta^r}[(\alpha+r+2)]\,; \quad r = 1,2,3,\dots,\theta > 0, \alpha > -2 \qquad (8.4.10)$$

The factorial moments of the PSBQL distribution is obtained very easily by using (8.4.10) as follows

$$\mu'_{(1)} = \frac{2!(\alpha+3)}{\theta(\alpha+2)} \quad \text{(Mean)}$$

$$\mu'_{(2)} = \frac{3!(\alpha+4)}{\theta^2(\alpha+2)}$$

$$\mu'_{(3)} = \frac{4!(\alpha+5)}{\theta^3(\alpha+2)}$$

$$\mu'_{(4)} = \frac{5!(\alpha+6)}{\theta^4(\alpha+2)} \quad \text{etc.}$$

8.4.4 Raw and Central moments

The r^{th} order moment about the origin (raw moment) of the PSBQL distribution can be obtained as

$$\mu'_r = E[E(X^r|\lambda)] \qquad (8.4.11)$$

From equation (8.2.2), we get

$$\mu'_r = \int_0^\infty \left[\sum_{x=o}^{\infty} x^r \frac{e^{-\lambda}\lambda^x}{\Gamma(1+x)}\right] \frac{\theta^2}{(\alpha+2)} \lambda(\alpha+\theta\lambda)e^{-\theta\lambda} d\lambda \qquad (8.4.12)$$

Obviously the expression under bracket is the r^{th} order moment about origin of the Poisson distribution. Taking $r = 1$ in (8.4.12) and then using the mean of the Poisson distribution, the mean of the PSBQL distribution is obtained as

$$\mu'_1 = \frac{\theta^2}{(\alpha+2)} \int_0^\infty \lambda^2(\alpha+\theta\lambda)e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^2}{(\alpha+2)} \left[\frac{\Gamma3(\alpha+3)}{\theta^3}\right]$$

$$= \frac{2(\alpha+3)}{\theta(\alpha+2)} \qquad (8.4.13)$$

Taking $r = 2$ in (8.4.12) and using the second moment about origin of the Poisson distribution, the second moment about origin of the PSBQL distribution becomes

$$\mu'_2 = \frac{\theta^2}{(\alpha+2)} \int_0^\infty (\lambda^2+\lambda)\lambda(\alpha+\theta\lambda)e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^2\Gamma3}{(\alpha+2)} \left[\frac{3(\alpha+4)}{\theta^4} + \frac{(\alpha+3)}{\theta^3}\right]$$

$$= \frac{2\theta(\alpha+3)+6(\alpha+4)}{\theta^2(\alpha+2)} \qquad (8.4.14)$$

Substituting $r = 3$ and $r = 4$ in (8.4.12) and using respective moments of the Poisson distribution we obtain the third and the fourth moments of the discrete PSBQL distribution as

$$\mu'_3 = \frac{2\theta^2(\alpha+3)+18\theta(\alpha+4)+24(\alpha+5)}{\theta^3(\alpha+2)} \qquad (8.4.15)$$

$$\mu_4' = \frac{2\theta^3(\alpha+3)+42\theta^2(\alpha+4)+144\theta(\alpha+5)+120(\alpha+6)}{\theta^4(\alpha+2)} \qquad (8.4.16)$$

It can be seen that, at $\alpha = \theta$ these moments reduces to the respective moments of the Poisson size-biased Lindley distribution mentioned in chapter 3. The moments about origin of the PSBQL distribution can be also obtained by using the relationship between factorial moments and moments about origin of the distribution.

The first three central moments of the PSBQL distribution may be obtained as

$$\mu_2 = \frac{2[\theta(\alpha+2)(\alpha+3)+(\alpha^2+6\alpha+6)]}{\theta^2(\alpha+2)^2} \text{ (Variance)} \qquad (8.4.17)$$

$$\mu_3 = \frac{2[\theta^2(\alpha+3)(\alpha+2)^2+3\theta(\alpha^3+8\alpha^2+18\alpha+12)+2(\alpha^3+9\alpha^2+18\alpha+12)]}{\theta^3(\alpha+2)^3} \qquad (8.4.18)$$

$$\mu_4 = \frac{2[\theta^3(\alpha+3)(\alpha+2)^3+2\theta^2(9\alpha^4+65\alpha^3+230\alpha^2+348\alpha+192)+24\theta(\alpha^4+11\alpha^3+39\alpha^2+42\alpha+30) -3(19\alpha^4+228\alpha^3+119\alpha^2+2106\alpha+1536)]}{\theta^4(\alpha+2)^4}$$

(8.4.19)

8.4.5 Coefficients of Skewness and Kurtosis

A fundamental task in many statistical analyses is to characterize the location and variability of a distribution. Further characterizations of the distribution include skewness and kurtosis. The coefficients of skewness $\sqrt{\beta_1}$ and kurtosis β_2 for the PSBQL distribution can be derived as

$$\sqrt{\beta_1} = \frac{[\theta^2(\alpha+3)(\alpha+2)^2+3\theta(\alpha^3+8\alpha^2+18\alpha+12)+2(\alpha^3+9\alpha^2+18\alpha+12)]}{\sqrt{2[\theta(\alpha+2)(\alpha+3)+(\alpha^2+6\alpha+6)]^3}} \qquad (8.4.20)$$

and $$\beta_2 = \frac{\begin{array}{r}[\theta^3(\alpha+3)(\alpha+2)^3+2\theta^2(9\alpha^4+65\alpha^3+230\alpha^2+348\alpha+192)+24\theta(\alpha^4+11\alpha^3+39\alpha^2+42\alpha+30) \\ -3(19\alpha^4+228\alpha^3+119\alpha^2+2106\alpha+1536)]\end{array}}{2[\theta(\alpha+2)(\alpha+3)+(\alpha^2+6\alpha+6)]^2}$$

(8.4.21)

Both $\sqrt{\beta_1}$ and β_2 are an increasing function in θ when α is fixed.

8.4.6 Coefficient of Variation

The coefficient of variation (CV) is the ratio of the standard deviation to the mean. The higher is the coefficient of variation, the greater the level of dispersion around the mean. The coefficient of variation (CV) of the PSBQL distribution is given by

$$CV = \frac{\sqrt{2[\theta(\alpha+2)(\alpha+3)+(\alpha^2+6\alpha+6)}}{(\alpha+3)\sqrt{2}} \qquad (8.4.22)$$

The coefficient of variation (CV) of the PSBQL distribution increases, when both the parameter θ and α increases.

8.4.7 Index of dispersion

The index of dispersion for the PSBQL distribution is obtained as

$$\gamma = \frac{\sigma^2}{\mu} = \frac{\theta(\alpha+2)(\alpha+3)+(\alpha^2+6\alpha+6)}{\theta(\alpha+2)(\alpha+3)}$$

$$= 1 + \frac{\alpha^2+6\alpha+6}{\theta(\alpha+2)(\alpha+3)} \qquad (8.4.23)$$

As the values of α increases, the coefficient of dispersion decreases slowly. Whereas the coefficient of dispersion increases to a certain point then it decreases as θ increases. Graphical representations of the index of dispersion for varying values of the parameters have been shown in **Fig. 8.3** and **8.4.**

Fig. 8.3 The index of dispersion increases to a certain point then it decreases as θ increases for different values of α for the PSBQL distribution.

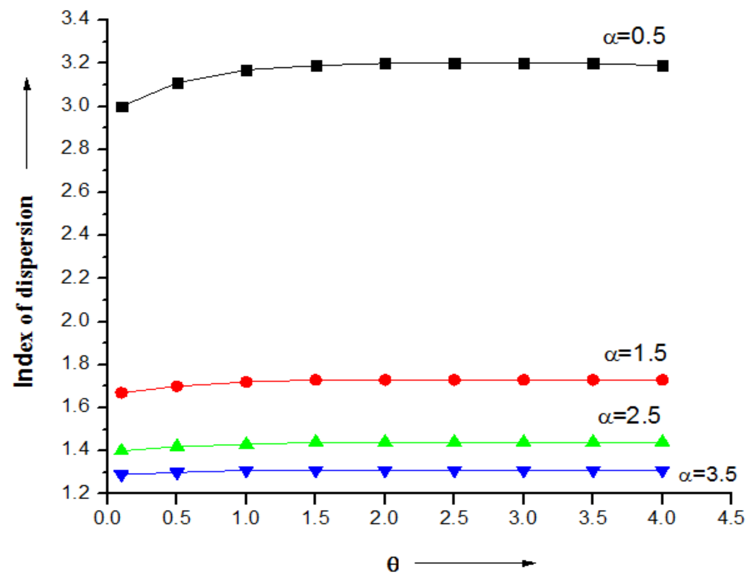
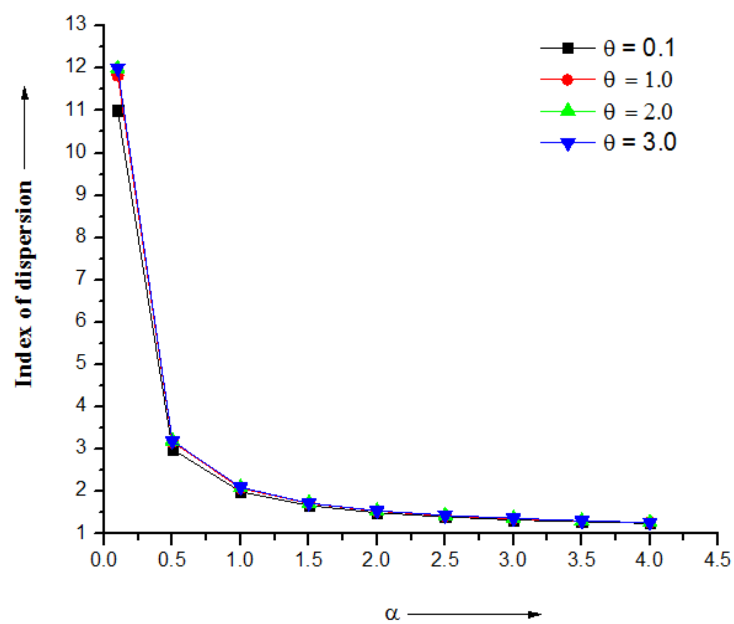


Fig. 8.4 The index of dispersion decreases sharply to a certain point then slowly decreases as the values of α increases for different values θ for the PSBQL distribution.



From the above figures it is observed that as the values of α increases, the index of dispersion decreases sharply to a certain point then slowly decreases. Whereas the index of dispersion increases to a certain point then it decreases as θ increases.

8.4.8 Parameter Estimation

The problem of parameter estimation is one of the most important properties of a distribution. One of the most important and oldest methods of estimation is the method of moments. To estimate the parameters of the PSBQL distribution, we have been used method of moment estimation procedure as follows:

8.4.8.1 Estimation based on the method of moments

The PSBQL distribution has two parameters, viz. α and θ. The first two moments are required to get the estimates of these parameters by the method of moments.

From equations (8.4.13) and (8.4.14), we have

$$\frac{\mu_2' - \mu_1'}{(\mu_1')^2} = \frac{6(\alpha+4)(\alpha+2)}{4(\alpha+3)^2} = K \text{ (say)} \qquad (8.4.24)$$

which gives a quadratic equation in α as

$$g(\alpha) = (3-2K)\alpha^2 + (18-12K)\alpha + (24-18K) = 0 \qquad (8.4.25)$$

Now, replacing the first two population moments by the respective sample moments in (8.4.24) an estimate k of K can be obtained and using it in (8.4.25), an estimate $\hat{\alpha}$ of α can be obtained.

Again, substituting the value of $\hat{\alpha}$ in (8.4.13) and replacing the population mean by the sample mean $\bar{x},$ an estimate of θ is obtained as

$$\hat{\theta} = \frac{2(\alpha+3)}{\bar{x}(\alpha+2)}; \quad \bar{x} \text{ be the sample mean.} \qquad (8.4.26)$$

8.5 Conclusion

Size-biased distributions arise in practice when observations from a sample are recorded with unequal probabilities having probability proportional to some measure of unit size. In this chapter, the Poisson size-biased quasi Lindley distribution is obtained by compounding the Poisson distribution with size-biased quasi Lindley distribution without considering its size-biased form. We have been studied some distributional properties of the distribution including problem of parameter estimation by method of moments also in this chapter.

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