Chapter 3

On Certain Recurrence Relations Arising in Different Forms of Size-biased Poisson-Lindley Distributions

3.1 Introduction

As mentioned in chapter 2, size-biased distributions arise in practice when observations from a sample are recorded with unequal probabilities, having probability proportional to some measure of unit size. In many situation investigators do not work with truly random sample from the population, in which they are interested. However, since the observations do not have an equal probability of entering the sample, the resulting sampled distribution does not follow the original distribution. Statistical models that incorporate these restrictions are called weighted models. The weighted distribution with weight function $w(x) = x$ is called size-biased distribution. Patil and Rao (1978) examined some general models leading to weighted distributions. A study of size-biased sampling and related form-invariant weighted distributions was made by Patil and Ord (1975).

Ghitany and Al-Mutairi (2008) proposed size-biased Poisson-Lindley distribution by considering the size biased form of Poisson-Lindley distribution and suggested its applications.

They obtained the probability mass function (pmf) of the size-biased Poisson-Lindley (SBPL) distribution as

$$P_1(x;\theta) = \frac{x}{\mu} P(\,x;\theta) = \frac{\theta^3}{(\theta+2)} \frac{x(x+\theta+2)}{(1+\theta)^{x+1}}\,, \qquad x = 1, 2, \ldots;\;\; \theta > 0. \tag{3.1.1}$$

where, $\mu_0 = \frac{\theta+2}{\theta(1+\theta)}$ is the mean and $P(\,x;\theta)$ be the probability mass function of the Poisson-Lindley distribution mentioned in chapter 2. The mean, variance, coefficient of skewness and coefficient of kurtosis for the SBPL were also obtained by them.

Adhikari and Srivastava (2013) proposed a new form of size-biased Poisson-Lindley (SBPL1) distribution obtained by compounding the size-biased Poisson distribution with Lindley distribution due to Lindley (1958) without considering its size-biased form, which is obtained as

$$P_2(x;\theta) = \int_0^\infty \left[\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\right] \frac{\theta^2}{(1+\theta)} (1+\lambda) e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^2}{(1+\theta)^{x+2}}\;\; (x+\theta+1),\;\; x = 1, 2, 3, \ldots, \theta > 0 \tag{3.1.2}$$

Since, the ratio

$$\frac{P_2(x+1;\,\theta)}{P_2(x;\,\theta)} = \frac{1}{(1+\theta)}\left(1 + \frac{1}{x+\theta+1}\right) \tag{3.1.3}$$

is a decreasing function of x , $P_2(x;\,\theta)$ is log-concave. Therefore, the new form of size-biased Poisson-Lindley distribution is unimodal and has an increasing failure rate (IFR).

The mean (μ_1), variance (μ_2), co-efficient of skewness ($\sqrt{\beta_1}$) and coefficient of kurtosis(β_2) for new form of size-biased Poisson-Lindley distribution proposed by Adhikari and Srivastava (2013) are as follows

$$\mu_1 = \frac{\theta^2+2\theta+2}{\theta(1+\theta)} \tag{3.1.4}$$

$$\mu_2 = \frac{\theta^3+4\theta^2+6\theta+2}{\theta^2(1+\theta)^2} \tag{3.1.5}$$

$$\sqrt{\beta_1} = \frac{\theta^5+7\theta^4+22\theta^3+32\theta^2+18\theta+4}{(\theta^3+4\theta^2 6\theta+2)^{3/2}} \tag{3.1.6}$$

$$\beta_2 = \frac{\theta^7+15\theta^6+87\theta^5+258\theta^4+406\theta^3+338\theta^2+144\theta+24}{(\theta^3+4\theta^2+6\theta+2)^2} \tag{3.1.7}$$

Adhikari and Srivastava (2014) also proposed the Poisson-size-biased Lindley (PSBL) distribution obtained by compounding the Poisson distribution with size-biased Lindley distribution and also investigated some properties of the proposed distribution. The pmf of the distribution is obtained as

$$P_3(x;\ \theta)=\int_0^\infty \frac{e^{\lambda}\lambda^x}{x!}\frac{\theta^3}{(\theta+2)}\lambda(1+\lambda)\mathrm{e}^{-\theta\lambda}\mathrm{d}\lambda$$

$$= \frac{\theta^3}{(1+\theta)^{x+3}}\frac{1}{(\theta+2)}(x+1)(x+\theta+3)\ ,\ x = 0,1,2,\dots\ ,\ \theta > 0 \tag{3.1.8}$$

They showed that, since the ratio

$$\frac{P_3(x+1;\theta)}{P_3(x;\theta)} = \left(\frac{1}{1+\theta}\right)\left(1+\frac{1}{x+1}\right)\left(1+\frac{1}{x+\theta+3}\right) \tag{3.1.9}$$

is a decreasing function in x, $P_3(x;\ \theta)$ is log-concave. Therefore the PSBL distribution is unimodal, has an increasing failure rate (IFR).

The organization of this chapter is as follows: In section 3.2, above mentioned two distributions SBPL1 and PSBL are further investigated by working out some recursive or recurrence relations arising in case of probabilities, moments and cumulants of these distributions. Section 3.3 deals with the cumulative distribution functions and moment generating functions of the SBPL1 and PSBL distributions. The problem of estimate the parameter of these distributions by using different methods is discussed in section 3.4. Finally, to see the suitability of the SBPL1 and PSBL distributions fitting these distributions to some reported data sets in section 3.5.

3.2 Recurrence or Recursive relations of probabilities, factorial moments and cumulants

The probability Generating function (pgf), factorial moment generating function (fmgf) and cumulant generating function (cgf) of the SBPL1 and PSBL distributions are listed below as

SBPL1 distribution	1. pgf	$g(t) = \frac{\theta^2 t(\theta+2-t)}{(\theta+1)(\theta+1-t)^2}\,;\, \theta > 0$ (3.2.1)
	2. fmgf	$m(t) = \frac{\theta^2(t+1)(\theta+1-t)}{(\theta+1)(\theta-t)^2}$ (3.2.2)
	3. cgf	$K(t) = log\left[\frac{\theta^2 e^t(\theta+2-e^t)}{(\theta+1)(\theta+1-e^t)^2}\right]$ (3.2.3)
PSBL distribution	4. pgf	$g(t) = \frac{\theta^3(\theta+3-t)}{(\theta+2)(\theta+1-t)^3}\,;\, \theta > 0$ (3.2.4)
	5. fmgf	$m(t) = \frac{\theta^3(\theta+2-t)}{(\theta+2)(\theta-t)^3}$ (3.2.5)
	6. cgf	$K(t) = log\left[\frac{\theta^3(\theta+3-e^t)}{(\theta+2)(\theta+1-e^t)^3}\right]$ (3.2.6)

3.2.1 Recursive relation of probabilities

The recursive relation of probabilities corresponding to the probability generating function (3.2.1) of the SBPL1 distribution may be obtained by equating the coefficients of

't^r' on both sides of (3.2.1) written as

$$P_r = \frac{1}{(1+\theta)^2}[2(1+\theta)P_{r-1} - P_{r-2}]; \;\; r > 2 \tag{3.2.7}$$

where, $P_1 = \frac{\theta^2(\theta+2)}{(1+\theta)^3}$

$P_2 = \frac{\theta^2(\theta+3)}{(1+\theta)^4}$

The general expression for r^{th} order probability of the SBPL1 distribution is written as

$$P_r = \frac{\theta^2(\theta+r+1)}{(1+\theta)^{r+2}} \;\; ; \; r = 1, 2, \ldots \tag{3.2.8}$$

The higher order probabilities of the SBPL1 distribution for calculation purpose may be obtained very easily by using either (3.2.7) or (3.2.8) as

$P_3 = \frac{\theta^2(\theta+4)}{(1+\theta)^5}$

$P_4 = \frac{\theta^2(\theta+5)}{(1+\theta)^6}$

$P_5 = \frac{\theta^2(\theta+6)}{(1+\theta)^7}$ etc.

Similarly, for the PSBL distribution the recursive relation of probabilities obtained from equation (3.2.4) by equating the coefficients of 't^r' on both sides of (3.2.4) as

$$P_r = \frac{1}{(1+\theta)^3}[3(1+\theta)^2 P_{r-1} - 3(1+\theta)P_{r-2} + P_{r-3}] \; ; \;\; r > 3 \tag{3.2.9}$$

where, $P_0 = \frac{\theta^3(\theta+3)}{(\theta+2)(1+\theta)^3}$ $\qquad P_1 = \frac{\theta^3 2(\theta+4)}{(\theta+2)(1+\theta)^4}$,

$P_2 = \frac{\theta^3 3(\theta+5)}{(\theta+2)(1+\theta)^5}$ $\qquad P_3 = \frac{\theta^3 4(\theta+6)}{(\theta+2)(1+\theta)^6}$ etc.

The general expression for r^{th} order probability of the PSBL distribution is expressed as

$$P_r = \frac{\theta^3(r+1)(\theta+r+3)}{(\theta+2)(1+\theta)^{r+3}} \quad ; r = 0, 1, 2, ... \tag{3.2.10}$$

The higher order probabilities of the PSBL distribution can be obtained very easily by using either relation (3.2.9) or (3.2.10).

3.2.2 Recursive relation of factorial moments and related measures

Equating the coefficient of $\frac{t^r}{r!}$ on both sides of equation (3.2.2), we get the recursive relation of factorial moments for the SBPL1 distribution as

$$\mu'_{(r)} = \frac{r}{\theta^2}\left[2\theta\mu'_{(r-1)} - (r-1)\mu'_{(r-2)}\right]; \; r > 2 \tag{3.2.11}$$

where, $\mu'_{(1)} = \frac{\theta^2+2\theta+2}{\theta(1+\theta)}$ and $\mu'_{(2)} = \frac{2(\theta^2+3\theta+3)}{\theta^2(1+\theta)}$

The general expression for r^{th} order factorial moment is given by

$$\mu'_{(r)} = \frac{r![(r+1)(1+\theta)]}{\theta^r(1+\theta)} \quad ; \; r = 1, 2, ... \tag{3.2.12}$$

After obtaining the first four factorial moments, by substituting $r = 1, 2, 3$ and 4 in (3.2.12) and then using the relationship between factorial moments and moments about origin, the first four moments about origin(raw moments) of the SBPL1 distribution were obtained as follows

$$\mu'_1 = \frac{\theta^2+2\theta+2}{\theta(1+\theta)}$$

$$\mu'_2 = \frac{\theta^3+4\theta^2+8\theta+6}{\theta^2(1+\theta)}$$

$$\mu_3' = \frac{\theta^4+8\theta^3+26\theta^2+42\theta+24}{\theta^3(1+\theta)}$$

$$\mu_4' = \frac{\theta^5+16\theta^4+80\theta^3+930\theta^2+984\theta+120}{\theta^4(1+\theta)} \quad .$$

where μ_r' denotes the r^{th} order raw moment of the SBPL1 distribution.

The mean (μ) and variance (σ^2) of the SBPL1 distribution are respectively given by

$$\mu = \frac{\theta^2+2\theta+2}{\theta(1+\theta)} \tag{3.2.13}$$

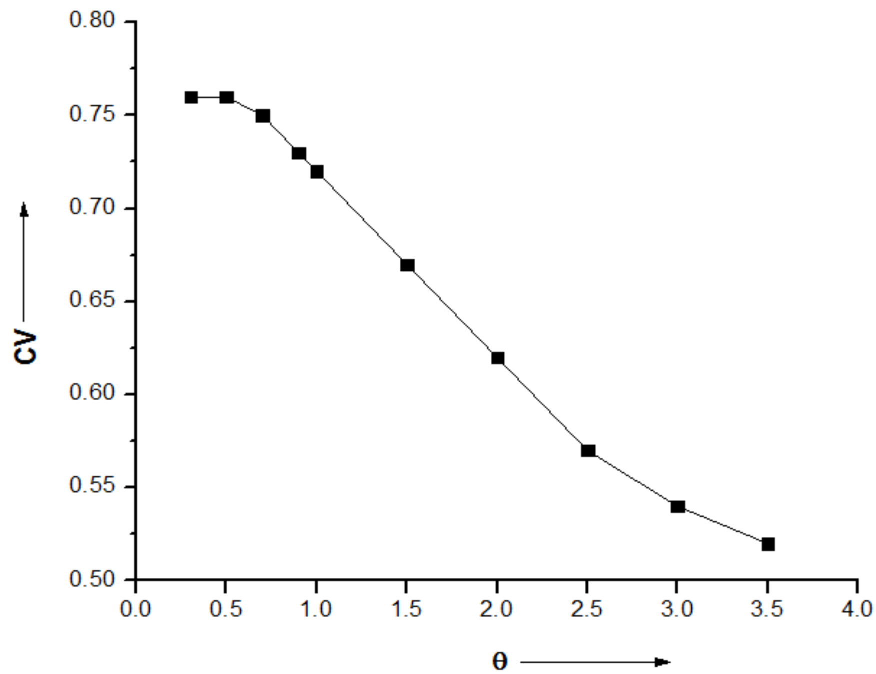
$$\sigma^2 = \frac{\theta^3+4\theta^2+6\theta+2}{\theta^2(1+\theta)^2} \tag{3.2.14}$$

The coefficient of variation (CV) is the ratio of the standard deviation to the mean. The higher is the coefficient of variation, the greater the level of dispersion around the mean. It is generally expressed as a percentage. The coefficient of variation of the SBPL1 distribution is given below

$$C_v = \frac{\sigma}{\mu} = \frac{\sqrt{\theta^3+4\theta^2+6\theta+2}}{\theta^2+2\theta+2} \tag{3.2.15}$$

To see the effect in changes of coefficient of variation of the SBPL1 distribution for varying values of the parameter θ is shown in **Fig. 3.1**.

Fig. 3.1 The coefficient of variation of the SBPL1 distribution for different values of θ.



From the above figure it is seen that, as the values of θ increases, the coefficient of variation of the SBPL1 distribution decreases slowly.

The factorial moment recursive relation for the PSBL distribution obtained from fmgf (3.2.5) may be written as

$$\mu'_{(r)} = \frac{r(r-1)}{\theta^3}\left[3\theta^2\mu'_{(r-1)} - (r-2)\mu'_{(r-2)} + (r-2)(r-3)\mu'_{(r-3)}\right];\ r > 3 \qquad (3.2.16)$$

where $\mu'_{(1)} = \frac{2(\theta+3)}{\theta(\theta+2)}$

$$\mu'_{(2)} = \frac{6(\theta+4)}{\theta^2(\theta+2)}$$

$$\mu'_{(3)} = \frac{24(\theta+5)}{\theta^3(\theta+2)} \quad \text{etc.}$$

The general form of r^{th} order factorial moment of the SBPL2 distribution is given as

$$\mu'_{(r)} = \frac{(r+1)!(\theta+2+r)}{\theta^r(\theta+2)} \;;\quad r = 1, 2, \ldots \tag{3.2.17}$$

Higher order factorial moments of the PSBL distribution can be obtained by using either relation (3.2.16) or (3.2.17).

The first four moments about origin (raw moments) of the PSBL distribution were obtained as follows

$$\mu'_1 = \frac{2(\theta+3)}{\theta(\theta+2)}$$

$$\mu'_2 = \frac{2\theta^2+12\theta+24}{\theta^2(\theta+2)}$$

$$\mu'_3 = \frac{2\theta^3+22\theta^2+96\theta+120}{\theta^3(\theta+2)}$$

$$\mu'_4 = \frac{2\theta^4+48\theta^3+312\theta^2+840\theta+720}{\theta^4(\theta+2)} \quad \text{respectively.}$$

The mean (μ) and variance (σ^2)of the PSBL distribution are given as

$$\mu = \frac{2(\theta+3)}{\theta(\theta+2)} \tag{3.2.18}$$

$$\sigma^2 = \frac{2\theta^3+12\theta^2+24\theta+12}{\theta^2(\theta+2)^2} \tag{3.2.19}$$

The coefficient of variation of the PSBL distribution is given as

$$C_v = \frac{\sigma}{\mu} = \frac{\sqrt{2\theta^3+12\theta^2+24\theta+12}}{2(\theta+3)} \;;\; \theta > 0 \tag{3.2.20}$$

It is seen from the above expression that, as the values of θ increases, the coefficient of variation (CV) of the PSBL distribution decreases.

3.2.3 Recursive relation of cumulants

Differentiating cumulant generating function (3.2.3) w.r.to‘t’ and then equating the coefficient of $\frac{t^r}{r!}$ on both sides gives the recursive relation of cumulants for the SBPL1 distribution as

$$K_{r+1} = \frac{1}{\theta(1+\theta)}\left[\sum_{j=1}^{r}\left(3+2\theta-2^j\right)\binom{r}{j}K_{r+1-j}-\theta\right] \;;\quad r=1,2,\ldots \qquad (3.2.21)$$

where, $K_1 = \frac{\theta^2+2\theta+2}{\theta(1+\theta)}$ (Mean)

Putting $r = 1,2,3,\ldots$ in equation (3.2.21), the higher order cumulants may be obtained very easily.

Similarly, the cumulant recursive relation for the PSBL distribution may be expressed as

$$K_{r+1} = \frac{1}{\theta(\theta+2)}\left[2(\theta+4-2^r)+\sum_{j=1}^{r}(4+2\theta-2^j)\binom{r}{j}K_{r+1-j}\right] \;;\; r=1,2,\ldots \qquad (3.2.22)$$

where, $K_1 = \frac{2(\theta+3)}{\theta(\theta+2)}$ (Mean)

Sometimes it is difficult to obtain the moments of a distribution by using moment generating function, in such situation cumulants is used to obtain the moments of a distribution.

3.3 Cumulative distribution function and Moment generating function

Let X is a random variable with SBPL1 distribution. Then the cumulative distribution function of X is written as

$$F_X(x) = \frac{\theta^2}{(1+\theta)^2} \sum_{j=1}^{x} \frac{(1+\theta+j)}{(1+\theta)^j} \; ; \quad j = 1, 2, 3, .., \theta > 0 \tag{3.3.1}$$

The distribution of a random variable is often characterized in terms of its moment generating function (mgf). Moment generating functions have great practical relevance not only because they can be used to easily derive moments, but also because a probability distribution is uniquely determined by its mgf. The mgf of the SBPL1 distribution can be derived as

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} P_2(x;\, \theta)$$

$$= \frac{\theta^2 e^t (\theta + 2 - e^t)}{(\theta+1)(\theta+1-e^t)^2} \; ; \; \theta > 0, t > 0 \tag{3.3.2}$$

Similarly, for the PSBL distribution the cumulative distribution function can be given as

$$F_X(x) = \frac{\theta^3}{(1+\theta)^3(\theta+2)} \sum_{j=o}^{x} \frac{(1+j)(3+\theta+j)}{(1+\theta)^j} \; ; j = 0, 1, 2, \ldots, \theta > 0 \tag{3.3.3}$$

and moment generating function can be derived as

$$M_x(t) = \sum_{x=0}^{\infty} e^{tx} P_3(x, \theta)$$

$$= \frac{\theta^3 (\theta + 3 - e^t)}{(\theta+2)(\theta+1-e^t)^3} \; ; \; \theta > 0, t > 0 \tag{3.3.4}$$

3.4 Estimation of Parameter

To estimate the parameters of the SBPL1 and PSBL distributions, we have been used method of moments (MoM) and method based on the first two relative frequencies of the distributions as follows.

3.4.1 Method of Moment

Given a random sample $x_1, x_2, \ldots, x_n$ of size n from the SBPL1 distribution with the pmf (3.1.2), then the method of moment estimate $\hat{\theta}$ of θ is given as

$$\hat{\theta} = \frac{(2-\bar{x})+\sqrt{\bar{x}^2+4\bar{x}-4}}{2(\bar{x}-1)} \;;\; \bar{x} > 1 \qquad (3.4.1)$$

Note that $\bar{x} = 1$ if and only if $x_i = 1$ for all $i = 1, 2, \ldots, n$. A data set where all observations are ones in not worth analyzing. This situation, of course, will not lead to any estimate of θ. However, such situation may arise in a simulation experiment when n is small. The method of moment estimate $\hat{\theta}$ of θ is positively biased for the SBPL1 distribution.

Similarly, given a random sample $x_1, x_2, \ldots, x_n$ of size n from the PSBL distribution with the pmf (3.1.3), the MoM estimate $\hat{\theta}$ of θ is obtained as

$$\hat{\theta} = \frac{(1-\bar{x})+\sqrt{2\bar{x}^2+2\bar{x}+2}}{\bar{x}} \;;\; \bar{x} > 1 \qquad (3.4.2)$$

3.4.2 Method based on the first two relative frequencies

The ratio of the first two relative frequencies of the SBPL1 distribution gives the estimate $\hat{\theta}$ of θ as

$$\hat{\theta} = \frac{(f_1-3f_2)+\sqrt{{f_1}^2+6f_1f_2+{f_2}^2}}{2f_2} \qquad (3.4.3)$$

where $\frac{f_1}{N} = \frac{\theta^2(\theta+2)}{(1+\theta)^3}$ and $\frac{f_2}{N} = \frac{\theta^2(\theta+3)}{(1+\theta)^4}$ are the first two relative frequencies of the SBPL1 distribution.

Similarly, for the PSBL distribution, the method based on the first two relative frequencies gives the estimate $\hat{\theta}$ of θ as

$$\hat{\theta} = \frac{(3f_1 - 10f_2) + \sqrt{9{f_1}^2 + 60f_1f_2 + 36{f_2}^2}}{4f_2} \qquad (3.4.4)$$

where $\frac{f_1}{N} = \frac{\theta^3 2(\theta+4)}{(\theta+2)(1+\theta)^4}$ and $\frac{f_2}{N} = \frac{\theta^3 3(\theta+5)}{(\theta+2)(1+\theta)^5}$ are the first two relative frequencies of the PSBL distribution.

3.5 Fitting of distribution to data

To examine the flexibility and applicability of the SBPL1 and PSBL distributions, fitting these distributions to some reported published data sets.

In **Table 3.1** and **3.2**, we have considered the immunogold assay data of Cullen et al. (1990) and animal abundance data of Keith and Meslow (1968) for which size-biased Poisson-Lindley distribution was fitted by Ghitany et al. (2008).

We need here to test the null hypothesis

H_0: The data fits SBPL1 versus H_1: the data doesn't fit SBPL1.

The χ^2-values and the p-value of the test for fitting the SBPL1 distribution to the two data sets in **Table 3.1** and **Table 3.2** shows that, the null hypothesis H_0 (distribution of the data is SBPL1) cannot be rejected; indeed, the close agreement between the observed and expected frequencies and suggest that the SBPL1 distribution provides a “good fit” to these data sets as compared with the other given distribution. In both the cases, method of moments gives a better fit than the other method of estimation.

In order to examine the flexibility of the PSBL distribution, a set of real data taken from Kemp and Kemp (1965) is considered in **Table 3.3**. The data represents the mistakes in copying groups of random digits for which Poisson-Lindley distribution was fitted by Sankaran (1970). Observing the χ^2-value and p-value of the test for fitting the PSBL distribution to the data in **Table 3.3**, it is clear that the null hypothesis H_0 (distribution of the data is PSBL) cannot be rejected. It is seen that the expected frequencies computed by PSBL distribution match satisfactory than the other distribution considered earlier. In case of **Table 3.3**, the method based on the first two relative frequencies does not give better fit, as the computed χ^2 value is quite large. Hence the result is not reported in this case.

In **Table 3.4**, we considered the observed data of Kendall (1961), on the number of strikes in 4-week periods in two leading industries in U.K during 1948-1959 and concluded that the aggregate data for the two industries of vehicle manufacturing and ship building agree with Poisson law. The distribution corresponding to Poisson size-biased Lindley (SBPL2) has been fitted to the observed data for the two industries. The results are given in **Table 3.4**. Based on observed and expected frequencies it is clear that the pattern of strikes in which manufacturing and ship building data describes the model very closely. In case of **Table 3.4**, method of moments estimator gives satisfactory fit to the data set than the method based on the first two relative frequencies.

Table 3.1 Comparison of observed frequencies for immunogold assay data with fitted Size-biased Poisson (SBP) and new form of size-biased Poisson-Lindley (SBPL1) distributions [data from Cullen et al. (1990)]

Number of attached particles	Observed frequencies	Expected frequencies		
		SBP(MoM) Ghitany et al. (2008)	Fitted SBPL1(MoM)	Fitted SBPL1(RF)
1	122	111.3	124.5	119.0
2	50	64.1	47.1	49.6
3	18	18.5	17.1	19.1
4	4	3.5	6.1	7.3
5	4	0.6	3.2	3.0
Total	198	198.0	198.0	198.0
Parameter estimates		$\hat{\lambda} = 0.576$	$\hat{\theta} = 2.267$	$\hat{\theta} = 2.043$
χ^2		4.642	0.457	0.592
$d.f$		1	2	2
$p - value$		0.031	0.796	0.743

Note: SBP: Size-biased Poisson.

SBPL1: New form of size-biased Poisson-Lindley distribution.

MoM: Method of moments.

RF: Method based on the first two relative frequencies.

Table 3.2 Observed and expected frequencies on the basis of size-biased Poisson (SBP) and new form of size-biased Poisson-Lindley (SBPL1) distributions to animal abundance data.[data from Keith and Meslow (1968)]

Number of Counts	Observed frequencies	Expected frequencies		
		SBP(MoM) Ghitany et al. (2008)	Fitted SBPL1(MoM)	Fitted SBPL1(RF)
1	184	170.6	182.4	183.8
2	55	72.5	55.6	54.9
3	14	15.4	16.4	16.0
4	4	2.2	4.2	4.6
5	4	0.3	2.4	1.7
Total	261	261.0	261.0	261.0
Parameter estimates		$\hat{\lambda} = 0.425$	$\hat{\theta} = 2.949$	$\hat{\theta} = 3.013$
χ^2		6.216	0.669	0.709
$d.f$		1	2	2
$p-value$		0.013	0.716	0.702

Note: MoM: Method of moments.

RF: Method based on the first two relative frequencies.

Table 3.3 Comparison of observed frequencies with expected frequencies of Poisson, Poisson-Lindley (PL) and Poisson size-biased Lindley (PSBL) distributions for the mistakes in copying groups of random digits. [data from Kemp and Kemp (1965)]

Number of error per group	Observed frequencies	Expected frequencies		
		Poisson(MoM)	PL(MoM) Sankaran(1970)	Fitted PSBL(MoM)
0	35	27.4	33.1	30.8
1	11	21.5	15.3	16.7
2	8	8.4	6.8	7.4
3	4	2.2	2.9	3.5
4	2	0.4	1.2	1.6
Total	60	60.0	60.0	60.0
Parameter estimates		$\hat{\lambda} = 0.783$	$\hat{\theta} = 1.743$	$\hat{\theta} = 3.073$
χ^2		8.05	2.20	2.69
$d.f$		1	1	1
$p-value$		0.005	0.138	0.101

Note: PL : Poisson-Lindley distribution.

PSBL: Poisson size-biased Lindley distribution.

MoM: Method of moments.

Table 3.4 Comparison of observed frequencies of the No. of Outbreaks of strike in U.K during 1948-1959 with the expected frequencies of fitted Poisson size-biased Lindley (PSBL) distribution. [data from Consul (1989)]

Vehicle manufacturing Industries				Ship building Industries		
No.of outbreaks	Observed frequency	Fitted distribution PSBL		Observed frequency	Fitted distribution PSBL	
		MoM $\hat{\theta} = 5.768$	RF $\hat{\theta} = 4.583$		MoM $\hat{\theta} = 7.239$	RF $\hat{\theta} = 5.677$
0	110	109.0	99.4	117	117.3	108.4
1	33	35.8	40.3	29	31.2	36.2
2	9	8.8	12.1	9	6.2	9.0
3	3	1.9	3.2	0	1.1	1.9
4	1	0.5	1.0	1	0.2	0.5
Total	156	156.0	156.0	156	156.0	156.0
χ^2		0.516	3.120	χ^2	0.989	2.285
$d.f$		1	1	$d.f$	1	1
$p-value$		0.473	0.077	$p-value$	0.319	0.131

Note: MoM: Method of moments.

RF: Method based on the first two relative frequencies.

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