A STUDY ON FUZZY TOPOLOGICAL RELATIONS OF SPATIAL OBJECTS WITH HOLES

A thesis submitted in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

By

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Dedicated to my Parents

Maa & Dauta

Abstract

Fuzziness is ubiquitous in modeling spatial relations, occurring in most situations at object description or relational representation. The central theme of our study is to develop a framework for representing topological relations of fuzzy regions with holes and suggesting their applications to practical situations. As boundary is one of the important topological inputs for studying fuzzy spatial objects, a comparative analysis of various forms of fuzzy boundary is carried out, as a part of our study. Some useful set-theoretic identities on fuzzy boundaries have been established in the process. The case of crisp fuzzy topological spaces is taken up first for development of a theoretical framework to define a fuzzy region with holes as a consistent generalization of the established Tang and Kainz's definition of a simple fuzzy region. Considering the content of intersection to be the eight basic topological relations between two fuzzy regions without hole, we have provided a methodology to determine the topological relations between fuzzy regions with holes and some of the basic fuzzy spatial objects. In the next phase the setting is extended to that of general fuzzy topological spaces. The approach in this setting, however, is different due to intrinsic difference in the structure of these spaces. A good number of additional topological entities are devised to be used for the same purpose.

Using node, arc and path consistency we introduce a set comprising of several geometric conditions in order to reduce the number of redundant topological relations between fuzzy region with a hole and some of the basic fuzzy sets in a

crisp fuzzy topological space. We redefine the proof-by-constraint and drawing method and apply it to eliminate the reducible topological relations between fuzzy regions with and without holes in a crisp fuzzy topological space. We also use the same method to identify the conditions for reducing the number of inconsistent topological relations between fuzzy regions each with a hole in a general fuzzy topological space.

Finally, we provide a novel methodology for application of our theoretical framework to determine the distribution of the occurrence of bird flu effect over a locality in which, a particular colony, having taken some precautionary measures to control the disease, acts as a hole. With this methodology we underline the possibility of applying the model as well as other models developed on similar lines to various real life situations

Declaration

I, **Dibyajyoti Hazarika**, hereby declare that the subject matter in this thesis entitled "**A study on fuzzy topological relations of spatial objects with holes**" is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the thesis has not been submitted by me for any research degree in any other university/institute.

This thesis is being submitted to the Tezpur University for the degree of Doctor of Philosophy in Mathematical Sciences.

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CERTIFICATE

This is to certify that the thesis entitled **A Study on Fuzzy Topological Relations of Spatial Objects** submitted to the School of Sciences of Tezpur University in partial fulfillment for the award of the degree of Doctor of Philosophy in Mathematical Sciences is a record of research work carried out by **Miss Dibyajyoti Hazarika** under my supervision and guidance.

All help received by her from various sources have been duly acknowledged.

No part of this thesis has been submitted elsewhere for award of any other degree.

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Chapter 1

Introduction

1.1 Background

1.1.1 Spatial relations

The term *spatial* pertains to positioning of an object in a given space Geographical spatial objects correspond to the real world elements such as rivers, mountains, valleys, buildings etc in their respective localities or neighbourhoods Depending upon the attribute under study spatial objects can be broadly categorized to be of two types - crisp spatial objects and fuzzy spatial objects Crisp spatial objects are those whose behaviour or attribute under study is determinate whereas fuzzy spatial objects are those whose attribute under consideration is uncertain Topological spatial relations are those relations amongst spatial objects that are invariant under topological transformations In real situations the spatial relations may vary with respect to time and other parameters For example, two islands in a river may be separated during the rainy season and connected in the dry season and during the transition from rainy to dry there may be many other relations between the islands Thus, spatial relations between the islands provide information about the intermediate cases between separatedness and connectedness of the two islands in the river in different seasons. The utility of the above information can be easily seen. The area flooded by water during the rainy season can be used for cultivation or other productive purposes during the period from dry season to the arrival of rainy season. Then again, based on the type of the spatial objects concerned, spatial relations are of two types - crisp topological relations and fuzzy topological relations. Crisp topological relations are those in which crisp spatial objects are involved and fuzzy topological relations are those in which fuzzy spatial objects are considered.

The importance of topological relations lies in the fact that they are used for assessing topological information for storage purposes such as keeping records of geographical maps, surveys etc. in one computer system and to transfer it to another computer system. This is a process which usually results in vital information loss and is also very expensive due to the requirement in storage space. It is also very difficult to store all these relations explicitly for ready usability. However, it is found to be much easier to infer these relations by their geometry and topological relations play a vital role in this regard. Topological relations are used by the Open Geographic Consortium (OGC) for developing industry standard software [3]. In simple terms, two geographic areas are considered to be planar spatial regions and a topologically based function is developed to expand the intersection values for coding how two geographic regions lie in relation to each other.

It is, however, commonly observed that a large number of phenomena occurring in nature have discontinuities in the boundary and exterior in the form of cavities which give rise to the study of spatial objects with holes. As simple examples, we can think of an island in a river or of puddles of water near a coastline where the first object is a hole on the second because the membership grade of the attribute of the second object inside the first object will be either zero or negligible. Crisp spatial objects without and with holes and their relations, and models for relations between fuzzy spatial objects without holes have been fairly extensively studied by various authors over the last few decades. However, study of fuzzy spatial objects with holes, their relations and applications remains a fairly untouched area of research, despite their potential for simulation of real life phenomenon. So, modeling of fuzzy spatial objects with explicit stimulation of holes and their relations emerges as an potential area of research.

In this thesis, we undertake a novel topological approach for the study of these objects, their relations as well as applications to simple real life like situations.

1.1.2 Review of classical topological relations

Classical topological relations are the relations that are invariant under homeomorphism. The basic model for interpretation of crisp topological relations is the 4-intersection model proposed by Egenhofer and Franzosa [30] based on the four possible intersections of interior and boundary of the two spatial objects. Considering the content of intersection to be empty and non-empty, a total of sixteen topological relations exist between any two sets in $\mathbb{R}^2$. However, if the sets are restricted to spatial regions, then there are only nine viable topological relations. Further, if the two regions have connected boundaries then only eight distinct topological relations between the regions are realizable. In [27] Egenhofer and Herring provided prototypes for the eight relations that exist between two regions with connected boundary and provided refinement of these relations by considering criteria such as number of segments of the four intersections or their dimensions. In [28] Egenhofer provided a formal definition of topological relations based on the simplicial complex of algebraic topology. Further, Egenhofer and Herring [31] provided the 9-intersection model as an extension of the 4-intersection model based on content of intersection of interior, boundary and exterior of the two spatial objects

Since then various other models have been developed to represent crisp spatial objects and their relations These models have shown that topological changes are qualitative in nature rather than being quantitative

In [32] Egenhofer and Al-Taha derived topological relations between spatial objects under deformations such as translation, rotation, reduction and expansion of an object and looked for answers to the questions such as what will be the next most likely state of the time and positional dependent spatial objects under these deformations Further, Egenhofer and Mark [35] provided a model to derive a conceptual neighborhood among the topological relations between a region and a line They developed two similarity models, namely the snapshot model and the smooth-transition model The snapshot model compares two snapshots of line-region relations without having any knowledge about the potential process and select neighborhoods based on the least noticeable differences Smooth-transition model derives neighborhood among the topological relations under slight deformation based on the knowledge of the kind of deformation and test the results using data from human subject test or human observational data

In 2007, Liu and Shi [56] provided an extended model to determine topological relations between two convex or non-convex regions by using the concepts of connectivity and that of a fundamental group They also provided a sequence of 4×4 matrices for two convex regions based on the intersection and difference operators of the interior and boundary of the two regions In his doctoral work, Paiva [71] provided applications of topological relations between spatial regions in multiple representation for assessing topological consistency as well as similarity measures

Sometimes it has been seen that spatial objects may have discontinuities in the boundary and exterior giving rise to the study of topological relations between regions with holes when they are used in application purposes such as GIS, robotics, artificial intelligence etc. It may be noted that regions with holes are visually similar to the broad boundary regions developed by Clementini and Di Filice [15], Cohn and Gott [18] but their topologies are different. The regions with broad boundary is the union of inner subregion (called as yolk) and outer subregion (called as white) whereas regions with holes is the union of the region and hole (the part of the region where attribute under study is zero but other attribute of the space remain same). Therefore, frameworks for broad boundary regions cannot be applied to represent topological relations between regions with holes. Egenhofer et al. [33] defined regions with holes in an $\mathbb{R}^2$ framework and derived topological relations between regions with 'n' and 'm' holes respectively. They provided an algorithm to minimize the number of redundant relations and applied their model for assessment of consistency in multiple representation by introducing the concept of dropping of holes. Egenhofer and Vasardani [36, 37, 38] provided models to derive topological relations between (i) regions with a hole and without holes (ii) regions each with a hole and (iii) region without holes and multi-holed region. Considering the content of invariant to be the eight topological relations between two regions with connected boundary in $\mathbb{R}^2$, they found that there are 23 distinct relations between regions without and with a hole and 152 relations between two regions each with a hole. In [93] Vasardani studied the compositional inference of the topological relations between regions with and without holes, regions each with a hole, region without hole and with multi-holes. Application of these relations in the study of similarity assessment was also discussed in the same work.

Huo et al. [47] provided a D9-intersection model for topological relations between holed regions by extending the 9-intersection model using binary codes. Zhang and Qin [104] provided a model for topological relations between objects with holes in 3D by decomposing complicated object into simple objects and proposed an algorithm to analyze complete sets of topological relations between two objects with holes in 3D

Crisp topological models have the usual shortcoming in representing real life situations as they model a situation using values 1 and 0 to indicate the presence and absence of an attribute As a result vital information may be lost during their transition in the modeling process In such circumstances, fuzzy topological relations prove to be a powerful tool to represent imprecision or uncertainty of the spatial phenomena

1.1.3 Fuzzy topological relational models

After the introduction of fuzzy sets by Prof Lotfy Zadeh in his seminal paper [100], C L Chang [13] initiated the study of topology using fuzzy sets which resulted in the branch of fuzzy topology [52 69, 70, 74, 77] The subject has been developed extensively, and in the course of development, it has been shown by various authors [7, 8, 9, 16, 21, 22, 25, 26, 39 85, 86, 87, 101, 103] that fuzzy topology is a potential tool to model the uncertainty or vagueness of spatial objects by assigning suitable membership values between 0 and 1 to indicate the degree of belongingness of an element in relation to the space As in many real life situations, it is not often possible to derive the exact relations between spatial objects due to intrinsic fuzziness In the sequel, we discuss some of these models

Zhan [102] developed a fuzzy analogue of the 4-intersection model of Egenhofer [30] He proposed a method for approximately analyzing the binary topological relations between fuzzy region without holes by dividing the region into number of α-cut regions and provided a formula to determine membership grade of eight elementary topological relations between two fuzzy regions Later on, Du et al [23, 24] attempted to fuzzify the 9-intersection model to describe uncertainty of position by defining membership functions for interior boundary and exterior, and derived their relations in a uniform framework by developing raster algorithm for computing fuzzy 9-intersection matrix between two crisp objects, between two fuzzy objects, and between a crisp and a fuzzy object

Winter [97, 98] determined topological relations between imprecise regions generalizing the concept of 4-intersection model of Egenhofer [30] in a probabilistic approach by using some semantic examples These models, however have a drawback due to lack of spatial abstractions

Schneider [79] provided definition of fuzzy regions in terms of open sets using the concepts of a regularization function and continuity gap He also determined topological relations between various complex spatial objects in his work [81] This approach of fuzzy regions however has the drawback that it is inconsistent with the crisp case In other words, it does not simply yield the crisp case as a particular case of itself

Most notions developed in fuzzy topology can be construed as a generalization of notions in classical topology Therefore, it is only but natural that fuzzy regions have also been attempted to be generalized as a proper extension of crisp regions in classical topological space

In [89] Tang observed that there are two kinds of fuzzy topological spaces (fts) - crisp and general that can be treated separately Tang and Kainz [88 89] proposed two definitions of a fuzzy region in a crisp and a general fuzzy topological space respectively and shown that their definitions of fuzzy regions are closed sets and provides a consistent generalization of the crisp case In [88, 91] the same authors generalized the concept of broad boundary in their work [15 16, 17, 18] and showed that in case of fuzzy topology the intersection of interior and boundary as well as boundary and exterior is non empty They proposed a more detailed definition of the boundary of spatial objects and proposed several other notions such as core, fringe, internal, internal boundary etc. and proved that these notions are topological. Based on these topological parts, a new 9-intersection matrix and a $4 * 4, 5 * 5$ intersection matrix were derived. Further, using the notions of node, arc and path consistency, sets of geometric conditions were derived to reduce the number of redundant relations between various spatial objects. Tang and Kainz also proposed a framework for dealing with fuzzy spatial object based on fuzzy cell complex structure [90]. Further, in [92] they developed a formal framework for generation of fuzzy spatial objects and utilized it in the analysis of land cover changes.

In a related development, Palshikar [73] provided a definition of fuzzy regions in a finite discrete fts and reformulated the Regional Connection Calculus (RCC) theory in the setting of this space. Around the same time, Bjorke [9] proposed a model for generating verbal terms for topological relations between fuzzy regions by providing a method to compute the fuzzy boundary of spatial objects and provided a simulation experiment to illustrate the theoretical development.

In their work, Liu and Shi [55, 58, 59] developed a computational fuzzy topology to practically implement conceptual topological relations in a computer environment based on interior and closure operators which further generated a coherent fuzzy topology and used it to determine the interior, boundary and exterior of an area effected by a harmful weed. They also provided a mathematical model for the topological relations between fuzzy spatial objects and introduced the concept of bound on the intersection of the boundary and interior as well as boundary and exterior of the computational fuzzy topology. In [57], the same authors proposed a model to determine topological relations between fuzzy region and fuzzy line using the concepts of quasi-coincidence and quasi-difference and used the same to determine effects of Severe Acute Respiratory Syndrome (SARS) over the people in a particular community. This is one of the early applications of fuzzy topological methods in a practical field of applications.

In a parallel development, Schmitz and Morris [82] provided definition of Region with Multiple Alpha-Cut (RMAC) to model a fuzzy region and described strategies for defining topological relations between two RMAC as well as to minimize the number of relations. Later, Schockaert et al. [85, 86] proposed the fuzzy extension of regional connection calculus (RCC) theory and show how spatial reasoning based on this theory is helpful in linear programming problem and representing information about vague topological information.

Alboody et al. [7] provided another framework for modeling topological relations between fuzzy regions based upon a new model known as fuzzy intersection and difference model. In this model fuzzy spatial objects are decomposed into four components and using these components, a new 4 * 4 intersection model and fuzzy intersection and difference model are derived. The main advantage of fuzzy intersection and difference model is that it reduces the cost of computation by replacing intersection operator by the difference operator.

In spite of the existence of various models in fuzzy topology to handle imprecision of the fuzzy spatial objects, the development has been in some sense scattered and some of the important related aspects too are required to be taken up for further study.

1.2 Motivational aspects

The frameworks of classical topological relations indeed have the inherent shortcoming due to their inability of representing uncertainty and vagueness which invariably occur in physical phenomenon due to the intrinsic imprecision of the objects with respect to various parameters associated with them. Topological relations based on fuzzy topology have the potential of removing these shortcoming with their capacity to accommodate uncertainty or vagueness occurring in different forms. As described in the preceding section, there are many frameworks due to Schneider [79], Tang and Kainz [88, 92], Liu and Shi [57] and various other authors which model topological relations of fuzzy spatial objects. Since there are a large number of spatial phenomena resembling fuzzy region with holes indicating treatment of fuzzy spatial objects with hole, it underlines the requirement of developing a model for fuzzy regions with incorporation of holes and their topological relations as well as utilizing these representations to deal with real life situations.

1.3 Objective and methodology

In broader terms the objective of our study is to develop fuzzy spatial objects and their relations particularly in the area of representation of topological approximate relations of fuzzy spatial objects with holes and modeling them based on topological relations.

The objectives of the thesis may therefore be summed up as follows

- *To define fuzzy spatial objects with the explicit incorporation of holes*
- *To develop theoretical frameworks for topological relations of simple fuzzy spatial objects with holes*
- *To explore the possibility of some application of the developed theoretical framework to simulated real life situation based on purely hypothetical data sets*

In order to deal with the above, we adopted the following course of approach. (i) Extensive study of basics of classical topology, fuzzy sets theory and fuzzy topology was taken up. (ii) Extracting the salient aspects of the basic theoretical frameworks for crisp spatial objects and their topological relations as regards to various existing models. (iii) Interpretation of Schneider's as well as Tang and Kainz's approaches of fuzzy regions and their topological relations.

(iv) Analysis of Liu and Shi's quantitative model of topological relations between fuzzy spatial objects. (v) Developing a model of fuzzy regions with holes in two separate cases of crisp and general fts's. (vi) Use node, arc and path consistency of network relations in an appropriate way to reduce the number of redundant relations. (vii) Adopting Zhan's formula for membership grade of topological relations between two fuzzy regions based on definition of α-cuts of fuzzy set theory.

1.4 Chapterwise overview of the thesis

This thesis is organized in six chapters followed by references used in the study. General introduction is provided in Chapter 1. It seeks to present an overview of the thesis including a brief background of the work after a detailed analysis of previous work. The motivational aspects of the research problem, objective of the work, methodology applied and outline of the work carried out in the thesis is presented in this chapter.

In Chapter 2, we present the basic aspects and results of fuzzy topology that are used for developing some properties of fuzzy spatial objects, their relations and applications in the subsequent chapters. The background of classical topological spaces is also included in this chapter.

In Chapters 3 and 4, theoretical frameworks are developed to model fuzzy spatial objects and their topological relations in some of the simple cases. Noting that fuzzy boundary is one of the most important inputs for studying topological relations of fuzzy spatial objects, we provide a detailed analysis of different types of fuzzy boundary at the beginning of Chapter 3. We then introduce fuzzy regions with holes in the setting of crisp fuzzy topological space. Since a crisp fuzzy topological space exhibits behavior which to a large extent is similar to that of a classical topological space, the collection of fuzzy topological spaces are treated separately into classes of crisp and general fuzzy topological spaces respectively for defining fuzzy regions with holes In Chapter 3 the case of crisp fuzzy topological space is taken up The proposed definition of fuzzy region with holes is found to be consistent as a generalization with the existing definition of crisp regions with holes in a classical topological space Further a general framework is developed for determining topological relations between fuzzy regions with arbitrary (finite) number of holes and basic fuzzy spatial object (viz fuzzy point fuzzy line and fuzzy region without hole) which are not restricted to single hole in any of the case under consideration As particular cases fuzzy regions with single hole has been considered in each of the situations which can then be easily visualized and assessed for feasibilty

In Chapter 4 definition of fuzzy regions with holes is proposed in the setting of general fuzzy topological space This development is independent of the development in the case of crisp fuzzy topological spaces Since general fuzzy topological spaces allow flexibility in membership grades than what is available in a crisp fuzzy topological spaces a general fuzzy topological space is obviously a better representative of the imprecision of the spatial objects though the setting is much more challenging In this setting the structures of crisp regions with and without holes are defined first and then proceeding to formulate conditions to maintain consistency of the proposed definition of fuzzy regions with holes with crisp regions with holes in general topological space Work is then carried out to determine the topological relations between fuzzy regions each with holes Using node arc and path consistency a set of geometric conditions have been derived which identifies and thereby reduces the number of redundant relations between fuzzy regions each with a hole

In Chapter 5 we attempt to provide an application of our theoretical tools developed in Chapters 3 and 4 The tool is applied for determination of severeness of bird flu over a particular vaccinated locality A methodology has been developed to indicate severeness over the locality when flu enters into a locality which has a vaccinated colony which is interpreted as a fuzzy region with a hole The stress however is on the development of a methodology and not on the accuracy or reliability of assessment as the methodology is based purely on a hypothetical data set Further, a point-wise model is suggested to determine the severeness of any point inside the locality w r t flu point from the center of the vaccinated colony of the locality as well as position of the point inside the locality

Finally, in Chapter 6 we have provided the conclusion and discussion on the outcome is presented We present the significance of the work done in the thesis and outline future scope of the work

A bibliography containing all the references is included at the end of the thesis

Chapter 2

Preliminaries

In this chapter we provide the preliminary results and definitions used in the thesis. It includes definitions and results in classical topology, fuzzy set theory and fuzzy topology. We also recollect the basic theoretical frameworks of topological and fuzzy topological relations. These are mostly available in the literature and are adopted in the subsequent chapters of this thesis.

2.1 Classical Point-Set Topology

2.1.1 Topology and topological space

Definition 2.1.1. Let X be any set and T be a collection of subsets of X such that

(i) $\phi, X \in T$

(ii) For any pair $A, B \in T$, $A \cap B \in T$

(iii) For $A_i \in T$, $\cup_i A_i \in T$

Then the collection T is called a topology on X and the pair (X, T) is called a topological space. The elements in T are called T-open sets or open sets and their complements are called T-closed or closed sets. Those sets which are both closed and open at the same time are called clopen sets.

A classical topological space is also referred to as a crisp topological space or simply as a topological space

2.1.2 Interior, closure, boundary and exterior

Let (X, T) be a topological space and let A be a subset of X

Definition 2.1.2. Interior of A is the union of all open sets contained in A Alternatively, it is the largest open set contained in A denoted by A° or int(A)

Definition 2.1.3. Closure of A is the intersection of closed sets containing A Alternatively, it is the smallest closed set containing A, denoted by $\overline{A}$ or cl(A)

Definition 2.1.4. Boundary of a subset A (denoted by ∂A) is defined as the difference between the closure and the interior of the set A (i e $\partial A = \overline{A} - A^{\circ}$) Equivalently it is the intersection of the closure of the set with the closure of the complement of the set ($\partial A = \overline{A} \cap \overline{A^c}$)

Definition 2.1.5. Exterior of A is defined as the complement of the closure of A and is denoted by A^e or A^-

The following proposition is an important feature of an interior, boundary and exterior of a classical topological space

Proposition 2.1.1. *Let A be a subset of a crisp topological space Then A°, ∂A A^e are mutually disjoint parts*

Definition 2.1.6. Let A be a subset of a topological space (X, T) and $x \in X$ be a point Then A is said to be neighbourhood of x if there exists a set $B \in T$ such that $x \in B \subseteq A$ The union of all the neighbourhoods of a point is called the neighbourhood of that system

Definition 2.1.7. A function from a topological space (X,T) to a topological space (Y,T') is said to be continuous if for every open set in T', the inverse image is open in T

Alternatively, a function $f \colon X \longrightarrow Y$ from a topological space X to a topological space Y is called continuous at a point x if for every open set B in Y containing $f(x)$, there is an open set A in X containing x such that the image of A is a subset of B i e $f(A) \subset B$ If f is continuous at every point of X then f is a continuous function on X

Definition 2.1.8. A function f from a topological space X to a topological space Y is said to be a homeomorphism if it is continuous, bijective and its inverse is also continuous A property that is preserved under homeomorphism is said to be a topological invariant (or a topological property)

Definition 2.1.9. (Topological relations are relations that are invariant under topological transformation such as homeomorphism) If R is a binary relation from a subset $A \subset X$ to a subset $B \subset Y$ where X and Y are topological spaces Then R is called a topological relation from A to B on $X \times Y$ if R is a topological invariant (i e if $f \colon X \times Y \longrightarrow X \times Y$ is a homeomorphism then R is topological invariant if $R(x,y) = R(f(x), f(y))$ where $x \in A$ and $y \in B$)

Definition 2.1.10. A subset A in a topological space $(X\ T)$ is said to be regular closed if $A = \overline{A^\circ}$ and is said to be regular open if $A = \overline{A}^\circ$

Definition 2.1.11. Let A and B be two sets in a topological space X then A and B are separated if there exist two open sets H and K such that $H \supseteq A$, $K \supseteq B$ and $H \cap B = \phi$, $A \cap K = \phi$

Definition 2.1.12. A topological space X is said to be connected if there do not exist two non empty sets A and B such that $X = A \cup B$ and $\overline{A} \cap B \neq \phi$ or $A \cap \overline{B} \neq \phi$.

Proposition 2.1.2. *The following are equivalent:*

1. X is connected.

2. The only clopen sets are X and ϕ.

3. X cannot be represented as the union of two disjoint non-empty open sets (or closed sets).

2.2 Crisp region with and without holes

A subset of a given classical topological space is said to be a crisp region if either it is a singleton or for each point of the subset there is another point in the main space.

Definition 2.2.1. A crisp region in a connected crisp topological space (X, T) is a non empty regular closed subset A of X such that A° is connected.

Definition 2.2.2. A region with holes in $\mathbb{R}^2$ is a region whose exterior is separated into one outer exterior and $n > 0$ inner exteriors. The outer exterior will be denoted by $A_0{}^-$ and inner exteriors by $A_1{}^-, A_2{}^-, ..., A_n{}^-$ such that their union makes the entire exterior as $A^- = \cup_{i=0}^{i=n} A_i$. Thus, a region with holes denoted by A is a non-empty subset of $\mathbb{R}^2$ with connected interior such that the closure of any two different inner exteriors are disjoint and A is equal to closure of A's interior. i.e. $\forall i, j = 0, 1, ..., n; i \neq j$, $\overline{A_i^-} \cap \overline{A_j^-} = \phi$ and $A = \overline{A^\circ}$.

Definition 2.2.3. A hole of A is the closure of an inner exterior denoted by H_A.

A hole is a connected set that is strictly contained in A and each hole H_{A_i} is disjoint from hole H_{A_j}, $i \neq j$

Definition 2.2.4. Suppose the region A has n holes $H_{A_1}, H_{A_2}, \ldots, H_{A_n}$ then the generalized region of A denoted by A^* is the union of A and all the holes contained in A

The concept of hole as the closure of inner exterior allows us to map any region with holes into a group of simple regions without holes By considering the holes as separate objects, modeling of topological relations between region with holes can be expressed in terms of topological relations between regions without holes as given in section 2 2 1

Definition 2.2.5. A spatial scene is a conceptual model for deriving the new sets of consistent relations using the binary relations between host-regions, between holes, between host-regions and holes

Definition 2.2.6. (Node consistency) If for every variable x the constraint on it coincides with the domain of x, then such consistency is referred to as node consistency

For example - If $\mathbb{N}$ is the set of Natural numbers and $\mathbb{Z}$ is the set of integers, then $\{x_1 \geq 0, \ldots, x_n \geq 0, x_1 \in \mathbb{N}, \ldots, x_n \in \mathbb{N}\}$ is node consistent whereas $\{x_1 \geq 0, \ldots, x_n \geq 0, x_1 \in \mathbb{N}, \ldots, x_{n-1} \in \mathbb{N}, x_n \in \mathbb{Z}\}$ is not node consistent

Definition 2.2.7. (Arc consistency) A constraint C on the variable x y with domains X and Y (so $C \subset X \times Y$) is arc consistent if

$$\forall a \in X \; \exists b \in Y \textit{ such that } (a, b) \in C$$

$$\forall b \in Y \; \exists a \in X \textit{ such that } (a, b) \in C$$

For example- $\{x < y \ x \in [2 \ 6], y \in [3 \ 7]\}$ is arc consistent but $\{x < y \ x \in [2 \ 7], y \in [3 \ 7]\}$ is not arc consistent

Definition 2.2.8. (Path consistency) If x, y, z are three variables in the domain X, Y, Z and $C_{x\,y}$, $C_{y\,z}$, $C_{z\,x}$ are constraints then they are said to be path consistent if

$$C_{x\,y} \subseteq C_{y\,z} \ \ C_{z\,x}$$

For example -$\{x < y, y < z, x < z, x \in [0 \ 4], y \in [1 \ 5], z \in [6 \ 10]\}$ is path consistent because $C_{x\,y} = \{(a, b)|a < b, a \in [0 \ 4], b \in [1 \ 5]\}$, $C_{y\,z} = \{(b, c)|b < c \ b \in [1 \ 5], c \in [6 \ 10]\}$, $C_{x\,z} = \{(a, c)|a < c, a \in [0 \ 4] \ c \in [6 \ 10]\}$

2.2.1 Analysis of crisp topological relations

The 4-intersection model

This model was developed by Egenhofer and Franzosa [30] in 1990 They characterized the topological relations between two sets in terms of the four intersections of the boundary and interior of the spatial object Thus, if A and B are two spatial objects with interior and boundary A°, B°, ∂A and ∂B respectively, then the 4-intersection matrix is given by

$$\begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B \\ \partial A \cap B^\circ & \partial A \cap \partial B \end{pmatrix}$$

Table 2 1 4-intersection matrix

Considering the content of the intersection to be empty (ϕ) or non empty ($-\phi$), there are a total of sixteen topological relations between two spatial objects in $\mathbb{R}^2$ which are listed in table 2 2

Relation	$(\partial \cap \partial, \circ \cap \circ, \partial \cap \circ, \circ \cap \partial)$
r_0	(ϕ, ϕ, ϕ, ϕ)
r_1	$(-\phi, \phi, \phi, \phi)$
r_2	$(\phi, -\phi, \phi, \phi)$
r_3	$(-\phi, -\phi, \phi, \phi)$
r_4	$(\phi, \phi, -\phi, \phi)$
r_5	$(-\phi, \phi, -\phi, \phi)$
r_6	$(\phi, -\phi, -\phi, \phi)$
r_7	$(-\phi, -\phi, -\phi, \phi)$
r_8	$(\phi, \phi, \phi, -\phi)$
r_9	$(-\phi, \phi, \phi, -\phi)$
r_{10}	$(\phi, -\phi, \phi, -\phi)$
r_{11}	$(-\phi, -\phi, \phi, -\phi)$
r_{12}	$(\phi, \phi, -\phi, -\phi)$
r_{13}	$(-\phi, \phi, -\phi, -\phi)$
r_{14}	$(\phi, -\phi, -\phi, -\phi)$
r_{15}	$(-\phi, -\phi, -\phi, -\phi)$

Table 2.2: Topological relations between two sets

Out of the 16 relations, only 8 relations are realizable between two objects with connected boundary in $\mathbb{R}^2$. They are given below in table 2.3.

Relation	Intersection value	Name
r_0	(ϕ, ϕ, ϕ, ϕ)	Disjoint
r_1	$(-\phi\ \varphi, \phi, \phi)$	Meet
r_3	$(-\phi, -\varphi\ \phi, \phi)$	Equal
r_6	$(\varphi\ -\phi, -\phi\ \phi)$	Inside
r_7	$(-\varphi\ -\varphi, -\phi, \phi)$	Coveredby
r_{10}	$(\phi\ -\phi, \phi, -\phi)$	Contain
r_{11}	$(-\phi\ -\phi, \phi, -\phi)$	Cover
r_{15}	$(-\phi\ -\phi, -\phi, -\phi)$	Overlap

Table 2.3 Topological relations between two regions with connected boundary

This set of 8 relations provide a complete coverage and are mutually exclusive relations so that exactly one relation holds good between two regions with connected boundary in $\mathbb{R}^2$

Crisp 9-intersection model

Another important model for analyzing binary topological relations between two crisp sets is the 9-intersection model. In this model, a spatial object A is decomposed into three parts: an interior (A°), boundary (∂A) and exterior (A^-). For two crisp objects A and B, their topological relations can be determined by the intersection of the interior, the boundary and the exterior of the objects. There are a total of nine intersections among the six parts of two objects. These nine intersections can be represented by the following intersection matrix given below

$$\begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix}$$

Table 2.4: 9-intersection matrix

Considering the content of intersection to be empty and non empty, a total of 512 relations are realizable between two objects with connected boundary in $\mathbb{R}^2$.

Intersection matrix for crisp regions with and without holes

The topological relations between a region A and a region B with a hole is modeled as a spatial scene considering region, region with hole and hole as separate spatial objects without hole and topological relations between two regions with connected boundaries will be considered as content of intersection. Thus, the relation matrix for this case is given below.

	A	B^*	H_B
A	$t(A,A)$	$t(A,B^*)$	$t(A,H_B)$
B^*	$t(B^*,A)$	$t(B^*,B^*)$	$t(B^*,H_B)$
H_B	$t(H_B,A)$	$t(H_B,B^*)$	$t(H_B,H_B)$

Table 2.5: Topological relation matrix for holed regions

Here, B^* is the generalized region, H_B is a hole in B^* and $t(A,B^*)$ represents the topological relation between A and B^*.
Using 8 topological relations between two regions with connected boundary as the content of intersection, the topological relations in the above intersection matrix are given by

	A	B^*	H_B
A	equal	$t(A, B^*)$	$t(A, H_B)$
B^*	$t(B^*, A)$	equal	contain
H_B	$t(H_B, A)$	inside	equal

Table 2.6: Equivalent topological relation matrix for holed regions

Here, the relation $t(A, B^*)$ is implied by the relation $t(B^*, A)$ and vice-versa. Therefore, the number of distinct relations in the matrix depends upon the relations $t(A, B^*)$ and $t(A, H_B)$. Since, content of intersection is the eight relations between two regions, so there are total of 8 choices for each of these two relations. Thus, the total number of relations in this matrix will be 8^2. However, under node consistency, arc consistency and path consistency, there are only 23 relations realizable between regions without and with hole in $\mathbb{R}^2$.

2.3 Fuzzy set theory

Fuzzy set theory is an extension of classical set theory which allows the membership of an element in the range [0,1].

Definition 2.3.1. Let X be a set. A fuzzy set or a fuzzy subset in X is a function A from X into the closed unit interval $[0, 1]$. The function A is called the membership function. For each $x \in X$, $A(x)$ is called the membership grade of x in the closed interval $[0, 1]$.

2.3.1 Basic fuzzy set theoretic operations

Definition 2.3.2. Let A and B be two fuzzy subsets of X. Then we have the following:

(i) Union : $(A \cup B)(x) = max\{A(x), B(x)\};\ for\ x \in X$

(ii) Intersection: $(A \cap B)(x) = min\{A(x), B(x)\};\ for\ x \in X$

(iii) Complement: The complement A^c of A is defined as

$$A^c(x) = \{A^c(x) : A^c(x) = 1 - A(x),\ for\ x \in X\}$$

(iv) Equality: $A = B$ iff $A(x) = B(x),\ \forall x \in X$

(v) Containment: $A \subseteq B$ iff $A(x) \leq B(x),\quad x \in X$

(vi) Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$

(vii) Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$

(viii) Idempotency: $A \cup A = A$, $A \cap A = A$

(ix) Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(x) Absorption: $A \cup \phi = A$, $A \cap X = A$

(xi) De Morgan's law: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

(xii) Involution: $(A^c)^c = A$

(xiii)Equivalence formula: $(A^c \cup B) \cap (A \cup B^c) = (A^c \cap B^c) \cup (A \cap B)$

(xiv) Symmetrical difference formula: $(A^c \cap B) \cup (A \cap B^c) = (A^c \cup B^c) \cap (A \cup B)$

Remark 2.3.1. *In fuzzy set theory, the law of contradiction and law of excluded middle doesnot holds i.e.* $A \cap A^c \neq \phi$ *and* $A \cup A^c \neq X$

2.3.2 Extended operations

Definition 2.3.3. For any two fuzzy sets A and B in X

(i) Fuzzy difference: $(A - B)(x) = (A \cap B^c)(x),\ \forall x \in X$

(ii) Simple difference: $(A - B)(x) = \{A(x) - B(x) :\ x \in X\}$

(iii) Bounded difference: $(A \nabla B)(x) = max(0, A(x) - B(x)),\quad \forall x \in X$

(iv) Absolute difference: $(A| - |B)(x) = |A(x) - B(x)|,\quad \forall x \in X$

(v) Product: $(A \cdot B)(x) = A(x) \cdot B(x), \quad \forall x \in X$

(vi) Bold intersection: $(A \cap B)(x) = max(0, A(x) + B(x) - 1), \quad \forall x \in X$

(vii) Probabilistic sum: $(A \hat{+} B)(x) = A(x) + B(x) - A(x).B(x), \quad \forall x \in X$

(viii) Bounded union: $(A \dot{\vee} B)(x) = min(1, A(x) + B(x)), \quad \forall x \in X$

Definition 2.3.4. A fuzzy set A on $\mathbb{R}$ is convex iff $A(\lambda x_1 + (1 - \lambda)x_2) \geq min\{A(x_1), A(x_2)\}$ for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0, 1]$, where min denotes the minimum operator.

Definition 2.3.5. An α-cut of a fuzzy set A (where $\alpha \in [0, 1]$) in X is defined as $A^{\alpha} = \{x \in X : A(x) \geq \alpha\}$. The set $A^{\alpha+} = \{x \in X : A(x) > \alpha\}$ is called the strong α-cut of A. Both α-cut and strong α-cut of a fuzzy set are crisp sets.

Definition 2.3.6. The support of a fuzzy set A is the collection of all those elements whose membership grades are greater than zero.

$$Supp(A) = \{x : A(x) > 0, \; x \in X\}$$

The support of a fuzzy set is always a crisp set.

Definition 2.3.7. The height of a fuzzy set A denoted by $H(A)$ is the highest membership values of its membership grades i.e.

$$H(A) = \max_{x \in X}\{A(x)\}$$

A fuzzy set is normal if $H(A) = 1$ and subnormal if $H(A) < 1$.

2.3.3 Fuzzy relations

A crisp relation represents the presence or absence of an association, interaction or interconnectedness between the elements of two or more sets. This concept is generalized to represent various degrees or strengths of association or interaction between elements which can be represented by membership grades in a fuzzy relation similar to the membership grade as in case of the fuzzy sets. Thus, a crisp relation can also be viewed as a restricted or particular case of a fuzzy relation. Each crisp relation can be defined by a characteristic function which assigns a value 1 to every tuple of the universal set belonging to the relation and 0 to every tuple not belonging to it. The membership of a tuple in a relation signifies that the elements of the tuple are related to or associated with one another or not.

Definition 2.3.8. Let X, Y be universal sets, then

$$R = \{((x,y), A(x,y)) | (x,y) \in X \times Y\}$$

where $A : X \times Y \longrightarrow [0,1]$ is called a binary fuzzy relation on $X \times Y$.
Let R be a fuzzy relation on $X \times Y$. Then R is reflexive if $R(x,x) = 1,\ x \in X$; R is irreflexive, if $R(x,x) = 0,\ \forall x \in X$. R is symmetric if $R(x,y) = R(y,x),\ x, y \in X$. R is perfectly antisymmetric if $\forall x, y \in X, x \neq y$ and $R(x,y) > 0$ implies $R(y,x) = 0$. R is antisymmetric if $x \neq y$ then $R(y,x) = R(x,y) = 0$, or $R(y,x) \neq R(x,y)$.

Definition 2.3.9. Let $X, Y \subset \mathbb{R}$ and $A = \{(x, A(x)) | x \in X\}$, $B = \{(y, B(y)) | y \in Y\}$ be two fuzzy sets. Then $R = \{((x,y), R(x,y)) | (x,y) \in X \times Y\}$ is a fuzzy relation on A and B if

$$R(x,y) \leq A(x)\ and\ R(x,y) \leq B(x),\ \forall (x,y) \in X \times Y$$

.

Definition 2.3.10. Let R and Z be two fuzzy relations in the same product space. The union and intersection of R with Z is defined as

$$(R \cup Z)(x,y) = max\{R(x,y), Z(x,y)\},\ \forall (x,y) \in X \times Y$$

$$(R \cap Z)(x, y) = min\{R(x, y), Z(x, y)\},\ \forall (x, y) \in X \times Y$$

Definition 2.3.11. (Max-Min composition:) Let $R_1(x, y)$, $(x, y) \in X \times Y$ and $R_2(y, z)$, $(y, z) \in Y \times Z$ be two fuzzy relations. The max-min composition of R_1 and R_2 is defined as

$$R_1 \circ R_2 = \{[(x, z), max_y\{min\{R_1(x, y), R_2(y, z)\}\}] | x \in X, y \in Y, z \in Z\}$$

where $R_1 \circ R_2$ is again the membership function of a fuzzy relation on fuzzy sets defined in definition 2.3.10.

Definition 2.3.12. (Zadeh extension principle) Any given function $f : X \longrightarrow Y$ induces two functions, $f^{\rightarrow} : F(X) \longrightarrow F(Y)$ and $f^{\leftarrow} : F(Y) \longrightarrow F(X)$ which are defined as $[f(A)](y) = sup_{x|y=f(x)}A(x)$ for all $A \in F(X)$ and $[f^{-1}(B)](x) = B(f(x))$ for all $B \in F(Y)$, where $F(X)$ and $F(Y)$ denote the class of fuzzy subsets of X and Y respectively. We simply denote $f^{\rightarrow}$ and $f^{\leftarrow}$ by f and f^{-1} respectively, when there is no scope of confusion.

2.4 Fuzzy topology

Fuzzy topology is constructed using fuzzy sets. It may be noted that due to existence of stratum structure, each notion in general topology usually has several counterparts in fuzzy topology.

Definition 2.4.1. A fuzzy topology is a family T of fuzzy sets in X which satisfies the following:

i) $0_X, 1_X \in T$

ii) If $A, B \in T$ then $A \cap B \in T$

iii) If $\{A_i \cdot i \in J\} \subset T$, where J is an index set, then $\cup_{i \in J} A_i \in T$.

Here, 0_X and 1_X respectively denotes the functions on X which are identically 0 and 1 respectively.

The pair (X, T) is called a fuzzy topological space (fts, in brief).

The elements in T are called the fuzzy open sets (or simply open sets) and their complements are the fuzzy closed sets (or closed sets). Those fuzzy sets in X which are both open and closed are called clopen sets.

Definition 2.4.2. Let A be a fuzzy set in (X, T) then

(i) The union of all the open sets contained in A is the interior of A, denoted by A°.

(ii) The intersection of all the closed sets containing A is the closure of A, denoted by $\overline{A}$.

Following are some of the properties of closure and interior of a fuzzy set:

Theorem 2.4.1. *Let A and B be fuzzy sets in a fuzzy topological space (X, T). Then,*

(i) A is fuzzy closed (resp. fuzzy open) $\Leftrightarrow \overline{A} = A$ (resp. $A^\circ = A$)

(ii) $A \leqslant B \Rightarrow \overline{A} \leqslant \overline{B}$ and $A^\circ \leqslant B^\circ$

(iii) $\overline{(\overline{A})} = \overline{A}$ and $(A^\circ)^\circ = A^\circ$

(iv) $\overline{A} \cup \overline{B} = \overline{A \cup B}$

(v) $\overline{A} \cap \overline{B} \geq \overline{A \cap B}$

(vi) $A^\circ \cup B^\circ \leq (A \cup B)^\circ$

(vii) $A^\circ \cap B^\circ = (A \cap B)^\circ$

(viii) $(A^c)^\circ = (\overline{A})^c$, $\overline{A^c} = (A^\circ)^c$, $(A^\circ)^c = \overline{A^c}$

Theorem 2.4.2. *For any fuzzy set A, $A^\circ = 1_X - \overline{1_X - A}$*

Definition 2.4.3. The exterior of A is the complement of the closure of A.

Definition 2.4.4. A fuzzy set is said to be fuzzy regular closed if $\overline{A} = \overline{(\overline{A})^{\circ}}$ and fuzzy regular open if $A^{\circ} = (\overline{(A^{\circ})})^{\circ}$.

2.4.1 Types of fuzzy boundary

In fuzzy topology there are three different useful definitions of fuzzy boundary proposed by Warren in 1977, Pu-Liu in 1980, Cuchillo-Ibanez in 1997. Let A be a fuzzy set in a fuzzy topological space (X, T). Then

Definition 2.4.5 (Warren). The fuzzy boundary of A is the infimum of all the closed fuzzy sets D in X with the property $D(x) \geqslant \overline{A(x)}$ for all $x \epsilon X$ for which $(\overline{A} \wedge \overline{A^c})(x) > 0$ or $A^{\circ}(x) \neq 1$.

Definition 2.4.6 (Pu and Liu). The fuzzy boundary of A is defined as $\overline{A} \cap \overline{A^c}$.

Definition 2.4.7 (Cuchillo-Ibanez and Tarres). Fuzzy boundary of A is the infimum of all closed fuzzy sets D in X with the property $D(x) \geqslant \overline{A(x)}$ for all $x \epsilon X$ for which $(\overline{A} - A^{\circ})(x) \geqslant 0$.

Notably, these definitions do not simultaneously satisfy properties that boundary of a set satisfies in case of classical topology. Some of the useful properties are discussed below. For convenience, we denote the fuzzy boundaries of A due to Warren, Pu and Liu and Cuchillo-Ibanez by $\partial_1 A, \partial_2 A$ and $\partial_3 A$ respectively. The following results are trivial.

Theorem 2.4.3. *(i) Boundary of an empty set as well as whole space is empty.*
(ii) Boundary of a fuzzy set is a closed fuzzy set.
(iii) Boundary of a fuzzy set is contained in the closure of the set (i.e. $\partial_i A \leq \overline{A}, i = 1, 2, 3$*).*
(iv) A *is a closed fuzzy set iff* $\partial_i A \leq A, i = 1, 2, 3$.

(v) For any fuzzy set A, boundary of the boundary of A is a subset of the boundary of A (i e $\partial_i(\partial_i A) \leq \partial_i A, i = 1, 2, 3$)

(vi) Boundary of the intersection of the set is less than or equal to the union of their boundary (i e $\partial_i(A \cap B) \leq \partial_i A \cup \partial_i B, i = 1, 2, 3$)

In addition, the following results hold for the specific fuzzy boundaries

Theorem 2.4.4. *(i) $\partial_1 A^\circ \leq \partial_1 A$ and $\partial_3 A^\circ \leq \partial_3 A$*

(ii) $\partial_1 \overline{A} \leq \partial_1 A$ and $\partial_3 \overline{A} \leq \partial_3 A$

(iii) $\overline{A} = \partial_1 A \cup A^\circ$, $\overline{A} = \partial_3 A \cup A^\circ$ and $\overline{A} \geq \partial_2 A \cup A^\circ$

(iv) $\partial_1 A(x) = \overline{A}(x)$ or 0, according as $(\overline{A} \cap \overline{A^c})(x)$ is $>$ or $= 0$

(v) $\partial_3 A(x) = \overline{A}(x)$ or 0 according as $(\overline{A} - A^\circ)(x)$ is $>$ or $= 0$

(vi) If $(\overline{A} \cap \overline{A^c}) = 0$ then $\partial_1 A(x) = \partial_1 A^c(x) = 0$

(vii) For any fuzzy set A, $\overline{A} = A \cup \partial_1 A$ $\overline{A} = A \cup \partial_3 A$, $\overline{A} \geq A \cup \partial_2 A$

(viii) $\partial_2 A = \partial_2 A^c$

In Chapter 3 some more properties of these three boundaries shall be discussed for a comparative evaluation

2.4.2 Fuzzy point and fuzzy neighborhood

Definition 2.4.8. A fuzzy point is a fuzzy subset of a set X with support x which is defined by

$$x_\lambda(y) = \begin{cases} \lambda, & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

In other words, a fuzzy set in X is called a fuzzy point iff it has membership degree zero for all $y \in X$ except one, say $x \in X$ We denote a fuzzy point by

$x_\lambda (0 < \lambda \leq 1)$. An example of fuzzy point P (x_λ) is given in figure 2.1.

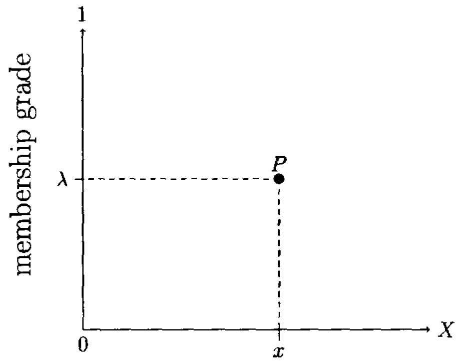


Figure 2.1: A fuzzy point

The fuzzy point x_λ is contained in a fuzzy set A or belongs to A denoted by $x_\lambda \in A$ iff $\lambda \leq A(x)$ the membership degree of x belongs to A. A fuzzy set with membership function

$$P(x) = \begin{cases} 1, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

is called a crisp point.

Definition 2.4.9. [69] A fuzzy set A in (X,T) is called a neighbourhood of a fuzzy point x_λ if there exists $B \in T$ such that $x_\lambda \in B$ and $B \subseteq A$. A fuzzy point $x_\lambda \in A^\circ$ iff x_λ has a neighborhood contained in A. Obviously, a fuzzy point $x_\lambda \notin A^o$ iff every neighborhood of x_λ is not contained in A. A fuzzy point x_λ is said to be quasi-coincident with A, denoted by $x_\lambda q A$, iff $\lambda > A^c(x)$ or $\lambda + A(x) > 1$. A fuzzy set A is said to be quasi-coincident with B if $A(x) > B^c(x)$ or $A(x) + B(x) > 1$, $\forall x \in X$. A fuzzy set A in (X,T) is called a quasi-neighborhood of x_λ if there exists $B \in T$ such that $x_\lambda q B$ and $B \subset A$. A fuzzy point $x_\lambda \in A^-$ iff each quasi-neighborhood of x_λ is quasi-coincident with A.

2.4.3 Connectedness in fuzzy topological space

Definition 2.4.10. [69] Two fuzzy sets A and B in an fts (X,T) are said to be separated if there exists $V, U \in T$, such that $U \supseteq A$, $V \supseteq B$ and $U \cap B = V \cap A = \phi$. Two fuzzy sets A and B in (X,T) are said to be Q-separated if there exist closed sets H, K such that $H \supseteq A$, $K \supseteq B$ and $H \cap B = K \cap A = \phi$.

In general, Q-separation and separation do not imply each other. However, two crisp sets in an fts are separated iff they are Q-separated.

Definition 2.4.11. [52] A fuzzy topological space X is called connected if there are no separated sets C and D such that $A = C \cup D$. A fuzzy set A is said to be open-connected if there are no separated sets C and D such that $A = C \cup D$. A fuzzy set A is said to be closed-connected if there are no Q-separated sets C and D such that $A = C \cup D$. A fuzzy set is said to be double-connected if it is both open-connected and closed-connected.

Definition 2.4.12. A connected component in a fts is a maximal connected subset.

2.4.4 Fuzzy homeomorphism and topological relations

Definition 2.4.13. A function $f : X \to Y$ is said to be fuzzy continuous, if for each fuzzy open set A in Y, $f^{-1}(A)$ is a fuzzy open set in X.

Definition 2.4.14. A mapping $f : X \to Y$ is said to be fuzzy open (fuzzy closed), if for each fuzzy open (fuzzy closed) set A in X, its image $f(A)$ is a fuzzy open (fuzzy closed) set in Y.

Definition 2.4.15. A mapping $f : X \longrightarrow Y$ from a fts (X,T) to fts (Y,T') is said to be fuzzy homeomorphism if f is bijective, continuous and open.

Fuzzy homeomorphism is a union preserving and crisp subset preserving. Those properties of fuzzy set that are invariant under fuzzy homeomorphism are said to be fuzzy topological invariant or simply topological invariant whenever there will be no confusion.

Definition 2.4.16. Let R be a binary fuzzy relation from fuzzy set $A \subset X$ to fuzzy set $B \subset X$ on fuzzy topological space X. R is called a fuzzy topological relation from A to B on X if R is a topological invariant under fuzzy homeomorphism. If $R(a, b)$, $a \in A, b \in B$ takes only value 0 and 1, then R is crisp. In other words fuzzy topological relations are relations that are invariant under fuzzy homeomorphism.

2.5 Fuzzy line

A fuzzy line is a curve with continuous transition of membership grades with neighboring points.
Let (X, T) be an fts. Then

Definition 2.5.1. [59] Let P and Q be two points in X. The non-fuzzy line joining PQ is defined as the image of a map $\alpha : [0,1] \longrightarrow X$ given by $\alpha(t) = P + t(Q - P)$, where [0,1] is a closed interval in $\mathbb{R}$ and $t \in [0,1]$.

Definition 2.5.2. [59] A fuzzy subset l in X is called a fuzzy line if support of l is a non-fuzzy line in X and its boundary has at most two supported connected components. ,

Remark 2.5.1. *In definition 2.5.2, Pu-Liu's notion of fuzzy boundary is considered.*

Example 2.5.2. *The following is an example of a fuzzy line in* $\mathbb{R}^2$:

$L : \mathbb{R}^2 \longrightarrow [0,1]$ *by*

$$L(x) = \begin{cases} 0, & \text{if } x \leq 0.25 \\ \epsilon, & \text{if } 0.25 < x < 0.5 \\ \frac{1}{2}, & \text{if } 0.5 \leq x \leq .75 \\ \frac{1}{2} - \epsilon, & \text{if } 75 < x \leq 1 \\ 0, & \text{if } x \geq 1 \end{cases}$$

where $\epsilon > 0$ *is any arbitrary real number.*

It is represented by figure 2.2.

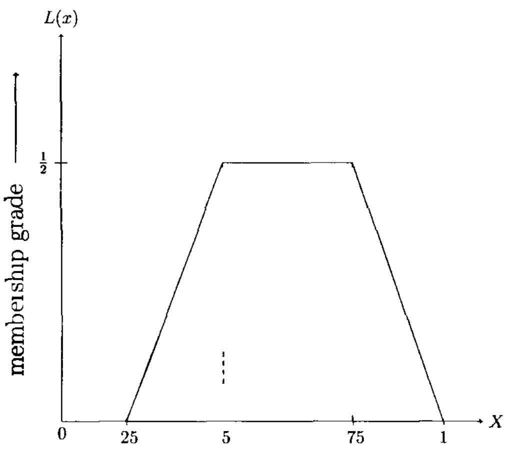


Figure 2.2: A fuzzy line in $\mathbb{R}^2$

2.6 Fuzzy regions

2.6.1 Reason for occurrence of fuzzy regions

A fuzzy region occurs due to a variety of causes. Two important reasons are the following:

i) *Indeterminate boundary*

Most of the spatial phenomena occurring in nature do not have sharp boundaries or precisely defined boundary which gives rise to fuzzy spatial objects and fuzzy regions

ii) *Temporal changes*

Due to temporal changes of a spatial object it frequently changes its position which give rise to position uncertainty Some of the attributes of the object also depend upon the passage of time and hence give rise to a temporal fuzzy region

Since their inception, fuzzy sets have been very effectively used in many branches of knowledge to deal with uncertainty or vagueness associated with various phenomenon Fuzzy regions are natural generalization of crisp regions to accommodate uncertainties involved in describing spatial objects particularly due to reasons (i) and (ii) above

2.6.2 Examples of fuzzy regions

Following are some typical examples of fuzzy regions that we encounter in our day to day life

i) Clouds of polluted air near a chemical factory

ii) Regions with different chances of contacting virus infections

iii) Weather maps

2.6.3 Mathematical approaches

Informally speaking, a fuzzy region is a region that associates a degree of membership (in the fuzzy region) to each point of the region w r t the space such that the support set of a fuzzy region is a crisp region

There are two noteworthy definitions of fuzzy region proposed by Schneider [79] and Tang and Kainz [88, 89] provided in the next section.

2.6.4 Different approaches to fuzzy regions

There are various approaches of defining fuzzy regions in an fts provided by various authors including Schneider [79], Tang and Kainz [88, 92], Palshikar [73] etc. In this work, we will consider Schneider's as well as Tang and Kainz's approach as these approaches consider fuzzy regions as open and closed sets respectively so that a fuzzy region shall be an extension of a crisp region. Other approaches discard this point.

Schneider's approach (1999)

(a) Schneider's fuzzy regions in reality: According to Schneider there are four types of fuzzy regions in reality:

i) *Core-boundary fuzzy regions*:

In such type of fuzzy regions one can differentiate core, boundary and the exterior of the region. It can be modeled by assigining membership value 1 to core, $\frac{1}{2}$ to boundary and 0 to exterior.

For example, a lake with minimal water level during dry periods (core) and maximal water level in rainy periods (boundary is the difference between maximal and minimal water level). It can be seen that dry periods entail puddles which are less flooded but more flooded (not completely flooded) in rainy season.

ii) *Finite valued fuzzy regions*:

In this case finite number of memberships are used for representation. If $n \in N$ is the number of possible "truth value", then an n-valued membership function is used to represent a wide range of belongingness of a point into the region.

For example, we can consider a region with different possibilities for virus infection where region will be divided into 'n' different risk levels extending from areas with extreme risk of infection over areas with average risk of infection to safe areas

iii) *Interval-valued fuzzy regions*

Here interval valued fuzzy sets are used in place of assigning single value to represent a region If there is an order set of n-arbitrary but disjoint values of the interval $[0, 1]$, then we assign any value say v to one of this values and assign values to other points of the component such that v is the lower bound of the set and each point has a successor If w is the greatest of all these successors, then all point of the component is mapped to the interval $[v, w]$

For example, the map about the population density of a country Here the country is sub-divided into regions showing the minimal guarantee population density per unit area for each region The density values of different regions can be rather different

iv) *Smooth fuzzy regions*

In this class of fuzzy region the distribution of attributes are smooth, which is achieved by predominantly continuous membership function

For example magnetic field, temperature zone, sun insolation etc can be termed as smooth fuzzy regions

It may be noted that core-boundary fuzzy regions and finite value fuzzy regions are qualitative in character, i e the number involved in membership function gives indication about the symbolic role but it does not provide any information regarding the size of the attribute or effect whereas interval-valued fuzzy regions and smooth fuzzy regions give emphasis on quantitative character of the effect

Schneider's approach [79]

Let T be a fuzzy topology on Euclidean space $\mathbb{R}^2$ and A be a fuzzy set in $(\mathbb{R}^2, T)$ The following topological inputs are required for formal definition of fuzzy regions:

Definition 2.6.1. Frontier of a fuzzy set A is defined as

$frontier_T(A) = \{((x,y), \mu_A(x,y)) : (x,y) \in supp(A) - supp(A^\circ)\}$

where μ_A is a membership function from $\mathbb{R}^2$ to [0,1].

Definition 2.6.2. A fuzzy set A is said to be spatially fuzzy regular set, if A° is a regular open set such that $frontier_T(A) \subseteq frontier_T((\overline{A})^\circ)$ and $frontier_T(A)$ is a partition of 'n' connected boundary parts.

Definition 2.6.3. Regularisation function of a fuzzy set A is defined as $reg_f(A) = (\overline{A})^\circ \cup (frontier_T(A) \cap frontier_T(\overline{A})^\circ))$

The interior operator eliminates dangling points and line features. The closure operator introduces fuzzy boundary similar to crisp boundary separating the points of closed set from exterior. The operator $frontier_T$ ensures the restriction of the boundary because Schneider considered that fuzzy regions are partially bounded.

Definition 2.6.4. A function f is said to contain a continuity gap at point x_o of its domain if it is semi-continuous (i.e we recall that a function $f : X \longrightarrow [0,1]$ is said to be semi-continuous if f is continuous for both upper and lower topology on [0,1] where upper and lower topologies are generated by the sets $\{[0,a) : 0 \leq a \leq 1\}$ and $\{(a,1] : 0 \leq a \leq 1\}$ respectively) but not continuous at x_o.

Definition 2.6.5. A function f is said to be predominantly continuous if it is continuous and has at most a finite number of continuity gaps.

Definition 2.6.6. A region R which is a subspace of spatially regular fuzzy set and its membership is predominantly continuous.

Tang and Kainz's approach [92]:

Definition 2.6.7. A fuzzy set A is called a simple fuzzy region in a connected crisp fts if it meets the following conditions:

(SR1) The closure of the simple fuzzy region is a regular closed subset in fts.

(SR2) The interior, the interior of the boundary and the exterior are non empty connected sets.

(SR3) The support of the simple fuzzy region is equal to the closure of A.

The first condition ensures that a fuzzy region is an generalization of a crisp simple region. The regularity condition ensures the removal of geometric anomalies (missing points and lines) in the region in the form of cuts and punctures. The first part of the second condition ensures that the interior should be in one piece. Connectedness of the interior of the boundary removes dangle points or break lines. The connectedness of boundary and exterior of a simple fuzzy region ensures that it shall not contain any hole or gap. The third condition requires that A should be equal to a crisp simple region when A is projected to 1 of [0,1].

Possible settings of a fuzzy region in the crisp fts

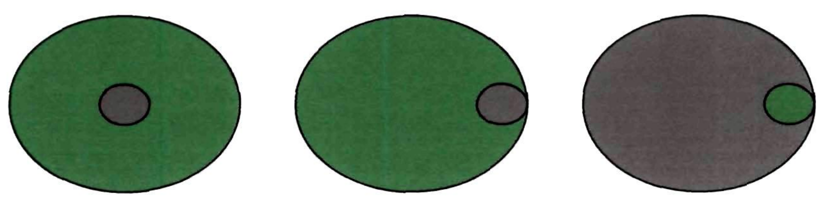


Figure 2.3: Possible setting of a simple fuzzy region

These three figures simultaneously satisfy all the conditions required for Tang and Kainz's simple fuzzy region.

Impossible settings of fuzzy region in a crisp fts

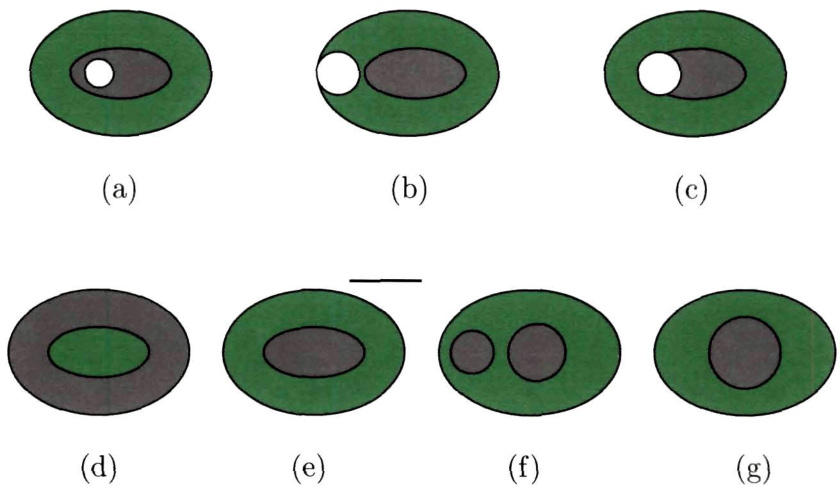


Figure 2.4: Impossible settings of simple fuzzy region

The figure 2.4 shows impossible settings for a simple fuzzy region as they do not satisfy some of the conditions of fuzzy regions. In the figures (a), (b) and (c) exterior is not connected, in figure (d) boundary is not connected. Figure (e) in the second row, closure of the set is not regular closed set, in the figure (f) in the same row, interior of the set is not connected and in the figure (g) in the same row, interior of the boundary is not connected. Hence these do not represent a simple fuzzy region in $(\mathbb{R}^2, C)$.

Tang and Kainz provided another definition of fuzzy regions in the setting of a general fts which will be discussed later in Chapter 4, using some more fuzzy topological notions.

Merits and demerits of the above approaches

Schneider's definition of fuzzy region is based on open sets and Tang and Kainz's definition is in terms of closed sets. Since, crisp region is considered to be a closed set, therefore, Tang and Kainz's definition is seen as more consistent than Schneider's definition. However, each of these approaches has certain advantages and disadvantages. We now enlist some of the merits and demerits of both the approaches.

Merits

i) Regularization function in the Schneider's definition of fuzzy region can alone work for the removal of geometric anomalies which decreases the complexity of the model.

ii) Schneider's definition can be applied even when exterior is not connected.

iii) Tang and Kainz's definition is consistent with the definition of crisp simple region.

Demerits

i) Schneider definition is inconsistent with the definition of crisp simple region.

ii) Tang and Kainz's definition is applicable only when exterior is connected.

The approach of Tang and Kainz's for defining a fuzzy region is seen to be more consistent than Schneider's approach because in the case of Schneider's definition regularization of the space only removes breaklines and dangle points but in case of Tang and Kainz's fuzzy region, as justified in the definition, regularization not only removes breaklines and dangle points, it also removes cuts and punctures. Furthermore, Tang and Kainz's definition is a direct extension of that of a crisp region in fuzzy setting. However, both the definitions of fuzzy regions would not work if the region contains holes.

2.6.5 Topological relations of fuzzy spatial objects

Topological relations between two fuzzy regions:

A classical topological space cannot accommodate fuzzy sets since all subsets are crisp. So, the study of topological relations between fuzzy sets in the setting of an fts emerge as an useful area of research. Many authors have studied the topological relations between fuzzy spatial objects in an fts. In fuzzy set theory due to the non existence of the law of contradiction and excluded middle, the intersection of interior and boundary as well as boundary and exterior is non-empty. Hence, further analysis of properties are required to study the topological relations between two fuzzy sets. It has been shown by many authors that the number of topological relations between two fuzzy sets will vary depending on the procedure of derivation. But there are eight elementary topological relations viz. disjoint, meet, overlap, equal, cover, covered by, inside, contain (similar to classical case) which will always exist between any two simple fuzzy regions A and B in any fts, in particular, in $(\mathbb{R}^2, T)$ where T is a topology on $\mathbb{R}^2$.

The mathematical definitions of these eight relations [55] are given below:

i) Disjoint: If $(A \cap B)(x) = 0$ for all $x \in \mathbb{R}^2$.

ii) Meet: If there exists some $x_0 \in \mathbb{R}^2$ such that $(A \cap B)(x_0) > 0$ and there is no $x \in X$ such that neither $A(x) < B(x)$ nor $A(x) > B(x)$.

iii) Overlap: If $(A \cap B)(x_0) > 0$, $A(x_1) < B(x_1)$ and $B(x_2) < A(x_2)$ for some $x_0, x_1, x_2 \in \mathbb{R}^2$.

iv) Equal: If $A(x) = B(x)$ for all $x \in \mathbb{R}^2$.

v) Cover: If $A(x) \geq B(x)$ for all $x \in \mathbb{R}^2$ satisfying $B(x) > 0$ and there exists $x_1 \in \mathbb{R}^2$ such that $A(x_1) = B(x_1) > 0$.

vi) Covered by: If $A(x) \leq B(x)$ for all $x \in \mathbb{R}^2$ satisfying $A(x) > 0$ and there exists $x_1 \in \mathbb{R}^2$ such that $A(x_1) = B(x_1) > 0$.

vii) Inside/contained in: If $A(x) < B(x)$ for all $x \in \mathbb{R}^2$ satisfying $A(x) > 0$.

viii) Contain: If $A(x) > B(x)$ for all $x \in \mathbb{R}^2$ satisfying $B(x) > 0$.

In an fts due to the occurrence of membership grades, each of these eight elementary relations has certain membership which can be calculated by using Zhan's formula [102].

Quantitative topological relations between simple fuzzy objects

Liu and Shi [55, 58, 59] developed a method to determine quantitative fuzzy topological relations between two fuzzy spatial objects. They have developed the following 3×3 integration model

	A_α	∂A	$(A^c)_\alpha$
B_α	$\int_X (A_\alpha \cap B_\alpha)$ dV	$\int_X (\partial A \cap B_\alpha)$ dV	$\int_X ((A^c)_\alpha \cap B_\alpha)$ dV
∂B	$\int_X (A_\alpha \cap \partial B)$ dV	$\int_X (\partial A \cap \partial B)$ dV	$\int_X ((A^c)_\alpha \cap \partial B)$ dV
$(B^c)_\alpha$	$\int_X (A_\alpha \cap (B^c)_\alpha)$ dV	$\int_X (\partial A \cap (B^c)_\alpha)$ dV	$\int_X ((A^c)_\alpha \cap (B^c)_\alpha)$ dV

Table 2.7: Quantitative topological relation matrix for fuzzy regions

where A and B are two fuzzy sets, A_α is α-interior of A where
$A_\alpha(x) = \begin{cases} A(x), & \text{if } A(x) > \alpha; \\ 0, & A(x) \leq \alpha. \end{cases}$, ∂A-boundary of A, $(A^c)_\alpha$-exterior of A, and
$\int_X (A \cap B) = (\int_X (A \cap B)(x)dx | \int_X (A \cup B)(x)dx)$.

The geometric meaning of $\int_X (A \cap B)dV$ is that it represents the ratio of the area (or volume) of the meet of two fuzzy spatial objects to the join of the two fuzzy spatial objects, where join means union and meet means intersection of the object parts. If the involved spatial objects are a fuzzy line and a fuzzy point, then volume integral is replaced by surface or line integral.

Using 3×3 integration matrix, Liu and Shi [59] found that there are 3 realizable relations between a fuzzy region and a fuzzy point, 16 realizable relations between a fuzzy region and a fuzzy line, 3 between a fuzzy line and a fuzzy point, and as many as 46 between two fuzzy lines in $\mathbb{R}^2$.

Further, from the mathematical definition of topological relations between a fuzzy region and a fuzzy point it can be seen that the set of topological relations between fuzzy region and fuzzy point is a subset of the set of topological relations between two fuzzy regions, that is, if A be the fuzzy region and P be the point then, the relevant mathematical definitions are

i) Disjoint: $A \cap P = \phi$

ii) Meet/overlap: $\partial A \cap P \neq \phi$

iii) Inside/contain/cover/covered by: $A^\circ \cap P \neq \phi$.

Chapter 3

Fuzzy region with holes and their topological relations in a crisp fts

3.1 Introduction

A fuzzy region is a region with imprecise boundary which allows flexibility of strict belongingness criteria of a point in space in relation to the region. Fuzzy regions were initially defined by Schneider [79] in terms of fuzzy open sets. A fuzzy region can be considered as an extension of a crisp region that allows flexibility in belongingness of a point in the region with respect to the space. Tang and Kainz [88, 89, 92] provided two definitions of fuzzy regions - one in a special type of fuzzy topological space viz. a crisp fuzzy topological space and the other in a general fuzzy topological space. They considered fuzzy regions to be closed sets making it consistent with its classical counterpart. However, incorporation of holes in a fuzzy region has not been considered by any of these approaches. A large number of real life phenomena exhibit discontinuity at boundary and exterior in the form of cavities giving rise to regions with holes. In order to capture these real phenomena, it becomes essential to incorporate

[1] Selected portions of this chapter have appeared in our papers [43] and [44]

them explicitly in the formulation of fuzzy regions. In this Chapter, we have defined fuzzy regions with holes in a crisp fts $(\mathbb{R}^2, C)$ which will be an extension of crisp region in a classical topological space in $(\mathbb{R}^2, C)$.

The main purpose behind the development of fuzzy regions is the derivation of topological relations. As discussed in Chapter 1, Zhan [102] formulated fuzzy analogues of the 4-intersection model in terms of the α-cut operation. Du et. al. [23] proposed the fuzzy extension of the 9-intersection matrix by defining the membership grades for interior, boundary and exterior. Tang and Kainz [88, 92], derived topological relations between fuzzy regions and used them in analysis of land cover changes. In the current chapter, we have derived the topological relations between fuzzy region with holes and various other basic fuzzy spatial objects in a crisp fts.

3.2 A comparative study of fuzzy boundaries

In Chapter 2, we have discussed various types of fuzzy boundaries available in the literature. It may be noted that the boundary of the spatial objects is one of the important aspects of studying and analyzing fuzzy topological relations. In this section we analyze and compare these definitions in terms of their set-theoretic properties. The purpose is to justifiably identify the fuzzy boundary suitable to be used in the context of application in this thesis.

3.2.1 Interrelationship among the boundaries

We recall the fuzzy boundaries in the sense of Warren, Pu-Liu and Cuchillo-Ibanez-Tarres. For a fuzzy set A, we denote these boundaries by $\partial_1 A$, $\partial_2 A$, $\partial_3 A$ respectively.

The following are obvious:

(i) $\overline{A} \geq \partial_1 A \geq \partial_3 A$ and (ii) $\partial_1 A \geq \partial_2 A$

i.e. $\partial_1 A$ contains the other two boundaries, which in turn is the boundary contained in the closure of the set.

Remark 3.2.1. *Fuzzy boundaries in the sense of Pu and Liu and Cuchillo-Ibanez are independent of each other.*

We consider the following example to establish the same:

Example 3.2.2. *Let $X = \{a, b\}$ and fuzzy topology on X is given by*
$T = \{0_X, \{a_{.4}, b_{.8}\}, \{a_{.6}, b_{.9}\}, \{a_{.5}, b_{.7}\}, \{a_{.8}, b_{.7}\}, \{a_{.3}, b_{.2}\}, \{a_{.4}, b_{.2}\}, \{a_{.5}, b_{.2}\}, \{a_{.6}, b_{.7}\},$
$\{a_{.5}, b_{.8}\}, \{a_{.8}, b_{.8}\}, \{a_{.4}, b_{.7}\}\{a_{.6}, b_{.8}\}, \{a_{.8}, b_{.9}\}, 1_X\}$
Let $A = \{a_{.4}, b_{.2}\}$ and $B = \{a_{.4}, b_{.7}\}$.
Then, $\partial_1 A = \{a_{.4}, b_{.2}\}$, $\partial_2 A = \{a_{.4}, b_{.2}\}$, $\partial_3 A = 0_X$ and
$\partial_1 B = \{a_{.5}, b_{.8}\}$, $\partial_2 B = \{a_{.5}, b_{.3}\}$, $\partial_3 B = \{a_{.5}, b_{.8}\}$
Therefore, $\partial_3 A \lneqq \partial_2 A$ and $\partial_2 B \lneqq \partial_3 B$.

3.2.2 Comparison of important set theoretic identities

Theorem 3.2.3. $\partial_i(\partial_i(\partial_i A)) \leq (\partial_i(\partial_i A))$, $i = 1, 2, 3$.

Proof For $i = 2$ and 3 proofs were provided by Ahmed and Athar [5] and Cuchillo-Ibanez [19] and for $i = 1$ the proof is trivial. □

Remark 3.2.4. *It was shown in [5] that $\partial_2 A = \partial_2 A^c$ but the result would not hold for other two definitions*

Example 3.2.5. *Let $X = \{a, b, c\}$. Let the fuzzy topology on X be given by :*
$T = \{0_X, \{a_{.4}, b_{.8}, c_{.2}\}, \{a_{.6}, b_{.9}, c_{.1}\}, \{a_{.5}, b_{.7}, c_{.3}\}, \{a_{.4}, b_{.8}, c_{.1}\}, \{a_{.4}, b_{.7}, c_{.2}\},$
$\{a_{.5}, b_{.7}, c_{.1}\}, \{a_{.4}, b_{.7}, c_{.1}\}, \{a_{.5}, b_{.8}, c_{.3}\}\{a_{.5}, b_{.8}, c_{.1}\}, \{a_{.6}, b_{.9}, c_{.2}\}, \{a_{.5}, b_{.8}, c_{.2}\},$
$\{a_{.6}, b_{.9}, c_{.3}\}, \{a_{.5}, b_{.7}, c_{.2}\}, 1_X\}$.

Let $A = \{a_{.5}, b_{.3}, c_{.7}\}$.

Then, $\partial_1 A = \{a_{.5}, b_{.3}, c_{.7}\}, \partial_1 A^c = 1_X, \partial_3 A = \{a_{.5}, b_{.3}, c_{.7}\}$ *and* $\partial_3 A^c = 1_X$

Hence, $\partial_1 A \neq \partial_1 A^c$ *and* $\partial_3 A \neq \partial_3 A^c$

Theorem 3.2.6. $\partial_i A \geq \overline{A} - A^\circ$ *(i=1,2,3).*

Proof. (i) If $(\overline{A} \cap \overline{A^c})(x) > 0$ then $\partial_1 A(x) = \overline{A}(x)$

It suffices to examine only those $x \in X$ for which $(\overline{A} \cap \overline{A^c})(x) > 0$.

If for all $x \in X$, we have $(\overline{A} \cap \overline{A^c})(x) > 0$ then $\partial_1 A(x) = \overline{A}(x)$.

Now, if for all $x \in X$, $(\overline{A} \cap \overline{A^c})(x) = 0$ then $\partial_1 A(x) = 0$.

As $\overline{A} \geq \overline{A} - A^\circ$, it follows that $\partial_1 A \geq \overline{A} - A^\circ$.

(ii) $\partial_2 A = \overline{A} \cap \overline{A^c} \geq \overline{A} - A^\circ$

(iii) If $(\overline{A} - A^\circ)(x) > 0$ then $\partial_3 A(x) = \overline{A}(x) \geq \overline{A}(x) - A^\circ(x)$.

On the other hand, if $(\overline{A} - A^\circ)(x) = 0$, then $\partial_3 A \geq \overline{A} - A^\circ = 0_X$. □

In [5], it was shown that if $f : X \to Y$ is a fuzzy continuous function, then $\partial_2 f^{-1}(A) \leq f^{-1}(\partial_2 A)$ for each fuzzy set A in Y. The same also holds for other two boundaries.

Theorem 3.2.7. *Let* $f : X \to Y$ *be a fuzzy continuous function and* A *be a fuzzy set in* Y*. Then* $\partial_i f^{-1}(A) \leq f^{-1}(\partial_i A)$ *(i=1,3).*

Proof. Here, $\partial_1 A$ is fuzzy closed in Y. Then, $f^{-1}(\partial_1 A)$ is fuzzy closed in X.

Since f is non-null, $(\overline{f^{-1}(A)} \cap \overline{(f^{-1}(A))^c})(x) > 0$.

Hence, $\partial_1 f^{-1}(A) \leq \overline{f^{-1}(\partial_1 A)} = f^{-1}(\partial_1 A)$.

Likewise, $\partial_3 f^{-1}(A) \leq f^{-1}(\partial_3 A)$. □

Theorem 3.2.8. *Let* $A \leq B$ *and* $B \in FC(X)$*. Then* $\partial_i A \leq B$ (i=1,2,3).

Proof. The proof is obvious as $\partial_i A \leq \overline{A}$, $i = 1, 2, 3$. □

It has been shown in [19, 95] that $\partial_i A^\circ \le \partial_i A$ and $\partial_i \overline{A} \le \partial_i A$, $i = 1, 3$.

Theorem 3.2.9. *(i)* $\partial_2 A^\circ \le \partial_2 A$ *(ii)* $\partial_2 \overline{A} \le \partial_2 A$

Proof. (i) $\partial_2 A^\circ = \overline{A^\circ} \cap \overline{(A^\circ)^c} = \overline{A^\circ} \cap \overline{\overline{A^c}} \le \overline{A} \cap \overline{A^c} = \partial_2 A$

(ii) $A \le \overline{A} \Rightarrow A^c \ge (\overline{A})^c \Rightarrow \overline{A^c} \ge \overline{(\overline{A})^c} \Rightarrow \overline{A} \cap \overline{A^c} \ge \overline{A} \cap \overline{(\overline{A})^c} \Rightarrow \partial_2 \overline{A} \le \partial_2 A$. □

Theorem 3.2.10. *In a crisp fts all three definitions of fuzzy boundary are equivalent.*

Proof. If $A \in X$ be any fuzzy set where X is a crisp fts. Then closed set containing A is a crisp set. By definition of the Warren boundary, $\partial_1 A$=infimum of closed sets D such that $D(x) \ge \overline{A}(x)$ for all $x \in X$ for which $(\overline{A} \cap \overline{A^c})(x) > 0$. Since in crisp fts if $(\overline{A} \cap \overline{A^c})(x) > 0$ then $(\overline{A} \cap \overline{A^c})(x) = 1$ that is $(\overline{A} \cap \overline{A^c})$ is the smallest closed set containing A. Hence $\partial_1 A = \overline{A} \cap \overline{A^c}$.
Similary $\partial_2 A = \overline{A} \cap \overline{A^c}$ and $\partial_3 A = \overline{A} \cap \overline{A^c}$. □

Corollary 3.2.11. *If the intersection of the closure of the set and closure of the set complement is zero then value of all the three forms of boundary are equal.*

Remark 3.2.12. *From Theorem 3.2.10 we can infer that if space under consideration is a crisp fts then Pu-Liu definition of fuzzy boundary shall be most suitable, it being analogous to definition of boundary in classical topology. All other forms are equivalent to this form under this setting. If the space under consideration is a general fts then we would prefer to choose Warren boundary because from Section 3.2.1 it contains the other two. Results derived using the same shall therefore be more general.*

3.3 Separate settings for crisp and general fts's

In the course of this thesis, the settings of a crisp fts and general fts are taken up separately for development of fuzzy regions with holes The reasons are as follows

A crisp fts exhibits behaviour similar to a classical topological space The difference however, lies in the fact that in a crisp fts fuzzy sets are allowed while the later does not allow it Further in a crisp fts, it was noted in 3 2 1 that all the three definitions of fuzzy boundary will be equivalent So Pu-Liu definition of a fuzzy boundary can be utilized as it is analogous to definition of a boundary in classical topological space Moreover, in this setting we can define a fuzzy region with holes using simple topological notions such as interior, boundary, exterior and the likes However, a crisp fts has the definite drawback that it overlooks the membership grades of open sets and consequently the membership grades of the closed sets as well As most of the real life phenomena shows variation due to intrinsic fuzziness, these can be better represented if we allow the membership grades of the open sets A general fts allows the intrinsic imprecision of the objects by allowing membership grade of the open sets But in that setting, we shall require to define some special topological notions to deal with complexity of the situation and to maintain the consistency of the definition of fuzzy regions with holes with crisp regions with holes in classical topology at the same time

The main advantage of treating a crisp and a general fts separately and independently, therefore, lies in the development of the framework, if the underlying space of the spatial objects does not show much variation in the behaviour of the attributes under study with respect to other time dependant attribute of the space, then we choose the setting of a crisp fts, reducing time as well as complexity of the computation in the process

3.4 Fuzzy region with holes

Fuzzy regions with holes occur in different real life situations:
Some typical example of fuzzy regions with holes are - the earth surrounded by oceans and inside the earth there are various water bodies such as lakes, pond etc. Generally, ocean water is not used for drinking purposes but water of lakes and ponds can be used for drinking. So we can say that lakes and ponds water are bounded as it contain fresh water which can be used for household purpose and ocean water as unbounded as it contains impurities and hence cannot be used in household activities. We can, therefore, consider oceans as an outer exterior of the earth and lakes as well as ponds as an inner exterior of the earth due to their differences in attributes. In this case the earth can be considered as a fuzzy region with holes where the inner and outer exterior can be distinguished. Occurrence of oil underground is another example of fuzzy regions with holes because the membership grade of fertility of soil varies from portion to portion and the pore that contain oil inside the ground are that portion where membership grade of fertility of the soil is zero. In each of these examples first object is a hole on the second. Holes are placed in host material that surround them and therefore cannot occur alone unless a surface for its occurrence is provided by the host.

In next subsection, we have provided a framework for defining a fuzzy region with holes in a crisp fts.

3.4.1 Framework for definition of fuzzy region with holes

We now generalize the definition of fuzzy regions in crisp fts. Since, a crisp subset is a special case of fuzzy set so a crisp region with and without holes is a special case of fuzzy region with holes. We require the following definitions for defining fuzzy regions with holes in the crisp fuzzy topological space $(\mathbb{R}^2, C)$:

Definition 3.4.1. Inner exterior of a fuzzy region with holes is the bounded exterior contained in the region

Definition 3.4.2. A hole is the closure of the inner exterior

Definition 3.4.3. Outer exterior of the fuzzy region corresponds to the unbounded exterior of the region

Definition 3.4.4. A component of boundary is a boundary of the hole which separates the interior and inner exterior of the fuzzy region

Remark 3.4.1. *The term 'component' used above is not being used in the usual meaning of the term It simply means the piecewise connected boundary of the region We will call the boundary of the fuzzy region with holes as main/outer boundary of the region*

3.4.2 Formal definition

For simplicity we assume that holes should contained in the region and are disjoint from each other Further the region should contain at most finite number of holes which are not along the main/outer boundary of the region

Definition 3.4.5. A fuzzy set A is called a fuzzy region with holes in a connected crisp fuzzy topological space in $(\mathbb{R}^2, C)$, if it satisfies the following conditions

(SR1) The closure of A is a connected regular closed set

(SR2) The interior of A is a connected set

(SR3) Boundary is the disjoint union of connected components

(SR4) Exterior is the disjoint union of inner and outer exteriors

(SR5) Inner and outer exterior are themselves connected

(SR6) The support of A is equal to the closure of A

A fuzzy region with holes with its different parts in a crisp fuzzy topological space is schematically illustrated in Figure 3.1.

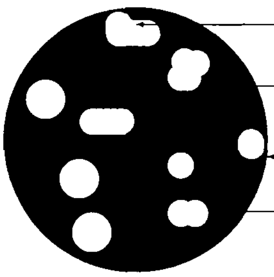


Figure 3.1: A fuzzy region with holes

Justification of the formal definition

The first condition is an extension of a crisp region in the fuzzy setting as crisp regions are considered as regular closed sets. In (SR2) we assume the connectedness of the interior to ensure that a crisp region (with holes) is a particular case of a fuzzy region (with holes). Conditions (SR3) and (SR4) ensure the existence of holes disjoint from each other and are not along the boundary of the fuzzy region. Connectedness condition in (SR5) ensures that the region does not contain spikes so that if the holes are eliminated, it becomes a fuzzy region without holes as provided by Tang [89]. Condition (SR6) signifies that when we draw a plane through 1 of the interval $[0, 1]$ then the projection of fuzzy region in this plane will be the crisp region with hole provided by Egenhofer et al. [33] in a classical topological space.

This definition therefore simultaneously generalizes Tang's [89] definition of fuzzy regions in crisp fts and Egenhofer et al.'s definition [33] of crisp regions with holes in a classical topological space.

Fuzzy region with holes cannot be defined in an analogous way of the definition of crisp region with holes in a classical topological space as defined by

Egenhofer et al. [33], as then it becomes impossible to ensure that the region and holes are disjoint topological invariants. Tang [89] had earlier shown that in an fts, the intersection of interior and boundary as well as intersection of boundary and exterior are, in general, non-empty due to non existence of the law of contradiction and the law of excluded middle in the fuzzy case. Hence, such a definition does not ensure that the region and holes are mutually disjoint topological parts and therefore in that case we are unable to derive topological relations without giving an explicit formula for interior, boundary, exterior of the region and their intersections which makes the procedure cumbersome and complex. The concept of hole as closure of inner exterior allows us to treat the fuzzy region with holes into a number of fuzzy regions without hole so that topological relations among these regions are expressed in terms of topological relations between fuzzy region without hole. However, this definition would not allow existence of infinite number of holes or a situation in which holes overlap with each other and also would not allow regions where holes would be lying along the boundary.

It may also be observed that holes are not isolated cases of points whose membership of fuzzy attribute is zero but concerns a connected neighbourhood with unsharp boundary whose membership is zero.

Remark 3.4.2. *The crisp intersection of interior and component of boundary as well as component of boundary and holes of a fuzzy region with holes in connected crisp fts* $(\mathbb{R}^2, C)$ *is empty.*

Definition 3.4.6. A generalized fuzzy region is union of the interior, the component of boundary and the holes.

Theorem 3.4.3. *If component of boundary and inner exterior are empty then fuzzy regions with hole becomes a simple fuzzy region.*

Proof. Let A be a fuzzy region with holes H_1, H_2, H_3 and let A^* be the generalized fuzzy region of A. Then, $A^* = A^\circ \cup (\cup_{i=1}^{3} \partial H_i) \cup (\cup_{i=1}^{3} H_i)$. Since $H_i = \overline{A_i^{ext}}$; $i = 1, 2, 3$ and ∂H_i and A_i^{ext} are empty, therefore, $A^* = A^\circ$. □

3.5 Topological relations

In this section, we have determined the topological relations between fuzzy region with holes and some of the basic fuzzy spatial objects. By definition of holes we know that holes are the bounded exterior of the region, so hole/inner exterior does not mean a void or empty set. Though it means that membership grade of attribute under study in the holes are zero but membership grade of other attribute of the region in the holes are not zero. Due to this reason we will consider the generalized fuzzy region and holes as separate spatial objects without hole for deriving topological relations otherwise considering hole as separate spatial object will be meaningless.

3.5.1 Topological relations between fuzzy region with holes and fuzzy point in a crisp fts

We have considered generalized fuzzy region, holes and fuzzy points as topological invariants to determine the relational matrix. We shall consider each of the topological invariants as a fuzzy region without holes and the following topological relations shall be considered (i) topological relations between generalized fuzzy region and holes (which are considered as simple fuzzy regions). (ii) topological relations between generalized fuzzy region and fuzzy point and (iii) topological relations between holes and fuzzy point as content of intersection to determine the topological relations between fuzzy region with holes and fuzzy point. We have seen in the last chapter that topological relations between fuzzy region and fuzzy point is a subset of the topological relations between two fuzzy regions We therefore, consider the content of intersection to be the eight topological relations between two fuzzy regions, i e , we consider the content of intersection to be the 8 relations of regional variations (disjoint, meet, overlap, covered by, inside, equal, covers, contains as defined in Subsection 2 6 5) depending on the nature of intersection As mentioned earlier, we assume that a fuzzy region with holes contains at most a finite number of holes Let A^* be the generalized fuzzy region consisting of 'n' holes H_1, H_2 , H_n P_A be a fuzzy point, then the intersection matrix will be of the form given in table 3 1

	A^*	H_1	H_2			H_n	P_A
A^*	$t(A^* A^*)$	$t(A^*, H_1)$	$t(A^*, H_2)$			$t(A^*, H_n)$	$t(A^*, P_A)$
H_1	$t(H_1, A^*)$	$t(H_1, H_1)$	$t(H_1, H_2)$			$t(H_1, H_n)$	$t(H_1, P_A)$
H_2	$t(H_2, A^*)$	$t(H_2, H_1)$	$t(H_2, H_2)$			$t(H_2, H_n)$	$t(H_2, P_A)$
H_n	$t(H_n, A^*)$	$t(H_n, H_1)$	$t(H_n, H_2)$			$t(H_n H_n)$	$t(H_n, P_A)$
P_A	$t(P_A, A^*)$	$t(P_A, H_1)$	$t(P_A, H_2)$			$t(P_A, H_n)$	$t(P_A, P_A)$

Table 3 1 Intersection matrix for fuzzy region with holes and fuzzy point

where $t(A^*, H_1)$ represents the topological relation between A^* and H_1 Similarly all other entries in the matrix have their usual meaning

There are a total of $(n+2)^2$ distinct entries in this matrix and each entry will be exactly one of the 8 relations that will be accounted to the number of consistent relations

Further this matrix follows a kind of converse relation and implied by relation in the following sense The relation $t(A^*, H_i)$, $i = 1, 2,$, n will be converse of the relation $t(H_i, A^*)$, $i = 1$ 2, n and these relations can be obtained if we know any of the two relations because if we say generalized fuzzy region contains the holes then it also means that holes are inside the generalized fuzzy region, such kind of relations are known as converse relations. Similarly, the relation $t(A^*, P_A)$ implies the relations for $t(H_i, P_A), i = 1, 2, ..., n$ because if the relation between generalized fuzzy region and fuzzy point is disjoint then holes being contained in the generalized fuzzy region the relation between holes and fuzzy point will trivially be disjoint, such relations are known as implied by relations. So, the topological relations in the above matrix reduces to an equivalent upper or lower triangular matrix given by table 3.2

	A^*	H_1	H_2	.	.	H_n	P_A
A^*	$t(A^*, A^*)$	$t(A^*, H_1)$	$t(A^*, H_2)$	.	.	$t(A^*, H_n)$	$t(A^*, P_A)$
H_1	.	$t(H_1, H_1)$	$t(H_1, H_2)$	.	.	$t(H_1, H_n)$	$t(H_1, P_A)$
H_2	.	.	$t(H_2, H_2)$	.	.	$t(H_2, H_n)$	$t(H_2, P_A)$
.	.	.	.	.	.	.	.
H_n	.	.		.	.	$t(H_n, H_n)$	$t(H_n, P_A)$
P_A	.	.	.	.	.	.	$t(P_A, P_A)$

Table 3.2. Upper triangular matrix for a fuzzy region with holes and a fuzzy point

This is a topological relation matrix, i.e. each entry in this matrix can have one of the eight choices viz. disjoint, overlap, equal, inside, contain, cover, covered by, as we have considered the content of intersection to be the topological relations between two fuzzy regions with connected boundary. The main advantage of considering upper or lower triangular matrix is that it will reduce the number of redundant relations (only mutually exclusive relations are considered). As a result the number of steps of computation of mutually disjoint relations in the matrix is reduced resulting in increased efficiency in computation

To determine the number of consistent relations we add up the entries with distinct relations in each row of the upper triangular matrix as follows

Number of elements in the 1^{st} row $= n + 1$

Number of elements in the 2^{nd} row $= n$

Number of elements in the $(n+1)^{th}$ row $= 1$

Number of elements in the $(n+2)^{th}$ row $= 0$

Therefore, total number of elements in this matrix $= (n+1) + n + \quad + 2 + 1$

$$= \frac{(n+1)(n+2)}{2}$$

We know that if the generalized fuzzy region have the equal relation then generically all the holes must have the equal relation So adding two (one due to the equal relations between the generalized fuzzy region and other due to the equal relations between the fuzzy point as each entry in the diagonal has the same relation i e equal relation) to total number of distinct entries, give us the required number of consistent relations

Therefore, the total number of distinct possible relations is $\frac{(n+1)(n+2)}{2} + 2$

As per our assumption (a) the relation between each of the hole and the generalized fuzzy region is that of containment and (b) each pair of holes are disjoint from each other Therefore, it is possible to further reduce the number of implied relations So that, in the first row of the above intersection matrix the relations $t(A^*, H_1)$, $t(A^*, H_2)$, , $t(A^*, H_n)$ are considered to be a single relation, in the second row $t(H_2, H_3)$, ,$t(H_2, H_n)$ are considered as a single relation Proceeding in this manner, we get,

Number of elements in the 1^{st} row $= 2$

Number of elements in the 2^{nd} row $= 2$

Number of elements in the $(n-1)^{th}$ row $= 2$

Number of elements in the n^{th} row $= 2$

Number of elements in the $(n+1)^{th}$ row $= 1$

Number of elements in the $(n+2)^{th}$ row $= 0$

Therefore, total number of elements $= 2n + 1$

Further, to obtain the total number of relations, as discussed above we add two (due to equal relation of diagonal entries of the matrix) to the total number of relations so that number of consistent relations becomes $2n + 3$.

The following example illustrates the specific case of the above deduction when we consider a fuzzy region with a single hole.

Example 3.5.1. *If there is only one hole then only* 5 *distinct topological relations are realizable between a fuzzy point and a fuzzy region with the hole in a crisp fts as shown in Figure 3.2.*

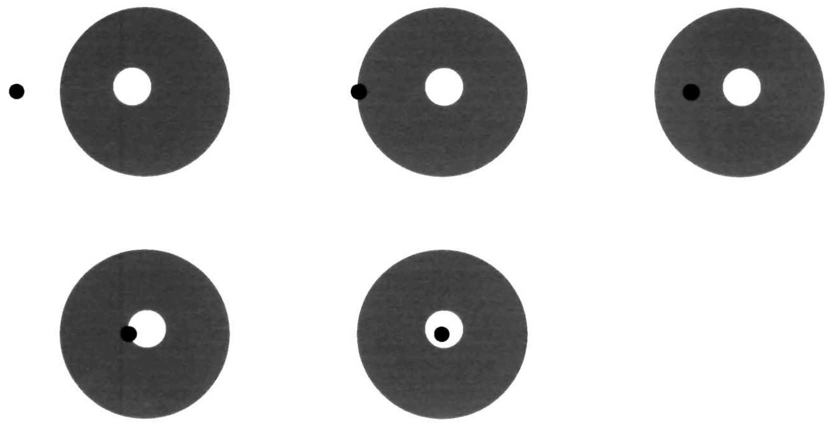


Figure 3.2: Possible topological relations between a single holed fuzzy region and a fuzzy point

The matrix form in this case is given by the matrix given in the Table 3.3.

	A^*	H	P
A^*	$A^* \cap A^*$	$A^* \cap H$	$A^* \cap P$
H	$H \cap A^*$	$H \cap H$	$H \cap P$
P	$P \cap A^*$	$P \cap H$	$P \cap P$

Table 3.3: Intersection matrix for single holed fuzzy region and fuzzy point

Here, A^* is the generalized region, H is hole and P is a fuzzy point.
Since we have considered a generalized fuzzy region, holes and fuzzy points for determining topological relation matrix, so naming of the topological predicates is possible if we consider the particular case of fuzzy region containing only one hole.

Relations between a single holed fuzzy region and a fuzzy point

The following five relations are possible.

1. Disjoint: $A^* \cap Supp(P) = \phi$ and $H \cap Supp(P) = \phi$.
2. Meet: $\partial A \cap Supp(P) \neq \phi$ and $H \cap Supp(P) = \phi$, where ∂A means outer boundary of the fuzzy region with hole.
3. Contain: $A^\circ \cap Supp(P) \neq \phi$ and $H \cap Supp(P) = \phi$.
4. Contain meet: $Supp(P) \in \partial_i A$ and $Supp(P) \notin H$, where $\partial_i A$ means boundary of holes.
5. Disjoint contain: $Supp(P) \notin A$ and $Supp(P) \subset H$.

Here, all the three considered sets viz., A^*, H, P are fuzzy sets.

3.5.2 Topological relations between fuzzy regions with holes and fuzzy line in crisp fts

Suppose a generalized fuzzy region A^* consists of n holes $H_1, H_2, ..., H_n$. Let L_A be a fuzzy line. Then the relation between the fuzzy line and the fuzzy region with 'n' holes is determined by a relational matrix as given in table 3.4.

	A^*	H_1	H_2	.	.	H_n	L_A
A^*	$t(A^*, A^*)$	$t(A^*, H_1)$	$t(A^*, H_2)$	.	.	$t(A^*, H_n)$	$t(A^*, L_A)$
H_1	$t(H_1, A^*)$	$t(H_1, H_1)$	$t(H_1, H_2)$	.	.	$t(H_1, H_n)$	$t(H_1, L_A)$
H_2	$t(H_2, A^*)$	$t(H_2, H_1)$	$t(H_2, H_2)$	.	.	$t(H_2, H_n)$	$t(H_2, L_A)$
.	.	.	.	.	.	.	.
H_n	$t(H_n, A^*)$	$t(H_n, H_1)$	$t(H_n, H_2)$	.	.	$t(H_n, H_n)$	$t(H_n, L_A)$
L_A	$t(L_A, A^*)$	$t(L_A, H_1)$	$t(L_A, H_2)$	.	.	$t(L_A, H_n)$	$t(L_A, L_A)$

Table 3.4: Intersection matrix for a fuzzy region with holes and a fuzzy line

Here, the symbols in the intersection matrix have their usual meaning as in case of fuzzy region with holes and fuzzy point.

Further, generalized fuzzy region and holes can be considered as simple fuzzy region without holes. From regional connection calculus of fuzzy region and fuzzy line given by Liu and Shi [59], we know that only 16 recognizable relations exist between fuzzy region without holes and a fuzzy line. So each entry in the intersection matrix can be filled in 16 ways. Therefore, the total number of relations between a fuzzy line and a fuzzy region with hole are 16^{n+2} where 'n' is the number of holes.

In particular, for $n = 1$ i.e. fuzzy region with a hole there are total 4096 relations. It has been, however, observed that under certain conditions only a few of them will actually be realized in $\mathbb{R}^2$.

Conditions for reducing redundant relations between fuzzy region with hole and fuzzy line

Egenhofer and Herring [31] listed 8 geometric conditions between a region and a line in a classical topological space $\mathbb{R}^2$. These conditions can be extended to determine the relations between a fuzzy line and a fuzzy region with hole in a crisp fts $\mathbb{R}^2$. But in the crisp case the content of intersection are considered as empty and non-empty intersection of interior, boundary and exterior. Whereas in the fuzzy setting, we have considered the content of intersection to be 16 relations between a fuzzy region without hole and a fuzzy line. So, to determine the conditions for obtaining feasible relations we have named the 16 relations as follows: *disjoint*, *meet-at-end*, *meet-at-ends*, *meet-at-center*, *overlap*, *inside*; relations other than these relations are named as *intersect*. Thus, using node, arc and path consistency we have derived a set of 9 conditions to reduce the eliminate relations between fuzzy region with hole and fuzzy line.

These conditions are

1. If the relation between fuzzy line and the generalized fuzzy region is *disjoint* then the relation between the fuzzy line and the hole is also *disjoint*.

2. If the relation between fuzzy line and the generalized fuzzy region is *meet-at-end/meet-at-ends/meet-at-center* then also the relation between the fuzzy line and the hole is *disjoint*.

3. If the relation between fuzzy line and the hole is *inside/meet-at-end/meet-at-ends/meet-at-center*, then the relation between fuzzy line and generalized fuzzy region must be *inside*.

4. If the relation between fuzzy line and the generalized fuzzy region is *inside*, then the relation between fuzzy line and the hole will be anyone of the sixteen relations.

5. If the relation between fuzzy line and the generalized fuzzy region is *overlap*, then the relation between fuzzy line and the hole will be *disjoint/intersect.*

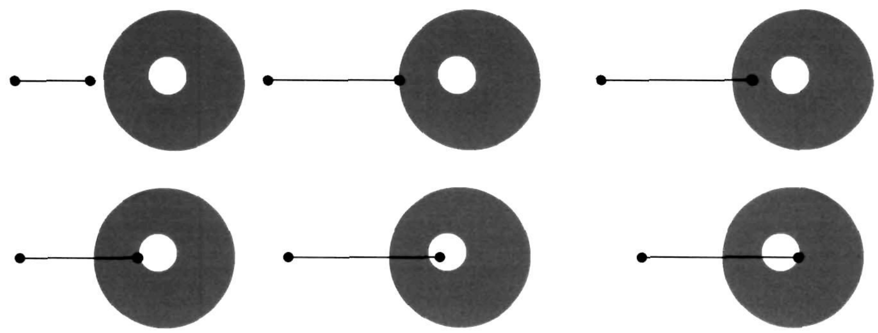
6. If the fuzzy line *intersects* the hole, then the relation between the fuzzy line and the generalized fuzzy region will be *overlap/inside/intersect.*

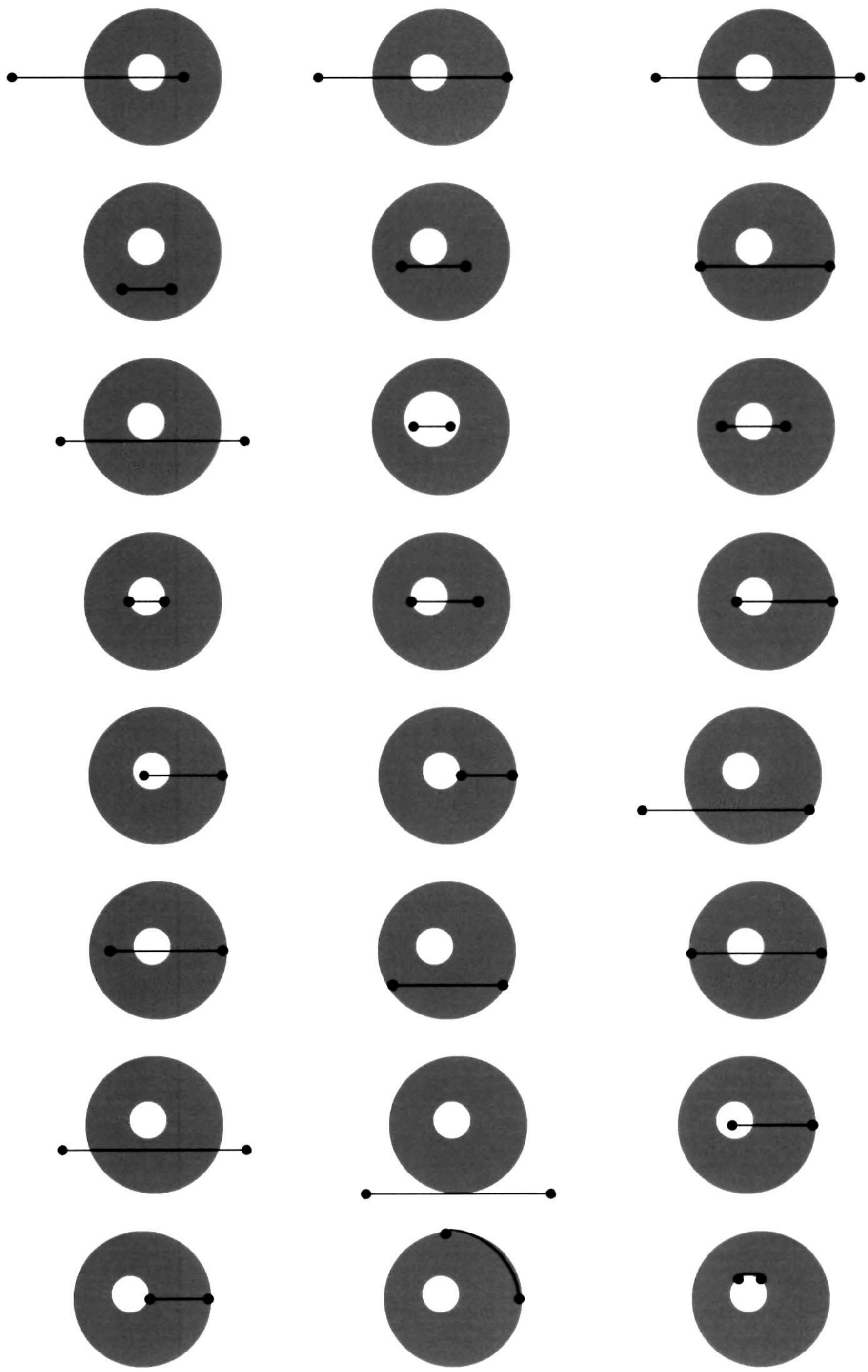
7. If the fuzzy line is along the boundary of the hole then the relation between generalized fuzzy region and the fuzzy line is *inside.*

8. If the fuzzy line is along the outer boundary of the fuzzy region with hole, then the relation between the hole and the fuzzy line is *disjoint.*

9. If the relation between the fuzzy line and the hole is *intersect*, then the relation between the fuzzy region with hole and the fuzzy line is *any one* of the sixteen relations other than disjoint and inside.

The relational matrix of the existing relations between a fuzzy line and a fuzzy region with hole can be determined by successively applying the conditions and canceling the corresponding non existing relations from the set of 4096 relations. Out of 4096 relations, only 52 relations satisfy these conditions. The geometrical representation of relations between a fuzzy line and a fuzzy region with hole is shown in figure 3.3.





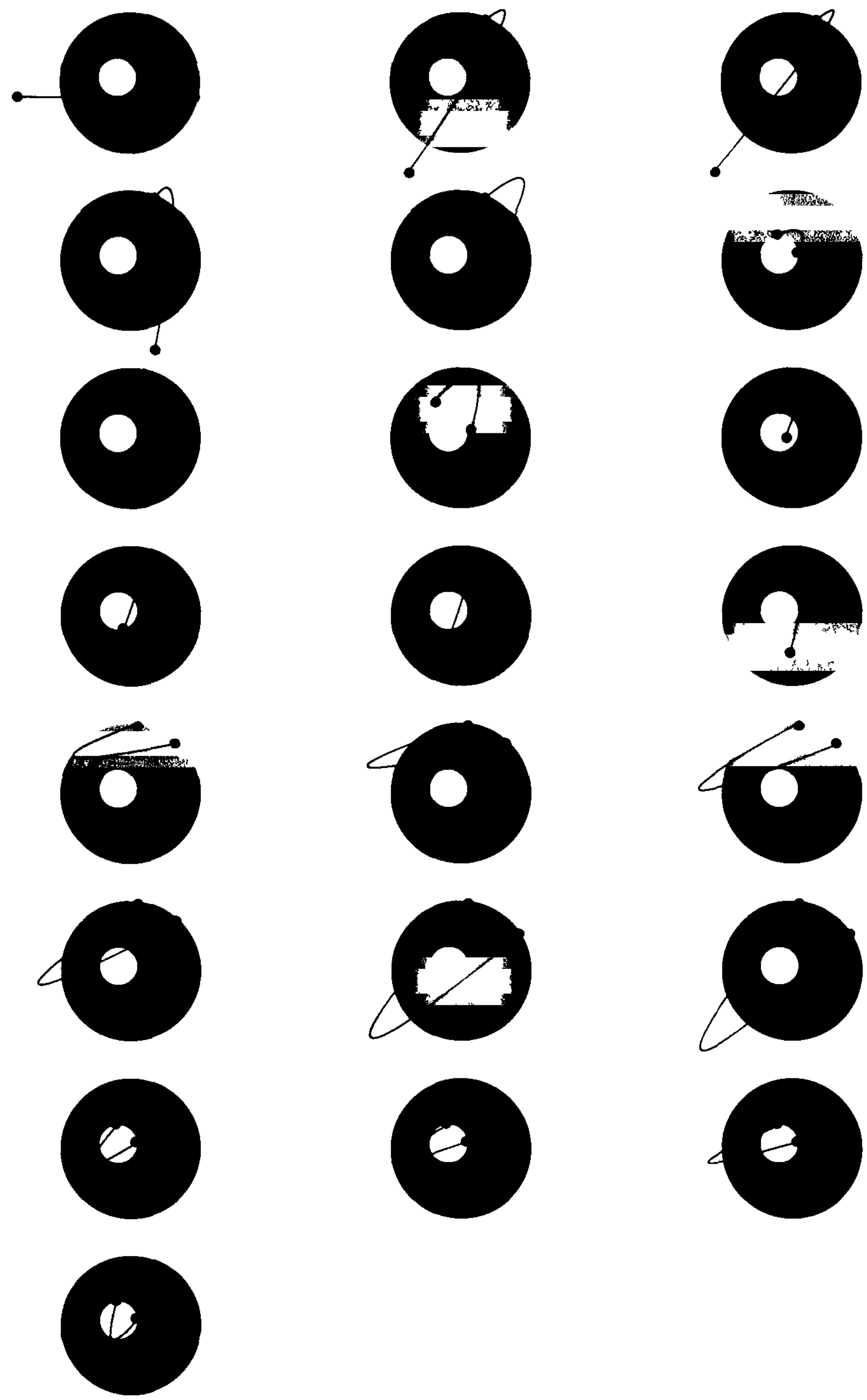


Figure 3.3: Topological relations between a single-holed fuzzy region and a fuzzy line

3.5.3 Topological relations between a fuzzy region with holes and a fuzzy region without hole

As in the previous cases, if A^* be a generalized fuzzy region consisting of 'n' holes $H_1, H_2, ..., H_n$ and B be a fuzzy region without holes in $\mathbb{R}^2$, then the topological relational matrix between the fuzzy regions with and without holes are given by Table 3.5

	A^*	H_1	H_2	.	.	H_n	B
A^*	$t(A^*, A^*)$	$t(A^*, H_1)$	$t(A^*, H_2)$	.	.	$t(A^*, H_n)$	$t(A^*, B)$
H_1	$t(H_1, A^*)$	$t(H_1, H_1)$	$t(H_1, H_2)$	.	.	$t(H_1, H_n)$	$t(H_1, B)$
H_2	$t(H_2, A^*)$	$t(H_2, H_1)$	$t(H_2, H_2)$	.	.	$t(H_2, H_n)$	$t(H_2, B)$
.	.	.	.	.	.	.	. .
H_n	$t(H_n, A^*)$	$t(H_n, H_1)$	$t(H_n, H_2)$	.	.	$t(H_n, H_n)$	$t(H_n, B)$
B	$t(B, A^*)$	$t(B, H_1)$	$t(B, H_2)$	.	.	$t(B, H_n)$	$t(B, B)$

Table 3.5: Intersection matrix for fuzzy regions with and without holes

where the symbol in each entry of the intersection matrix has their usual meaning as given in Subsection 3.5.1.

Here we have used a spatial scene (i.e. considering the fuzzy region without hole, the generalized fuzzy region and holes together with eight binary topological relations amongst these regions) to know exactly which spatial relation exists between a fuzzy region without hole and a fuzzy region with 'n' holes. Therefore, to obtain the number of feasible relations we will consider the variations (or possible choices for the number of feasible relations) of the fuzzy region B with the generalized fuzzy region A^* and possible choices of B with the holes of A w.r.t the eight basic relations between two fuzzy regions. Thus, the total number of relations between fuzzy regions with and without holes is 8^{n+1}.

In particular, if $n = 1$, i.e. fuzzy region with one hole there are total of $8^2 = 64$ relations.

Next, we shall determine certain conditions to reduce the number of redundant relations in $\mathbb{R}^2$.

Conditions to reduce redundant relations between fuzzy regions with and without hole:

We now identify conditions to reduce the number of redundant topological relations between a fuzzy region without hole and a fuzzy region with a hole either by proofing it or discussing the validity of the argument by proof-by-constraint and drawing method or by node, arc and path consistency.

Proof-by-constraint and drawing method [81]

Proof-by-constraint and drawing method is a method to determine consistency of topological relations between two spatial objects. It involves the following two steps:

1. Each combination of topological relations can be formulated in terms of existing relationship, that is, the set of topological relations can be reduced by evaluating the relation which does not fulfill the rule of the set of eight elementary relations between two regions.

2. The existence of topological relations are given by realizing prototypical spatial configurations in $\mathbb{R}^2$, that is, the consistency of the configuration can be determined by drawing it in the plane.

If A be a fuzzy region with hole A_H, A^* be the generalized fuzzy region and B be the fuzzy region without hole then

Lemma 3.5.2. *If the relation between generalized fuzzy region and fuzzy region without hole is disjoint then relation between hole and fuzzy region without hole is also disjoint, that is,*

$$t(A^*, B) = disjoint \;\; then \;\; t(A_H . B) = disjoint.$$

Lemma 3.5.3. *If the relation between generalized fuzzy region and fuzzy region without hole is meet then relation between hole and fuzzy region without hole is disjoint, that is,*

$$t(A^*, B) = meet \;\; then \;\; t(A_H, B) = disjoint$$

Lemma 3.5.4. *If the relation between generalized fuzzy region and fuzzy region without hole is equal then relation between hole and fuzzy region without hole is inside, that is,*

$$t(A^*, B) = equal \;\; then \;\; t(A_H, B) = inside$$

Lemma 3.5.5. *If the relation between generalized fuzzy region and fuzzy region without hole is overlap then relation between hole and fuzzy region without hole is not anyone of equal/contain/cover, that is*

$$t(A^*, B) = overlap \;\; then \;\; t(A_H, B) \neq equal/contain/cover$$

Lemma 3.5.6. *If the relation between generalized fuzzy region and fuzzy region without hole is inside then relation between hole and fuzzy region without hole is also inside, that is*

$$t(A^*, B) = inside \;\; then \;\; t(A_H . B) = inside.$$

Lemma 3.5.7. *If the relation between generalized fuzzy region and fuzzy region without hole is covered by then relation between hole and fuzzy region without hole is inside, that is,*

$$t(A^*, B) = covered\,by \;\; then \;\; t(A_H, B) = inside$$

Lemma 3.5.8. *If the relation between generalized fuzzy region and fuzzy region without hole is cover then relation between hole and fuzzy region without hole is any one of disjoint/meet/overlap/inside, that is,*

$$t(A^*, B) = cover \;\; then \;\; t(A_H, B) = disjoint/meet/overlap/inside$$

Lemma 3.5.9. *If the relation between generalized fuzzy region and fuzzy region without hole is contain then relation between hole and fuzzy region without hole is anyone of U (where U is the set consisting of eight relations between two fuzzy regions), that is,*

$$t(A^*, B) = contain \;\; then \;\; t(A_H, B) = U$$

Lemma 3.5.10. *If the relation between hole and fuzzy region without hole is equal then relation between fuzzy region without hole and generalized fuzzy region is inside, that is,*

$$t(A_H, B) = equal \;\; then \;\; t(B, A^*) = inside$$

Theorem 3.5.11. *Based on the intersection matrix for holed regions, 23 different topological relations are identified between two fuzzy regions each with and without hole in* $\mathbb{R}^2$

Proof For each of the condition of lemmas 3 5 2, 3 5 3 3 5 4, 3 5 6, 3 5 7 and 3 5 10 there is one distinct relation between fuzzy regions with and without hole, under lemma 3.5.5 there are five different relations between fuzzy regions with and without hole; under lemma 3.5.8 there are four distinct relations and under lemma 3.5.9 there are eight consistent relations between fuzzy regions with and without hole. Therefore, adding the number of distinct relations due to each lemma, there are a total of 23 distinct relations exist between fuzzy regions with and without hole in $\mathbb{R}^2$. □

The set of 23 distinct relations between fuzzy regions with and without hole in $\mathbb{R}^2$ is shown in figure 3.4.

3.6 Conclusion

We have proposed a definition of fuzzy regions with holes in a crisp fts while making it a consistent generalization of the definition of a crisp region with holes in classical topology. Further, we have provided general frameworks for topological relations between fuzzy region with hole and fuzzy point, fuzzy region with hole and fuzzy line as well as fuzzy regions with and without hole. In case of topological relations between fuzzy region with holes and fuzzy point, fuzzy regions with and without holes we have considered the content of intersection to be the eight binary topological relations between two fuzzy regions with connected boundary and derive relational matrix which is related to the 9-intersection matrix (but not based on it). After calculation we have found that in case of fuzzy region with hole and fuzzy point there are only 5 distinct realizable relations. Liu and Shi's model [59] shows that there are only three consistent relations between fuzzy region (without hole) and fuzzy point. Thus, the number of relations between fuzzy region with hole and fuzzy point exceeds the number of relations between fuzzy region (without hole) and fuzzy point. In case of topological relations between fuzzy region with holes and a fuzzy line we

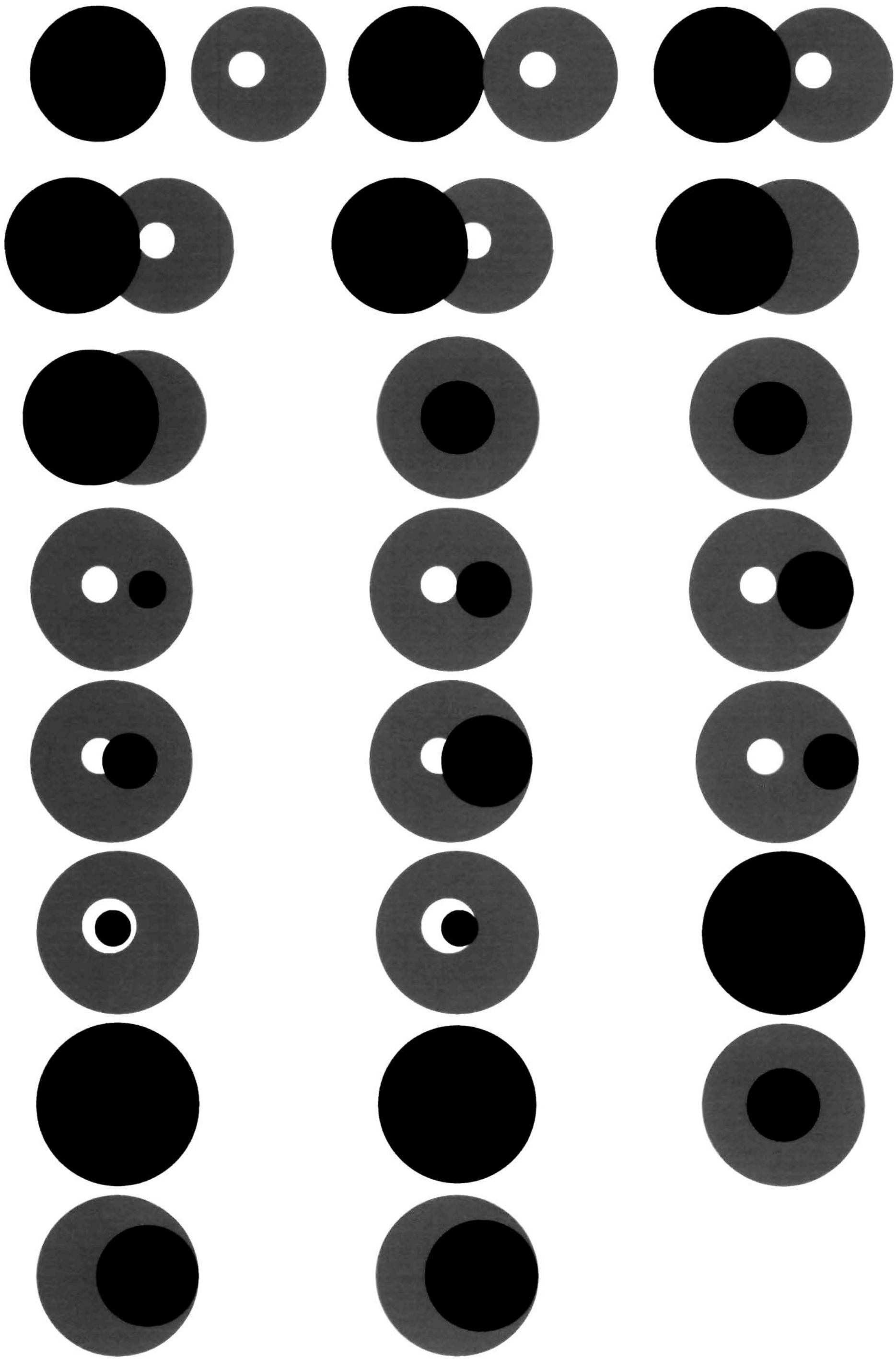


Figure 3.4: Topological relations between fuzzy regions with and without hole in a crisp fts

have considered the content of intersection to be sixteen topological relations between fuzzy region (without holes) and a fuzzy line and found that there are 52 recognizable relations. For fuzzy regions without and with a hole, our model gives only 23 consistent relations while Tang and Kainz's approach [88] and, Liu and Shi model [59] have shown that there are 44 recognizable relations between two fuzzy regions without hole. The number of relations between fuzzy regions with and without hole are same as the number of relations between crisp regions with and without hole as determined by Egenhofer and Vasardani [36] in classical topology. The challenge, however remains to fully accommodate the membership grades of points in the fuzzy region with holes and for the same it is required to formulate a suitable fuzzy region with holes in the general framework in fts. The same is undertaken in the next chapter.

Chapter 4

Topological relations of fuzzy regions with holes in a general fts

4.1 Introduction

In the previous chapter, we have provided a formal definition of a fuzzy region with hole in the setting of a crisp fts. Since a general fts allows flexibility and relaxation of membership grades of points in space. As shown by Tang and Kainz [88, 89], additional topological invariants are required for defining fuzzy regions without hole in a general fts. Thus, fuzzy regions with holes in this space cannot be defined analogous to its definition in a crisp topological space. Furthermore, it is required that the definition of fuzzy regions with holes should ensure that the hole/s and the region will be disjoint topological parts so that the generalized region can be defined as union of the region and the holes, and these definitions would be used in the derivation of topological relations. In this Chapter, we have proposed a definition of fuzzy regions with holes in the setting of general fts utilizing the framework for fuzzy regions without hole given by Tang and Kainz [88] and obtained the topological relations between fuzzy

[1]Selected results of this chapter have been published in [45].

regions with holes under general and restricted conditions.

This chapter is structured as follows: in section 4.2 we have revisited some of the important topological notions which are required for defining fuzzy regions as provided by Tang and Kainz [88]. In section 4.3 we have provided the definition of fuzzy regions with holes in the setting of a general fts and developed some of its mathematical properties. In section 4.4 we have derived topological relations between two fuzzy regions with holes. Section 4.5 summarizes the chapter.

4.2 Fuzzy regions

In the last chapter, we have defined fuzzy regions with holes in a crisp fts. In this chapter, we take up fuzzy regions with holes in a general fts. We shall require some of the special topological notions as given below, which were introduced by Tang [89]. Some of the required results are reproduced with proofs.

4.2.1 Topological notions for defining fuzzy regions

Definition 4.2.1. Let (X, T) be an fts and A be a fuzzy set in X. Then

i) The core of A (denoted by $A^{\oplus}$) is the subset of $\overline{A}$ defined as

$$A^{\oplus}(x) = \begin{cases} \overline{A}(x), & \text{if } (\overline{A} \cap \overline{A^c})(x) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

ii) The fringe (denoted by lA) is the subset of $\overline{A}$ where

$$lA(x) = \begin{cases} \overline{A}(x), & \text{if } (\overline{A} \cap \overline{A^c})(x) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.2.1. *$A^{\oplus}$ is the only crisp subset of A°.*

Proof. According to the definition of core, $\forall x \in X$, $A^{\oplus}(x) = \overline{A}(x) > 0$ iff $(\overline{A} \cap \overline{A^c})(x) = 0$. Then either $\overline{A}(x) = 0$ or $\overline{A^c}(x) = 0$. (i) When $\overline{A}(x) = 0$, then $A^{\oplus}(x) = 0$ which is a contradiction.

(ii) When $\overline{A^c}(x) = 0$, then $A^{\circ c}(x) = 0$. So, $A^{\circ}(x) = \overline{A}(x) = 1$. It shows $\forall x \in X$, if $A^{\circ}(x) > 0$, then $A^{\oplus}(x) = 1$ and $A^{\circ}(x) = 1$. On the other hand, if $A^{\circ}(x) = 1$, then $A^{\circ c} = 0$, so $\overline{A^c} = 0$. Therefore $(\overline{A} \cap \overline{A^c})(x) = 0$.

It follows that if $A^{\circ}(x) = 1$, then $A^{\oplus}(x) = 1$. □

Proposition 4.2.2. *$A^{\oplus} \cup B^{\oplus} = (A \cup B)^{\oplus}$ where A and B are fuzzy sets in X.*

Proof. From theorem 4.2.1 we know that $A^{\oplus}$ and $B^{\oplus}$ are crisp subsets of A° and B° respectively, then $A^{\oplus} \cup B^{\oplus}$ is a crisp subset of $A^{\circ} \cup B^{\circ}$. Since $(A^{\circ} \cup B^{\circ}) \leq (A \cup B)^{\circ}$, so $A^{\oplus} \cup B^{\oplus}$ is also a crisp subset of $(A \cup B)^{\circ}$. By theorem 4.2.1, $(A \cup B)^{\oplus}$ is the only crisp subset of $(A \cup B)^{\circ}$.

Therefore, we have $A^{\oplus} \cup B^{\oplus} = (A \cup B)^{\oplus}$. □

Remark 4.2.3. *lA is a subset of ∂A. If A is closed then $lA = \partial A$.*

Definition 4.2.2. Let (X, T) be an fts. Then

i) The frontier of A (denoted by $l^c A$) is a subset of $\overline{A}$, where

$$l^c A(x) = \begin{cases} \overline{A}(x), & \text{if } \overline{A(x)} > A^{\circ}(x); \\ 0, & \text{otherwise.} \end{cases}$$

ii) The internal of A (denoted by $A^{\imath}$) is a subset of the closure of the fuzzy set A where

$$A^{\imath}(x) = \begin{cases} \overline{A}(x), & \text{if } \overline{A(x)} = A^{\circ}(x); \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.2.4. *i) $A^{\oplus} \subset A^{\imath} \subset A^{\circ}$;*

ii) $\partial A \supset lA \supset l^c A$.

Proof. (i) We know that $A^{\oplus}$ is the crisp subset of A°. According to the definition of A^{i}, it is a subset of A° and it contains the crisp subset of A°.
(ii) If $\overline{A}(x) > A^{\circ}(x)$ for all $x \in X$, then $A^{\circ}(x) < 1$ and $\overline{A^c} = A^{\circ c} > 0$. Therefore $(\overline{A} \cap \overline{A^c})(x) = \overline{A}(x) \cap \overline{A^c}(x) > 0$, therefore $lA(x) > 0$. So, $lA \supseteq l^cA$. □

Definition 4.2.3. Let (X, T) be an fts. Then
(i) Internal fringe of A (denoted by l^iA) is defined as

$$l^iA(x) = \begin{cases} A^i(x), & \text{if } (A^i \cap (A^{\oplus})^c)(x) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Outer of fuzzy set A denoted by $A^{=}$ is defined as

$$A^{=}(x) = (Supp(\overline{A}))^c(x)$$

which is obviously a crisp set.

Properties of core, fringe, internal, frontier, internal fringe and outer:

We shall recall some of the properties of core, fringe, internal, frontier, internal fringe, outer etc. without proof. Proofs are available in Tang and Kainz [88, 89].

Proposition 4.2.5. *Let A be fuzzy set in an fts (X, T) . Then $\overline{A} = A^{\oplus} \cup \partial A = A^{\oplus} \cup lA$.*

Proposition 4.2.6. *Let A and B be a fuzzy sets in fts (X, T)*
(i) $A^{\oplus\oplus} \subseteq A^{\oplus}$
(ii) $A^{\circ\oplus} = A^{\oplus}$; $\overline{A}^{\oplus} = A^{\oplus}$
(iii) If $A \supseteq B$ *then* $A^{\oplus} \supseteq B^{\oplus}$
(iv) $A^{\oplus} \cap B^{\oplus} = (A \cap B)^{\oplus}$
(v) If $A^{\oplus}$ *is open then* $A^{\oplus} \cap l(A^{\oplus}) = \phi$

(vi) $A^{\oplus\ominus} = A^{\ominus}$ *iff* $A^{\ominus}$ *is open*

(vii) $lA = \overline{A} \cap A^{\oplus c}$

(viii) $(lA)^{\oplus} = \phi$ *and* $l(lA) = \partial A$

(ix) If $A^{\oplus}$ *is open then* $((lA)^{\oplus}) \subseteq lA$

(x) $\partial A \cap A^{\oplus} = \phi$ *iff* $\partial A = lA$

(xi) If $A^{\oplus}$ *is open then* $\partial A = lA$

(xii) $l^c A$ *and* $A^{\imath}$ *are disjoint with each other,* $l^c A \cap A^{\imath} = \phi$

(xiii) $\overline{A} = l^c A \cup A^{\imath}$

(xiv) $lA = l^c A \cup l^{\imath} A$ *and* $A^{\imath} = A^{\oplus} \cup l^{\imath} A$

(xv) $l^{\imath} A = \phi$ *and* $l^{\imath}(A^{\oplus}) = \phi$

(xvi) $l^{\imath}(A^{\oplus}) = \phi$

(xvii) If $A^{\oplus}$ *is open then* $l^c(A^{\oplus}) = \phi$

Proposition 4.2.7. *Let* A *and* $l^c A$ *be closed sets in fts* (X, T) *then* $l^c A = l^c(l^c A)$.

Theorem 4.2.8. *Let* A *be a fuzzy set in an fts* (X, T). *Then* $A^{=}$, $l^c A$, $l^{\imath} A$, $A^{\oplus}$ *are mutually disjoint and they are topological invariants*

Formal definition of a fuzzy region

Definition 4.2.4. A fuzzy set is called a simple fuzzy region in a connected fts if it satisfies the following conditions.

i) It is a non-empty proper double-connected closed set.

ii) The interior, the core and the outer are double-connected regular open.

iii) The support is equal to the support of the closure of the interior.

iv) The fringe is double connected and the internal fringe is a double-connected open set.

v) The frontier is a non-empty closed set.

Condition (i) provides an extension of the definition of simple crisp region. Condition (ii) implies that core being equal to the interior of fuzzy region and since in a crisp region interior is considered to be regular open set so core of a fuzzy region should be open. Condition (iii) signifies that if we draw a plane through 1 of the interval $[0, 1]$, then the projection of the fuzzy region in this plane will be simple crisp region in a classical topological space. Condition (iv) ensures the non existence of holes. Condition (v) ensures the non existence of spikes. Double connectedness of the internal signifies that interior is not separated into pieces.

An schematic illustration of a fuzzy region without hole is given below in the figure 4.1.

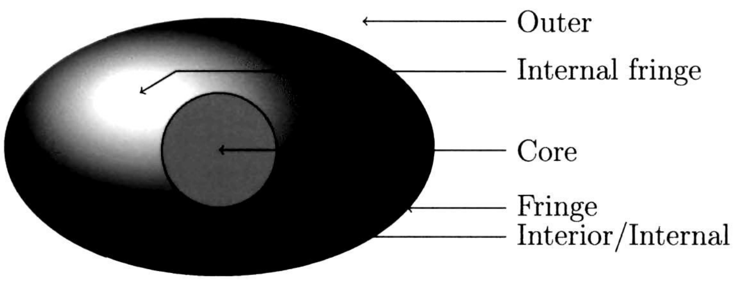


Figure 4.1: Fuzzy region without hole in a general fts

4.3 Fuzzy region with holes

Initially, a crisp region was defined in a crisp topological space but a crisp region being the special case of a fuzzy region it has been seen that crisp regions are also accommodated in an fts. So, for defining a fuzzy region with holes in a general fts, we should adopt the following principles:

i) A crisp subset of a simple fuzzy region should exhibit the same behaviour as a subset of simple crisp region in a general topological space.

ii) A crisp region with holes in an fts should exhibit behaviour similar to a region with holes in a general topological space.

iii) When holes are eliminated, it should be fuzzy region without hole in a general fts.

The first condition implies that crisp sets are special cases of fuzzy sets. The second condition stipulates extension of crisp regions in classical topological space to fuzzy regions in an fts. The third condition signifies that fuzzy regions without holes is a special case of fuzzy regions with holes in an fts.

In order to define fuzzy regions with holes which will satisfy the above three conditions, we shall at first define the structure of crisp region with holes in an fts. In 1994, Egenhofer et. al. [33] provided a definition of crisp regions with holes in such a way that the holes and the region are disjoint topological parts so that if we eliminate the holes it should be the definition of crisp region without holes in classical topological space. To maintain the consistency with the classical definition, next we have to define the structure of crisp regions with holes in an fts.

Definition 4.3.1. A subset is called a simple crisp region with hole in a fts if it meets the following conditions

i) Its interior is a non-empty proper double connected open crisp set.

ii) It boundary and exterior are not closed as a whole but are the union of disjoint connected components of the fringes which are themselves double connected regular crisp sets.

In a classical topological space, connected sets are those which cannot be separated by two disjoint closed/open sets. In fuzzy topology, Q-separation does not imply separation because in this case open connectedness does not implies closed connectedness as shown in Chapter 2. We have, therefore, adopted the notion of double connectedness for representing connectedness in an fts. Next, we consider boundary and exterior not to be connected as whole because if they are connected then fringe being similar to the boundary in a general fts it does not allow the existence of hole and above definition becomes the definition of crisp regions without holes in an fts. We further require the following definitions:

Definition 4.3.2. Main outer is the outer of the fuzzy region.

Definition 4.3.3. Inner outer is the outer inside the interior of the fuzzy region.

Definition 4.3.4. Hole of the fuzzy region is the closure of the inner outer.

Definition 4.3.5. Connected component of fringe (frontier) is a fringe (frontier) which separates interior (internal) of the fuzzy region and the inner outer.

4.3.1 Formal definition of a fuzzy region with holes

Throughout our discussion, we consider that holes are disjoint from each other and are not along the boundary of the region with holes and the region should have at most finite number of holes.

Definition 4.3.6. A fuzzy set A in a connected fts (X, T) is called a simple fuzzy region with holes if it meets the following requirements:

i) The interior and the core are double-connected regular open sets.

ii) The outer as a whole is not double connected but is the disjoint union of main outer and inner outer which are itself double connected.

iii) Inner outers are double connected regular open sets.

iv) The support is equal to the support of the closure of the interior.

v) The fringe and the frontier as a whole are not double connected but they are the union of disjoint connected components such that each component itself is closed.

vi) The internal fringe is a double connected open set.

Condition (i) is a direct extension of a crisp region in the fuzzy setting as connectedness is extended into double connectedness (i.e. both open connectedness and closed connectedness) in fuzzy topology. Conditions (ii) and (v) signify the existence of holes as the outer is equal to an exterior of the fuzzy region and the fringe as well as frontier is equal to the boundary of the region. Double connectedness of inner outer and fringe signifies the non existence of cuts and punctures. Condition (iii) removes the irregular points or spikes etc. Condition (iv) signifies that shadow of the region will be a crisp region with holes in a crisp topological space. Internal fringe being a subset of the internal which in turn is a subset of the interior, meaning thereby that the internal fringe is a subset in the interior of the fuzzy region. Thus, if internal fringe is not double connected, then it will separate the interior of the fuzzy region into several pieces, as a result of which our definition would cease to be a proper extension of a crisp region in which the interior is connected. We, therefore, consider the internal fringe to be a double connected open set.

Figure 4.2 provides an illustration of a fuzzy region with holes in a general fts which allows the membership of a point in the region in relation to the space.

Definition 4.3.7. The generalized fuzzy region is the union of the double connected interior and the holes.

The above definition is seen as a more suitable representative of the uncertainty of the spatial objects and can be used in the derivation of topological relations between fuzzy regions with holes and basic spatial objects.

Since derivation of topological relations between fuzzy regions with holes and

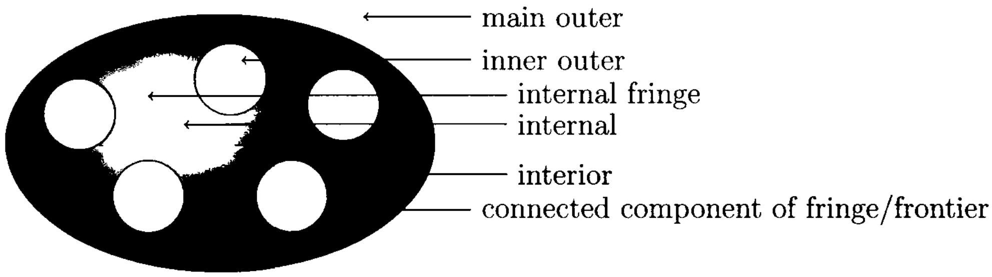


Figure 4.2: Fuzzy region with hole in a general fts

a fuzzy point as well as fuzzy regions with holes and a fuzzy line and fuzzy regions with and without holes in a general fts is similar to their derivation in a crisp fts so in this chapter we have restricted ourselves only to the derivation of topological relations between fuzzy regions each with a hole.

Remark 4.3.1. *Unlike in a crisp fts the definition of fuzzy region with holes in a general fts is not formulated in terms of interior, boundary and exterior. This is because if we do so, then the hole and the generalized region would cease to be disjoint. We therefore define fuzzy regions with holes in terms of core, fringe, internal, frontier, internal fringe, outer etc. Also, we know that in an fts intersection of interior with boundary as well as intersection of boundary with exterior are not empty in general A fuzzy region with holes is defined in such a way that it is a consistent generalization of a crisp region in a classical topology. Further, it would ensure that the intersection of holes and the fuzzy region with holes are disjoint from each other so that we can use it in the derivation of topological relations.*

4.3.2 Properties of simple fuzzy regions with holes

Theorem 4.3.2. *Let A be a simple fuzzy region with holes $A_1, A_2, \ldots, A_n$ in a connected fts (X, T)*

(i) The boundary of A is equal to the union of the fringe of the region and holes

(ii) The internal boundary is equal to the union of fringes of A_i

(iii) The frontier of the boundary is equal to the union of the frontier of A_i

(iv) The fringe of the inner (main) outer is equal to the boundary of the inner (main) outer

(v) Interior of the component of fringe/frontier is regular open

(vi) The core of the component of fringe/frontier is empty

(vii) The closure of the boundary is equal to boundary of the boundary of A_i

(viii) The interior of the fringe of the core of A_i's are empty i e $(l(A_i^{\oplus}))^{\circ} = \phi$

Proof (i) By definition of fuzzy region with holes, the core is open so fringe being its complement is closed as core and fringe are mutually disjoint parts We know from Remark 4 2 3 that if the fringe is closed, then it is equal to the boundary Hence, the boundary of a fuzzy region with hole is equal to the fringe and by definition of a fuzzy region with hole, the fringe is the union of the fringe of the region and the holes Therefore, boundary of fuzzy region with holes is the union of the fringe of the region and holes

(ii) We know that $l^i(A)(x) = A^i(x)\ \forall x \in X$ for which $(A^i \cap A^{\oplus c})(x) > 0$ and $\partial^i A(x) = \partial A(x)\ \forall x \in X$ for which $\overline{A(x)} = A^{\circ}(x)$ Now, $\partial^i A(x) = \partial A(x) = \cup_j l^i A_j(x) = \cup_j A_j^i(x)$ (using (i))

In particular, $\partial^i A(x) = \partial A(x) = \cup_j A_j^i(x)\ \forall x \in X$ for which $(A^i \cap A^{\oplus c})(x) > 0$

(iii) Since the fringe contains the frontier and from (i) boundary is equal to the union of fringe of the region and the holes so $\partial A = lA \cup lA_i \implies \partial A \supset lA_i$ and

$lA_i \supset l^cA_i$. So $\partial A \supset l^cA_i \Rightarrow l^c(\partial A) \supset l^c(l^cA_i) = l^cA_i \Rightarrow l^c(\partial A) \supset \cup_i(l^cA_i)$.
Next, since $\partial A_i \subset A_i \Rightarrow l^c(\partial A_i) \subset l^cA_i\ \forall i$ So, $l^c(\partial A_i) \subset \cup_i(l^cA_i)$.
(iv) Since $A^=$ is open so $A^{=\oplus}$ is also open Then, $lA = \partial A$ (by Proposition 4.2.6 part (xi)), we therefore have $l_i(A^=) = \partial_i(A^=)$
(v) Since, $A^\oplus$ is regular open so $A^{\oplus c}$ is regular closed Hence, $A^{\oplus co}$ is regular open.
Now, $(l^iA_j)^\circ = (\overline{A} \cap A^{\oplus c})^\circ = (A \cap A^{\oplus c})^\circ(A$ being closed$) = A^\circ \cap A^{\oplus co}$, which is regular open. Hence, the fringe of a hole is regular open.
Similarly $(l^cA_j)^\circ = (\overline{A} \cap A^{\oplus c})^\circ = (A \cap A^{\oplus c})^\circ = A^\circ \cap A^{\oplus co}$ which is regular open.
(vi) Since $lA = \cup_i lA_i$. $(lA)^\oplus = (\cup_i lA_i)^\oplus = \cup_i(lA_i)^\oplus$ (using proposition 4.2.2).
We know that $(lA)^\oplus = \phi$ so $\cup_i(lA_i)^\oplus = \phi \Rightarrow (lA_i)^\oplus = \phi\ \forall i$. Likewise, $(l^cA_i)^\oplus = \phi\ \forall i$.
(vii) We have $\overline{\partial A_i} = (\partial A_i)^\oplus \cup \partial(\partial A_i) = \partial(\partial A_i)$ (by using (i) and (vi)).
(viii) Since $A_i^\ominus$ is open so $l(A_i^\oplus) = \partial(A_i^\oplus)$ and $A_i^{\oplus\ominus} \cap l(A_i^\oplus) = A_i^\oplus \cap l(A_i^\oplus) = \phi$.
Suppose $(l(A_i^\oplus))^\circ \neq \phi$ then $A_i^{\oplus\circ} \cup (l(A_i^\oplus))^\circ \supsetneq A_i^{\oplus\circ}$ and $A_i^\oplus$ is not regular open which is a contradiction. □

4.4 Topological relations between fuzzy regions with holes

We shall consider the generalized region and holes as mutually disjoint topological invariants to determine topological relations between two fuzzy regions with holes. From regional connection calculus, we know that there are only 8 recognizable relations between two spatial objects with connected boundary. As each generalized region and holes can be considered as separate spatial objects without holes and topological relations between them is determined by considering spatial scene (i e , we consider each generalized region and holes as separate spatial objects without holes and 8 topological relations between them are considered as content of the topological invariants) Let A^* and B^* be two generalized regions with 'n' and 'm' holes $H_{A_1}, H_{A_2}, \ \ , H_{A_n}$ and $H_{B_1}, H_{B_2}, \ \ , H_{B_m}$ respectively Then the topological relations between these two regions are given by the following relational matrix

$t(R\ R)$	A^*	H_{A_1}			H_{A_n}	B^*	H_{B_1}			H_{B_m}
A^*	$t(A^*\ A^*)$	$t(A^*\ H_{A1})$			$t(A^*\ H_{An})$	$t(A^*\ B^*)$	$t(A^*\ H_{B1})$			$t(A^*\ H_{Bm})$
H_{A_1}	$t(H_{A_1}\ A^*)$	$t(H_{A_1}\ H_{A_1})$			$t(H_{A_1}, H_{A_n})$	$t(H_{A_1}, B^*)$	$t(H_{A_1}, H_{B_1})$			$t(H_{A_1}, H_{B_m})$
H_{A_n}	$t(H_{A_n}, A^*)$	$t(H_{A_n}\ H_{A_1})$			$t(H_{A_n}\ H_{A_n})$	$t(H_{A_n}\ B^*)$	$t(H_{A_n}, H_{B_1})$			$t(H_{A_n}, H_{B_m})$
B^*	$t(B^*, A^*)$	$t(B^*\ H_{A_1})$			$t(B^*\ H_{A_n})$	$t(B^*\ B^*)$	$t(B^*\ H_{B_1})$			$t(B^*\ H_{B_m})$
H_{B_1}	$t(H_{B_1}\ A^*)$	$t(H_{B_1}\ H_{A_1})$			$t(H_{B_1}\ H_{A_n})$	$t(H_{B_1}\ B^*)$	$t(H_{B_1}\ H_{B_1})$			$t(H_{B_1}\ H_{B_m})$
H_{B_m}	$t(H_{B_m}\ A^*)$	$t(H_{B_m}\ H_{A_1})$			$t(H_{B_m}\ H_{A_n})$	$t(H_{B_m}, B^*)$	$t(H_{B_m}\ H_{B_1})$			$t(H_{B_m}, H_{B_m})$

Table 4 1 Topological relational matrix for fuzzy regions with holes

Here, $t(A^*, H_{A_1})$ denotes the topological relation between the object parts Further, $t(A^*, H_{A_i})$ $i = 1, 2, \ \ , n$ is implied by $t(H_{A_i}, A^*)$ $i = 1, 2, \ \ , n$ as a kind of converse relation and $t(A^*, H_{B_i})$ $i = 1, 2 \ \ , m$ is implied by $t(A^*, B^*)$ as a kind of implied by relation

Therefore, the intersection matrix in table 4 1 can be reduced to an equivalent upper triangulai or lower triangular matiix

Since the relation between each pair of the holes is disjoint, relation to itself is equal and holes of the generalized region should inside the region Therefore, the

$t(R\ R)$	A^*	H_{A_1}			H_A	B^*	H_{B_1}			H_{B_m}
A^*	$t(A^*\ A^*)$									
H_{A_1}	$t(H_{A_1}\ A^*)$	$t(H_{A_1}\ H_{A_1})$								
H_{A_n}	$t(H_{A_n}\ A^*)$	$t(H_{A_n}\ H_{A_1})$			$t(H_{A_n}\ H_{A_n})$					
B^*	$t(B^*\ A^*)$	$t(B^*\ H_{A_1})$			$t(B^*\ H_{A_n})$	$t(B^*\ B^*)$				
H_{B_1}	$t(H_{B_1}\ A^*)$	$t(H_{B_1}\ H_{A_1})$			$t(H_{B_1}\ H_{A_n})$	$t(H_{B_1}\ B^*)$	$t(H_{B_1}\ H_{B_1})$			
H_{B_m}	$t(H_{B_m}\ A^*)$	$t(H_{B_m}\ H_{A_1})$			$t(H_{B_m}\ H_{A_n})$	$t(H_{B_m}\ B^*)$	$t(H_{B_m}\ H_{B_1})$			$t(H_{B_m}\ H_{B_m})$

Table 4 2 Lower triangular matrix for fuzzy regions with holes

number of redundant relations in the relational matrix can be reduced further Under this assumption the number of distinct relations in 1^{st} row, 2^{nd} row, , n^{th} row is zero but the number of distinct elements in $(n+1)^{th}$, $(n+2)^{th}$, , $(m+n+1)^{th}$ row are given as follows

Number of distinct elements in the $(n+1)^{th}$ row $= 8^{n+1}$

Number of distinct elements in the $(n+2)^{th}$ row $= 8^{n+1}$

Number of distinct elements in the $(m+n+2)^{th}$ row $= 8^{n+1}$

Now, the number of consistent relations will be equal to the sum of the number of distinct relations in each row of the relation matrix

Therefore, the total number of consistent relations will be $8^{(n+1)(m+1)}$

As a particular case, if $n = 1$ and $m = 1$ i e each region with a single hole there will be $8^4 = 4096$ relations but under certain conditions only a few subsets of them are realized between two fuzzy regions each with a hole

4.4.1 Formulation of conditions for reducing redundant relations

Topological relations between two fuzzy regions each with a hole can be identified by using intersection matrix considering the generalized regions and holes as separate spatial objects without holes as discussed above We now identify some of the geometric conditions to reduce the number of redundant relations between two fuzzy regions each with a hole by proving it or discussing the argument by proof-by-constraint and drawing method.

If A and B are two fuzzy regions with holes A_H and B_H respectively and let A^*, B^* be the two corresponding generalized regions. We, then have the following:

Lemma 4.4.1. *If 'disjoint'/'meet' is the relation between the generalized regions then 'disjoint' is the relation between holes, that is,*

$$t(A^*, B^*) = disjoint/meet \;\; then \;\; t(A_H, B_H) = disjoint.$$

Lemma 4.4.2. *If 'inside' is the relation between the generalized regions then 'disjoint' is the relation between the holes of both the regions and the relation between first generalized region and the hole of the second is U (where U denote the set of eight elementary relations between two fuzzy regions), that is,*

$$t(A^*, B^*) = inside, \;\; then \;\; t(A_H, B_H) = disjoint \;\; and \;\; t(A^*, B_H) = U.$$

Lemma 4.4.3. *If 'contain' is the relation between the generalized regions then 'disjoint' is the relation between the holes of both the regions and U is the relation between first generalized region and the hole of the second region, that is,*

$$t(A^*, B^*) = contain \;\; then \;\; t(A_H, B_H) = disjoint \;\; and \;\; t(A^*, B_H) = U.$$

Lemma 4.4.4. *If 'equal' is the relation between generalized regions then U is the relation between the holes, that is,*

$$t(A^*, B^*) = equal \ \ then \ \ t(A_H, B_H) = U$$

Lemma 4.4.5. *If 'equal' is the relation between the holes of both the regions then the relation between generalized regions is any relation other than 'disjoint' and 'meet', that is,*

$$t(A_H, B_H) = equal \ \ then \ \ t(A^*, B^*) \neq disjoint/meet$$

Lemma 4.4.6. *If the relation between generalized regions is 'overlap' and relation between first generalized region and hole of the second is 'contain' and relation between hole of the first and second generalized region is 'inside, then relation between the holes will be anyone of the eight relations in U, that is,*

$$t(A^*, B^*) = overlap, \ t(A^*, B_H) = contain, \ t(A_H, B^*) = inside \ then \ t(A_H, B_H) = U.$$

Lemma 4.4.7. *If 'inside/contain' is the relation between first generalized region and hole of the second then relation between first generalized region and second fuzzy region with hole is any one of U, that is,*

$$t(A^*, B_H) = inside/contain \ \ then \ \ t(A^*, B) = U$$

Lemma 4.4.8. *If 'inside'/'contain' is the relation between generalized regions and hole of one overlap with the generalized region of the other then U is the relation between holes, that is,*

$$t(A^*, B^*) = inside/contain, \ \ t(A_H, B^*) = overlap \ \ then \ \ t(A_H, B_H) = U$$

Lemma 4.4.9. *If 'cover'/'covered by' is the relation between generalized regions and relation between the holes is 'disjoint' then U is the relation between hole of the first and the second generalized region, that is,*

$$t(A^*, B^*) = cover/coveredby, \ \ t(A_H, B_H) = disjoint \ \ then \ \ t(A_H, B^*) = U$$

Lemma 4.4.10. *If relation between holes of the regions is 'overlap' then relation between generalized regions cannot be 'disjoint'/'meet', that is,*

$$t(A_H, B_H) = overlap \ \ then \ \ t(A^*, B^*) \neq disjoint/meet$$

Lemma 4.4.11. *If relation between generalized regions is 'cover'/'covered by' and relation between one generalized region and hole of the other is 'inside' then relation between holes of both the regions is any one of U, that is,*

$$t(A^*, B^*) = cover/coveredby, \ \ t(B^*, A_H) = inside, \ \ then \ \ t(A_H, B_H) = U$$

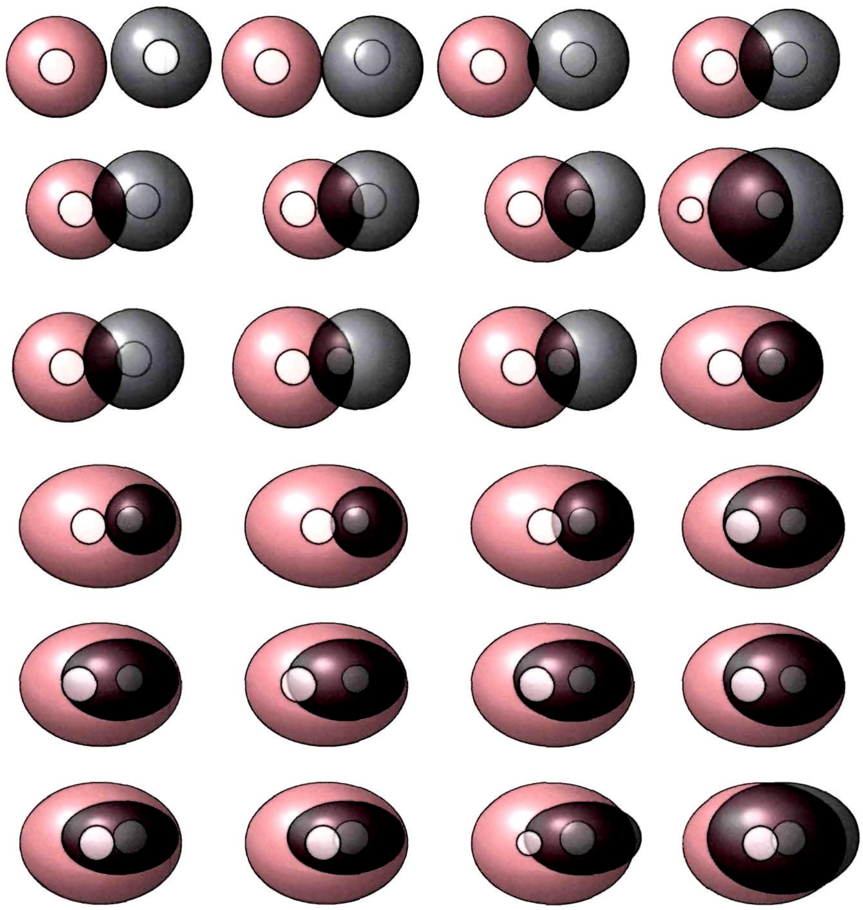
Lemma 4.4.12. *If 'overlap' is the relation between generalized regions and relation between one generalized region and hole of the other is also 'overlap' and relation between hole of first with the generalized region of the second is 'inside' then relation between holes of both is any one of U other than equal, that is,*

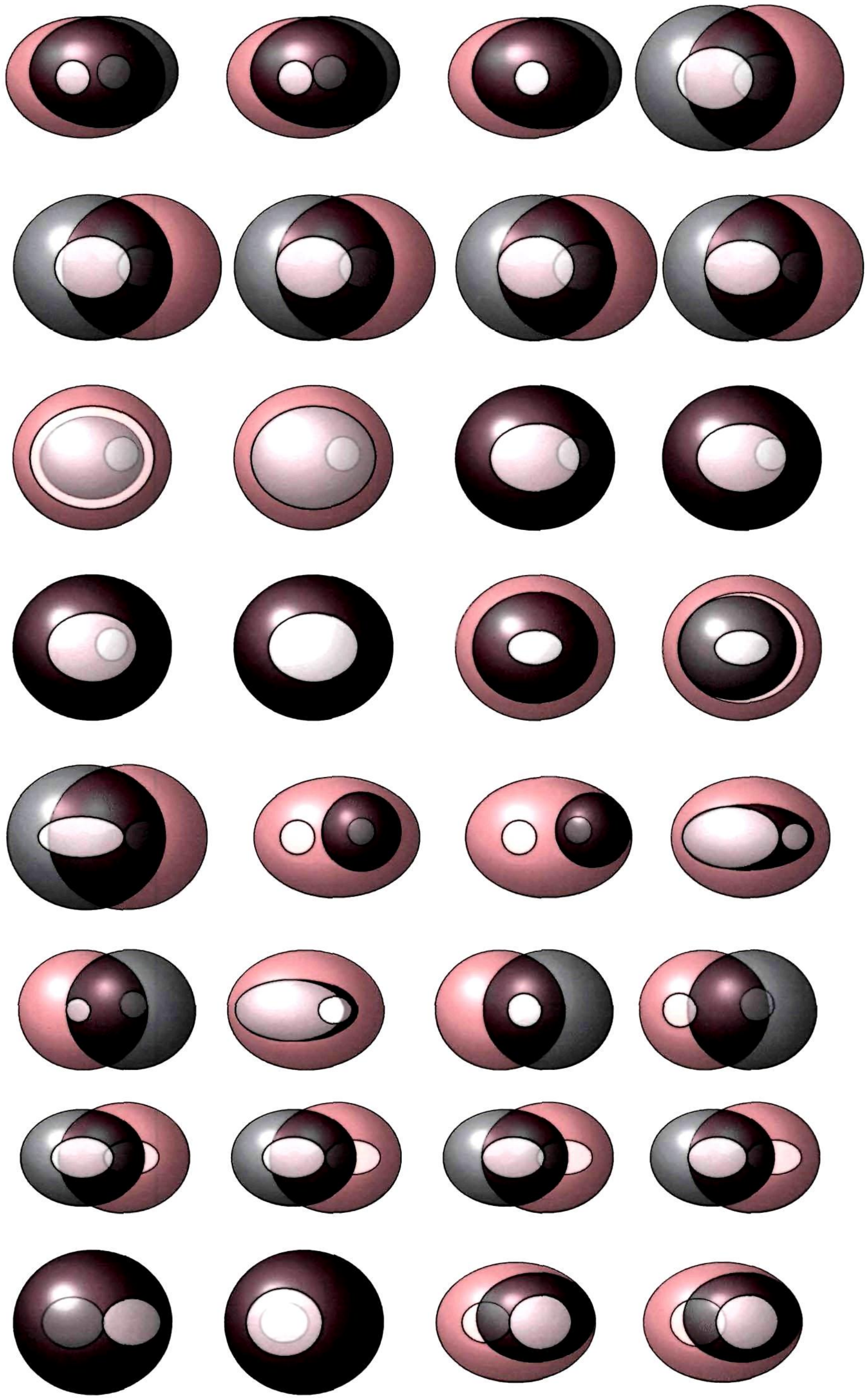
$$t(A^*, B^*) = overlap = t(A^*, B_H), \ t(A_H, B^*) = inside \ then \ t(A_H, B_H) \neq equal$$

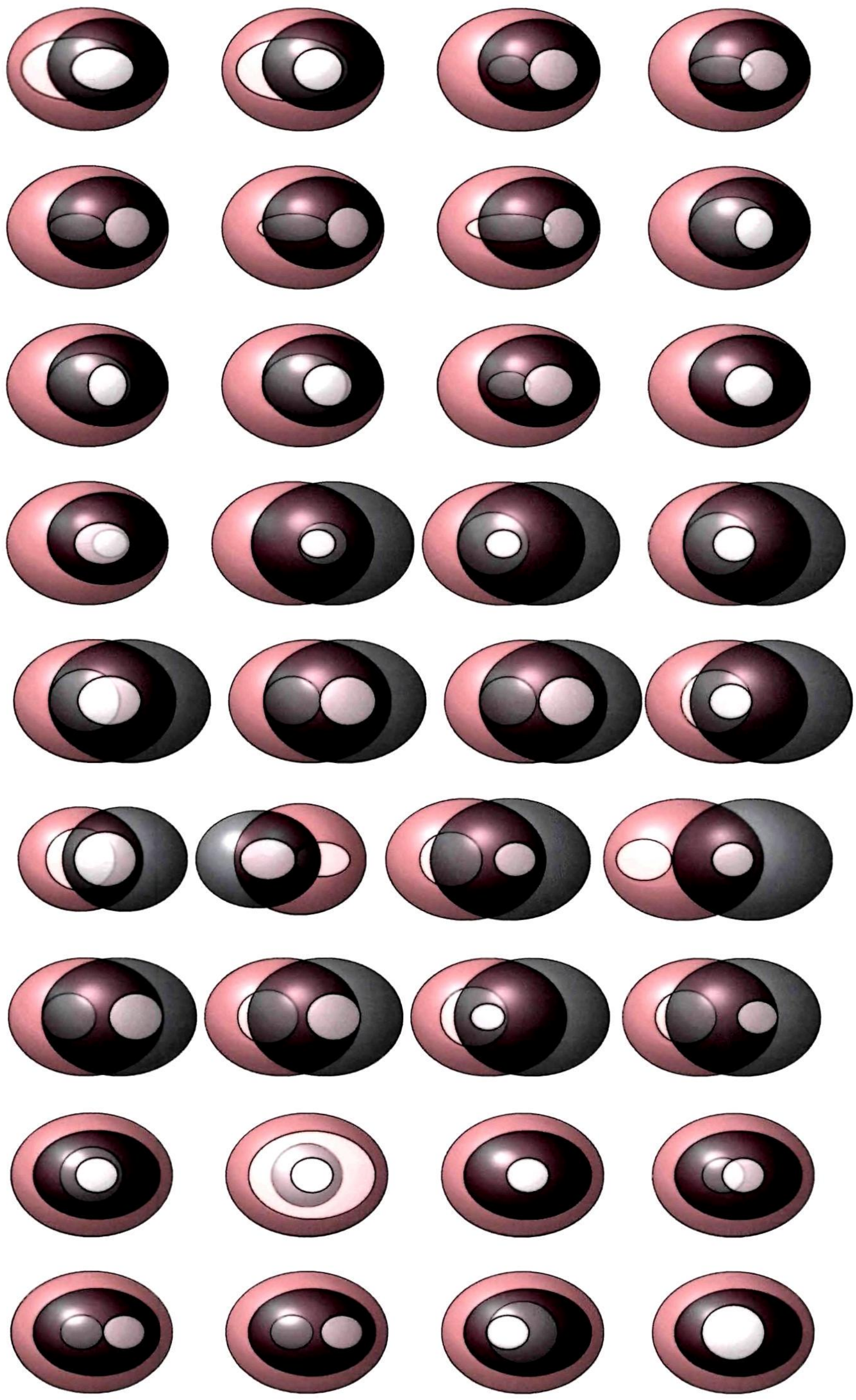
Theorem 4.4.13. *If we consider the content of intersection to be the eight topological relations, there are only 117 different consistent relations between two fuzzy regions each with a hole*

Proof Under the constraint rule of Lemma 4.4.1, only two distinct relations exist between two fuzzy regions each with a hole. Lemmas 4.4.2, 4.4.3, 4.4.4, 4.4.6 imply that there are eight different relations between two single-holed fuzzy regions. Under Lemmas 4.4.5 and 4.4.10 there are six consistent relations for each condition, under Lemmas 4.4.7, 4.4.8, 4.4.9 and 4.4.11, there are sixteen different relations for each condition whereas under Lemma 4.4.12, there are seven relations. Thus, adding the number of distinct relations implied by the twelve lemmas, there are total of 117 relations that can be identified between two single-holed fuzzy regions. □

The set of 117 distinct relations between two fuzzy regions each with a hole is shown in figure 4.3.







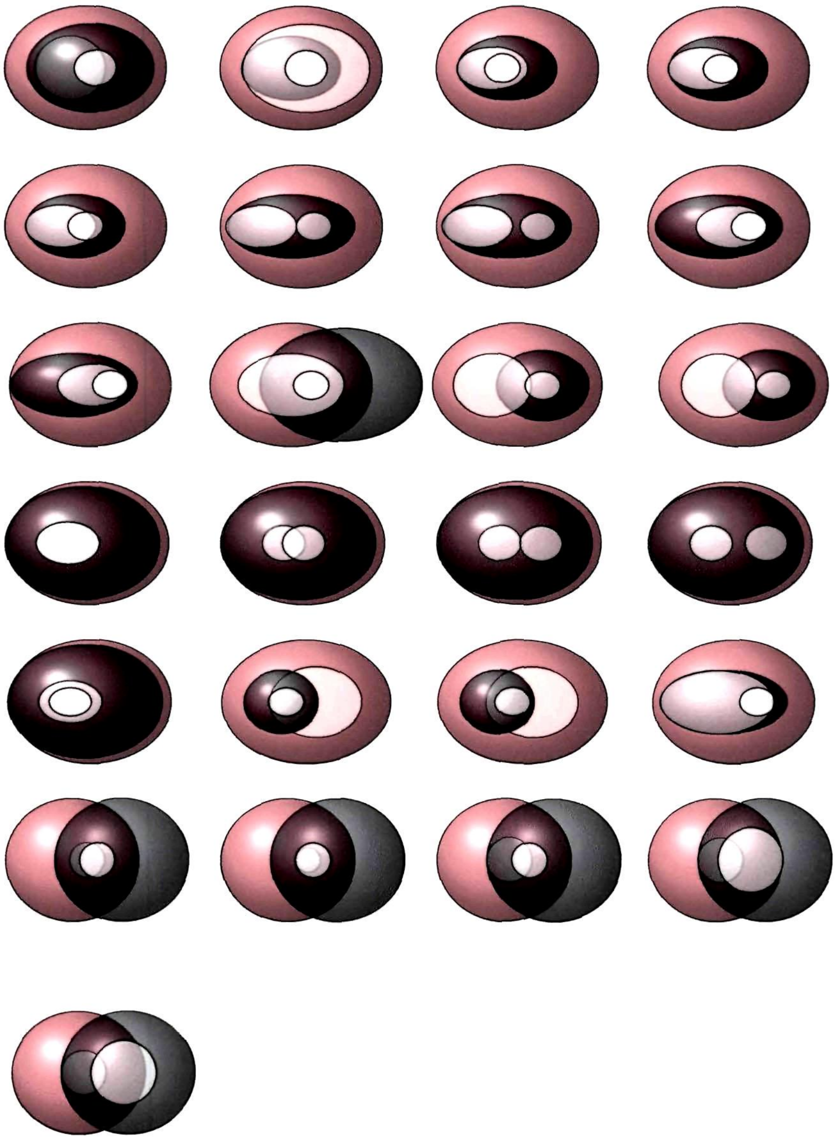


Figure 4.3: Topological relations between two single holed fuzzy regions

4.5 Conclusion

In this chapter, we have proposed a formal definition of a fuzzy region with holes in a general fts and derived the topological relations between two fuzzy regions each with holes. As a particular case topological relations between two single holed fuzzy regions are derived. This definition provides a more general framework to deal with imprecision of the objects such as dealing with complex spatial objects. We have seen that in case of general fts there are only 117 distinct topological relations recognizable between two single-holed fuzzy regions. Egenhofer et. al. [33, 37], considered his framework in a crisp topological space and determined topological relations between two regions each with single hole (by considering the generalized region and the hole as topological invariant) as well as regions each with 2 and 3 holes respectively. Here, we have considered our space to be a general fts and defined fuzzy region with holes in such a way that the intersection between the fuzzy region and the holes is disjoint. We have discussed the topological relations between two fuzzy regions each with 'm' and 'n' holes respectively and found that there are total of $8^{(n+1)(m+1)}$ relations. In particular, if we consider $m = 1 = n$ i.e., both fuzzy regions with single hole, then the number of relations between two fuzzy regions each with a hole is 4096. Further, we have deduced a set of twelve geometric conditions to reduce the number of redundant relations between two fuzzy regions each with a hole in $\mathbb{R}^2$ by proving it or discussing the argument with proof-by-constraint and drawing method. After applying the conditions we have found that the number of consistent relations reduces to only 117.

Chapter 5

An application of topological relations of fuzzy regions with holes

5.1 Introduction

In the last two chapters, we have provided theoretical frameworks for fuzzy regions with holes in terms of closed sets in the settings of crisp and general fts respectively while maintaining consistency of the definition with crisp region with holes in classical topological space. In the present chapter, we aim to develop an application of topological relations of fuzzy regions with holes to a real life situation. In this context, there are various models including Liu and Shi's model [57, 58], Tang and Kainz's model [92] etc. which present applications of topological relations among the fuzzy spatial objects. Liu and Shi [58] developed a computable fuzzy topology by defining new interior and closure operator and used it to determine interior, boundary and exterior of an area affected by a harmful weed. In [57], the same authors provided another model to show an application of topological relations of a fuzzy region and a fuzzy line for investigating the effect of the distribution of SARS over a particular community. In another model, Tang and Kainz [58] provided an application of topological relations between fuzzy regions in land cover changes in China. In this chapter, we will discuss a simple model that seeks to utilize the concepts of topological relations between fuzzy region with hole and a fuzzy point developed in last two chapters for assessing bird flu distribution over a particular locality in which a particular colony is vaccinated. Whereas, the people in a vaccinated colony are usually not affected by the disease, the people in the non-vaccinated colonies are always at a higher risk of being infected. Thus, the whole locality can be considered as a fuzzy region with hole with reference to the effect/risk factor of the disease Severity is then calculated using topological relations between fuzzy region with holes and fuzzy point. The data used in the model is purely hypothetical and the physical implications obtained are not in a rigorous sense. The basic purpose of the model is simply to explore possibilities of application of the topological model.

There are three sections in this chapter. In the Section 2, we have formulated vaccinated locality as a fuzzy region with hole and have determined the effect of flu on the people at different position of the locality using topological relations. In Section 3, we have formulated a method to determine qualitative information about the severeness of the disease using membership grades of topological relations considering a set of hypothetical data. In Section 4, a point-wise model is provided to determine the severity and position of any point of the locality w.r.t. the flu point.

5.2 A model application

We assume that there is a vaccinated locality near a locality infected by bird flu. The term vaccinated is being used in the sense that a particular colony of the locality takes some precautions for prevention or spread of the disease, for instance, such as killing and burial of affected/likely to be affected poultry, imposing restriction on selling and buying exposed poultry etc. Bird flu is a well known viral disease which affects the life of the people residing in the nearby areas and spreads even when a single bird migrates from the infected locality to the nearby regions. Though the people in the vaccinated colony are at a lower risk of being infected by the disease but they are still susceptible to the disease. So, when people in the vaccinated colony are susceptible to the disease, the people in the non vaccinated colonies are at much higher risk of being affected by the disease. The fuzzy set under consideration is the risk of disease at each point of the locality for each day which varies from individual to individual in the vaccinated and non-vaccinated colonies of the locality. We have formulated the vaccinated locality as a fuzzy region with a hole considering the vaccinated colony as the hole and the union of vaccinated and non-vaccinated colonies of the locality as the generalized fuzzy region in a crisp fts $(\mathbb{R}^2, C)$. A carrier is considered as a fuzzy point due to variation in level of infection. This is illustrated in figure 5.1.

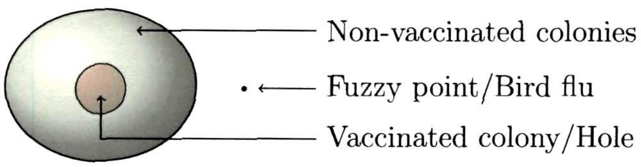


Figure 5.1: A vaccinated locality and a fuzzy point

We then use topological relations between fuzzy region with hole and fuzzy point to determine the various stages of infection as given in Table 5.1.

Here, A^* represents the locality which is considered as a generalized fuzzy region, H represents the vaccinated colony in the locality considered as a hole, P is bird flu infection considered as a fuzzy point. $t(A^*, H)$ represents the topological

	A^*	H	P
A^*	$t(A^*, A^*)$	$t(A^*, H)$	$t(A^*, P)$
H	$t(H, A^*)$	$t(H, H)$	$t(H, P)$
P	$t(P, A^*)$	$t(P, H)$	$t(P, P)$

Table 5.1: Topological relations between vaccinated locality and bird flu

relation between A^* and H.

As discussed in Chapter 3, there are only 5 distinct consistent relations between a fuzzy region with a hole and a fuzzy point. The physical significance of these five relations over the locality w.r.t. bird flu infection is given in the observations below:

Observations:

The following cases may arise:

- The relation between fuzzy region with hole and flu point is 'disjoint' (i.e., $t(H, P)$ and $t(A^*, P)$ are disjoint). It then implies that infection has not reached the area or locality. They are, however, susceptible to the flu.

- The relation between the fuzzy region with hole and flu point is 'meet' (i.e., $t(A^*, P)$ is meet and $t(H, P)$ is disjoint). It implies that the people in the locality are at a lower risk of infection. The flu has just entered the locality.

- The relation between the fuzzy region with hole and flu point is 'contain' (i.e., $t(A^*, P)$ is contain and $t(H, P)$ is disjoint), then the people in the locality are at considerable risk of being infected. The flu has affected the locality to some extent.

- The relation between the fuzzy region with hole and flu point is 'contain meet' (i.e. $t(A^*, P)$ = contain, $t(H, P)$ = meet). People in the locality are at greater risk of being infected. The flu covers the maximum extent of the locality.

- The relation between the fuzzy region with hole and flu point is 'disjoint inside' (i.e. $t(A^*, P)$ = disjoint, $t(H, P)$ = inside). The flu covers the entire locality with incidence of maximum risk in the non-vaccinated colony.

Now, membership grades of these topological relations shall be used to determine the severeness of the disease at different position of the locality. The membership grade of topological relations for the above five cases are determined by slightly modifying Zhan's formula considering a set of hypothetical data in Section 5.3.

5.2.1 Zhan's formula for membership grade of topological relations

Let A and B be two fuzzy regions in $\mathbb{R}^2$. If 'n' is the number of α-cut regions of a given region which are nested and $\tau(A^{\alpha_i}, B^{\alpha_j})$ denotes the membership value of topological relations between two α-cut regions A^{α_i} and B^{α_j}, then membership value of topological relation between two fuzzy regions A and B in $\mathbb{R}^2$ is given by

$$\tau_k(A, B) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i - \alpha_{i+1})(\alpha_j - \alpha_{j+1})\tau_k(A^{\alpha_i}, B^{\alpha_j})$$

where $k \in \{disjoint, meet, overlap, equal, contain, inside, cover, covered\ \ by\}$ and $\alpha_i, \alpha_j \in [0, 1]$.

5.3 A case study

5.3.1 Objective and Methodology

Our aim is to propose a methodology to find the degree of severeness of the disease over the locality when flu reaches a particular distance from the center of the vaccinated locality and severeness of flu at any point within the locality w.r.t flu point in a given time interval. We use a hypothetical data set for the number of infected people in the vaccinated and non vaccinated colonies, assuming that the number of infected people on each day should occur either in increasing or decreasing order while the vaccinated colony remains in the center of the locality.

Fuzzy set of the problem: Here, our fuzzy set under consideration is the risk of infection or the proportion of the number of person infected by the disease at any position of the locality on each day. The number of persons infected over vaccinated and non-vaccinated colony will increase with time and risk of the already infected person will also fluctuate so that fuzziness of the problem is time and positional dependent.

Adaptation of Zhan's formula for relative membership: Zhan's formula is applicable for determining membership grade of topological relations between two fuzzy regions without holes. But here, as we consider the locality as fuzzy region with hole, Zhan's formula cannot be applied directly to determine the membership grade of topological relations. To apply Zhan's formula we utilize the theoretical framework on topological relations between fuzzy region with holes and fuzzy point developed in Chapter 3. As this framework considers generalized region and hole as two separate fuzzy objects without holes and topological relations are determined considering generalized region, hole and point as distinct spatial object without holes which is related to 9-intersection matrix. Hence, Zhan's formula can be extended for determining relative membership grade of topological relations between fuzzy region with hole and flu point as

$$\tau_k' = \sum_i \sum_j (\alpha_i - \alpha_{i+1})(\beta_j - \beta_{j+1}) t(H, P) + \sum_i \sum_j (\alpha_i - \alpha_{i+1})(\beta_j - \beta_{j+1}) t(A^*, P)$$

where α_i and β_j are α-cut regions of the vaccinated and non-vaccinated colonies in the locality respectively; i, j are the positional number of α-cut of vaccinated and non-vaccinated colony, $k \in \{disjoint, meet, contain, contain\ meet, disjoint\ contain\}$ and τ_k is the membership grade of the topological relation k.

In the above formula, since we are using the α-cut of both the vaccinated and non-vaccinated colonies for each relations, this formula will give relative membership grades of the topological relations w.r.t vaccinated and non-vaccinated colony of the locality. To apply the modified Zhan's formula we required the value of α and the value of corresponding crisp topological relation between fuzzy region with hole and fuzzy point, and between hole and fuzzy point. Then to determine severeness at any point inside the locality, we first find the relative severeness of vaccinated and non-vaccinated colony on each day and then multiply the difference of relative severeness of two consecutive days by inverse of distance.

Value of α for relative severeness: The relative severeness will be calculated considering the value of α to be the ratio of the number of individuals infected in the vaccinated/non-vaccinated colonies each day to the total of the number of people residing in that vaccinated/non-vaccinated colonies so that the corresponding crisp set will be the set of all infected person in vaccinated and non-vaccinated colonies of the locality.

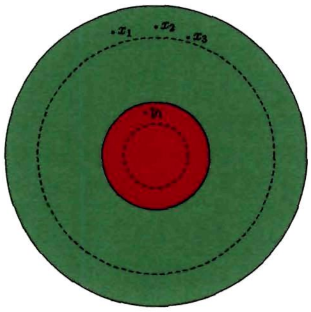


Figure 5.2: Relative α-cut regions of a generalized region and hole on the 1^{st} day

5.3.2 Relative membership grades of topological relations

Our theoretical framework for topological relations between fuzzy region with holes and fuzzy point will give qualitative information about the situation. To obtain the information on severeness at a particular distance of flu point from the center of the vaccinated colony of the locality we need to find relative membership grade of the relations (i.e. quantitative information about the relations) which can be calculated by modified Zhan's formula. The relative membership grades of the disease at any position of the vaccinated and the non-vaccinated colony of the locality for each day (as shown in Figure 5.2) is given as follows:

$$\tau_k = \alpha_i \beta_i t(H, P) + \alpha_i \beta_i t(A^*, P)$$

where α_i and β_i are α-cuts of the vaccinated and non-vaccinated colony respectively on i^{th} day.

Theorem 5.3.1. *For any $k \in \{disjoint, meet, contain, disjoint\,contain, contain\,meet\}$ the value of $\tau_k = \alpha_i \beta_i t(H, P) + \alpha_i \beta_i t(A^*, P)$ lies in [0,1].*

Proof. For the relations 'disjoint', 'meet', 'contain', we have $t(H, P) = 0$. So, $\tau_k \in [0, 1]$. Similarly, for 'disjoint contain', we have $t(A^*, P) = 0$. Consequently, $\tau_k \in [0, 1]$. But if the relation is 'contain meet' then both the relations $t(H, P) \neq 0$ and $t(A^*, P) \neq 0$, then also $\tau_k \in [0, 1]$ as initially when point meets the hole crossing the generalized region, its value is less than 0.5.
Hence, in any case $\tau_k \in [0, 1]$. □

Since, we are considering the α-cuts of both the vaccinated and non-vaccinated colonies for determining relative membership grade, therefore, $t(H, P)$ and $t(A^*, P)$ will be crisp relations taking values 0 and 1 only, depending on whether the relations between H and P as well as A^* and P belong to the topological relations under consideration or not.

The relative membership grade of vaccinated and non-vaccinated colony of the locality for each day is given by

$$Relative\ severeness\,(R.S.) \;=\; Max\{\tau_k\}$$

where $k \in \{disjoint, meet, contain, contain\ meet, disjoint\ contain\}$

The data set considered below is purely hypothetical. As mentioned earlier, the primary stress is on the methodology and not on the authenticity of the data or implied outcome.

Consider a locality of 1386 sq. kms(approx) consisting of 16000 people, of which 1000 are residing in vaccinated colony which is 63.585 sq. kms (approx). The hypothetical data of infected by the disease in seven days over the vaccinated and non-vaccinated colonies of the locality are given in Table 5.2.
The relative membership grade of the five relations between fuzzy region with hole and point for the first day are
$\tau_{disjoint} = 0$.

Day	vaccinated region(out of 1000)	non-vaccinated region(out of 15000)
1	5	1000
2	7	1050
3	8	1070
4	9	1100
5	11	1200
6	17	1450
7	25	1650

Table 5.2: Number of infected people

$\tau_{meet} = \alpha_1\beta_1 t(A^*, P) = 0.005 \times .0667 = .000334.$

$\tau_{contain} = \alpha_1\beta_1 t(A^*, P) = 0.005 \times .0667 = .000334.$

$\tau_{contain\ meet} = 2\alpha_1\beta_1 = 2 \times 0.005 \times .0667 = .000667.$

$\tau_{disjoint\ contain} = \alpha_1\beta_1 t(H, P) = 0.005 \times .0667 = .000334.$

Hence, the relative severeness for the first day is 0.000667.

Similarly, relative membership grades of severeness of infection in the vaccinated and non-vaccinated colonies for the remaining six days are given in Table 5.3.

5.3.3 Severeness at a particular distance of flu point in the locality from the center of the vaccinated colony

From the observation in Section 5.2, we know that when people in the vaccinated colony are susceptible to the disease then the people in the non-vaccinated

Day	Relative severeness (R. S.)
1	.000668
2	.00098
3	.001143
4	.00132
5	.00176
6	.003287
7	.0055

Table 5.3: Relative degree of severeness

colonies are at a much higher risk of being infected. That is, severity of the disease decreases as the distance of the carrier of infection increases from the vaccinated colony. The elements in Table 5.3, indicates the maximum of membership grade of the disease in the people w.r.t. the vaccinated and non-vaccinated colony of the locality on each day of the survey. In reality, however, the effect of flu at different position of the locality will be different although sometime it may be equal at some points. To determine the severity at a particular distance of flu point in the locality in the present Subsection we have provided a basic formula for severeness using relative membership grade of topological relations and distance of the flu point from the center of the vaccinated colony. We, therefore, consider the relative membership grades of flu over vaccinated and non-vaccinated colonies for each day to divide whole locality into number of α-cut regions as shown in the Figure 5.3.

Thus, the membership grade of severeness when the fuzzy point is at a distance 'r' from the center of the locality on ith day of the survey is given by

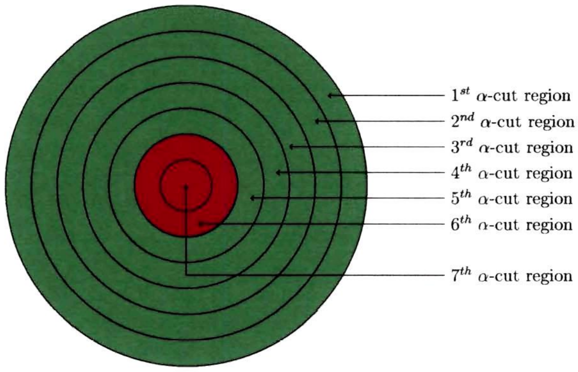


Figure 5.3: α-cut regions of the vaccinated locality

Severity at any point P at a distance 'r' on i^{th} day

$$= \frac{R.\ S.\ of\ i^{th}\ day - R.\ S.\ of\ (i-1)^{th}\ day}{total\ of\ R.\ S.} \times \frac{1}{r}$$

Here, we consider the difference of relative membership grade of severeness of two consecutive days because the number of infected persons on each day is double counted in the number of infected person in the succeeding/next day (i.e. in the considered data the number of infected for the 1^{st} day are 5 infected and for the 2^{nd} day the number of infected are 7 that is the number of infected on the second day are only 2 which would contributed to the severity of the locality on the 2^{nd} day). Then, we divide it by the sum of all the relative severities for given period. Finally, we multiply it with the inverse of distance so that it will give severeness of that point of the locality where flu had reached as the severeness increase as flu entered toward the center of the locality.

In the subsequent section this formula is modified to determine severeness at any point w.r.t flu point as well as position of the point in the locality as the current formula does not provide any information about the position of the point. Severeness of the bird flu at a distance 8km and 13km from the center of the vaccinated colony inside the locality for seven days is given in Table 5.4.

Distance	8km	13km
Day 1	.005697	.003506
Day 2	.002661	.001637
Day 3	.00139	.000855
Day 4	.001509	.000929
Day 5	.0030018	.002309
Day 6	.013022	.008013
Day 7	.018872	.011614

Table 5.4: Severeness at a distance

5.4 Pointwise/Local severeness model

Suppose the bird flu enters the locality from east to west and the point at which we want to find severeness is along the same side of the flu point. We then use vector difference to calculate the position and the value of severeness of the point from the flu point.

As shown in Figure 5.4, 'P_1' is at a distance 'a' from the center of the vaccinated colony and lies in the upper region at which we want to find the severity when the flu reached the point 'P' at a distance 'r' from the center of the vaccinated locality. Then severeness at the point P_1 on i^{th} day is given by

$$S = \frac{R.\ S.\ on\ i^{th}\ day - R\ S.\ on\ (i-1)^{th}\ day}{total\ of\ R.\ S.} \times \frac{1}{|a-r|} \tag{5.4.1}$$

Here, we consider the vector difference in the formula because we want to find the severeness of any point of the locality w.r.t flu point. Then vector difference will gives the relative distance between the points if the point lies in the same direction.

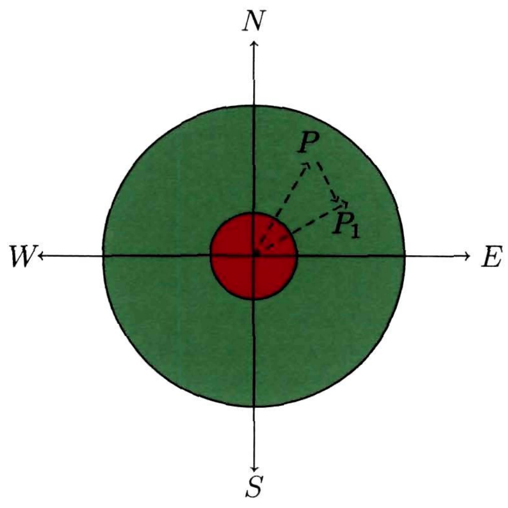


Figure 5.4: Point lies in the same side of fuzzy point but upper region

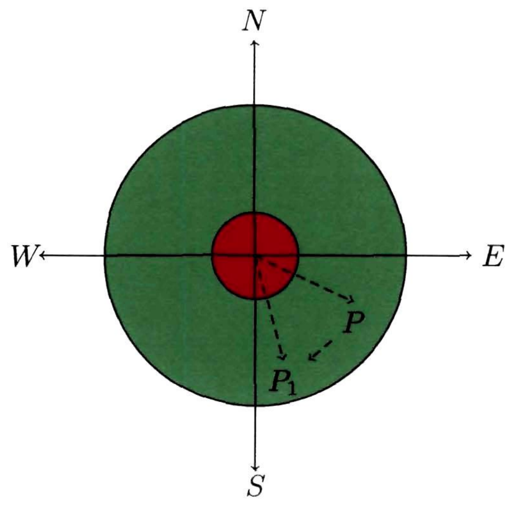


Figure 5.5: Point lies in the same side of fuzzy point but lower region

In Figure 5 4 , direction of $\overrightarrow{PP_1}$ is towards P_1 which is in anticlockwise direction so it is towards the upper half or in the left side of the locality

Similarly, if the point lies at P_2 which is along the flu point and in the lower region at which we want to calculate the severity as shown in Figure 5 5, we shall use the same formula as given in equation (5 4 1) But in this case, the direction of the $\overrightarrow{PP_2}$ is towards P_2 which is in clockwise direction so it is towards the lower half of the region or towards the right side of the region

Likewise, we can find the position and membership of any point along the same side of thc flu point if the flu enter the locality from the left side

Next, if bird flu entered the locality from the right and the point at which we want to find severeness is along the opposite direction of flu point (as shown in Figure 5 6 and 5 7) Then we use resultant of the vector sum to calculate the position and the value of severeness of the point from the flu point

Then the severeness at the point is given by

$$S = \frac{R\ S\ \ on\ i^{th}\ day - R\ S\ \ on\ (i-1)^{th}\ day}{total\ of\ R\ S} \times \frac{1}{|a+r|} \qquad (5\ 4\ 2)$$

The direction of the resultant gives the position of the point whether it is along the upper or lower, left or right side of the locality and magnitude of vector will denote the severeness of the flu at the target point For instance, if $OP =$ $5Km = r$ towards east and $OP_1 = 10Km = a$ towards north-east in the above discussed locality with the hypothetical data set, then $PP_1 = 5Km$ east to north i e anticlockwise direction and the value of severencss at any point P_1 when flu reach P on the i^{th}-day is given by the Table 5 5

Next, if $OP = 5Km = r$ towards east-south and $OP_2 = 10Km = a$ towards

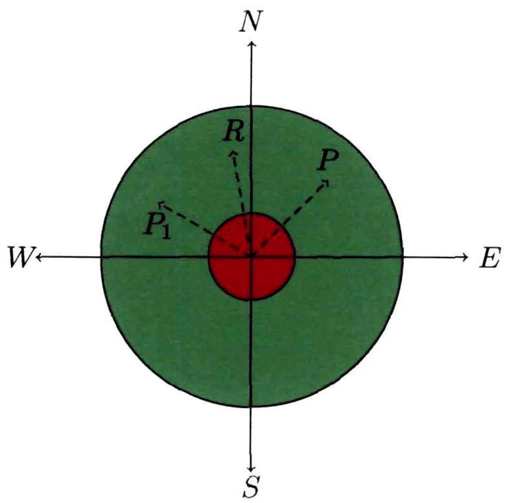


Figure 5.6: Point lies on opposite side of fuzzy point in upper region

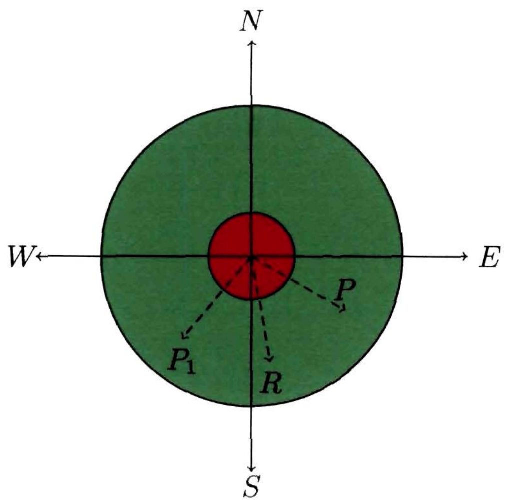


Figure 5.7: Point lies in the opposite side of fuzzy point but lower region

Day	Severeness at point P_1
1	.22786
2	.106425
3	.0556
4	.060375
5	.15009
6	.520875
7	.754475

Table 5.5: Severeness at any point along the same side of the flu point

east south in the above locality with the considered data then $PP_2 = 5Km$ towards south-east and the value will remains same as given in the Table 5.5.

Further, if $OP = 5Km = r$ north-east and $OP_1' = 10Km = a$ towards north-west in the considered locality with considered data then $R = 15Km$ towards east-west and the severeness at the point when flu reach point P_1' is given in the Table 5.6

Similarly, if $OP = 5Km$ towards east-south and the $OP_2' = 10km$ towards west-south in the above discussed hypothetical locality then $R = 15Km$ towards the north-south and the membership grades of the severeness on different days at the given point from the bird flu point is same as in Table 5.6.

From the two Tables 5.5 and 5.6, it can be seen that points which are at same distance from the center of the vaccinated locality and lie in the either side of the locality, the severeness decrease from the side where flu entered the locality to the point which lies opposite or far away from the flu point.

Day	Severeness at point P_1
1	.0030397
2	.0014197
3	.0007417
4	.000805
5	.002002
6	.006949
7	.010065

Table 5.6: Severeness at the opposite side of flu point

5.4.1 Limitations

The following two assumptions that

(i) *the number of infected people over the vaccinated and non-vaccinated colonies of the locality should be either in increasing or in decreasing order*, and that

(ii) *the vaccinated colony should be in the center*

are the two main limitations of the model. However, most of the part of the model have been deduced using theoretical derivations and assumptions supported by valid arguments, so the model is applicable on real life situation. Here, we are using a purely hypothetical data set instead of real life data, the objective being just to exhibit as to how this framework can be applicable to a real life situation.

5.5 Conclusion

In this chapter, we have proposed an application of our theoretical model on topological relations between fuzzy region with holes and their relations for the determination of effect of bird flu over a particular locality in which certain colony is vaccinated. The basic idea is to formulate the locality as fuzzy region with hole and carrier of infection as fuzzy point. Subsequently, the topological relations between a fuzzy region with hole and a fuzzy point are used to determine the various level of risk over the different positions of the locality. The membership grades of the relations are calculated by modifying Zhan's formula based on our framework of topological relations between fuzzy region with hole and a fuzzy point. The number of infected people and intensity of affliction at different position of the vaccinated and non-vaccinated colony will vary from day to day. Hence, the fuzziness is time and positional dependent. Therefore, the modified Zhan's formula will give relative membership grade of severeness. Then value of relative severeness will be used for dividing the whole locality into number of α-cut regions. Finally, the value of relative membership grades are used to determine the severeness of the people at a given distance of the flu point from the center of the locality. But as it does not give idea about severeness of any point of the locality w.r.t. the flu point and position of the point in the locality, we have further provided a point-wise model which determines severeness of a point w.r.t flu point as well as position of the point. This model will be helpful in analyzing different real life situations by determining relative membership grades of topological relations using α-cuts, then using these values determine severeness of any point in the affected region. This model may be used to classify extent of severeness or intensity of any harmful infectious disease/epidemics over different colonies of an area so that remedial measures can be taken according to necessity. Models for multi-holed regions may be developed as an extension of this work.

Chapter 6

Conclusion

6.1 Outcome

Study of spatial relations between spatial objects is one of the important aspects in upcoming fields including branches of spatial reasoning, artificial intelligence, cognitive sciences etc. Of the two kinds of spatial objects - crisp and fuzzy, the crisp spatial objects and their relations have been extensively studied over last two decades. Fuzzy spatial objects carry vast potential due to their ability to present real situations more meaningfully but at the same time pose higher degree of difficulty of being modeled. In this front though there are various models to deal with fuzzy regions without hole, no model for fuzzy regions provided for intrinsic incorporation of holes, which is an unavoidable necessity. In this thesis, we provided a formal framework for modeling fuzzy regions with holes and their topological relations in the frameworks of general and crisp fuzzy topological spaces respectively. A model is also proposed as an application of the developed framework to real life situations. Our study is expected provide a step towards the development of optimal solution of topological relations of fuzzy spatial objects with holes.

6.2 Future directions

Modeling complex real life phenomenon for useful application oriented purposes presents a great challenge to researchers working in different interdisciplinary fields of knowledge. Despite the recognition of its immense potential and highly developed theoretical stature, serious applications of topology in directly applicable fields have started less than two decades ago. Research on topological relations of fuzzy spatial objects is one area of work carrying vast potential and at the same time posing a hard challenge to researchers. As an attempt in this direction, we have, in this work, endeavoured to develop a theoretical framework to define fuzzy regions with holes containing a finite number of holes upon the assumption that holes are not along the boundary of the region and are not overlapping. It is however readily accepted that most of the real life phenomena may not always or perhaps more often than not satisfy the above restrictions, calling for further work to be carried out to formulate fuzzy regions to take care of these situations. We have attempted to provide a simple application of topological relations between fuzzy region with hole and fuzzy point. Given the occurrence of such situations in real life, application of topological relations of fuzzy regions which containing more than one holes are bound to be useful in different contexts. Further, future research can be done to represent application of topological relations between fuzzy region with holes and fuzzy line, fuzzy region with and without holes, fuzzy region each with a holes and thereby representing different complex real life situations. It is expected that our work shall open up newer avenues from the theoretical as well as application perspectives.

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