MODELLING THE EFFECT OF POLLUTANTS ON ECOSYSTEM

A Thesis submitted
in partial fulfilment of the Requirements
for the degree of

DOCTOR OF PHILOSOPHY

by

MD. JAMAL HUSSAIN

to the

DEPARTMENT OF MATHEMATICAL SCIENCES
SCHOOL OF SCIENCE & TECHNOLOGY
TEZPUR UNIVERSITY

SEPTEMBER, 1999

Dedicated to

My Beloved Parents

Karima Begum

&

Md. Tazamul Ali

CERTIFICATE

This is to certify that the matter embodied in the thesis entitled **"Modelling the Effect of Pollutants on Ecosystem"** by Md. Jamal Hussain for the award of degree of Doctor of Philosophy of the Tezpur University is a record of bonafied research work carried out by him under my supervision and guidance. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

September, 1999

(B. DUBEY)
Lecturer
Dept. of Mathematical Sciences
School of Science & Technology
Tezpur University
Tezpur, Assam, INDIA.

ACKNOWLEDGEMENTS

I wish to express my deep gratitude to my supervisor, Dr. Balram Dubey, who has guided me through out this work and without whose invaluable inspiration, encouragement and constructive criticism the investigations contained in this thesis would not have been possible.

I would like to thank all the faculties of the Dept. of Mathematical Sciences. In particular I would like to thank Professor A. K. Borkakati, Dr. S. K. Laskar and Dr. Munindra Borah for their inspiration and suggestions from time to time.

I express my sincere thanks to Dr. D. K. Saikia, Professor and Head, Dept. of Computer Science, Tezpur University for providing all the facilities at the Computer Centre.

I would like to thank the faculties of the Dept. of Computer Science for helping me in preparing my thesis. In particular, I would like to thank Dr. D. K. Bhattacharya, Dr. Rajib Das and Mr. Utpal Kr. Sharma for teaching me Latex from time to time. My sincere thanks goes to Mr. Dhiraj Kr. Sharma for his help at the computer centre.

I am grateful to Mrs. Barnali Das for her encouragement and cooperation throughout my research work.

Though it is beyond the scope of any acknowledgment for what I have received from my parents and my brother and sister in the form of inspiration, cooperation and patience, yet I make an effort to express my heartfelt gratitude to them.

I take this opportunity to express my profound respect to Dr. Uma S. Dubey whose inspiration, encouragement and moral support at many critical hours have been a great source of strength to me.

I take this opportunity to express my sincere thanks to Mr. Azizur Rahman and his family for their affection and encouragement throughout my research work at Tezpur University.

I owe my thanks to all my friends especially Pallabi, Sangeeta, Zakir and Anjan for their enthusiastic help throughout my stay at Tezpur University making it a memorable and pleasant one.

Jamal Hussain

Contents

Chapter 1

GENERAL INTRODUCTION

1.1 Introduction

Ecology is the branch of science that deals with the relationships of life forms with each other and with their surroundings. The basic unit in ecology is the ecosystem which is a fairly self contained system of plants and animals living in a particular kind of environment. Every ecosystem has four components:

1. The nonliving environment: This includes sunlight, water, oxygen, minerals, and dead plant and animal matter.

2. Producers: These are green plants which range in size from the microscopic phytoplankton to giant redwood trees. They have the unique ability to absorb the sun's energy and use it to produce foods.

3. Consumers: These are animals: both herbivores, which feed on plants and carnivores, which eat other animals.

4. Decomposers: These include bacteria, fungi, and insects that break down dead plants and animals. In the process they release energy into the environment and return matter to the soil. The matter provides nourishment that is absorbed by green plants and started through the cycle again.

In theory the ecosystem is a closed cycle. But in practice ecosystems are seldom in a state of balance. Natural changes, which gradually shift the composition of the ecosystem, occur continuously. An ecosystem that supports many kinds of green plants and animals is not likely to be disrupted by such changes. If one species is lost, many others remain to continue the cycling of materials and energy. On the other hand, an ecosystem with only a few species may collapse if the environment changes suddenly, killing one or two species. Throughout the biosphere same principle applies; wherever diversity is lacking, ecosystems tend to be unstable and fragile.

Air pollution has been a problem ever since fire was discovered by cave dwellers. With the Industrial Revolution, the intensive burning of coal and oil in centralized locations began. The problem was compounded because the population of the Earth had also been rapidly growing. The addition of motor vehicles caused more and more serious problems, until finally a series of dangerous air pollution episodes occurred. The three most notorious episodes were all associated with light winds and reduced vertical mixing that persisted for several days. Many deaths were recorded in 1930 in the Meuse Valley in Belgium, in 1948 in Donora, Pennsylvania, and in 1952 in London.

One of the important problems that society faces today is the pollution of our environment affecting the quality of life in the form of diseases, epidemics etc. The abnormal level of green house gases in the atmosphere is affecting the climate, which has already changed to a considerable extent due to deforestation and manmade projects, bringing prolonged drought, abnormal temperature in one region and occurrence of floods in the other (Treshow, 1968; Woodwell, 1970; Davis, 1972; Maugh, 1979; Smith, 1981; Reish et al., 1982; Reish et al., 1983; Kormondy, 1986; Parry and Carter, 1988; Veeman, 1988; Woodman and Cowling, 1987; Sahani, 1998).

The depletion of resources such as forestry, fisheries, fertile topsoil, crude oil, minerals, etc. is causing great concern for the mankind. These resources are being depleted due to rapid industrialization, fast urbanization and rising population. These factors have deteriorated our ecology and environment to such an extent that if concrete steps are not taken soon to conserve these resources, many undesirable effects will occur leading to disastrous consequences for the mankind (Frevert et al., 1962; Detwyler, 1971; Smith, 1972; Pimental et al., 1976; Annon, 1977; Das, 1977; Gadgil and Prasad, 1978; Karamchandani, 1980; Brown, 1981; Gadgil et al., 1983; Larson et al., 1983; Repetto and Holmes, 1983; Brown and Wolf, 1984; Haigh, 1984; Gadgil, 1985; Waring and Schiessinger 1985; Biswas and Biswas, 1986; Khoshoo, 1986; Munn and Fedorov, 1986; Shukla et al., 1987; Gadgil, 1987; Shukla et al., 1988; Gadgil and Chandran, 1989; Shukla et al., 1989; Banerjee and Banerjee, 1997).

Forests play a very important role in maintaining the environment and in supplying the essential requirements of people. But forests are suffering rapid depletion due to diversion of forest lands to other uses such as industrialization and cultivation, the inadequacy of protection measures and the attitude of our people to look upon forests as revenue earning resource (Singh, 1993). There are many ecologically unstable regions around the world and the Doon Valley in the northern part of Uttar Pradesh in India is one such example where the main reasons for the depletion of forest biomass are limestone quarrying; growth of wood based industries and associated pollution, growth of human and livestock populations, etc. (Munn and Fedorov, 1986; Shukla et al., 1989). Other ecologically unstable areas include uplands of Western Amazonia, the Atlantic Coast of Brazil, the Madagascar Islands, the Malaysian rain forest zones etc., (Wilson, 1989).

It is, therefore, absolutely essential to study the effects of various factors such as industrialization, pollution and population responsible for the depletion of resources so that appropriate measures for conservation are taken and the desired level of the resource biomass can be maintained without harming our ecology and environment (Ghosh and Lohani, 1972; Pathak, 1974; Das, 1977; Karamchandani, 1980; Martino, 1983; Khoshoo, 1986; Lamberson, 1986; Munn and Fedorov, 1986; Shukla et al., 1987, 1988, 1989).

In the following an account of the literature related to pollutant diffusion and migration of the species and its effects on their evolution and co-existence is presented.

1.2 Effect of Diffusion

Air pollutants, such as sulphur dioxide, carbon dioxide, etc., are dispersed in the environment by the process of molecular diffusion which arises due to changes in concentration and depends upon various factors such as types and number of sources, stack heights, meteorological conditions and the topography of the terrain. A great deal of attention has been devoted to study the molecular diffusion process by using the well known Fick's law of diffusion and these have been well documented by Sutton (1953), Pasquill (1962), Scorer (1968), Stern (1968), Deininger (1974) and Crank (1975). Due to environmental factors such as overcrowding, anticlimate, predator chasing prey and more importantly due to resource limitation in the habitat and other related effects biological species living in a habitat has a tendency to migrate to better suited regions for their survival and existence (Rosen, 1974, 1975; Verma, 1980).

The evolution and existence of species has been the subject of scientific investigation since the days of Darwin. Earlier studies were mainly concerned with experimental observations and it is only in the beginning of twentieth century that attempts have been made to predict the evolution and existence of species mathematically. The first major attempt in this direction is due to Volterra and Lotka which constitute the main basis of the deterministic theory of population dynamics in theoretical Biology even today. Over the last fifty years, many complex models for two or more interacting species have been proposed on the basis of Lotka and Volterra models by taking into account the effects of crowding, age structure, time delay, functional response, switching etc. (Holling, 1965; Rescigno, 1968; Rosen, 1970; May, 1971; Maynard Smith, 1974; Gomatam, 1974; Freedman, 1976; Cushing, 1976; Brauer, 1977; Harada and Fukao, 1978; Tansky, 1978; Freedman, 1979; Gopalsamy, 1980, 1981).

It may be noted that Lotka-Volterra model focuses on population interactions at a point in space ignoring movement (migration/diffusion) which means a perfect mixing of the species in a given region. Mathematically, this is equivalent to assuming that the dispersal rates are sufficiently high and the population in the habitat are well mixed.

Without assuming so, one ignores the essential aspects of species response to environmental and ecological changes it encounters in the habitat. Thus, Lotka-Volterra type models describe the situations which correspond to only laboratory conditions rather than real situations arising in natural environment. It may be noted here that even in the laboratory spatial variations may be essential for the coexistence of the species (Huffaker, 1958, 1963).

In recent years many researchers have studied the effect of diffusion in ecological models. The classical Volterra model for the evolution of two interacting species ignores the effects of migration which may arise due to environmental and ecological gradients in the habitat. These may be studied by taking into account the dispersive and convective migration terms in population models. Skellam (1951) was probably the first to study the effects of dispersive migration on the growth of populations. Later, several investigators studied this effect by considering various models (Landahl, 1959; Segal and Jackson, 1972; Levin, 1974; Hadeler et al., 1974; Comins and Blatt, 1974; Hadeler and Rothe, 1975; Chow and Tam, 1976; Freedman and Waltman, 1977; Gopalsamy, 1977; Rosen, 1977; McMurtrie, 1978; Caisson, 1978; Fife, 1979; Okubo, 1980; Cohen and Murray, 1981; Nallaswamy and Shukla, 1982; Cosner and Laser, 1984; Bergerud et al., 1984; Anderson and Arthur, 1985; Freedman et al., 1986; Takeuchi, 1986; Bergerud and Page, 1987; Freedman, 1987; Cantrell and Cosner, 1987, 1989; Freedman and Shukla, 1989; Shukla et al., 1989; Freedman and Wu, 1992; Angulo and Linares, 1995). It has been pointed out that an unstable equilibrium state may become stable with dispersion under certain conditions (Levins and Culver, 1971; Smith, 1972; Gopalsamy, 1977). The importance of density dependent dispersal coefficients in the case of single species model has also been studied (Gurney and Nisbet, 1975).

The evolution of interacting species in a certain environment depends on the nature of their interactions, the age structure, the size of the habitat and the environmental gradients which might induce the convective and dispersive migrations in the species. In recent years, the effects of environmental gradients on the interacting species have been studied by taking dispersion into account (Levins and Culver, 1971; Vandermeer, 1973; May, 1974; Roff, 1974; Chewning, 1975; Gurtin and MacCamy, 1977). McMurtric (1978) surveyed the effects of diffusion on some prey-predator systems, and it has been noted that diffusion of interacting species stabilized the otherwise unstable equilibrium states (Levins and Culver, 1971; Smith, 1972; Vandermeer, 1973; May, 1974; Roff, 1974). However this is not always true, and in certain cases diffusion can make a stable equilibrium state into an unstable one (Segel and Jackson, 1972; Levin, 1974; Chewning, 1975). This case is known as diffusive instability which may not be a rare event specially in prey-predator systems (Levin, 1976; Casten and Holland 1978; Wollkind et al., 1991; Timm and Okubo, 1992; Chattopadhyay et al., 1996; Raichaudhury et al., 1996). But, this analysis is applicable only to systems with unbounded domain. In fact, the boundedness of the domain and the nonlinearity cannot be negligible. In model with reservoir type boundary condition proposed by Gopalsamy (1977), boundedness of the domain is necessary for the coexistence of competing species, which is unstable without diffusion. Moreover, Levin (1974) showed boundedness of the domain and nonlinearity are requisite for the coexistence of the competing species.

In this thesis we have noted the stabilizing effect of diffusion on the system. It has been shown that an unstable equilibrium can be made stable by increasing diffusion coefficients to sufficiently large values.

1.3 Objectives of the Thesis

The main objective of this thesis is to study the survival of biological species dependent on resource, which is being depleted due to industrialization and pollution, using mathematical modelling. Specifically the following types of problems have been proposed and analysed in this thesis using mathematical models.

1. Allelopathic effect on two competing plant species.

2. Survival of species dependent on resource in industrial and polluted environments.

3. Effect of time delay on the depletion of forestry resources and their conservation.

4. Effect of pollutants formed by precursors in the atmosphere on population.

In the following we give an overview of the relevant literature so that the research work carried out in the thesis related to above mentioned problems can be seen in its proper perspective.

1.3.1 Allelopathic effect on competing plant species

The discovery that many plants and some animals contain or secrete chemicals injurious to competitors or natural enemies has led to development of the study of allelopathy; the chemicals are called allelochemics or allelochemicals. This phenomenon - the suppression of some higher plants by chemicals released by another higher plant has been extended to include chemical defenses of plants against herbivores, phytophagous insects against predators, and the resistance of hosts to parasitoids.

Two types of allelopathy are distinguished: (1) the production and release of an allelochemical by one species inhibiting the growth of only other adjacent species, which may confer competitive advantage for the allelopathic species; and (2) autoallelopathy, in which both the species producing the allelochemical and unrelated species are indiscriminately affected.

Examples of plant-to-plant antibiosis based on allelochemicals include the chaparral plants, whose toxic phenolic secretions are washed by rains into the soil, where they inhibit the germination and growth of herb seeds close enough to provide competition. The black walnut tree (*Juglans nigra*) produces a potent allelochemical, juglone (5-hydroxynaphthoquinone) that inhibits many annual herbs. Tomato and alfalfa under or near black walnut trees wilt and die. For plants growing in habitats with extreme climates, such as a desert, competition for the limited resources is critical, and allelopathy may have survival value. Desert shrubs are often surrounded by a bare zone; thus, all the moisture of that zone remains available to the shrub and is not shared with other plants. In the Mojave Desert of California incienso (*Encelia farinosa*) inhibits the growth of desert annuals. From the decomposing leaf litter, 5-

acetyl-1-2-methoxybenzaldehyde is released and persists in the desert soil, functioning as an allelochemical. Incienso is apparently not affected by its own toxin (Rice, 1984; Thompson, 1985; Putnam and Tang, 1986; Waller, 1987).

To study such type of interactions among biological species using mathematical modelling, Maynard Smith (1974) proposed a mathematical model in which he considered two competing species and assumed that each species produces a substance toxic to the other, but only when the other is present. Then Chattopadhyay (1996) analysed the above model under the same assumptions. He considered linear growth rates of the two competing species and their carrying capacities as constants. He showed that stability of the system depends upon the ratio of the two toxicants.

In view of the above in chapter 2, we have proposed and analysed a mathematical model to study the allelopathic effect on two competing plant species. The growth rates and carrying capacities of the competing species are taken as nonlinear functions. Further, the effect of diffusion is also incorporated in the model.

1.3.2 Survival of species dependent on resource in industrial and polluted environments

The rapid industrialization, rising population and increasing energy requirements have caused a great concern to mankind. The depletion of various resources such as forestry biomass, oil and natural gas, fisheries, fertile topsoil, minerals etc. due to their over exploitation at an alarming rate has caused a great concern in both developed and developing countries. It is, therefore, important to study the effects of industrialization and environmental pollution on ecosystem so that appropriate measures for conservation of resources and to control the environmental pollution are taken and the desired level of the resource biomass can be maintained.

Some investigation have been made to study the effect of pollutants on biological species using mathematical models (Hallam and Clark, 1982; Hallam et. al., 1983; Hallam and De Luna, 1984; De Luna and Hallam, 1987; Freedman and Shukla, 1991; Huaping and Ma, 1991). In particular, Hallam et. al. (1983b) studied the effects of toxicant on a directly exposed population using mathematical modelling. Hallam and De Luna (1984) further proposed a model and discussed the effects of a toxicant on a population when exposed via environmental and food chain pathways. They focused mainly on effects of the toxicant on a population and found persistence and extinction criteria. De Luna and Hallam (1987) also proposed and analysed a mathematical model to study the effect of a toxicant on population and showed that if the population exhibits a potential for growth and if there is a input of resource, then the population will persist. Shukla et. al. (1989) proposed a mathematical model to study the cumulative effect of industrialization and pollution on depletion of resources and have shown that if the pressures of industrialization and population increase without control, the resource will not last long. However, if appropriate measures for conservation are taken, the resources can be maintained at a desired level even under the sustained pressure of industrialization and population. Huaping and Ma (1991) proposed a mathematical model to study the effects of toxicants on naturally stable two species communities. They studied the persistence-extinction thresholds for populations in toxicant stressed Lotka-Volterra model of two interacting species. In the above investigations, the growth rate of population density depends linearly upon the concentration of toxicant in the population and the effect of environmental concentration of toxicant on the carrying capacity of the population has not been considered.

It may be noted here that in the above studies the concentration of toxicant was defined with respect to the biomass of the total population. Freedman and Shukla (1991), however, felt that if the biomass of the population, toxicant uptaken by the population and toxicant in the environment are defined with respect to mass or volume of the total environment in which the population lives, the model becomes more visible. Keeping this in view, Freedman and Shukla (1991) proposed models to study the effect of a single toxicant on single-species and predator-prey systems. In case of single species growth they found conditions for local as well as global stability and in case of predator-prey systems, they determined the existence of steady states for a small constant influx of toxicant. Chattopadhyay (1996) proposed a model to study the effect of toxic substances on a two-species competitive system. He considered the linear growth rate of the competing species and their carrying capacities as constants. Shukla and Dubey (1996a) studied the effect of two toxicants, when one is more toxic than the other, on the growth and survival of a biological species. Shukla and Dubey (1997) studied the depletion of resources in a forest habitat due to the increase of both population and pollution. Dubey (1997a) proposed a mathematical model to study the depletion and conservation of forestry resources which is affected by a toxicant. Dubey (1997b) investigated a mathematical model in which two species share a common resource, and one of the species is itself an alternative food for the other. But in the above investigation the survival of the species population dependent on resource which is affected by a toxicant has not been considered.

Keeping in view the above literature survey, in Chapter 3, we have proposed and analysed a mathematical model to study the survival of a single species population dependent on resource which is affected by a pollutant present in the environment. It is assumed that the population depends partially or wholly on the resource or just predating on the resource.

Chapter 4 of this thesis is devoted to study the survival of two biological species competing for a single resource under industrialization pressure with and without diffusion.

Chapter 5 of this thesis deals with the interaction of two biological species in a polluted environment. Three types of interaction between the two species have been considered, namely, competition, cooperation and predator-prey. The effect of diffusion on the system is also studied.

Chapter 6 of this thesis is devoted to study the effects of industrialization and pollution on forestry resources in a diffusive system.

1.3.3 Effect of time delay on the depletion of forestry resources and their conservation

Time delay systems are those systems in which time delays exist between the application of input or control to the system and their resulting effect on it. They arise either as a result of inherent delays in the components of the system or as a deliberate introduction of time delay into the system for control purposes. Time delays occur in various systems including biological and chemical systems. The mathematical formulation of a time delay system results in a system of delay-differential equations. A particular class of these equations, the integro-differential equations, was first studied by Volterra (1959) who developed a theory for them and investigated time delay phenomena in different systems. Others have made significant contributions to the development of the general theory of functional differential equations of Volterra type (Krasovskii, 1957; Driver, 1961, 1962; Hale, 1961, 1962, 1963, 1964; Lakshmikantham, 1962, 1964, 1987).

Several investigations related to ecological models with delay effects can be found in the literature (Wangersky and Cunningham, 1957; Caswell, 1972; May, 1973; Cushing, 1976; McDonald, 1976, 1977, 1978; Brauer, 1978; Leung, 1979; Burton, 1983; Freedman and Rao, 1983; Erbe et al., 1986; Freedman and Gopalsamy, 1986; Stepan, 1986; Leung and Zhou, 1988; Rao and Sivasundaram, 1988; Gyori and ladas, 1991; Gopalsamy, 1992; Rao and Pal, 1992; Murakami and Hamaya, 1995; Cavani and Avis, 1995; Wang and Yi, 1995; Dubey, 1997c). In particular, Rao and Pal (1992) proposed and analysed a general model for grazing a grassland on the pattern of a prey-predator system by considering the effect of delay in the growth rate of a cattle population. They discussed linear and nonlinear systems and found sufficient conditions for asymptotic stability of a positive equilibrium of these systems. Wang and Yi (1995) studied the global asymptotic stability of Volterra-Lotka systems with infinite delay together with global exponential stability of Volterra-Lotka systems with bounded delay. Criteria for stability are also obtained. Dubey (1997c) proposed a mathematical model with delay to study the cumulative effect of industrialization and population on the degradation of forestry resources. He obtained criteria for local stability, instability and global stability of the system and showed that it is worthwhile to incorporate the time delay factor for the friendly technology of industrialization dependent on forestry resources.

The effect of time delay on depletion of forestry resources in a polluted environment does not appear in the above investigations. In chapter 7, we therefore, propose and analyse a mathematical model to study the effect of environmental pollution on forestry resource biomass with time delay in a diffusive system.

1.3.4 Effect of pollutants formed by precursors in the atmosphere on population

The menace of environmental pollution is well known. As pointed out in section 1.3.2, some investigations have been conducted to study the effect of environmental pollution on biological species using mathematical modelling. But in these studies, the role of a precursor pollutant has not been taken into account. However, some attempts have been made to study the effect of a precursor pollutant (Rescigno and Richardson, 1967; Forrester, 1971; Meadows, 1972; Borsillino and Torre, 1974; Resigno, 1977). In particular, Rescigno (1977) studied the general properties of the equations describing a single species living in a limited environment in the presence of its own pollutant. The effect of pollutants formed by precursors in the atmosphere on population with diffusion does not appear in the above investigations. In chapter 8, therefore, we propose and analyse a mathematical model to study the effect of a pollutant on a population which is living in an environment polluted by its own activities. Effect of diffusion is also incorporated in the model.

1.4 Mathematical tools used in the Thesis

In this thesis the following two methods have been used to analyse the mathematical models.

1.4.1 The method of Characteristic roots

The conclusion regarding asymptotic stability of the systems depend on the eigenvalues of the variational matrix, a Jacobian matrix of first order derivatives of interaction-functions. As this Jacobian is determined by Taylor expansion of the interaction-functions and neglecting nonlinear higher order terms, this method studies only the local stability of the system in the neighbourhood of its equilibrium state. Routh-Hurwitz criterion (Sanchetz, 1968) and Gershgorin's theorem (Lancaster and Tismanetsky, 1985) are very useful to study the local stability of wide range of systems in homogeneous environments. This method establishes stability only relative to small perturbations of the initial state. Hence it is called local stability. An eigenvalue analysis is only a small initial step in understanding the dynamical behavior of an ecosystem model.

1.4.2 Liapunov's Direct Method

In the previous section, we have described methods which are mainly related to the study the linearized version of nonlinear models. But to get the real insight of problems, the nonlinear system as a whole must be investigated. In the real world ecosystems are subjected to large perturbations of the initial state and system dynamics. The most powerful analytical method for studying stability relative to finite perturbations of the initial state of an ecosystem model is the direct method of Liapunov (LaSalle and Lefschetz, 1961; Rao, 1981). This method requires the construction of certain functions called Liapunov functions. For a physical system the direct method of Liapunov generalizes the principle that a system, which continuously dissipates energy until it attains an equilibrium, is stable. The two basic theorems on stability can be found in La Salle and Lefschetz (1961). This method has also been used even to study the linear stability of the equilibrium state of interacting systems (Gatto and Rinaldi, 1977).

In population dynamics, to study the nonlinear stability of the equilibrium state Liapunov's second method has been used by several investigators (Gilpin, 1974; Goh, 1976, 1977; Jorne, 1977; Jorne and Carmi, 1977; Goh, 1978; Hastings, 1978; Hsu, 1978; Takeuchi et al., 1978; Harrison 1979; Goh, 1980; Shukla et al., 1981). In particular, the nonlinear stability of diffusive Lotka Volterra system has been carried out by Jorne and Carmi (1977), Gopalsamy and Aggarwalla (1980) and it has been shown that the otherwise stable system remains stable with positive dispersion coefficients under zero flux boundary conditions. Harrison (1979) has given a Liapunov function which generalizes the functions used by Goh (1976, 1977) and Hsu (1978) and can be used to study the nonlinear stability of various types of models even with functional response.

1.5 Summary of the Thesis

The thesis consists of eight chapters.

Chapter 1 contains a general introduction with relevant literature which provides a necessary background required for the forthcoming chapters.

In chapter 2, a mathematical model has been proposed and analysed to study the interaction of two plant species competing for nutrients. It has been assumed that each plant species produces a toxicant, which reaches to the other through diffusive process and affects its growth.

In the case of no diffusion, it has been shown that the two competing plant species settle down to their respective equilibrium levels, the magnitude of which are lower than their corresponding initial density independent carrying capacities. In the case when the two plant species do not produce any toxicant, it has been shown that the two plant species again settle down to their respective equilibrium levels, the magnitudes of which are higher than their corresponding values in the case when they produce toxicant. To illustrate the above facts a numerical example has also been presented in this chapter. It has also been found that the rate of decrease in the growth rates of plant species is faster in the case when each plant species produces a substance toxic to the other.

By incorporating diffusion in the system it has been shown that diffusion is playing the general role of stabilizing the system. It has been shown that if the interior equilibrium of the system with no diffusion is globally asymptotically stable, then the corresponding uniform steady state of the system with diffusion must be globally asymptotically stable. Further, an unstable steady state in the absence of diffusion can be made stable by increasing diffusion coefficients sufficiently large. In a particular case of rectangular habitat it has been shown that stability is more plausible in the case of diffusion.

In chapter 3, a mathematical model for the survival of a single-species population dependent on resource biomass which is affected by a pollutant present in the environment has been proposed and analysed. The rate of introduction of pollutant into the environment has been considered to be constant, instantaneous or periodic. It has been assumed that the population depends partially or wholly on the resource or just predating on the resource. It has also been assumed that the growth rate of the population increases as the density of the resource biomass increases while its carrying capacity increases with the increase in the density of the resource biomass, and decreases with the increase in the environmental concentration of the pollutant. It has been further assumed that the growth rate of the resource biomass decreases as the uptake concentration of the pollutant and the density of the population increase while its carrying capacity decreases as the environmental concentration of the pollutant increases.

In the case of no diffusion the model has been analysed using stability theory of ordinary differential equations. When the population depends partially on the resource, it has been shown that in the case of constant introduction of pollutant into the environment, both the population and the resource biomass settle down to their respective steady states. The magnitude of the equilibrium level of the population decreases as the equilibrium level of the resource biomass density decreases and the environmental concentration of the pollutant increases. The magnitude of the equilibrium level of the resource biomass decreases as the equilibrium levels of the population, the pollutant present in the environment and in the body increase. It has also been noted that the resource biomass may tend to zero for large influx of the pollutant into the environment affecting the survival of the species. In the case of instantaneous introduction of toxicant into the environment similar results have been found. In particular, it has been noted that the population and the resource biomass after initial decrease in their densities will settle down to their respective steady states but after a long time if the washout rate of the pollutant is small. In this case magnitudes of densities of the population and the resource biomass are larger than their respective densities in the case of constant introduction of pollutant. In the case of the periodic emission of the pollutant into the environment it has been found that a periodic behavior occurs in the system for a small amplitude of the influx of the pollutant.

The equilibrium levels of the population and the resource biomass have been compared in three different cases: (1) when the population partially depends upon the resource, (2) when the population wholly depends upon the resource, and (3) when the population is predating on the resource. It has been noted that the density of the population is maximum in the partially dependent case and minimum in the predating case, consequently the density of the resource biomass is minimum in the partially dependent case and maximum in the predating case, keeping other parameters same in the system. Thus, an increase in the density of the population will also lead to decrease in the density of the resource biomass. It has also been noted that the survival of the population will be threatened even in the partially dependent case if the continuous emission of pollutant into the environment is not controlled. In the wholly dependent case the population will doom to extinction if the environmental concentration of pollutant reaches at a threshold value. In case of predation it has been noted that the survival of the population is highly threatened.

In the case of diffusion, a complete analysis of the model has been carried out. It has been shown that if the positive equilibrium of the system with no diffusion is globally asymptotically stable, then it remains globally asymptotically stable in the case of diffusion. Further, if the positive equilibrium of the system with no diffusion is unstable, then the unstable equilibrium can be stabilized by increasing diffusion coefficients to sufficiently large values. Thus, it has been concluded that in the case of diffusion, solutions of the system approaches to the equilibrium state faster than the case of no diffusion.

A model to conserve the resource biomass and to control the undesired level of environmental pollutants has also been proposed and analysed. It has been shown that if suitable efforts are made, an appropriate level of the resource biomass can be maintained.

In chapter 4, a mathematical model has been proposed and analysed to study the survival of two biological species competing for a single resource under industrialization pressure with and without diffusion. The competing species are assumed to be either partially dependent, wholly dependent or predating on the resource. In the partially dependent case, criteria for survival and extinction of competing species have been derived. It has also been shown that the resource biomass settles down to its equilibrium level, the magnitude of which depends upon the equilibrium levels of the competing species and the industrialization pressure. This magnitude decreases as the densities of the competing species and the pressure due to industrialization increase and may driven to extinction if these factors increase without control. It has also been noted that the competing species may coexist even in the absence of the resource biomass in the partially dependent case, whereas in the wholly dependent case the two species will die out in the absence of the resource biomass. In the case when the competing species are predating on the resource, similar results have been found. It has been noted that the damage of the resource biomass density is maximum in partially dependent case, and is minimum in the predation case. This has also been established by numerical examples.

A model to study the effect of diffusion on the system under consideration has also been proposed and analysed. It has been found that diffusion has stabilizing effect on the system.

By analysing the conservation model it has been shown that if suitable efforts are made to conserve the resource biomass and to control the undesired level of the industrialization pressure, a desired level of the resource biomass can be maintained and the survival of the competing species may be ensured.

In chapter 5, a mathematical model has been proposed and analysed to study the survival of two interacting species in a polluted environment, the mode of interaction being competition, cooperation and predation. The model has been analysed with and without diffusion. When there is no diffusion it has been shown that in the case of constant introduction of pollutant into the environment the competing species settle down to their respective equilibrium levels, the magnitude of which depends upon the equilibrium levels of washout and uptake rates of pollutant. It has also been noted that if the concentration of pollutant increase unabatedly, then the survival of the species would be threatened. In the case of instantaneous introduction of pollutant into the environment, it has been found that the competing species again settle down to their respective equilibrium levels whose magnitude is higher than the case of constant introduction of pollutant into the environment. In case of periodic emission of pollutant into the environment, it has been found that a periodic influx of pollutant with small amplitude causes a periodic behaviour in the system.

The effect of diffusion on the interior equilibrium state of the system has also been investigated. It has been shown that if the positive equilibrium of the system without diffusion is globally asymptotically stable, then the corresponding uniform steady state of the system with diffusion is also globally asymptotically stable. It has further been noted that if the positive equilibrium of the system with no diffusion is unstable, then the corresponding uniform steady state of the system with diffusion can be made stable by increasing diffusion coefficients to sufficiently large values.

A model to control the undesired level of environmental pollutants has been proposed and analysed. It has been shown that the existence of the two interacting biological species can be ensured if the undesired level of the environmental concentration of pollutant is controlled by some mechanism.

In chapter 6, a mathematical model has been proposed and analysed to study the effects of industrialization and pollution on forestry resources with diffusion. The rate of introduction of pollutant into the environment is considered to be industrialization dependent, constant, zero or periodic. The model has been analysed with and without diffusion.

When there is no diffusion in the system, it has been shown that in the case of industrialization dependent introduction of pollutant into the environment the resource biomass settles down to its equilibrium level, whose magnitude depends upon the equilibrium level of industrialization, influx and washout rates of pollutant present in the environment. The magnitude of the resource biomass density decreases as the density of industrialization and influx rate of pollutant increase, and even it may tend to zero if these factors increase without control. In the case of constant introduction of pollutant, similar results have been found. In the case of instantaneous spill of pollutant into the environment, it has been noted that the pollutant may be washed out conmpeletely and the resource biomass may settle down to a lower equilibrium level than its original carrying capacity whose magnitude depends only upon the equilibrium level of the industrialization pressure. Even in this case the resource biomass may vanish if industrialization pressure increase unabatedly. In the case of periodic emission of pollutant into the environment it has been found that a small periodic influx of pollutant causes a periodic behaviour in the system.

Analysing the model with diffusion it has been shown that diffusion has a stabilizing effect on the system. It has been concluded that solutions of the system with diffusion converge towards its equilibrium state faster than the case of no diffusion.

A mathematical model to conserve the resource biomass by plantation, irrigation, fencing, fertilization etc., and to control the undesired levels of industrialization pressure and concentration of pollutant in the environment by some mechanisms has also been proposed. By analysing this model it has been shown that if suitable efforts are made, an appropriate level of resource biomass density can be maintained.

In chapter 7, a mathematical model has been proposed and analysed to study the effect of environmental pollution on forestry resource biomass with time delay. It has been considered that the environmental pollutant does not affect the forestry resource biomass directly, but the pollutant after entering into the biomass gets converted to a substance that is toxic to resource biomass, and consequently the growth rate of the resource biomass decreases. This conversion causes a time delay in the depletion of forest biomass. The model has been analysed with and without diffusion. When there is no diffusion it has been shown that in the case of constant emission of pollutant into the environment the resource biomass settles down to its equilibrium level, the magnitude of which depends upon the washout and uptake rates of pollutant. It has further been noted that if the concentration of pollutant increases unabatedly, the density of the resource biomass may tend to zero. The effect of time delay due to the formation of the chemical pollutants on decreasing the equilibrium level of resource biomass is determined by the rate of formation of the chemical pollutants and the depletion of the resource biomass. If the delay in formation of the pollutant is large, then this may help in reducing over all effect of the pollutant provided other parameters remain same.

By analysing the diffusion model it has been shown that an unstable steady state can be made stable by increasing diffusion coefficients to sufficiently large values. It has been noted that in the case of diffusion the resource biomass converges towards its carrying capacity faster than the case of no diffusion.

A conservation model has also been proposed and analysed. It has been shown that if suitable efforts are adopted to conserve the resource biomass and to control the undesired level of environmental concentration of pollutant, the forestry resource biomass can be maintained at an appropriate level.

In chapter 8, a mathematical model is proposed and analysed to study the effect of a pollutant on a population which is living in an environment polluted by its own activities. It has been assumed that the pollutant enters into the environment not directly, but by a precursor produced by the population itself. It has been considered that the larger the population, the faster the precursor is produced, and the larger the precursor, the faster the pollutant is produced. The model has been studied with and without diffusion. In case of no diffusion it has been shown that population density settles down to its equilibrium level, the magnitude of which depends upon the equilibrium levels of emission and washout rates of environmental pollutant as well as on the rate of precursor formation and its depletion. It has been noted that the rate of precursor formation is crucial in affecting the population. It has further been noted that if the concentration of pollutant increase unabatedly, the survival of the population would be threatened.

The effect of diffusion on the interior equilibrium of the system has also been investigated. It has been found that global stability is more plausible in the case of diffusion than the case of no diffusion.

By analysing conservation model it has been shown that if the formation of the precursor pollutant is controlled by some external means, its affect on the population can be minimised.

It is hoped that the models investigated in this thesis will be fruitful in developing environment friendly technology for industrialization, methods for control of pollution and conservation of resources. The work carried out here will also serve as a basis for further study of a very important problem of pollution and its effect on ecosystem.

Chapter 2

A MODEL FOR THE ALLELOPATHIC EFFECT ON TWO COMPETING SPECIES

2.1 Introduction

The decline in the growth rate of biological species is a major cause of concern in both developed and developing countries due to rapid pace of industrialization and associated pollution. In recent decades, some investigations have been made to study the effect of toxicant on biological species using mathematical models (Hallam et. al., 1983; Hallam and De Luna, 1984; De Luna and Hallam, 1987; Freedman and Shukla, 1991; Huaping and Ma, 1991; Shukla and Dubey, 1996a; Chattopadhyay, 1996; Dubey, 1997a; Shukla and Dubey, 1997). In particular, Freedman and Shukla (1991) studied the effect of toxicant in a single-species and predator-prey system. In the case of single species growth they obtained local and global dynamics of the system, and in the case of predator-prey system, they investigated the existence of steady states for a small influx of toxicant. Huaping and Ma (1991) studied the effects of toxicant on naturally stable two-species communities and obtained persistence-extinction thresholds for the species. Shukla and Dubey (1996a) studied the effect of two toxicants, one being more toxic than the other, on the growth and survival of a biological species. Chattopadhyay (1996) proposed a model to study the effect of toxic substances on a two-species competitive system. He considered the linear growth rate of the competing species and their carrying capacities as constants. Dubey (1997a) proposed a mathematical model to study the depletion and conservation of forestry resources which is affected by a toxicant. Shukla and Dubey (1997) investigated the depletion of resources in a forest habitat due to the increase of both population and pollution. In the above studies the allelopathic effect of toxicant with diffusion on two plant species has not been considered. Keeping the above in view in this chapter we propose a mathematical model to study the allelopathic effect on two competing plant species in which growth rates and carrying capacities of the competing species are taken as nonlinear functions. Further, the effect of diffusion is also incorporated in the model. In the case of diffusion our results agree with those in Shukla and Verma (1981), Hastings (1982), Shukla and Shukla (1982), Freedman and Shukla (1989), Dubey and Das (1999). Stability theory of differential equations is used to analyse the model (La Salle and Lefschetz, 1961).

We assume that all the functions utilized in the model are sufficiently smooth so that solutions to the initial-boundary value problems exist uniquely and are continuous for all positive time. Where there is no confusion, the prime denotes the derivative of a function with respect to its arguments.

2.2 Mathematical Model

Consider an ecosystem where we wish to model the interaction of two plant species competing for survival in a closed region D with smooth boundary ∂D. We also consider the allelopathic effect on the model where each species produces a different toxicant, the concentration of which is a function of its own density. In particular, it may be taken as proportional to its own density. It is further assumed that the toxicant produced by one species decreases the growth rate of the other. The dynamics of the system may be governed by the following autonomous differential equations:

$$\begin{aligned}
\frac{\partial N_1}{\partial t} &= N_1 r_1(N_2) - \frac{r_{10}N_1^2}{K_1(N_2)} - \beta_{12}N_1T_2,\\
\frac{\partial N_2}{\partial t} &= N_2 r_2(N_1) - \frac{r_{20}N_2^2}{K_2(N_1)} - \beta_{21}N_2T_1,\\
\frac{\partial T_1}{\partial t} &= \alpha_1 N_1 - \alpha_0 T_1 + D_1\nabla^2 T_1,\\
\frac{\partial T_2}{\partial t} &= \beta_1 N_2 - \beta_0 T_2 + D_2\nabla^2 T_2.
\end{aligned} \tag{2.1}$$

We impose the following initial and boundary conditions on the system:

$$\begin{aligned}
&N_1(x,y,0) = \phi(x,y) \geq 0,\ N_2(x,y,0) = \psi(x,y) \geq 0,\\
&T_1(x,y,0) = \xi(x,y) \geq 0,\ T_2(x,y,0) = \chi(x,y) \geq 0,\ (x,y)\epsilon D\\
&\frac{\partial T_1}{\partial n} = \frac{\partial T_2}{\partial n} = 0,\ (x,y)\ \epsilon\ \partial D, t \geq 0,
\end{aligned} \tag{2.2}$$

where n is the unit outward normal to ∂D.

In model (2.1), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian diffusion operator. $N_1(x,y,t)$ and $N_2(x,y,t)$ are the densities of the two species at coordinates $(x,y)\epsilon D$ and time $t \geq 0$. $T_1(x,y,t)$ is the concentration of the toxicant produced by the species N_1, which is toxic to the species N_2. $T_2(x,y,t)$ is the concentration of the toxicant produced by the species N_2, which is toxic to the species N_1. D_1 and D_2 are the diffusion rate coefficients of T_1 and T_2 respectively in D.

The functions $r_1(N_2)$ and $r_2(N_1)$ are the specific growth rates of the species of densities N_1 and N_2 respectively. Since the two species are competing with each other, hence $r_1(N_2)$ and $r_2(N_1)$ are decreasing functions of their arguments, i.e.,

$$\begin{aligned}
&r_1(0) = r_{10} > 0,\ r_1'(N_2)\ <\ 0\ for\ N_2 \geq 0,\\
&and\\
&r_2(0) = r_{20} > 0,\ r_2'(N_1)\ <\ 0\ for\ N_1 \geq 0.
\end{aligned} \tag{2.3}$$

The functions $K_1(N_2)$ and $K_2(N_1)$ are the maximum densities of N_1 and N_2 respectively which the environment can support. $K_1(N_2)$ and $K_2(N_1)$ are decreasing functions of

N_2 and N_1 respectively, i.e.,

$$K_1(0) = K_{10} > 0, K_1'(N_2) < 0 \ for \ N_2 \geq 0, K_1(N_{2a}) = 0 \ for \ some \ N_2 = N_{2a} > 0,$$

$$and \tag{2.4}$$

$$K_2(0) = K_{20} > 0, K_2'(N_1) < 0 \ for \ N_1 \geq 0, K_2(N_{1a}) = 0 \ for \ some \ N_1 = N_{1a} > 0.$$

In model (2.1), β_{12} and β_{21} are the depletion rate coefficients of N_1 and N_2 respectively due to toxicant produced by N_2 and N_1 respectively. α_1 and β_1 are the growth rate coefficients of T_1 and T_2 respectively and α_0 and β_0 are their respective natural depletion rate coefficients.

2.3 Model Without Diffusion

In this case we take $D_1 = D_2 = 0$ in model (2.1). Then the model reduces to

$$\begin{aligned}
\frac{dN_1}{dt} &= N_1 r_1(N_2) - \frac{r_{10}N_1^2}{K_1(N_2)} - \beta_{12}N_1T_2, \\
\frac{dN_2}{dt} &= N_2 r_2(N_1) - \frac{r_{20}N_2^2}{K_2(N_1)} - \beta_{21}N_2T_1, \\
\frac{dT_1}{dt} &= \alpha_1 N_1 - \alpha_0 T_1, \\
\frac{dT_2}{dt} &= \beta_1 N_2 - \beta_0 T_2. \\
N_1(0) &\geq 0, \ N_2(0) \geq 0, \ T_1(0) \geq 0, \ T_2(0) \geq 0,
\end{aligned} \tag{2.5}$$

It can be checked that model (2.5) has six non negative equilibria, namely, $E_0(0,0,0,0)$, $E_1(K_{10},0,0,0)$, $E_2(0,K_{20},0,0)$, $E_3(K_{10},0,\frac{\alpha_1 K_{10}}{\alpha_0},0)$, $E_4(0,K_{20},0,\frac{\beta_1 K_{20}}{\beta_0})$ and $E^*(N_1^*,N_2^*,T_1^*,T_2^*)$. The equilibria $E_0 - E_4$ obviously exist. We shall show the existence of E^* as follows.

Existence of $E^*(N_1^*,N_2^*,T_1^*,T_2^*)$:

Here N_1^*, N_2^*, T_1^* and T_2^* are the positive solutions of the following algebraic equations:

$$r_{10}N_1 = K_1(N_2)\{r_1(N_2) - \beta_{12}T_2\}, \tag{2.6}$$

$$r_{20}N_2 = K_2(N_1)\{r_2(N_1) - \beta_{21}T_1\}, \tag{2.7}$$

$$T_1 = \frac{\alpha_1}{\alpha_0} N_1, \tag{2.8}$$

$$T_2 = \frac{\beta_1}{\beta_0} N_2. \tag{2.9}$$

It can be checked that E^* exists, provided

$$K_{10} < N_{1a} \textit{ and } K_{20} < N_{2a} \tag{2.10}$$

or

$$K_{10} > N_{1a} \textit{ and } K_{20} > N_{2a} \tag{2.11}$$

hold, otherwise E^* does not exist and if it exists, then it is not in the positive orthant.

By computing the variational matrices corresponding to each equilibrium it can be checked that E_0 is a saddle point with unstable manifold locally in the $N_1 - N_2$ plane and stable manifold locally in the $T_1 - T_2$ plane. E_1 is also a saddle point with stable manifold locally in the $N_1 - T_1 - T_2$ space and unstable manifold along the N_2 direction. E_2 is also a saddle point with stable manifold locally in the $N_2 - T_1 - T_2$ space and unstable manifold locally along the N_1 direction. E_3 is also a saddle point with stable manifold locally in the $N_1 - T_1 - T_2$ space and unstable manifold locally along the N_2 direction (Here $r_2(K_{10}) - \beta_{21}\frac{\alpha_1 K_{10}}{\alpha_0}$ is taken to be positive). E_4 is also a saddle point with stable manifold locally in the $N_2 - T_1 - T_2$ space and unstable manifold locally along the N_1 direction (Here $r_1(K_{20}) - \beta_{12}\frac{\beta_1 K_{20}}{\beta_0}$ is taken to be positive).

In the following theorem it is shown that E^* is locally asymptotically stable.

Theorem 2.3.1 *Let the following inequalities hold*

$$\{r_1'(N_2^*) + r_2'(N_1^*) + \frac{r_{10}N_1^*}{K_1^2(N_2^*)}K_1'(N_2^*) + \frac{r_{20}N_2^*}{K_2^2(N_1^*)}K_2'(N_1^*)\}^2 < \frac{4}{9}\frac{r_{10}}{K_1(N_2^*)}\frac{r_{20}}{K_2(N_1^*)}, \tag{2.12}$$

$$\beta_{21}^2 < \frac{2}{3}c_1\alpha_0\frac{r_{20}}{K_2(N_1^*)}, \tag{2.13}$$

$$\beta_{12}^2 < \frac{2}{3}c_2\beta_0\frac{r_{10}}{K_1(N_2^*)}, \tag{2.14}$$

where

$$c_1 = \frac{1}{3}\frac{\alpha_0}{\alpha_1^2}\frac{r_{10}}{K_1(N_2^*)},$$
$$c_2 = \frac{1}{3}\frac{\beta_0}{\beta_1^2}\frac{r_{20}}{K_2(N_1^*)}.$$

Then E^* is locally asymptotically stable.

Proof: We first linearize system (2.5) by taking the transformations,

$$N_1 = N_1^* + n_1,\ N_2 = N_2^* + n_2,\ T_1 = T_1^* + \tau_1,\ T_2 = T_2^* + \tau_2.$$

Then taking the following positive definite function in the linearized form of model (2.5),

$$V(n_1, n_2, \tau_1, \tau_2) = \frac{1}{2}\{\frac{n_1^2}{N_1^*} + \frac{n_2^2}{N_2^*} + c_1\tau_1^2 + c_2\tau_2^2\}$$

it can be checked that the derivative of V with respect to t is negative definite under conditions (2.12), (2.13) and (2.14), proving the theorem.

To investigate the global stability behaviour of E^* we need the following lemma which establishes a region of attraction for system (2.5). The proof of this lemma is easy hence is omitted.

Lemma 2.3.1 *The set*

$$\Omega_1 = \{(N_1, N_2, T_1, T_2) : 0 \le N_1 \le K_{10},\ 0 \le N_2 \le K_{20}, 0 \le T_1 \le \frac{\alpha_1 K_{10}}{\alpha_0},$$
$$0 \le T_2 \le \frac{\beta_1 K_{20}}{\beta_0}\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant.

In the following theorem global stability behaviour of E^* is studied.

Theorem 2.3.2 *In addition to assumptions (2.3) and (2.4), let $r_1(N_2), r_2(N_1), K_1(N_2)$ and $K_2(N_1)$ satisfy the following conditions in Ω_1*

$$0 \le -r_1'(N_2) \le \rho_1, 0 \le -r_2'(N_1) \le \rho_2, 0 \le -K_1'(N_2) \le k_1, 0 \le -K_2'(N_1) \le k_2,$$
$$K_{m1} \le K_1(N_2) \le K_{10} \text{ and } K_{m2} \le K_2(N_1) \le K_{20}, \tag{2.15}$$

for some positive constants $\rho_1, \rho_2, k_1, k_2, K_{m1}$ and K_{m2}. Let the following inequalities hold:

$$\{\rho_1 + \rho_2 + \frac{r_{10}K_{10}k_1}{K_{m1}^2)} + \frac{r_{20}K_{20}k_2}{K_{m2}^2)}\}^2 < \frac{4}{9}\frac{r_{10}}{K_1(N_2^*)}\frac{r_{20}}{K_2(N_1^*)}, \tag{2.16}$$

$$\beta_{21}^2 < \frac{2}{3}c_1\alpha_0\frac{r_{20}}{K_2(N_1^*)}, \tag{2.17}$$

$$\beta_{12}^2 < \frac{2}{3}c_2\beta_0\frac{r_{10}}{K_1(N_2^*)}, \tag{2.18}$$

where

$$c_1 = \frac{1}{3}\frac{\alpha_0}{\alpha_1^2}\frac{r_{10}}{K_1(N_2^*)},$$

$$c_2 = \frac{1}{3}\frac{\beta_0}{\beta_1^2}\frac{r_{20}}{K_2(N_1^*)}.$$

Then E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: We define the following positive definite function around E^*,

$$\begin{aligned} V_1(N_1, N_2, T_1, T_2) &= N_1 - N_1^* - N_1^* \ ln(\frac{N_1}{N_1^*}) + N_2 - N_2^* - N_2^* \ ln(\frac{N_2}{N_2^*}) \\ &\quad + \frac{1}{2}\{c_1(T_1 - T_1^*)^2 + c_2(T_2 - T_2^*)^2\}. \end{aligned} \tag{2.19}$$

Differentiating V_1 with respect to t along the solutions of system (2.5), a little algebraic manipulation yields

$$\begin{aligned} \frac{dV_1}{dt} &= -\frac{r_{10}}{K_1(N_2^*)}(N_1 - N_1^*)^2 - \frac{r_{20}}{K_2(N_1^*)}(N_2 - N_2^*)^2 - c_1\alpha_0(T_1 - T_1^*)^2 - c_2\beta_0(T_2 - T_2^*)^2 \\ &\quad + \{\eta_1(N_2) + \eta_2(N_1) - r_{10}N_1\xi_1(N_2) - r_{20}N_2\xi_2(N_1)\}(N_1 - N_1^*)(N_2 - N_2^*) \\ &\quad + c_1\alpha_1(N_1 - N_1^*)(T_1 - T_1^*) - \beta_{12}(N_1 - N_1^*)(T_2 - T_2^*) \\ &\quad - \beta_{21}(N_2 - N_2^*)(T_1 - T_1^*) + c_2\beta_1(N_2 - N_2^*)(T_2 - T_2^*), \end{aligned} \tag{2.20}$$

where

$$\eta_1(N_2) = \begin{cases} \frac{r_1(N_2) - r_1(N_2^*)}{N_2 - N_2^*}, & N_2 \neq N_2^* \\ r_1'(N_2^*), & N_2 = N_2^* \end{cases}$$

$$\eta_2(N_1) = \begin{cases} \frac{r_2(N_1)-r_2(N_1^*)}{N_1-N_1^*}, & N_1 \neq N_1^* \\ \\ r_2'(N_1^*), & N_1 = N_1^* \end{cases}$$

$$\xi_1(N_2) = \begin{cases} \{\frac{1}{K_1(N_2)} - \frac{1}{K_1(N_2^*)}\}/(N_2 - N_2^*), & N_2 \neq N_2^* \\ \\ -\frac{1}{K_1^2(N_2^*)}K_1'(N_2^*), & N_2 = N_2^* \end{cases}$$

$$\xi_2(N_1) = \begin{cases} \{\frac{1}{K_2(N_1)} - \frac{1}{K_2(N_1^*)}\}/(N_1 - N_1^*), & N_1 \neq N_1^* \\ \\ -\frac{1}{K_2^2(N_1^*)}K_2'(N_1^*), & N_1 = N_1^* \end{cases}$$

From (2.15) and the mean value theorem, we note that

$$|\eta_1(N_2)| \leq \rho_1, \ |\eta_2(N_1)| \leq \rho_2, \ |\xi_1(N_2)| \leq \frac{k_1}{K_{m1}^2} \ and \ |\xi_2(N_1)| \leq \frac{k_2}{K_{m2}^2}. \tag{2.21}$$

Now Eq. (2.20) can be written as the sum of the quadratics

$$\begin{aligned} \frac{dV_1}{dt} = & -\frac{1}{2}a_{11}(N_1 - N_1^*)^2 + a_{12}(N_1 - N_1^*)(N_2 - N_2^*) - \frac{1}{2}a_{22}(N_2 - N_2^*)^2 \\ & -\frac{1}{2}a_{11}(N_1 - N_1^*)^2 + a_{13}(N_1 - N_1^*)(T_1 - T_1^*) - \frac{1}{2}a_{33}(T_1 - T_1^*)^2 \\ & -\frac{1}{2}a_{11}(N_1 - N_1^*)^2 + a_{14}(N_1 - N_1^*)(T_2 - T_2^*) - \frac{1}{2}a_{44}(T_2 - T_2^*)^2 \\ & -\frac{1}{2}a_{22}(N_2 - N_2^*)^2 + a_{23}(N_2 - N_2^*)(T_1 - T_1^*) - \frac{1}{2}a_{33}(T_1 - T_1^*)^2 \\ & -\frac{1}{2}a_{22}(N_2 - N_2^*)^2 + a_{24}(N_2 - N_2^*)(T_2 - T_2^*) - \frac{1}{2}a_{44}(T_2 - T_2^*)^2, \end{aligned}$$

where

$$a_{11} = \frac{2}{3}\frac{r_{10}}{K_1(N_2^*)}, \ a_{22} = \frac{2}{3}\frac{r_{20}}{K_2(N_1^*)}, \ a_{33} = c_1\alpha_0, \ a_{44} = c_2\beta_0,$$
$$a_{12} = \eta_1(N_2) + \eta_2(N_1) - r_{10}N_1\xi_1(N_2) - r_{20}N_2\xi_2(N_1),$$
$$a_{13} = c_1\alpha_1, \ a_{14} = -\beta_{12}, \ a_{23} = -\beta_{21}, \ and \ a_{24} = c_2\beta_1.$$

Sufficient conditions for $\frac{dV_1}{dt}$ to be negative definite are that the following conditions hold:

$$a_{12}^2 < a_{11}a_{22}, \tag{2.22}$$

$$a_{13}^2 < a_{11}a_{33}, \tag{2.23}$$

$$a_{14}^2 < a_{11}a_{44}, \tag{2.24}$$

$$a_{23}^2 < a_{22}a_{33}, \tag{2.25}$$

$$a_{24}^2 < a_{22}a_{44}. \tag{2.26}$$

By choosing

$$c_1 = \frac{1}{3}\frac{\alpha_0}{\alpha_1^2}\frac{r_{10}}{K_1(N_2^*)} \; and \; c_2 = \frac{1}{3}\frac{\beta_0}{\beta_1^2}\frac{r_{20}}{K_2(N_1^*)}$$

we note that Eqs. (2.23) and (2.26) are satisfied automatically. We also note that (2.16) $\Rightarrow$ (2.22), (2.17) $\Rightarrow$ (2.24) and (2.18) $\Rightarrow$ (2.25). Hence V_1 is a Liapunov function (La Salle and Lefschetz, 1961) with respect to E^* whose domain contains the region of attraction Ω_1, proving the theorem.

It is interesting to note here that after linearizing the conditions (2.22), (2.24) and (2.25), we get conditions (2.12), (2.13) and (2.14) respectively, as expected.

The above analysis shows that in the absence of diffusion the competing species settle down to their respective equilibrium levels under conditions (2.16)-(2.18). The magnitude of each species depends upon the equilibrium level of other species and on the concentration of toxicant produced by the other species, and it is lower than its initial density independent carrying capacity. It may be noted here that if the competing species of density N_1 reaches to critical level $N_1 = N_{1a}$, then the other competitor becomes extinct, and if the competing species of density N_2 reaches to a critical level $N_2 = N_{2a}$, then the first competitor becomes extinct. Further, both competing species survive under parametric condition (2.10) or (2.11).

2.4 Special Case: When the plant species do not produce any toxicant

In this case, model (2.5) reduces to

$$\begin{aligned} \frac{dN_1}{dt} &= N_1 r_1(N_2) - \frac{r_{10} N_1^2}{K_1(N_2)}, \\ \frac{dN_2}{dt} &= N_2 r_2(N_1) - \frac{r_{20} N_2^2}{K_2(N_1)}, \\ N_1(0) &\geq 0, \ N_2(0) \geq 0. \end{aligned} \tag{2.27}$$

It can be checked that model (2.27) has four nonnegative equilibria, namely, $\bar{E}_0(0,0)$, $\bar{E}_1(K_{10},0)$, $\bar{E}_2(0,K_{20})$ and $\bar{E}(\bar{N}_1,\bar{N}_2)$. The equilibria $\bar{E}_0$, $\bar{E}_1$ and $\bar{E}_2$ obviously exist. In $\bar{E}$, we note that $\bar{N}_1$ and $\bar{N}_2$ are the positive solutions of the following algebraic equations:

$$r_{10} N_1 = r_1(N_2) K_1(N_2), \tag{2.28}$$

$$r_{20} N_2 = r_2(N_1) K_2(N_1). \tag{2.29}$$

It can be checked that $\bar{E}$ exists, provided condition (2.10) or (2.11) is satisfied, otherwise $\bar{E}$ does not exist and if it exists, then it is not in the positive quadrant.

By computing the variational matrix corresponding to each equilibrium it can be checked that $\bar{E}_0$ is locally unstable in the $N_1 - N_2$ plane. $\bar{E}_1$ is a saddle point with stable manifold locally in the N_1-direction and unstable manifold locally in the N_2-direction. $\bar{E}_2$ is also a saddle point with unstable manifold locally in the N_1-direction and stable manifold locally in the N_2-direction.

In the following theorem it is shown that $\bar{E}$ is locally asymptotically stable. The proof of this theorem follows from Routh-Hurwitz criteria and hence is omitted.

Theorem 2.4.1 *Let the following inequality holds:*

$$\{r_1'(\bar{N}_2) + \frac{r_{10}\bar{N}_1}{K_1^2(\bar{N}_2)} K_1'(\bar{N}_2\}\{r_2'(\bar{N}_1) + \frac{r_{20}\bar{N}_2}{K_2^2(\bar{N})} K_2'(\bar{N}_1)\} < \frac{r_{10} r_{20}}{K_1(\bar{N}_2) K_2(\bar{N}_1)}. \tag{2.30}$$

Then $\bar{E}$ is locally asymptotically stable. Also $\bar{E}$ is unstable if inequality (2.30) is reversed.

To investigate the global stability behaviour of $\bar{E}$ we need the following lemma which establishes a region of attraction for the system under consideration. The proof of this lemma is easy and hence is omitted.

Lemma 2.4.1 *The set*

$$\Omega_2 = \{(N_1, N_2) : 0 \leq N_1 \leq K_{10},\ 0 \leq N_2 \leq K_{20}\}$$

attracts all solutions initiating in the interior of the positive quadrant.

In the following theorem global stability behaviour of $\bar{E}$ is studied, the proof of which is similar to the proof of Theorem 2.3.2 and hence is omitted.

Theorem 2.4.2 *In addition to assumptions (2.3) and (2.4), let $r_1(N_2), r_2(N_1), K_1(N_2)$ and $K_2(N_1)$ satisfy the following conditions in Ω_2*

$$\begin{aligned} &0 \leq -r_1'(N_2) \leq \bar{\rho}_1,\ 0 \leq -r_2'(N_1) \leq \bar{\rho}_2,\ 0 \leq -K_1'(N_2) \leq \bar{k}_1,\ 0 \leq -K_2'(N_1) \leq \bar{k}_2, \\ &\bar{K}_{m1} \leq K_1(N_2) \leq K_{10} \text{ and } \bar{K}_{m2} \leq K_2(N_1) \leq K_{20}, \end{aligned} \tag{2.31}$$

for some positive constants $\bar{\rho}_1$, $\bar{\rho}_2$, $\bar{k}_1$, $\bar{k}_2$, $\bar{K}_{m1}$ and $\bar{K}_{m2}$. If the following condition holds

$$\{\bar{\rho}_1 + \bar{\rho}_2 + \frac{r_{10}K_{10}\bar{k}_1}{\bar{K}_{m1}^2} + \frac{r_{20}K_{20}\bar{k}_2}{\bar{K}_{m2}^2}\}^2 < \frac{4r_{10}r_{20}}{K_1(\bar{N}_2)K_2(\bar{N}_1)}, \tag{2.32}$$

then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the positive quadrant.

2.5 Model With Diffusion

In this section we consider the complete model (2.1)-(2.2) and state the main results in the form of the following theorem.

Theorem 2.5.1 *(i) If the equilibrium E^* is globally asymptotically stable, then the corresponding uniform steady state of the initial-boundary value problems (2.1)-(2.2) is also globally asymptotically stable.*

(ii) If the equilibrium E^* is unstable, then the uniform steady state of the initial-boundary value problems (2.1)-(2.2) can be made stable by increasing diffusion coefficients appropriately.

Proof: Let us consider the following positive definite function

$$V_2(N_1(t), N_2(t), T_1(t), T_2(t)) = \int\int_D V_1(N_1(t), N_2(t), T_1(t), T_2(t))dA$$

where V_1 is given in equation (2.19).

We have,

$$\begin{aligned}\frac{dV_2}{dt} &= \int\int_D \{\frac{\partial V_1}{\partial N_1}\frac{\partial N_1}{\partial t} + \frac{\partial V_1}{\partial N_2}\frac{\partial N_2}{\partial t} + \frac{\partial V_1}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial V_1}{\partial T_2}\frac{\partial T_2}{\partial t}\}dA \\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \int\int_D \frac{dV_1}{dt}dA \text{ and } I_2 = \int\int_D \{D_1\frac{\partial V_1}{\partial T_1}\nabla^2 T_1 + D_2\frac{\partial V_1}{\partial T_2}\nabla^2 T_2\}dA. \qquad (2.33)$$

We note the following properties of V_1, namely,

$$\left.\frac{\partial V_1}{\partial T_1}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial T_2}\right]_{\partial D} = 0$$

and for all points of D,

$$\frac{\partial^2 V_1}{\partial N_1 \partial N_2} = \frac{\partial^2 V_1}{\partial N_1 \partial T_1} = \frac{\partial^2 V_1}{\partial N_1 \partial T_2} = \frac{\partial^2 V_1}{\partial N_2 \partial T_1} = \frac{\partial^2 V_1}{\partial N_2 \partial T_2} = \frac{\partial^2 V_1}{\partial T_1 \partial T_2} = 0,$$

$$\frac{\partial^2 V_1}{\partial N_1^2} > 0, \ \frac{\partial^2 V_1}{\partial N_2^2} > 0, \ \frac{\partial^2 V_1}{\partial T_1^2} > 0 \text{ and } \frac{\partial^2 V_1}{\partial T_2^2} > 0.$$

We now consider I_2 and determine the sign of each term. We utilize the following formula known as Green's first identity in the plane,

$$\int\int_D F\nabla^2 G \, dA = \oint_{\partial D} F\frac{\partial G}{\partial n}ds - \int\int_D (\nabla F.\nabla G) \, dA,$$

where $\frac{\partial G}{\partial n}$ is the directional derivative in the direction of the unit outward normal to ∂D and s is the arc length.

Then with $F = \frac{\partial V_1}{\partial T_1}$ and $G = T_1$, we get

$$\int\int_D \{\frac{\partial V_1}{\partial T_1}\nabla^2 T_1\} \; dA = \oint_{\partial D} \frac{\partial V_1}{\partial T_1}\frac{\partial T_1}{\partial n} ds - \int\int_D \{\nabla(\frac{\partial V_1}{\partial T_1}).\nabla T_1\} \; dA$$
$$= -\int\int_D \{\nabla(\frac{\partial V_1}{\partial T_1}).\nabla T_1\} \; dA, \; since \; \frac{\partial T_1}{\partial n} = 0.$$

Now

$$\nabla(\frac{\partial V_1}{\partial T_1}) = \frac{\partial^2 V_1}{\partial T_1^2}\frac{\partial T_1}{\partial x}\hat{i} + \frac{\partial^2 VV_1}{\partial T_1^2}\frac{\partial T_1}{\partial y}\hat{j}$$

Hence

$$\int\int_D \{\frac{\partial V_1}{\partial T_1}\nabla^2 T_1\} \; dA = -\int\int_D (\frac{\partial^2 V_1}{\partial T_1^2})\{(\frac{\partial T_1}{\partial x})^2 + (\frac{\partial T_1}{\partial y})^2\} \; dA \leq 0$$

similarly

$$\int\int_D \{\frac{\partial V_1}{\partial T_2}\nabla^2 T_2\} \; dA \; \leq \; 0.$$

i.e.,

$$I_2 \; \leq \; 0. \tag{2.34}$$

Thus we note that if $I_1 \leq 0$, i.e., if the interior equilibrium E^* of model (2.5) is globally asymptotically stable in the absence of diffusion, then the uniform steady state of the initial-boundary value problems (2.1)-(2.2) also must be globally asymptotically stable. This proves the first part of Theorem 2.5.1.

We further note that if $\frac{dV_1}{dt} > 0$, i.e., if $I_1 > 0$, then E^* will be unstable in the absence of diffusion. But Eqs. (2.33) and (2.34) show that by increasing diffusion coefficients D_1 and D_2 sufficiently large, $\frac{dV_2}{dt}$ can be made negative even if $I_1 > 0$. This proves the second part of Theorem 2.5.1.

We shall explain the above theorem for a rectangular habitat D defined by

$$D = \{(x, y) : 0 \leq x \leq a, \; 0 \leq y \leq b\} \tag{2.35}$$

in the form of the following theorem.

Theorem 2.5.2 *In addition to the assumptions (2.3) and (2.4) let $r_1(N_2)$, $r_2(N_1)$, $K_1(N_2)$ and $K_2(N_1)$ satisfy the inequalities in (2.15). If the following inequalities hold:*

$$\{\rho_1 + \rho_2 + \frac{r_{10}K_{10}k_1}{K_{m1}^2} + \frac{r_{20}K_{20}k_2}{K_{m2}^2}\}^2 < \frac{4}{9}\frac{r_{10}}{K_1(N_2^*)}\frac{r_{20}}{K_2(N_1^*)}, \tag{2.36}$$

$$\beta_{21}^2 < \frac{2}{3}c_1\frac{r_{20}}{K_2(N_1^*)}\{\alpha_0 + \frac{D_1\pi^2(a^2+b^2)}{a^2b^2}\}, \quad (2.37)$$

$$\beta_{12}^2 < \frac{2}{3}c_2\frac{r_{10}}{K_1(N_2^*)}\{\beta_0 + \frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\}, \quad (2.38)$$

where

$$c_1 = \frac{1}{3}\frac{1}{\alpha_1^2}\frac{r_{10}}{K_1(N_2^*)}\ \{\alpha_0 + \frac{D_1\pi^2(a^2+b^2)}{a^2b^2}\},$$

and

$$c_2 = \frac{1}{3}\frac{1}{\beta_1^2}\frac{r_{20}}{K_2(N_1^*)}\ \{\beta_0 + \frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\},$$

then the uniform steady state of the initial boundary value problems (2.1)-(2.2) is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Let us consider the rectangular region D given by Eq. (2.35). In this case I_2 can be written as

$$I_2 = -D_1\int\int_D(\frac{\partial^2 V_1}{\partial T_1^2})\{(\frac{\partial T_1}{\partial x})^2 + (\frac{\partial T_1}{\partial y})^2\}\ dA - D_2\int\int_D(\frac{\partial^2 V_1}{\partial T_2^2})\{(\frac{\partial T_2}{\partial x})^2 + (\frac{\partial T_2}{\partial y})^2\}\ dA.$$

From Eq. (2.19) we get

$$\frac{\partial^2 V_1}{\partial T_1^2} = c_1,$$

and

$$\frac{\partial^2 V_1}{\partial T_2^2} = c_2.$$

Hence

$$I_2 = -D_1c_1\int\int_D\{(\frac{\partial T_1}{\partial x})^2 + (\frac{\partial T_1}{\partial y})^2\}\ dA - D_2c_2\int\int_D\{(\frac{\partial T_2}{\partial x})^2 + (\frac{\partial T_2}{\partial y})^2\}\ dA.$$

Now

$$\begin{aligned}\int\int_D(\frac{\partial T_1}{\partial x})^2\ dA &= \int\int_D\{\frac{\partial(T_1 - T_1^*)}{\partial x}\}^2\ dA \\ &= \int_0^b\int_0^a\{\frac{\partial(T_1 - T_1^*)}{\partial x}\}^2\ dxdy\end{aligned}$$

Let $z = \frac{x}{a}$, then

$$\int\int_D(\frac{\partial T_1}{\partial x})^2\ dA = \frac{1}{a}\int_0^b\int_0^1\{\frac{\partial(T_1 - T_1^*)}{\partial z}\}^2\ dzdy$$

Now utilizing the inequality (Denn (1975), pp. 225)

$$\int_0^1 (\frac{\partial T_1}{\partial x})^2\, dx \;\geq\; \pi^2 \int_0^1 T_1^2\, dx$$

we get

$$\begin{aligned}\int\int_D (\frac{\partial T_1}{\partial x})^2\, dA \;&\geq\; \frac{\pi^2}{a}\int_0^b \int_0^1 (T_1 - T_1^*)^2 dz dy \\ &=\; \frac{\pi^2}{a^2}\int_0^b \int_0^a (T_1 - T_1^*)^2 dx dy \\ &=\; \frac{\pi^2}{a^2}\int\int_D (T_1 - T_1^*)^2 dA.\end{aligned}$$

Similarly,

$$\int\int_D (\frac{\partial T_1}{\partial y})^2\, dA \;\geq\; \frac{\pi^2}{b^2}\int\int_D (T_1 - T_1^*)^2 dA.$$

Thus,

$$I_2 \leq\; -\frac{D_1 c_1 \pi^2 (a^2+b^2)}{a^2 b^2}\int\int_D (T_1 - T_1^*)^2 dA - \frac{D_2 c_2 \pi^2 (a^2+b^2)}{a^2 b^2}\int\int_D (T_2 - T_2^*)^2 dA$$

Now from (2.20) and (2.33) we get

$$\begin{aligned}\frac{dV_2}{dt} \;\leq\; \int\int_D \{ &\frac{r_{10}}{K_1(N_2^*)}(N_1 - N_1^*)^2 - \frac{r_{20}}{K_2(N_1^*)}(N_2 - N_2^*)^2 \\ &- c_1[\alpha_0 + \frac{D_1 c_1 \pi^2 (a^2+b^2)}{a^2 b^2}](T_1 - T_1^*)^2 \\ &- c_2[\beta_0 + \frac{D_2 c_2 \pi^2 (a^2+b^2)}{a^2 b^2}](T_2 - T_2^*)^2 \\ &+ \{\eta_1(N_2) + \eta_2(N_1) - r_{10}N_1\xi_1(N_2) - r_{20}N_2\xi_2(N_1)\}(N_1 - N_1^*)(N_2 - N_2^*) \\ &+ c_1\alpha_1(N_1 - N_1^*)(T_1 - T_1^*) - \beta_{12}(N_1 - N_1^*)(T_2 - T_2^*) \\ &- \beta_{21}(N_2 - N_2^*)(T_1 - T_1^*) + c_2\beta_1(N_2 - N_2^*)(T_2 - T_2^*)\}\, dA, \end{aligned} \tag{2.39}$$

where $\eta_1(N_2)$, $\eta_2(N_1)$, $\xi_1(N_2)$ and $\xi_2(N_1)$ are defined in Eq. (2.20).

Now Eq. (2.39) can be written as the sum of the quadratics

$$\begin{aligned}\frac{dV_2}{dt} \;\leq\; \int\int_D \{ &-\frac{1}{2}b_{11}(N_1 - N_1^*)^2 + b_{12}(N_1 - N_1^*)(N_2 - N_2^*) - \frac{1}{2}b_{22}(N_2 - N_2^*)^2 \\ &-\frac{1}{2}b_{11}(N_1 - N_1^*)^2 + b_{13}(N_1 - N_1^*)(T_1 - T_1^*) - \frac{1}{2}b_{33}(T_1 - T_1^*)^2 \\ &-\frac{1}{2}b_{11}(N_1 - N_1^*)^2 + b_{14}(N_1 - N_1^*)(T_2 - T_2^*) - \frac{1}{2}b_{44}(T_2 - T_2^*)^2 \\ &-\frac{1}{2}b_{22}(N_2 - N_2^*)^2 + b_{23}(N_2 - N_2^*)(T_1 - T_1^*) - \frac{1}{2}b_{33}(T_1 - T_1^*)^2 \\ &-\frac{1}{2}b_{22}(N_2 - N_2^*)^2 + b_{24}(N_2 - N_2^*)(T_2 - T_2^*) - \frac{1}{2}b_{44}(T_2 - T_2^*)^2\}dA,\end{aligned}$$

where

$$
\begin{aligned}
b_{11} &= \frac{2}{3}\frac{r_{10}}{K_1(N_2^*)},\ b_{22} = \frac{2}{3}\frac{r_{20}}{K_2(N_1^*)}, \\
b_{33} &= c_1\{\alpha_0 + \frac{D_1\pi^2(a^2+b^2)}{a^2b^2}\},\ b_{44} = c_2\{\beta_0 + \frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\}, \\
b_{12} &= \eta_1(N_2) + \eta_2(N_1) - r_{10}N_1\xi_1(N_2) - r_{20}N_2\xi_2(N_1), \\
b_{13} &= c_1\alpha_1,\ b_{14} = -\beta_{12},\ b_{23} = -\beta_{21} \textit{ and } b_{24} = c_2\beta_1.
\end{aligned}
$$

Sufficient conditions for $\frac{dV_2}{dt}$ to be negative definite are that the following conditions hold:

$$b_{12}^2 < b_{11}b_{22}, \tag{2.40}$$

$$b_{13}^2 < b_{11}b_{33}, \tag{2.41}$$

$$b_{14}^2 < b_{11}b_{44}, \tag{2.42}$$

$$b_{23}^2 < b_{22}b_{33}, \tag{2.43}$$

$$b_{24}^2 < b_{22}b_{44}. \tag{2.44}$$

By choosing

$$c_1 = \frac{1}{3}\frac{1}{\alpha_1^2}\frac{r_{10}}{K_1(N_2^*)}\ \{\alpha_0 + \frac{D_1\pi^2(a^2+b^2)}{a^2b^2}\},$$

and

$$c_2 = \frac{1}{3}\frac{1}{\beta_1^2}\frac{r_{20}}{K_2(N_1^*)}\ \{\beta_0 + \frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\},$$

we note that conditions (2.41) and (2.44) are satisfied automatically. We also note that (2.36) $\Rightarrow$ (2.40), (2.37) $\Rightarrow$ (2.42) and (2.38) $\Rightarrow$ (2.43). Hence V_2 is a Liapunov function with respect to E^* whose domain contains the region of attraction Ω_1, proving the theorem.

It may be noted here that if $D_1 = D_2 = 0$, then Theorem 2.5.2 reduces to Theorem 2.3.2. We further note that inequalities (2.37) and (2.38) may be satisfied by increasing D_1 and D_2 to sufficiently large values. This implies that stability is more plausible in the case of diffusion.

2.6 Numerical Examples

In this section we present numerical examples to illustrate the results obtained in sections 2.3 and 2.4 by choosing the following particular form of functions in model (2.5):

$$
\begin{aligned}
r_1(N_2) &= r_{10} - r_{11}N_2, \\
r_2(N_1) &= r_{20} - r_{21}N_1, \\
K_1(N_2) &= K_{10} - K_{11}N_2, \\
K_2(N_1) &= K_{20} - K_{21}N_1,
\end{aligned}
\tag{2.45}
$$

where all coefficients are positive. We now choose the following values of the parameters in equation (2.45):

$$
\begin{aligned}
&r_{10} = 10.0,\ r_{11} = 0.03,\ r_{20} = 12.0,\ r_{21} = 0.04, \\
&K_{10} = 30.0,\ K_{11} = 0.05,\ K_{20} = 32.0,\ K_{21} = 0.08.
\end{aligned}
\tag{2.46}
$$

Example 1 In this example we consider system (2.5). In addition to the values of parameters given by Eq. (2.46), we choose the following values of the parameters in model (2.5):

$$
\begin{aligned}
&\beta_{12} = 0.10,\ \beta_{21} = 0.13,\ \alpha_1 = 0.80, \\
&\beta_1 = 0.90,\ \alpha_0 = 0.60,\ \beta_0 = 0.70.
\end{aligned}
\tag{2.47}
$$

With the above values of the parameters, it can be checked that condition (2.10) for the existence of E^* is satisfied, i.e.,

$$K_{10} = 30.00 < N_{1a} = 400.00 \ \textit{and}\ K_{20} = 32.00 < N_{2a} = 600.00.$$

Thus, the interior equilibrium E^* exists and is given by

$$N_1^* = 20.00705,\ N_2^* = 19.58694,\ T_1^* = 26.67606,\ T_2^* = 25.18320. \tag{2.48}$$

It can also be checked that *conditions* (2.12)-(2.14) *in Theorem* 2.3.1 *are satisfied*, which shows that E^* is locally asymptotically stable.

Further, by choosing $K_{m1} = 15.0$ and $K_{m2} = 20.0$ in Theorem 2.3.2, it can be verified that conditions (2.16)-(2.18) are satisfied and hence E^* is globally asymptotically stable.

Example 2 In this example we consider the model when the plant species do not produce any toxicant. We take the same set of functions as given in Eq. (2.45) and the same set of values of the parameters as given in Eq. (2.46).

It can be checked that $\bar{E}$ exists and is given by

$$\bar{N}_1 = 26.29291, \bar{N}_2 = 27.27633. \tag{2.49}$$

It can also be checked that condition (2.30) in Theorem 2.4.1 is satisfied, which shows that $\bar{E}$ is locally asymptotically stable.

By choosing $K_{m1} = 15.0$ and $K_{m2} = 20.0$ in Theorem 2.4.2, it can be verified that condition (2.32) is satisfied showing global stability character of $\bar{E}$.

Comparing Eqs. (2.48) and (2.49) we note that the values of $\bar{N}_1$ and $\bar{N}_2$ are considerably higher than their previous values N_1^* and N_2^*. This shows that the equilibrium levels of the plant species, when they produce toxicant, are lower than the case when they do not produce toxicants.

2.7 Conclusions

In this chapter, a mathematical model has been proposed and analysed to study the interaction of two plant species competing for nutrients. It has been assumed that each plant species produces toxicant, which reaches to the other through diffusive process and affects its growth.

In the case of no diffusion, it has been shown that the two competing plant species settle down to their respective equilibrium levels, the magnitude of which are lower than their corresponding initial density independent carrying capacities. In the case when the two plant species do not produce any toxicant, it has been shown that the two plant species again settle down to their respective equilibrium levels, the magnitudes of which are higher than their corresponding values in the case when they produce toxicant. To illustrate the above facts a numerical example has also been presented in this chapter. It has also been found that the rate of decrease in the growth rates of plant species is faster in the case when each plant species produces a substance toxic to the other.

By incorporating diffusion in the system it has been shown that diffusion is playing the general role of stabilizing the system. It has been shown that if the interior equilibrium of the system with no diffusion is globally asymptotically stable, then the corresponding uniform steady state of the system with diffusion must be globally asymptotically stable. Further, an unstable steady state in the absence of diffusion can be made stable by increasing diffusion coefficients sufficiently large. In a particular case of rectangular habitat it has been shown that stability is more plausible in the case of diffusion.

Chapter 3

MODELLING THE SURVIVAL OF SPECIES DEPENDENT ON RESOURCE IN A POLLUTED ENVIRONMENT

3.1 Introduction

Various kinds of industrial discharges and chemical spills in the form of smokes, poisonous gas fumes, hazardous wastes have polluted the air and contaminated the streams, rivers, lakes and oceans with varieties of chemicals and toxicants such as arsenic, cadmium, lead, zinc, copper, iron, mercury etc. causing damage to both terrestrial and aquatic environment (Jensen and Marshall, 1982; Nelson, 1970).

In recent years some investigations have been made to study the effect of toxicants on biological species (Chattopadhyay, 1996; De Luna and Hallam, 1987; Dubey, 1997a; Freedman and Shukla, 1991; Hallam and Clark, 1982; Hallam et. al., 1983; Hallam and De Luna, 1984; Huaping and Ma, 1991; Shukla and Dubey, 1996a; Shukla and Dubey, 1997). In particular, Freedman and Shukla (1991) studied the effect of toxicant on a single-species population and predator-prey systems. They showed that if the emission rate of the toxicant into the environment increases, the equilibrium level of the population decreases, the magnitude of which depends upon the influx and washout rates of the toxicant. Huaping and Ma (1991) also studied the effect of toxicant on naturally stable two species communities and found the persistence and extinction criteria for populations. Shukla and Dubey (1996a) studied the effects of two toxicants when one is more toxic than the other, on a single species population. Chattopadhyay (1996) studied the effect of toxic substances on a two-species competitive system and showed that toxic substances have some stabilizing effect on a two-species competitive system. Dubey (1997a) proposed a model for control of toxicant and conservation of forestry resources. The survival (growth and existence) of resource biomass dependent species in a forested habitat, which is being depleted due to industrialization pressure, has also been studied (Shukla et al., 1996). Shukla and Dubey (1997) studied the depletion of a forestry resource in a habitat, which is caused by increase in population density and pollutant emission into the environment. The pollutant emission rate is either population dependent, constant, periodic or instantaneous. But in the above investigations the survival of species population dependent on resource which is affected by pollutant has not been considered. Further, in the above studies the effect of diffusion has not been considered. Recently, Dubey and Das (1999) studied the survival of wildlife species dependent on resource in an industrial environment with diffusion. They showed that the increasing industrialization may lead to decrease in the density of resource biomass and consequently the survival of the species may be threatened, but diffusive migration may prevent extinction of the species.

Keeping the above in view, in this chapter, a mathematical model is proposed and analysed to study the survival of a single-species population dependent on resource which is affected by a toxicant present in the environment with diffusion. It is assumed that the population depends partially or wholly on the resource or just predating on the resource. Stability theory of ordinary differential equations (La Salle and Lefschetz, 1961) is used for the model analysis to study the equilibrium levels of the species population and the resource biomass density by taking into account the constant, instantaneous or periodic emission of toxicant into the environment.

3.2 Mathematical Model

We consider an ecosystem where we wish to model the survival of a biological species dependent on resource which is affected by a pollutant present in the environment in a closed region D with smooth boundary ∂D. It is assumed that the growth rate of the biological species increases as the density of the resource biomass increases while the carrying capacity increases as the resource biomass density increases and decreases as the concentration of the environmental pollutant increases. It is further assumed that the growth rate of the resource biomass decreases as the density of the species population and the uptake concentration of the pollutant increase but its carrying capacity decreases only with the increase in environmental concentration of the pollutant. Following Freedman and Shukla (1991), Huaping and Ma (1991) and Dubey (1997a), the system is assumed to be governed by the following differential equations:

$$\begin{aligned}
\frac{\partial N}{\partial t} &= r(B)N - \frac{r_0 N^2}{K(B,T)} + D_1 \nabla^2 N, \\
\frac{\partial B}{\partial t} &= r_B(U,N)B - \frac{r_{B0} B^2}{K_B(T)} + D_2 \nabla^2 B, \\
\frac{\partial T}{\partial t} &= Q(t) - \delta_0 T - \alpha BT + \theta_1 \delta_1 U + \pi \nu BU + D_3 \nabla^2 T, \qquad (3.1) \\
\frac{\partial U}{\partial t} &= \beta B + \theta_0 \delta_0 T - \delta_1 U + \alpha BT - \nu BU, \\
0 &\leq \theta_0,\ \theta_1,\ \pi \leq 1.
\end{aligned}$$

We impose the following initial and boundary conditions on the system:

$$\begin{aligned}
& N(x,y,0) = \phi(x,y) \geq 0,\ B(x,y,0) = \psi(x,y) \geq 0, \\
& T(x,y,0) = \xi(x,y) \geq 0,\ U(x,y,0) = \chi(x,y) \geq 0,\ (x,y) \epsilon D \qquad (3.2) \\
& \frac{\partial N}{\partial n} = \frac{\partial B}{\partial n} = \frac{\partial T}{\partial n} = 0,\ (x,y)\ \epsilon\ \partial D, t \geq 0,
\end{aligned}$$

where n is the unit outward normal to ∂D.

In model (3.1), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian diffusion operator. $N(x, y, t)$ is the density of the biological species, $B(x, y, t)$ the density of the resource biomass, $T(x, y, t)$ the concentration of pollutant present in the environment and $U(x, y, t)$ the concentration of pollutant uptaken by the population at coordinates $(x, y)\epsilon D$ and time $t \geq 0$. $Q(t)$ is the rate of introduction of pollutant into the environment which may be constant, zero or periodic. δ_0 is the depletion rate coefficient of pollutant from the environment, perhaps from biological transformation, chemical hydrolysis, volatilization, microbial degradation or photosynthetic degradation, and a fraction θ_0 of it may reenter into the resource biomass with the uptake of pollutant. δ_1 is the natural depletion rate coefficient of $U(t)$ due to ingestion and depuration of pollutant, and a fraction θ_1 of it may reenter into the environment due to recycling. Also the uptake concentration of the pollutant may decrease with the rate coefficient ν due to resource biomass and a fraction π of which may reenter into the environment. α is the depletion rate coefficient of the pollutant present in the environment due to its uptake by the resource biomass. β is the net intake of the pollutant by the resource biomass via food chain. D_1, D_2 and D_3 are the diffusion rate coefficients of N, B and T respectively in D.

In model (3.1), the function $r(B)$ denotes the specific growth rate of the biological species which increases as the density of the resource biomass increases. The function $r(B)$ may satisfy the following conditions:

$$r(0) > 0, \ r'(B) > 0 \ for B \geq 0. \tag{3.3}$$

In this case the species depends partially on the resource biomass i.e., B(x,y,t) may be thought of as an alternative resource for the population N(x,y,t).

$$r(0) = 0, \ r'(B) > 0 \ for B \geq 0. \tag{3.4}$$

In this case the species depends wholly on the resource.

$$\begin{aligned} &r(0) < 0, \ r'(B) > 0 \ for B \geq 0, \\ &and\ there\ exists\ a\ B = B_a\ such\ that\ r(B_a) = 0. \end{aligned} \tag{3.5}$$

In this case $N(x, y, t)$ acts as a predator on the resource.

The function $K(B,T)$ denotes the maximum density of the species population which the environment can support in the presence of the resource biomass and the environmental pollutant. It increases as the density of the resource biomass increases, and decreases as the environmental concentration of pollutant increases, i.e.,

$$K(0,0) = K_0 > 0, \ \frac{\partial K}{\partial B} > 0, \ \frac{\partial K}{\partial T} < 0, \ for \ B \geq 0, \ T \geq 0. \tag{3.6}$$

The function $r_B(U,N)$ denotes the specific growth rate of the resource biomass which decreases as the uptake concentration of the pollutant and the density of the population increase, i.e.,

$$r_B(0,0) = r_{B0} > 0, \ \frac{\partial r_B(U,N)}{\partial U} < 0, \ \frac{\partial r_B(U,N)}{\partial N} < 0, \ for \ U \geq 0, \ N \geq 0. \tag{3.7}$$

The function $K_B(T)$ denotes the maximum density of the resource biomass which the environment can support in the presence of the pollutant and it decreases as the environmental concentration of the pollutant increases, i.e.,

$$\begin{aligned} &K_B(0) = K_{B0} > 0, \ K'_B(T) < 0 \ for \ T \geq 0, \\ &and \ there \ exists \ a \ T = T_a \ such \ that \ K_B(T_a) = 0. \end{aligned} \tag{3.8}$$

We analyse model (3.1) for three different values of $Q(t)$, namely, $Q(t) = Q_0 > 0$, $Q(t) = 0$, and Q(t) is periodic in three different cases (3.3), (3.4) and (3.5).

3.3 Model Without Diffusion

In this section we take $D_1 = D_2 = D_3 = 0$ in model (3.1). Then the model reduces to

$$\begin{aligned} \frac{dN}{dt} &= r(B)N - \frac{r_0 N^2}{K(B,T)}, \\ \frac{dB}{dt} &= r_B(U,N)B - \frac{r_{B0}B^2}{K_B(T)}, \\ \frac{dT}{dt} &= Q(t) - \delta_0 T - \alpha BT + \theta_1 \delta_1 U + \pi \nu BU, \\ \frac{dU}{dt} &= \beta B + \theta_0 \delta_0 T - \delta_1 U + \alpha BT - \nu BU, \\ &N(0) \geq 0, B(0) \geq 0, T(0) \geq 0, U(0) \geq 0. \end{aligned} \tag{3.9}$$

Case I: When the species partially depends on the resource

In this case the function $r(B)$ satisfies (3.3). We first analyse this case when the rate of introduction of pollutant into the environment is constant i.e., $Q(t) = Q_0$ is a positive constant. It is noted here that model (3.9) has four nonnegative equilibria, namely, $E_{11}(0, 0, \frac{Q_0}{\delta_0(1-\theta_0\theta_1)}, \frac{Q_0\theta_0}{\delta_1(1-\theta_0\theta_1)})$, $E_{12}(N_c, 0, T_c, U_c)$, $E_{13}(0, \bar{B}, \bar{T}, \bar{U})$, and $E_{14}(N^*, B^*, T^*, U^*)$. It may be noted that in equilibrium E_{12}, N_c, T_c and U_c are given by

$$N_c = K(0, T_c),\ T_c = \frac{Q_0}{\delta_0(1-\theta_0\theta_1)},\ and\ U_c = \frac{\theta_0 Q_0}{\delta_1(1-\theta_0\theta_1)}.$$

Here we shall show the existence of E_{14} only, and the existence of E_{13} can be concluded form the existence of E_{14}.

To establish the existence of E_{14}, we note that N^*, B^*, T^* and U^* are the positive solutions of the system of following algebraic equations:

$$N = h_1(B), \tag{3.10}$$

$$r_{B0}B = r_B(g(B), h_1(B))K_B(h(B)), \tag{3.11}$$

$$T = h(B), \tag{3.12}$$

$$U = g(B), \tag{3.13}$$

where

$$h_1(B) = \frac{r(B)K(B, h(B))}{r_0}$$

$$h(B) = \frac{Q_0 + (\theta_1\delta_1 + \pi\nu B)g(B)}{\delta_0 + \alpha B},$$

$$g(B) = \frac{\beta B(\delta_0 + \alpha B) + Q_0(\theta_0\delta_0 + \alpha B)}{f(B)},$$

$$f(B) = \delta_0\delta_1(1-\theta_0\theta_1) + \delta_1\alpha(1-\theta_1)B + \nu\delta_0(1-\theta_0\pi)B + \nu\alpha(1-\pi)B^2.$$

Taking

$$F(B) = r_{B0}B - r_B(g(B), h_1(B))K_B(h(B)), \tag{3.14}$$

we note that $F(0) < 0$, $F(K_{B0}) > 0$. This shows that there exists a B^* in the interval $0 < B^* < K_{B0}$ such that $F(B^*) = 0$. For B^* to be unique, we must have

$$r_{B0} - K_B(h(B))(\frac{\partial r_B}{\partial U}\frac{dg}{dB} + \frac{\partial r_B}{\partial N}\frac{dh_1}{dB}) - r_B(g(B), h_1(B))\frac{\partial K_B}{\partial T}\frac{dh}{dB} > 0. \tag{3.15}$$

Thus knowing the value of B^*, the values of N^*, T^* and U^* can be computed from (3.10), (3.12) and (3.13) respectively.

It may be noted here that if $\frac{dg}{dB} > 0$, $\frac{dh}{dB} > 0$ and $\frac{dh_1}{dB} > 0$, then inequality (3.15) is automatically satisfied.

By computing the variational matrices (Freedman, 1987b) corresponding to each equilibrium it can be seen that E_{11} is a saddle point whose unstable manifold is locally in the $N-B$ plane and whose stable manifold is locally in the $T-U$ plane. E_{12} is also a saddle point with stable manifold locally in the $N-T-U$ space and with unstable manifold locally in the B-direction. E_{13} is unstable in the N-direction.

In the following theorem the local stability behavior of E_{14} is studied. First we write the following notations:

$$c_1 = -\frac{r'(B^*) + \frac{r_0 N^*}{K^2(B^*,T^*)}\frac{\partial K(B^*,T^*)}{\partial B}}{\frac{\partial r_B(U^*,N^*)}{\partial N}} > 0,$$

$$c_2 = \frac{2}{\delta_0 + \alpha B^*}\frac{r_0 N^{*2}}{K^3(B^*,T^*)}\{\frac{\partial K(B^*,T^*)}{\partial T}\}^2 > 0,$$

$$c_3 = -\frac{c_1\frac{\partial r_B(U^*,N^*)}{\partial U}}{\beta + \alpha T^* + \nu U^*} > 0.$$

Theorem 3.3.1 *Let the following inequalities hold*

$$\{c_1\frac{r_{B0}B^*}{K_B^2(T^*)}\frac{\partial K_B(T^*)}{\partial T} + c_2(\alpha T^* + \pi\nu U^*)\}^2 < \frac{4}{9}c_1c_2\frac{r_{B0}}{K_B(T^*)}(\delta_0 + \alpha B^*), \tag{3.16}$$

$$\{c_2(\theta_1\delta_1 + \pi\nu B^*) + c_3(\theta_0\delta_0 + \alpha B^*)\}^2 < \frac{2}{3}c_2c_3(\delta_0 + \alpha B^*)(\delta_1 + \nu B^*). \tag{3.17}$$

Then E_{14} is locally asymptotically stable.

Proof: Linearizing system (3.9) by substituting

$$N = N^* + n,\ B = B^* + b,\ T = T^* + \tau,\ U = U^* + u,$$

and using the following positive definite function

$$V = \frac{1}{2}\{\frac{n^2}{N^*} + c_1\frac{b^2}{B^*} + c_2\tau^2 + c_3u^2\}, \tag{3.18}$$

it can be checked that the derivative of V with respect to t along the solutions of (3.9) is negative definite under conditions (3.16)-(3.17), proving the theorem.

The following lemma establishes a region of attraction for all solutions initiating in the interior of the positive orthant. The proof of this lemma is similar to Hsu (1978a), Shukla and Dubey (1997) and hence is omitted.

Lemma 3.3.1 *The set*

$$\Omega_1 = \{(N, B, T, U) : 0 \le N \le N_c,\ 0 \le B \le K_{B0},\ 0 \le T + U \le Q_c\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$N_c = \frac{r(K_{B0})K(K_{B0}, 0)}{r_0},$$
$$Q_c = \frac{Q_0 + \beta K_{B0}}{\delta},$$
$$\delta = min\{\delta_0(1 - \theta_0), \delta_1(1 - \theta_1)\}.$$

In the following theorem global stability behaviour of E_{14} is studied.

Theorem 3.3.2 *In addition to the assumptions (3.3), (3.6)-(3.8) let $r(B)$, $K(B,T)$, $r_B(U, N)$ and $K_B(T)$ satisfy the following conditions in Ω_1*

$$\begin{aligned} &0 \le r'(B) \le \rho_1, 0 \le -\frac{\partial r_B(U,N)}{\partial U} \le \rho_2, 0 \le -\frac{\partial r_B(U,N)}{\partial N} \le \rho_3, \\ &K_{m1} \le K(B,T) \le K(K_{B0}, 0),\ K_{m2} \le K_B(T) \le K_{B0}, \\ &0 \le \frac{\partial K(B,T)}{\partial B} \le \kappa_1,\ 0 \le -\frac{\partial K(B,T)}{\partial T} \le \kappa_2,\ 0 \le -K'_B(T) \le \kappa_3, \end{aligned} \tag{3.19}$$

for some positive constants $\rho_1, \rho_2, \rho_3, K_{m1}, K_{m2}, \kappa_1, \kappa_2$ and κ_3. Then if the following inequalities hold:

$$\{\rho_1 + \rho_3 + \frac{r_0 N_c \kappa_1}{K_{m1}^2}\}^2 < \frac{2}{3}\frac{r_0}{K(B^*, T^*)}\frac{r_{B0}}{K_B(T^*)}, \tag{3.20}$$

$$\{\frac{r_0 N_c \kappa_2}{K_{m1}^2}\}^2 < \frac{2}{3}\frac{r_0}{K(B^*, T^*)}(\delta_0 + \alpha B^*), \tag{3.21}$$

$$\{\frac{r_{B0} K_{B0} \kappa_3}{K_{m2}^2} + (\alpha + \pi\nu)Q_c\}^2 < \frac{4}{9}\frac{r_{B0}}{K_B(T^*)}(\delta_0 + \alpha B^*), \tag{3.22}$$

$$\{\rho_2 + \beta + (\alpha + \nu)Q_c\}^2 < \frac{2}{3}\frac{r_{B0}}{K_B(T^*)}(\delta_1 + \nu B^*), \tag{3.23}$$

$$\{\theta_0\delta_0 + \theta_1\delta_1 + (\alpha + \pi\nu)B^*\}^2 < \frac{2}{3}(\delta_0 + \alpha B^*)(\delta_1 + \nu B^*), \tag{3.24}$$

E_{14} is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Consider the following positive definite function around E_{14},

$$V_1 = N - N^* - N^* \ ln(\frac{N}{N^*}) + B - B^* - B^* \ ln(\frac{B}{B^*}) + \frac{1}{2}\{(T - T^*)^2 + (U - U^*)^2\}. \tag{3.25}$$

Differentiating V_1 with respect to t along the solutions of system (3.9), a little algebraic manipulation yields

$$\begin{aligned}\frac{dV_1}{dt} &= -\frac{r_0}{K(B^*, T^*)}(N - N^*)^2 - \frac{r_{B0}}{K_B(T^*)}(B - B^*)^2 \\ &\quad -(\delta_0 + \alpha B^*)(T - T^*)^2 - (\delta_1 + \nu B^*)(U - U^*)^2 \\ &\quad +[\eta_1(B) + \eta_3(U^*, N) - r_0N\xi_1(B, T)](N - N^*)(B - B^*) \\ &\quad +[-r_0N\xi_2(B^*, T)](N - N^*)(T - T^*) \\ &\quad +[-r_{B0}B\xi_3(T) - \alpha T + \pi\nu U](B - B^*)(T - T^*) \\ &\quad +[\beta + \eta_2(U, N) + \alpha T - \nu U](B - B^*)(U - U^*) \\ &\quad +[\theta_0\delta_0 + \theta_1\delta_1 + (\alpha + \pi\nu)B^*](T - T^*)(U - U^*),\end{aligned} \tag{3.26}$$

where

$$\eta_1(B) = \begin{cases} \frac{r(B) - r(B^*)}{B - B^*}, & B \neq B^* \\ r'(B^*), & B = B^* \end{cases}$$

$$\eta_2(U, N) = \begin{cases} \frac{r_B(U, N) - r_B(U^*, N)}{U - U^*}, & U \neq U^* \\ \frac{\partial r_B(U^*, N)}{\partial U}, & U = U^* \end{cases}$$

$$\eta_3(U^*,N) = \begin{cases} \frac{r_B(U^*,N)-r_B(U^*,N^*)}{N-N^*}, & N \neq N^* \\ \frac{\partial r_B(U^*,N^*)}{\partial N}, & N = N^* \end{cases}$$

$$\xi_1(B,T) = \begin{cases} \{\frac{1}{K(B,T)} - \frac{1}{K(B^*,T)}\}/(B-B^*), & B \neq B^* \\ -\frac{1}{K^2(B^*,T)}\frac{\partial K}{\partial B}(B^*,T), & B = B^* \end{cases}$$

$$\xi_2(B^*,T) = \begin{cases} \{\frac{1}{K(B^*,T)} - \frac{1}{K(B^*,T^*)}\}/(T-T^*), & T \neq T^* \\ -\frac{1}{K^2(B^*,T^*)}\frac{\partial K}{\partial T}(B^*,T^*), & T = T^* \end{cases}$$

$$\xi_3(T) = \begin{cases} \{\frac{1}{K_B(T)} - \frac{1}{K_B(T^*)}\}/(T-T^*), & T \neq T^* \\ -\frac{K'_B(T^*)}{K_B^2(T^*)}, & T = T^* \end{cases}$$

From (3.19) and the mean value theorem, we note that

$$|\eta_1(B)| \leq \rho_1,\ |\eta_2(U,N)| \leq \rho_2,\ |\eta_3(U^*,N)| \leq \rho_3,\ |\xi_1(B,T)| \leq \frac{\kappa_1}{K_{m1}^2},$$
$$|\xi_2(B^*,T)| \leq \frac{\kappa_2}{K_{m1}^2} \text{ and } |\xi_3(T)| \leq \frac{\kappa_3}{K_{m2}^2}.$$

Now $\frac{dV_1}{dt}$ can further be written as the sum of the quadratics

$$\begin{aligned} \frac{dV_1}{dt} = & -\frac{1}{2}a_{11}(N-N^*)^2 + a_{12}(N-N^*)(B-B^*) - \frac{1}{2}a_{22}(B-B^*)^2 \\ & -\frac{1}{2}a_{11}(N-N^*)^2 + a_{13}(N-N^*)(T-T^*) - \frac{1}{2}a_{33}(T-T^*)^2 \\ & -\frac{1}{2}a_{22}(B-B^*)^2 + a_{23}(B-B^*)(T-T^*) - \frac{1}{2}a_{33}(T-T^*)^2 \\ & -\frac{1}{2}a_{22}(B-B^*)^2 + a_{24}(B-B^*)(U-U^*) - \frac{1}{2}a_{44}(U-U^*)^2 \\ & -\frac{1}{2}a_{33}(T-T^*)^2 + a_{34}(T-T^*)(U-U^*) - \frac{1}{2}a_{44}(U-U^*)^2, \end{aligned}$$

where

$$a_{11} = \frac{r_0}{K(B^*,T^*)},\ a_{22} = \frac{2}{3}\frac{r_{B0}}{K_B(T^*)},\ a_{33} = \frac{2}{3}(\delta_0 + \alpha B^*),$$
$$a_{44} = \delta_1 + \nu B^*,\ a_{12} = \eta_1(B) + \eta_3(U^*, N) - r_0 N\xi_1(B,T),\ a_{13} = -r_0 N\xi_2(B^*,T),$$
$$a_{23} = -r_{B0}B\xi_3(T) - \alpha T + \pi\nu U,\ a_{24} = \beta + \eta_2(U,N) + \alpha T - \nu U,$$
$$a_{34} = \theta_0\delta_0 + \theta_1\delta_1 + (\alpha + \pi\nu)B^*.$$

Sufficient conditions for $\frac{dV_1}{dt}$ to be negative definite are that the following conditions hold:

$$a_{12}^2 < a_{11}a_{22}, \quad (3.27)$$
$$a_{13}^2 < a_{11}a_{33}, \quad (3.28)$$
$$a_{23}^2 < a_{22}a_{33}, \quad (3.29)$$
$$a_{24}^2 < a_{22}a_{44}, \quad (3.30)$$
$$a_{34}^2 < a_{33}a_{44}. \quad (3.31)$$

It is noted here that (3.20) $\Rightarrow$ (3.27), (3.21) $\Rightarrow$ (3.28), (3.22) $\Rightarrow$ (3.29), (3.23) $\Rightarrow$ (3.30), and (3.24) $\Rightarrow$ (3.31). Hence V_1 is a Liapunov function with respect to E_{14} whose domain contains the region of attraction Ω_1, proving the theorem.

The above analysis shows that when the pollutant is emitted into the environment with a constant rate, the biological species and the resource biomass settle down to their respective equilibrium levels. The magnitude of the species depends upon the equilibrium level of the resource biomass and the concentration of environmental pollutant, which decreases as the equilibrium level of the resource biomass decreases and the concentration of environmental pollutant increases. The magnitude of the resource biomass depends upon the equilibrium level of the species, the environmental and the uptake concentrations of the pollutant which decreases as these factors increase, and even may tend to zero if the environmental concentration of pollutant becomes very high. It may be noted here that the survival of the species will be threatened if the environmental concentration of the pollutant is very high and the density of the resource biomass is very small.

Remark 1 When $Q(t) = 0$, i.e., in the case of instantaneous emission of pollutant into the environment, the corresponding results can be obtained from case I by substituting $Q_0 = 0$. In particular it is noted that the resource biomass and the population settle down to their respective equilibrium levels under certain conditions whose magnitudes are greater than their respective magnitudes in case I.

3.4 Periodic introduction of pollutant into the environment, i.e., $Q(t) = Q_0 + \varepsilon\phi(t)$, $\phi(t + \omega) = \phi(t)$.

In this case, model (3.9) can be written in the vector matrix form as

$$\frac{dx}{dt} = A(x) + \varepsilon C(t), \; x(0) = x_0 \tag{3.32}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} N \\ B \\ T \\ U \end{bmatrix}, \; x_0 = \begin{bmatrix} N(0) \\ B(0) \\ T(0) \\ U(0) \end{bmatrix}, \; C(t) = \begin{bmatrix} 0 \\ 0 \\ \phi(t) \\ 0 \end{bmatrix},$$

$$A(x) = \begin{bmatrix} r(x_2)x_1 - \frac{r_0 x_1^2}{K(x_2, x_3)} \\ r_B(x_1, x_1)x_2 - \frac{r_{B0} x_2^2}{K_B(x_3)} \\ Q_0 - \delta_0 x_3 - \alpha x_2 x_3 + \theta_1 \delta_1 x_4 + \pi\nu x_2 x_4 \\ \beta x_2 - \theta_0 \delta_0 x_3 - \delta_1 x_4 - \alpha x_2 x_3 - \nu x_2 x_4 \end{bmatrix}.$$

Let M^* be the variational matrix corresponding to the positive equilibrium $E_{14}(N^*, B^*, T^*, U^*)$. Then under an analysis similar to Freedman and Shukla (1991), we can state the following two theorems.

Theorem 3.4.1 *If M^* has no eigenvalues with zero real parts, then system (3.9) with $Q(t) = Q_0 + \varepsilon\phi(t), \phi(t + \omega) = \phi(t)$ has a periodic solution of period ω, $(B(t,\varepsilon),\ T(t,\varepsilon),\ U(t,\varepsilon),\ W(t,\varepsilon))$ such that $(B(t,0),\ T(t,0),\ U(t,0),\ W(t,0)) = (N^*,\ B^*,\ T^*,\ U^*)$.*

Theorem 3.4.2 *If M^* has no eigenvalues with zero real parts, then for sufficiently small ε, the stability behaviour of system (3.9) is same as that of E^*.*

Moreover, a periodic solution up to order ε can be computed as

$$x(t,\xi,\varepsilon) = x^* + e^{M^*t}[\int_0^t e^{M^*s}C(s)ds - (e^{M^*\omega} - I)^{-1}e^{M^*\omega}\int_0^\omega e^{M^*s}C(s)ds]\varepsilon + O(\varepsilon) \quad (3.33)$$

where I is the identity matrix.

The above results imply that a small periodic influx of pollutant causes a periodic behaviour in the system.

Case II: When the species wholly depends on the resource.

In this case, the function $r(B)$ satisfies condition (3.4). In the case of constant emission of pollutant into the environment it can be seen that model (3.9) has three nonnegative equilibria, namely, $E_{21}(0, 0, \frac{Q_0}{\delta_0(1-\theta_0\theta_1)} \frac{Q_0\theta_0}{\delta_1(1-\theta_0\theta_1)})$, $E_{22}(0, \tilde{B}, \tilde{T}, \tilde{U})$ and $E_{23}(\hat{N}, \hat{B}, \hat{T}, \hat{U})$. The equilibrium E_{23} exists under the same condition (3.15) as discussed in case I by replacing E_{14} by E_{23}. The model can be analysed in the similar way as in case I and the corresponding theorems can be deduced. In particular, it may be noted here that if the environmental concentration of the pollutant approaches to a critical level $T = T_a$, then the density of the resource biomass may tend to zero and thus the species will be driven to extinction.

Case III: When the species is predating on the resource.

In this case, the function $r(B)$ satisfies condition (3.5). Here model (3.9) has again three nonnegative equilibria, namely, $E_{31}(0, 0, \frac{Q_0}{\delta_0(1-\theta_0\theta_1)} \frac{Q_0\theta_0}{\delta_1(1-\theta_0\theta_1)})$, $E_{32}(0, B, T, U)$ and $E_{33}(N, B, T, U)$. The equilibrium E_{33} exists under the same condition (3.15) as discussed in case I by replacing E_{14} by E_{33}. The analysis can be carried out in the similar fashion as in case I. In particular, it is noted here that the survival of the species in this case is highly threatened.

remark 2 If N_{11}^*, N_{12}^* and N_{13}^* be the equilibrium levels of the population in case I, case II and case III respectively, then from (3.10) it follows that $N_{11}^* > N_{12}^* > N_{13}^*$. Further in case III, it is also noted that the species population will survive only if $B > B_a$, where B_a is defined in (3.5). Now if B^*_{11}, B^*_{12} and B^*_{13} be the equilibrium levels of the resource biomass in case I, case II and case III respectively, then from (3.11) it follows that under same environmental and uptake concentration of the toxicant, $B^*_{11} < B^*_{12} < B^*_{13}$. Thus if the density of the population increases, the density of the resource biomass decreases.

3.5 Model With Diffusion

In this section we consider the compelete model (3.1)-(3.2) and state the main results in the form of the following theorem.

Theorem 3.5.1 *(i) If the equilibrium E_{14} is globally asymptotically, then the corresponding uniform steady state of the initial-boundary value problems (3.1)-(3.2) is also globally asymptotically stable.*
(ii) If the equilibrium E_{14} is unstable, then the uniform steady state of the initial-boundary value problems (3.1)-(3.2) can be made stable by increasing diffusion coefficients appropriately.

Proof: Let us consider the following positive definite function

$$V_2(N(t), B(t), T(t), U(t)) = \int\int_D V_1(N, B, T, U)dA,$$

where V_1 is given in equation (3.18).

We have,

$$\begin{aligned} \frac{dV_2}{dt} &= \int\int_D \{\frac{\partial V_1}{\partial N}\frac{\partial N}{\partial t} + \frac{\partial V_1}{\partial B}\frac{\partial B}{\partial t} + \frac{\partial V_1}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial V_1}{\partial U}\frac{\partial U}{\partial t}\}dA \\ &= I_1 + I_2, \end{aligned} \tag{3.34}$$

where

$$I_1 = \int\int_D \frac{dV_1}{dt}dA$$

$$I_2 = \int\int_D \{D_1 \frac{\partial V_1}{\partial N} \nabla^2 N + D_2 \frac{\partial V_1}{\partial B} \nabla^2 B + D_3 \frac{\partial V_1}{\partial T} \nabla^2 T\} dA$$

We note the following properties of V_1, namely,

$$\left.\frac{\partial V_1}{\partial N}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial B}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial T}\right]_{\partial D} = 0$$

and for all points of D,

$$\frac{\partial^2 V_1}{\partial N \partial B} = \frac{\partial^2 V_1}{\partial N \partial T} = \frac{\partial^2 V_1}{\partial N \partial U} = \frac{\partial^2 V_1}{\partial B \partial T} = \frac{\partial^2 V_1}{\partial B \partial U} = \frac{\partial^2 V_1}{\partial T \partial U} = 0,$$
$$\frac{\partial^2 V_1}{\partial N^2} > 0, \ \frac{\partial^2 V_1}{\partial B^2} > 0, \ \frac{\partial^2 V_1}{\partial T^2} > 0 \ and \ \frac{\partial^2 V_1}{\partial U^2} > 0.$$

We now consider I_2 and determine the sign of each term.

Under an analysis similar to Chapter 2, it can be seen that

$$\begin{aligned} \int\int_D \{\frac{\partial V_1}{\partial N} \nabla^2 N\} \ dA &= -\int\int_D (\frac{\partial^2 V_1}{\partial N^2})\{(\frac{\partial N}{\partial x})^2 + (\frac{\partial N}{\partial y})^2\} \ dA \le 0, \\ \int\int_D \{\frac{\partial V_1}{\partial B} \nabla^2 B\} \ dA &= -\int\int_D (\frac{\partial^2 V_1}{\partial B^2})\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\} \ dA \le 0, \\ \int\int_D \{\frac{\partial V_1}{\partial T} \nabla^2 T\} \ dA &= -\int\int_D (\frac{\partial^2 V_1}{\partial T^2})\{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\} \ dA \le 0, \end{aligned} \quad (3.35)$$

i.e., $I_2 \le 0$.

Thus we note that if $I_1 \le 0$, i.e., if the positive equilibrium E_{14} of model (3.9) is globally asymptotically stable, then the uniform steady state of the initial-boundary value problems (3.1)-(3.2) also must be globally asymptotically stable. This proves the first part of Theorem 3.5.1.

We further note that if $\frac{dV_1}{dt} > 0$, i.e., if $I_1 > 0$, then E_{14} will be unstable in the absence of diffusion. But Eqs. (3.34) and (3.35) show that by increasing diffusion coefficients D_1, D_2 and D_3 sufficiently large, $\frac{dV_2}{dt}$ can be made negative even if $I_1 > 0$. This proves the second part of Theorem 3.5.1.

We shall explain the above theorem for a rectangular habitat D defined by

$$D = \{(x, y) : \ 0 \le x \le a, \ 0 \le y \le b\} \quad (3.36)$$

in the form of the following theorem.

Theorem 3.5.2 *In addition to assumptions (3.3), (3.6)-(3.8) let* $r(B)$, $K(B,T)$, $r_B(U,N)$ *and* $K_B(T)$ *satisfy the inequalities in (3.19). If the following inequalities hold:*

$$\{\rho_1 + \rho_3 + \frac{r_0 N_c \chi_1}{K_{m1}^2}\}^2 < \frac{2}{3}\{\frac{r_0}{K(B^*,T^*)} + \frac{D_1 N^* \pi^2 (a^2+b^2)}{a^2 b^2 N_c^2}\} \times \{\frac{r_{B0}}{K_B(T^*)} + \frac{D_2 B^* \pi^2 (a^2+b^2)}{a^2 b^2 K_{B0}^2}\}, \tag{3.37}$$

$$\{\frac{r_0 N_c \chi_2}{K_{m1}^2}\}^2 < \frac{2}{3}\{\frac{r_0}{K(B^*,T^*)} + \frac{D_1 N^* \pi^2 (a^2+b^2)}{a^2 b^2 N_c^2}\} \times \{\delta_0 + \alpha B^* + \frac{D_3 \pi^2 (a^2+b^2)}{a^2 b^2}\}, \tag{3.38}$$

$$\{\frac{r_{B0} K_{B0} \chi_3}{K_{m2}^2} + (\alpha + \pi\nu) Q_c\}^2 < \frac{4}{9}\{\frac{r_{B0}}{K_B(T^*)} + \frac{D_2 B^* \pi^2 (a^2+b^2)}{a^2 b^2 K_{B0}^2}\} \times \{\delta_0 + \alpha B^* + \frac{D_3 \pi^2 (a^2+b^2)}{a^2 b^2}\}, \tag{3.39}$$

$$\{\rho_2 + \beta + (\alpha + \nu) Q_c\}^2 < \frac{2}{3}\{\frac{r_{B0}}{K_B(T^*)} + \frac{D_2 B^* \pi^2 (a^2+b^2)}{a^2 b^2 K_{B0}^2}\} \times (\delta_1 + \nu B^*), \tag{3.40}$$

$$\{\theta_0 \delta_0 + \theta_1 \delta_1 + (\alpha + \pi\nu) B^*\}^2 < \frac{2}{3}\{\delta_0 + \alpha B^* + \frac{D_3 \pi^2 (a^2+b^2)}{a^2 b^2}\} \times (\delta_1 + \nu B^*), \tag{3.41}$$

then the uniform steady state of the initial boundary value problems (3.1)-(3.2) is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Let us consider the rectangular region D given by Eq. (3.36). In this case I_2, which is defined in Eq. (3.34), can be written as

$$I_2 = - D_1 \int\int_D (\frac{\partial^2 V_1}{\partial N^2})\{(\frac{\partial N}{\partial x})^2 + (\frac{\partial N}{\partial y})^2\}\, dA - D_2 \int\int_D (\frac{\partial^2 V_1}{\partial B^2})\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\} - D_3 \int\int_D (\frac{\partial^2 V_1}{\partial T^2})\{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\}\, dA. \tag{3.42}$$

From Eq. (3.25) we get

$$\frac{\partial^2 V_1}{\partial N^2} = \frac{N^*}{N^2},$$
$$\frac{\partial^2 V_1}{\partial B^2} = \frac{B^*}{B^2},$$

and

$$\frac{\partial^2 V_1}{\partial T^2} = 1.$$

Hence

$$\begin{aligned} I_2 \leq & - \frac{D_1 N^*}{N_c^2} \int\int_D \{(\frac{\partial N}{\partial x})^2 + (\frac{\partial N}{\partial y})^2\} \, dA - \frac{D_2 B^*}{K_{B0}^2} \int\int_D \{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\} \, dA \\ & - D_3 \int\int_D \{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\} \, dA. \end{aligned}$$

Now

$$\begin{aligned} \int\int_D (\frac{\partial N}{\partial x})^2 \, dA &= \int\int_D \{\frac{\partial (N - N^*)}{\partial x}\}^2 \, dA \\ &= \int_0^b \int_0^a \{\frac{\partial (N - N^*)}{\partial x}\}^2 \, dx dy \end{aligned}$$

Letting $z = \frac{x}{a}$, it can be seen under an analysis similar to chapter 2 that

$$\begin{aligned} \int\int_D (\frac{\partial N}{\partial x})^2 \, dA &\geq \frac{\pi^2}{a^2} \int\int_D (N - N^*)^2 dA \\ \int\int_D (\frac{\partial N}{\partial y})^2 \, dA &\geq \frac{\pi^2}{b^2} \int\int_D (N - N^*)^2 dA \end{aligned}$$

Thus,

$$\begin{aligned} I_2 \leq & - \frac{D_1 N^* \pi^2 (a^2 + b^2)}{a^2 b^2 N_c^2} \int\int_D (N - N^*)^2 dA - \frac{D_2 B^* \pi^2 (a^2 + b^2)}{a^2 b^2 K_{B0}^2} \int\int_D (B - B^*)^2 dA \\ & - \frac{D_3 \pi^2 (a^2 + b^2)}{a^2 b^2} \int\int_D (B - B^*)^2 dA. \end{aligned}$$

Now from (3.26) and (3.34) we get

$$\begin{aligned}
\frac{dV_2}{dt} \leq & \iint_D [-\{\frac{r_0}{K(B^*,T^*)} + \frac{D_1 N^* \pi^2(a^2+b^2)}{a^2 b^2 N_c^2}\}(N-N^*)^2 \\
& -\{\frac{r_{B0}}{K_B(T^*)} + \frac{D_2 B^* \pi^2(a^2+b^2)}{a^2 b^2 K_{B0}^2}\}(B-B^*)^2 \\
& -\{\delta_0 + \alpha B^* + \frac{D_3 \pi^2(a^2+b^2)}{a^2 b^2}\}(T-T^*)^2 \\
& -(\delta_1 + \nu B^*)(U-U^*)^2 \\
& +\{\eta_1(B) + \eta_3(U^*,N) - r_0 N \xi_1(B,T)\}(N-N^*)(B-B^*) \\
& +\{-r_0 N \xi_2(B^*,T)\}(N-N^*)(T-T^*) \\
& +\{-r_{B0} B \xi_3(T) - \alpha T + \pi \nu U\}(B-B^*)(T-T^*) \\
& +\{\frac{d_1 \theta \beta}{a_1} + \eta_2(U,N) + \alpha T - \nu U\}(B-B^*)(U-U^*) \\
& +\{\theta_0 \delta_0 + \theta_1 \delta_1 + \alpha B^* + \pi \nu B^*\}(T-T^*)(U-U^*)]dA, \qquad (3.43)
\end{aligned}$$

where $\eta_1(B)$, $\eta_2(U,N)$, $\eta_3(U^*,N)$, $\xi_1(B,T)$ $\xi_2(B^*,T)$ *and* $\xi_3(T)$ are defined in Eq. (3.26).

Now Eq. (3.43) can be written as the sum of the quadratics

$$\begin{aligned}
\frac{dV_2}{dt} \leq & \iint_D \{-\frac{1}{2}b_{11}(N-N^*)^2 + b_{12}(N-N^*)(B-B^*) - \frac{1}{2}b_{22}(B-B^*)^2 \\
& -\frac{1}{2}b_{11}(N-N^*)^2 + b_{13}(N-N^*)(T-T^*) - \frac{1}{2}b_{33}(T-T^*)^2 \\
& -\frac{1}{2}b_{22}(B-B^*)^2 + b_{23}(B-B^*)(T-T^*) - \frac{1}{2}b_{33}(T-T^*)^2 \\
& -\frac{1}{2}b_{22}(B-B^*)^2 + b_{24}(B-B^*)(U-U^*) - \frac{1}{2}b_{44}(U-U^*)^2 \\
& -\frac{1}{2}b_{33}(T-T^*)^2 + b_{13}(T-T^*)(U-U^*) - \frac{1}{2}b_{44}(U-U^*)^2\}dA,
\end{aligned}$$

where

$$\begin{aligned}
b_{11} &= \frac{r_0}{K(B^*,T^*)} + \frac{D_1 N^* \pi^2(a^2+b^2)}{a^2 b^2 N_c^2}, \\
b_{22} &= \frac{r_{B0}}{K_B(T^*)} + \frac{D_2 B^* \pi^2(a^2+b^2)}{a^2 b^2 K_{B0}^2}, \\
b_{33} &= \delta_0 + \alpha B^* + \frac{D_3 \pi^2(a^2+b^2)}{a^2 b^2}, \\
b_{44} &= \delta_1 + \nu B^*, \; b_{12} = \eta_1(B) + \eta_3(U^*,N) - r_0 N \xi_1(B,T),
\end{aligned}$$

$$b_{13} = -r_0 N \xi_2(B^*, T), \; b_{23} = -r_{B0} B \xi_3(T) - \alpha T + \pi \nu U,$$
$$b_{24} = \beta + \eta_2(U, N) + \alpha T - \nu U,$$
$$b_{34} = \theta_0 \delta_0 + \theta_1 \delta_1 + \alpha B^* + \pi \nu B^*.$$

Sufficient conditions for $\frac{dV_2}{dt}$ to be negative definite are that the following conditions hold:

$$b_{12}^2 < b_{11} b_{22}, \tag{3.44}$$
$$b_{13}^2 < b_{11} b_{33}, \tag{3.45}$$
$$b_{23}^2 < b_{22} b_{33}, \tag{3.46}$$
$$b_{24}^2 < b_{22} b_{44}, \tag{3.47}$$
$$b_{34}^2 < b_{33} b_{44}. \tag{3.48}$$

We note that (3.37) $\Rightarrow$ (3.44), (3.38) $\Rightarrow$ (3.45), (3.39) $\Rightarrow$ (3.46), (3.40) $\Rightarrow$ (3.47) and (3.41) $\Rightarrow$ (3.47). Hence V_2 is a Liapunov function with respect to E_{14} whose domain contains the region of attraction Ω_1, proving the theorem.

From the above theorem we note that inequalities (3.37)-(3.41) may be satisfied by increasing D_1, D_2 and D_3 to sufficiently large values. This implies that in the case of diffusion stability is more plausible than the case of no diffusion. Thus in the case of diffusion the resource biomass converges towards its carrying capacity faster than the case of no diffusion, and hence the survival of resource dependent species may be ensured.

3.6 Conservation Model

It has been noted that uncontrolled environmental pollution may lead to the extinction of resource biomass. Therefore, some kind of effort must be adopted to conserve the resource (Munn and Fedorov, 1986; Huttl and Wisniewski, 1987; Lamberson, 1986; Shukla et al., 1989; Reed and Heras, 1992; Shukla and Dubey, 1997; Dubey, 1997a). In this section a mathematical model is proposed to conserve the resource biomass by some efforts and by controlling environmental pollution by some mechanism. It is assumed that the effort applied to conserve the resource is proportional to the depleted level of the resource from its carrying capacity, and effort applied to control the environmental pollution is proportional to its undesired level. Following Shukla et al. (1989), Dubey (1997a) and Shukla and Dubey (1997), differential equations governing the system may be written as

$$\begin{aligned}
\frac{\partial N}{\partial t} &= r(B)N - \frac{r_0 N^2}{K(B,T)} + D_1\nabla^2 N, \\
\frac{\partial B}{\partial t} &= r_B(U,N)B - \frac{r_{B0}B^2}{K_B(T)} + r_1F_1 + D_2\nabla^2 B, \\
\frac{\partial T}{\partial t} &= Q(t) - \delta_0 T - \alpha BT + \theta_1\delta_1 U + \pi\nu BU - r_2F_2 + D_3\nabla^2 T, \\
\frac{\partial U}{\partial t} &= \beta B + \theta_0\delta_0 T - \delta_1 U + \alpha BT - \nu BU, \qquad (3.49)\\
\frac{\partial F_1}{\partial t} &= \mu_1(1 - \frac{B}{K_{B0}}) - \nu_1 F_1, \\
\frac{\partial F_2}{\partial t} &= \mu_2(T - T_c)H(T - T_c) - \nu_2 F_2, \\
0 &\le \theta_0,\ \theta_1,\ \pi \le 1.
\end{aligned}$$

The following initial and boundary conditions are imposed on the system:

$$\begin{aligned}
&N(x,y,0) = \phi(x,y) \ge 0,\ B(x,y,0) = \psi(x,y) \ge 0, \\
&T(x,y,0) = \xi(x,y) \ge 0,\ U(x,y,0) = \chi(x,y) \ge 0, \\
&F_1(x,y,0) = \chi_1(x,y) \ge 0,\ F_2(x,y,0) = \chi_2(x,y) \ge 0,\ (x,y)\epsilon D \qquad (3.50)\\
&\frac{\partial N}{\partial n} = \frac{\partial B}{\partial n} = \frac{\partial T}{\partial n} = 0,\ (x,y)\ \epsilon\ \partial D, t \ge 0,
\end{aligned}$$

where n is the unit outward normal to ∂D.

In model (3.49), $F_1(x,y,t)$ is the density of effort applied to conserve the resource biomass and $F_2(x,y,t)$ the density of effort applied to control the undesired level of environmental pollutants. r_1 is the growth rate coefficient of the resource biomass due to effort F_1 and r_2 is the depletion rate coefficient of $T(x,y,t)$ due to effort F_2. μ_1 and μ_2 are the growth rate coefficients of F_1 and F_2 respectively and ν_1 and ν_2 are their respective depreciation rate coefficients. T_c is the critical level of the environmental pollutant which is assumed to be harmless to the resource biomass. $H(t)$ denotes the unit step function which takes into account the case when $T \leq T_c$. We shall analyse the conservation model (3.49) only for the case when the rate of introduction of pollutant into the environment is constant.

3.7 Conservation Model Without Diffusion

In this case we take $D_1 = D_2 = D_3 = 0$ in the model (3.49). Then the model (3.49) has only interior equilibrium $\bar{E}(\bar{N}, \bar{B}, \bar{T}, \bar{U}, \bar{F}_1, \bar{F}_2)$, where $\bar{N}$ $\bar{B}$, $\bar{T}$, $\bar{U}$, $\bar{F}_1$ and $\bar{F}_2$ are the positive solutions of the following algebraic equations:

$$\begin{aligned}
r_0 N &= r(B)K(B, f_1(B)) = f_2(B), \ (say) \\
r_{B0} B &= \{r_B(f_3(B), f_2(B)) + \frac{r_1 F_1}{B}\} K_B(f_1(B)) \\
T &= \frac{Q_0 \nu_2(\delta_1 + \nu B) + \beta \nu_2 B(\theta_1 \delta_1 + \pi \nu B) + r_2 \mu_2 (\delta_1 + \nu B) T_c}{\nu_2\{\delta_0 \delta_1 (1 - \theta_0 \theta_1) + \delta_0 \nu B(1 - \theta_0 \pi) + \alpha \delta_1 B(1 - \theta_1) + \pi \nu B^2 (1 - \pi)\} + r_2 \mu_2 (\delta_1 + \nu B} \\
&= f_1(B), (say) \\
U &= \frac{\beta B + (\theta_0 \delta_0 + \alpha B) f_1(B)}{\delta_1 + \nu B} = f_3(B), (say) \\
F_1 &= \frac{\mu_1}{\nu_1}(1 - \frac{B}{K_{B0}}), \\
F_2 &= \frac{\mu_2}{\nu_2}(T - T_c) H(T - T_c) = \begin{cases} \{\frac{\mu_2}{\nu_2}(T - T_c), & T > T_c \\ 0, & T \leq T_c \end{cases}
\end{aligned}$$

It may be noted here that for F_1 to be positive we must have

$$K_{B0} > B.$$

It is easy to check that $\bar{E}$ exists if and only if the following inequality holds at $\bar{E}$,

$$\begin{aligned}
r_{B0} &- \{r_B(f_3(B), f_2(B)) + \frac{r_1 \mu_1}{\nu_1 B}(1 - \frac{B}{K_{B0}})\} K'_B(T) f'_1(B) \\
&- \{\frac{\partial r_B}{\partial U} f'_3(B) + \frac{\partial r_B}{\partial N} f'_2(B) - \frac{r_1 \mu_1}{\nu_1 B^2}\} K_B(f_1(B)) > 0. \qquad (3.51)
\end{aligned}$$

In the following theorem it is shown that $\bar{E}$ is locally asymptotically stable, the proof of which is similar to Theorem 3.3.1 and hence is omitted.

Theorem 3.7.1 *Let the following inequalities hold:*

$$\{c_1 \frac{r_{B0}\bar{B}}{K_B^2(\bar{T})} K'_B(\bar{T}) + c_2(\pi\nu\bar{U} + \alpha\bar{T})\}^2 < \frac{c_1 c_2}{4}\{\frac{r_{B0}}{K_B(\bar{T})} + \frac{r_1 \bar{F}_1}{\bar{B}^2}\} \times (\delta_0 + \alpha\bar{B}), \quad (3.52)$$

$$\{c_2(\theta_1\delta_1 + \pi\nu\bar{B}) + c_3(\theta_0\delta_0 + \alpha\bar{B})\}^2 < \frac{1}{2} c_2 c_3 (\delta_0 + \alpha\bar{B})(\delta_1 + \nu\bar{B}), \quad (3.53)$$

where

$$c_1 = -\frac{r'(\bar{B}) + \frac{r_0\bar{N}}{K^2(\bar{B},\bar{T})}\frac{\partial K}{\partial B}}{\frac{\partial r_B}{\partial N}} > 0,$$

$$c_2 = \frac{3}{\delta_0 + \alpha\bar{B}} \frac{r_0\bar{N}^2}{K^3(\bar{B},\bar{T})} (\frac{\partial K}{\partial T})^2 > 0,$$

$$c_3 = -\frac{c_1 \frac{\partial r_B}{\partial U}}{\beta + \alpha\bar{T} + \nu\bar{U}} > 0.$$

Then equilibrium $\bar{E}$ is locally asymptotically stable.

In the following lemma, a region of attraction for system (3.49) without diffusion is established. The proof of this lemma is similar to Lemma 3.3.1 and hence is omitted.

Lemma 3.7.1 *The set*

$$\Omega_2 = \{(N,B,T,U,F_1,F_2) \; : \; 0 \le N \le \bar{N}_c, \; 0 \le B \le \bar{K}_c, \; 0 \le T + U \le \bar{Q}_c, \; 0 \le F_1 \le \frac{\mu_1}{\nu_1}, \; 0 \le F_2 \le \frac{\mu_2}{\nu_2} Q_c\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$\bar{N}_c = \frac{r(\bar{K}_c)K(\bar{K}_c, 0)}{r_0},$$

$$\bar{K}_c = \frac{K_{B0}}{2}\{1 + \sqrt{1 + \frac{4 r_1 \mu_1}{\nu_1 K_{B0} r_0}}\},$$

$$\bar{Q}_c = \frac{Q_0 + \beta\bar{K}_c}{\delta},$$

$$\delta = min\{\delta_0(1-\theta_0), \delta_1(1-\theta_1)\}.$$

The following theorem gives criteria for $\bar{E}$ to be globally asymptotically stable, whose proof is similar to Theorem 3.3.2 and hence is omitted.

Theorem 3.7.2 *In addition to the assumptions (3.3), (3.6)-(3.8) let $r(B)$, $K(B,T)$, $r_B(U,N)$ and $K_B(T)$ satisfy the following conditions in Ω_2*

$$0 \le r'(B) \le \bar{\rho}_1, 0 \le -\frac{\partial r_B(U,N)}{\partial U} \le \bar{\rho}_2, 0 \le -\frac{\partial r_B(U,N)}{\partial N} \le \bar{\rho}_3,$$
$$K_{m1} \le K(B,T) \le K(\bar{K}_c, 0), \; K_{m2} \le K_B(T) \le K_{B0}, \tag{3.54}$$
$$0 \le \frac{\partial K(B,T)}{\partial B} \le \bar{\kappa}_1, \; 0 \le -\frac{\partial K(B,T)}{\partial T} \le \bar{\kappa}_2, \; 0 \le -K'_B(T) \le \bar{\kappa}_3,$$

for some positive constants $\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{K}_{m1}, \bar{K}_{m2}, \bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$. Then if the following inequalities hold:

$$\{\bar{\rho}_1 + \bar{\rho}_3 + \frac{r_0 \bar{N}_c \bar{\kappa}_1}{\bar{K}_{m1}^2}\}^2 < \frac{1}{2} \frac{r_0}{K(\bar{B},\bar{T})} \frac{r_{B0}}{K_B(\bar{T})}, \tag{3.55}$$
$$\{\frac{r_0 \bar{N}_c \bar{k}_2}{\bar{K}_{m1}^2}\}^2 < \frac{2 r_0 (\delta_0 + \alpha \bar{B})}{K(B^*,T^*)}, \tag{3.56}$$
$$\{\frac{r_{B0} \bar{K}_c \bar{\kappa}_3}{\bar{K}_{m2}^2} + (\alpha + \pi\nu)\bar{Q}_c\}^2 < \frac{r_{B0}}{K_B(\bar{T})}(\delta_0 + \alpha \bar{B}), \tag{3.57}$$
$$\{\bar{\rho}_2 + \beta + (\alpha + \nu)\bar{Q}_c\}^2 < \frac{1}{2} \frac{r_{B0}}{K_B(\bar{T})}(\delta_1 + \nu \bar{B}), \tag{3.58}$$
$$\{\theta_0 \delta_0 + \theta_1 \delta_1 + (\alpha + \pi\nu)\bar{B})\}^2 < 2(\delta_0 + \alpha \bar{B})(\delta_1 + \nu \bar{B}), \tag{3.59}$$

then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Theorems 3.7.1 and 3.7.2 show that if suitable efforts are made to conserve the resource biomass and to control undesired level of environmental pollution, an appropriate level of the resource biomass density can be maintained and consequently the survival of the species may be ensured.

3.8 Conservation Model With Diffusion

We now consider the case when $D_i > 0 (i = 1,2,3)$ in model (3.49). We shall show that the uniform steady state $N(x,y,t) = N^*, B(x,y,t) = B^*, T(x,y,t) = T^*, U(x,y,t) = U^*, F_1(x,y,t) = F_1^*$ and $F_2(x,y,t) = F_2^*$ is globally asymptotically stable. For this, we consider the following positive definite function

$$V_3(N(t), B(t), T(t), U(t), F_1(t), F_2(t)) = \int\int_D V_2(N, B, T, U, F_1, F_2) \, dA,$$

where

$$V_2(N,B,T,U,F_1,F_2) = N - N^* - N^* \ln \frac{N}{N^*} + B - B^* - B^* \ln \frac{B}{B^*}$$
$$+\frac{1}{2}(T-T^*)^2 + \frac{1}{2}(U-U^*)^2 + \frac{c_1}{2}(F_1 - F_1^*)^2 + \frac{c_2}{2}(F_2 - F_2^*)^2$$

and

$$c_1 = \frac{r_1 K_{B0}}{\mu_1 \bar{B}}, \ c_2 = \frac{r_2}{\mu_2}$$

Then as earlier, it can be checked that if $\frac{dV_2}{dt} < 0$, then $\frac{V_3}{dt} < 0$. This implies that if E^* is globally asymptotically stable for system (3.49) without diffusion, then the corresponding uniform steady state of system (3.49)-(3.50) is also globally asymptotically stable with respect to solutions such that $\phi(x,y) > 0, \psi(x,y) > 0, \xi(x,y) > 0, \zeta(x,y) > 0, \zeta_1(x,y) > 0, \zeta_2(x,y) > 0,\ (x,y) \epsilon D$.

3.9 Numerical Examples

In this section we present a numerical example to explain the applicability of the results discussed in section (3.3) and (3.7). We take the following particular form of the functions in model (3.9):

$$\begin{aligned} r(B) &= r(0) + r_1 B, \\ K(B,T) &= K_0 + K_1 B - K_2 T, \\ r_B(U,N) &= r_{B0} - r_{B1} U - r_{B2} N, \\ K_B(T) &= K_{B0} - K_{B1} T. \end{aligned} \tag{3.60}$$

We take the following values of the various parameters in model (3.9) and in equation (3.60):

$$\begin{aligned} & r_0 = 2.0,\ r_{B0} = 1.51,\ K_0 = 60.0,\ Q_0 = 2.0,\ \delta_0 = 0.21, \\ & \alpha = 0.01,\ \theta_1 = 0.03,\ \delta_1 = 3.50,\ \pi = 0.03, \nu = 0.039, \\ & \beta = 0.01,\ \theta_0 = 0.3,\ r_1 = 0.09,\ K_1 = 0.02,\ K_2 = 0.03, \\ & r_{B1} = 0.04,\ r_{B2} = 0.01,\ K_{B0} = 3.0, K_{B1} = 0.05, \end{aligned} \tag{3.61}$$

Example 1 In this example we consider the case I, i.e., when the species depends partially on the resource. In this case we take $r(0) = 2.0$. Then with the above set of parameters given in (3.61) it can be verified that the interior equilibrium $E_{11}^*(N_{11}^*, B_{11}^*, T_{11}^*, U_{11}^*)$ exists, and is given by

$$N_{11}^* = 63.68792,\ B_{11}^* = 1.46093,\ T_{11}^* = 8.99959,\ U_{11}^* = 0.20047. \qquad (3.62)$$

It can be checked that conditions (3.16) and (3.17) in Theorem 3.3.1 are satisfied. This shows that E_{11}^* is locally asymptotically stable.

By choosing $K_{m1} = 50.0$ and $K_{m2} = 2.0$ in Theorem 3.3.2 it can be checked that conditions (3.20)-(3.24) are satisfied showing the global stability character of E_{11}^*.

Example 2 In this example, we consider the case II, i.e., when the species wholly depends upon the resource and we take $r(0) = 0$. Then with the set of values of parameters in (3.61), it can be seen that the interior equilibrium $E_{12}^*(N_{12}^*, B_{12}^*, T_{12}^*, U_{12}^*)$ exists, and is given by

$$N_{12}^* = 6.57022, B_{12}^* = 2.44197, T_{12}^* = 8.63140, U_{12}^* = 0.21667. \qquad (3.63)$$

It can also be verified that E_{12}^* is globally asymptotically stable.

Example 3 In this example, we consider the case III, i.e., when the species is predating on the resource and we take r(0)=-0.15. Then with the same set of values of parameters in (3.61), it can be checked that the interior equilibrium $E_{13}^*(N_{13}^*, B_{13}^*, T_{13}^*, U_{13}^*)$ exists, and is given by

$$N_{13}^* = 2.28523, B_{13}^* = 2.51599, T_{13}^* = 8.60484, U_{13}^* = 0.21783. \qquad (3.64)$$

It can also be seen that E_{13}^* is globally asymptotically stable.

Comparing (3.62), (3.63) and (3.64), we note that $N_{11}^* > N_{12}^* > N_{13}^*$ and $B_{11}^* < B_{12}^* < B_{13}^*$, which supports our result in Remark 2.

Example 4 In this example, we consider the conservation model without diffusion. Here we have considered only one case, namely, when the species depends partially on the resource. In addition to the values of parameters given in (3.61), we choose the following values of parameters in model (3.49) with no diffusion:

$$r_1 = .10,\ r_2 = 0.03,\ \mu_1 = 3.14,\ \nu_1 = 0.06,$$
$$\mu_2 = 3.30,\ \nu_2 = 0.06,\ T_c = 1.5. \tag{3.65}$$

Then it can be checked that condition (3.51) for the existence of the interior equilibrium $\bar{E}$ is satisfied, and $\bar{E}$ is given by

$$\bar{N} = 66.49411,\ \bar{B} = 2.41469,\ \bar{T} = 2.37838,\ \bar{U} = 0.05767,$$
$$\bar{F}_1 = 10.21035,\ \bar{F}_2 = 48.31092. \tag{3.66}$$

It can easily be verified that conditions (3.52)-(3.53) in Theorem 3.7.1 are satisfied which shows that $\bar{E}$ is locally asymptotically stable.

Further, by choosing $\bar{K}_{m1} = 50.0$ and $\bar{K}_{m2} = 2.0$ in Theorem 3.7.2, it can be checked that conditions (3.55)-(3.59) are satisfied. This shows that $\bar{E}$ is globally asymptotically stable.

By comparing equilibrium levels E_{11}^{*} and $\bar{E}$ in Eqs. (3.62) and (3.66) respectively, we note that due to efforts F_1 and F_2, the equilibrium level of the resource biomass has increased whereas equilibrium levels of the concentration of pollutant in the environment and in the resource biomass have decreased. As a consequence of increase in the resource biomass, the equilibrium level of the species has also increased, ensuring the survival of the species.

3.10 Conclusions

In this chapter, a mathematical model for the survival of a single species population dependent on resource biomass which is affected by a pollutant present in the environment has been proposed and analysed. It has been assumed that the population depends partially or wholly on the resource or just predating on the resource. It has also been assumed that the growth rate of the population increases as the density of the resource biomass increases while its carrying capacity increases with the increase in the density of the resource biomass, and decreases with the increase in the environmental concentration of the pollutant. It has been further assumed that the growth rate of the resource biomass decreases as the uptake concentration of the pollutant and density of the population increase while its carrying capacity decreases as the environmental concentration of the pollutant increases.

In the case of no diffusion the model has been completely analysed using stability theory of ordinary differential equations. When the population depends partially on the resource, it has been shown that in the case of constant introduction of pollutant into the environment, both the population and the resource biomass settle down to their respective steady states. The magnitude of the equilibrium level of the population decreases as the equilibrium level of the resource biomass density decreases and the environmental concentration of the pollutant increases. The magnitude of the equilibrium level of the resource biomass decreases as the equilibrium levels of the population, the pollutant present in the environment and in the body increase. It has also been noted that the resource biomass may tend to zero for large influx of the pollutant into the environment affecting the survival of the species. In the case of instantaneous introduction of pollutant into the environment similar results have been found. In particular, it has been noted that the population and the resource biomass after initial decrease in their densities settle down to their respective steady states but after a long time if the washout rate of the pollutant is small. In this case magnitudes of densities of the population and the resource biomass are larger than their respective densities in the case of constant introduction of pollutant. In the case of the periodic emission of the pollutant into the environment it has been found that a periodic behavior occurs in the system for a small amplitude of the influx of the pollutant.

The equilibrium levels of the population and the resource biomass has been compared in three different cases: (i) when the population partially depends upon the resource, (ii) when the population wholly depends upon the resource, and (iii) when the population is predating on the resource. It has been noted that the density of the population is maximum in the partially dependent case and minimum in the predating case and consequently the density of the resource biomass is minimum in the partially dependent case and maximum in the predation case, keeping other parameters same in the system. Thus an increase in the density of the population will also lead to decrease in the density of the resource biomass. It has also been noted that the survival of the population will be threatened even in the partially dependent case if the continuous emission of pollutant into the environment is not controlled. In the wholly dependent case the population will doom to extinction if the environmental concentration of pollutant reaches at a critical value, $T = T_a$. In the case of predation it has been noted that the survival of the population is highly threatened.

In the case of diffusion, a complete analysis of the model has been carried out. It has been shown that if the positive equilibrium of the system with no diffusion is globally asymptotically stable, then it remain globally asymptotically stable in the case of diffusion. Further, if the positive equilibrium of the system with no diffusion is unstable, then it can be stabilized by increasing diffusion coefficients to sufficiently large values. Thus it has been concluded that in the case of diffusion, solutions of the system approaches to the equilibrium level faster than the case of no diffusion.

A model to conserve the resource biomass and to control the undesired level of environmental pollution is proposed and analysed. It has been shown that if suitable efforts are made an appropriate level of the resource biomass can be maintained and the survival of the species may be ensured.

Chapter 4

SURVIVAL OF TWO COMPETING SPECIES DEPENDENT ON RESOURCE IN INDUSTRIAL ENVIRONMENTS: A MATHEMATICAL MODEL

4.1 Introduction

In recent years there has been considerable interest in the study of competition between two or more species using mathematical models (Gomatam, 1974; Hsu, 1978a; Hsu and Hubbell, 1979; Gopalsamy and Aggarwalla, 1980; Hsu, 1981b; Butler et al., 1983; Hsu and Huang, 1995). During the last two decades increasing interest has been shown to study the consumer-resource interactions, with the aim to construct more theories of interspecies competition. The question of two or more competitors living on a single resource has received much attention and has helped to understand competitive processes. Many authors have tried to answer this question using mathematical models. All these focus mainly on the coexistence of the species with respect to their resource utilization (Armstrong and McGehee, 1976; De Jong, 1976; Miller, 1966, 1976; Armstrong and McGehee, 1980; Hsu, 1981a; Gopalsamy, 1986; Mitra et al., 1992; Shukla et al., 1996). In particular, Goh (1976) found sufficient conditions for the global stability of two species. This result was extended for nonlinear two species model by Hastings (1978b). Hallam et al. (1979) derived sufficient conditions for persistence and extinction of three species in a competitive system. But in these studies the effect of resource was not included in the model. In this regard Hsu (1981a) developed a resource based competition model with interference. Gopalsamy (1986) proposed a resource based competition model and found sufficient conditions for the convergence of three-species system to an equilibrium point. Shukla et al. (1989) proposed a dynamical model to assess the effects of industrialization on the degradation of forestry biomass with diffusion. Mukherjee and Roy (1990) obtained persistence conditions of a two prey-predator system linked by competition. Mitra et al. (1992) studied the permanent coexistence and global stability of a single Lotka-Volterra type mathematical model of a living resource supporting two competing predators. Shukla et al. (1996) proposed a mathematical model to study the growth and existence of resource dependent species in a forested habitat which is being depleted due to the pressure of industrialization. Dubey (1997b) investigated a mathematical model in which two species share a common resource, and one of the species is itself an alternative food for the other. Recently, Dubey and Das (1999) investigated the survival of wildlife species dependent on resource in an industrial environment with diffusion. However, in the above investigations the survival of two competing species dependent on resource under industrialization pressure in a diffusive system has not been considered.

In this chapter we consider a dynamical model in which two species compete with each other and depend on a common resource either partially, wholly or predating on the resource and the growth of industrialization pressure depends wholly on the resource. The effect of diffusion on the stability of the system is also studied. In presence of diffusion our results agree with those in Hastings (1978a), Shukla and Verma (1981), Shukla and Shukla (1982), Shukla et al. (1989), Freedman and Shukla (1989). The stability theory of ordinary differential equations (La Salle and Lefchetz, 1961) is used to analyse the model.

4.2 Mathematical Model

Consider an ecosystem where two biological species are competing for a single resource in an industrial environment in a closed region D with smooth boundary ∂D. It is assumed that the dynamics of the resource biomass and the competing species are governed by the generalized logistic type equations. It is also assumed that the growth rate of the resource biomass and the corresponding carrying capacity decrease with the increase in industrialization pressure. The two competing species are assumed to be either partially dependent, wholly dependent or just predating on the resource. The growth rate of industrialization pressure is assumed to be wholly dependent on the resource and its dynamics is of predator-prey type. In view of these arguments, the system is assumed to be governed by the following differential equations:

$$\begin{aligned}
\frac{\partial B}{\partial t} &= r(I)B - \frac{r_0 B^2}{K(I)} - \delta_1 B N_1 - \delta_2 B N_2 + D_1 \nabla^2 B, \\
\frac{\partial N_1}{\partial t} &= r_1(B)N_1 - \frac{r_{10} N_1^2}{K_1} - \alpha_{21} N_1 N_2 + D_2 \nabla^2 N_1, \\
\frac{\partial N_2}{\partial t} &= r_2(B)N_2 - \frac{r_{20} N_2^2}{K_2} - \alpha_{12} N_1 N_2 + D_3 \nabla^2 N_2, \\
\frac{\partial I}{\partial t} &= -\beta_0 I - \beta_1 I^2 + \beta_2 I B + D_4 \nabla^2 I.
\end{aligned} \tag{4.1}$$

We impose the following initial and boundary conditions on system (4.1):

$$\begin{aligned}
& B(x,y,0) = \phi(x,y) \geq 0, \;\; N_1(x,y,0) = \psi(x,y) \geq 0, \\
& N_2(x,y,0) = \xi(x,y) \geq 0, \;\; I(x,y,0) = \chi(x,y) \geq 0, \;\; (x,y)\epsilon D \\
& \frac{\partial B}{\partial n} = \frac{\partial N_1}{\partial n} = \frac{\partial N_2}{\partial n} = \frac{\partial I}{\partial n} = 0, \;\; (x,y)\;\epsilon\;\partial D, t \geq 0,
\end{aligned} \tag{4.2}$$

where n is the unit outward normal to ∂D.

In model (4.1), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian diffusion operator. $B(x,y,t)$ is the density of the resource biomass, $N_1(x,y,t)$ and $N_2(x,y,t)$ are densities of the competing species 1 and 2 respectively and $I(x,y,t)$ the density of industrialization pressure at coordinates $(x,y) \in D$ and at time $t \geq 0$. $D_i (i = 1,2,3,4)$ are the diffusion rate coefficients of $B(x,y,t)$, $N_1(x,y,t)$, $N_2(x,y,t)$ and $I(x,y,t)$ respectively in D. α_{ij} is the interference coefficient measuring the damage effect of species i on species j. β_0 is the natural depletion rate coefficient of the industrialization pressure, β_1 the intraspecific interference coefficient of industrialization pressure and β_2 the growth rate coefficient of industrialization pressure due to resource biomass. K_i is the carrying capacity of the species i. δ_1 and δ_2 are the depletion rate coefficients of the resource biomass due to the species 1 and 2 respectively.

The coefficients β_0, β_1, β_2, K_i *and* δ_i are strictly positive and α_{21} and α_{12} are nonnegative.

In model (4.1), the function $r(I)$ denotes the specific growth rate of resource biomass which decreases as I increases, i.e.

$$r(0) = r_0 > 0, \ r'(I) < 0 \ for \ I > 0. \tag{4.3}$$

The function $K(I)$ is the maximum density of resource biomass which the environment can support and it also decreases as I increases, i.e.

$$K(0) = K_0 > 0, \ K'(I) < 0 \ for \ I > 0. \tag{4.4}$$

The function $r_i(B)$ denotes the growth rate coefficient of the species i, which increases as biomass density increases. We consider the following three types of conditions satisfied by $r_i(B)$.

$$(i) \ r_i(0) > 0, \ r_i'(B) > 0 \ for \ B \geq 0, \ i = 1,2. \tag{4.5}$$

In this case, the resource biomass is an alternative resource for the species i.

$$(ii) \ r_i(0) = 0, \ r_i'(B) > 0 \ for \ B \geq 0, \ i = 1,2. \tag{4.6}$$

In this case, the species i wholly depends upon the resource.

$$(iii)\ r_i(0) < 0,\ r_i'(B) > 0\ for\ B \geq 0, \tag{4.7}$$

and there exists a $B = B_i$ such that $r_i(B_i) = 0, i = 1, 2$.

In this case, the species i is predating on the resource.

In the next section we analyse system (4.1)-(4.2) without diffusion.

4.3 Model Without diffusion

In the case of no diffusion (i.e., when $D_i = 0$, i=1,2,3,4), model (4.1) reduces to

$$\begin{aligned}
\frac{dB}{dt} &= r(I)B - \frac{r_0 B^2}{K(I)} - \delta_1 BN_1 - \delta_2 BN_2, \\
\frac{dN_1}{dt} &= r_1(B)N_1 - \frac{r_{10}N_1^2}{K_1} - \alpha_{21}N_1N_2, \\
\frac{dN_2}{dt} &= r_2(B)N_2 - \frac{r_{20}N_2^2}{K_2} - \alpha_{12}N_1N_2, \\
\frac{dI}{dt} &= -\beta_0 I - \beta_1 I^2 + \beta_2 IB, \\
& B(0) \geq 0,\ N_1(0) \geq 0,\ N_2(0) \geq 0,\ I(0) \geq 0.
\end{aligned} \tag{4.8}$$

Now we shall analyse the above model in three different cases, namely, when the competing species are partially dependent, wholly dependent or predating on the resource.

Case I: When the competing species partially depend on the resource

In this case the function $r_i(B)$ satisfies condition (4.5) and we take $r_i(0) = r_{i0} > 0, i = 1, 2$. We note that model (4.8) has twelve nonnegative equilibria, namely, $E_0(0,0,0,0)$, $E_1(K_0,0,0,0)$, $E_2(0,K_1,0,0)$, $E_3(0,0,K_2,0)$, $E_4(\bar{B},\bar{N}_1,0,0)$, $E_5(\hat{B},0,\hat{N}_2,0)$, $E_6(\tilde{B},0,0,\tilde{I})$, $E_7(0,N_{1p},N_{2p},0)$, $E_8(B_q,N_{1q},N_{2q},0)$, $E_9(B_r,N_{1r},0,I_r)$, $E_{10}(B_s,0,N_{2s},I_s)$ and $E(B^*,N_1^*,N_2^*,I^*)$.

The equilibria $E_0 - E_3$ obviously exist. We shall show the existence of other equilibria as follows.

Existence of $E_4(\bar{B}, \bar{N}_1, 0, 0)$:

Here $\bar{B}$ and $\bar{N}_1$ are the positive solutions of the following algebraic equations:

$$r_0 B = r_0 K_0 - \delta_1 K_0 N_1, \tag{4.9}$$

$$r_{10} N_1 = K_1 r_1(B). \tag{4.10}$$

It is easy to check that the isoclines (4.9) and (4.10) intersect at a unique point $(\bar{B}, \bar{N}_1)$ iff

$$r_0 > \delta_1 K_1. \tag{4.11}$$

The inequality (4.11) gives the necessary and sufficient condition for the survival of species 1 dependent on resource in the absence of the species 2 and the industrialization pressure.

Existence of $E_5(\hat{B}, 0, \hat{N}_2, 0)$:

Here $\hat{B}$ and $\hat{N}_2$ are the positive solutions of the following algebraic equations:

$$r_0 B = r_0 K_0 - \delta_2 K_0 N_2, \tag{4.12}$$

$$r_{20} N_2 = K_2 r_2(B). \tag{4.13}$$

Again it can be verified that isoclines (4.12) and (4.13) intersect at a unique point $(\hat{B}, \hat{N}_2)$ iff

$$r_0 > \delta_2 K_2. \tag{4.14}$$

The inequality (4.14) gives the necessary and sufficient condition for the survival of species 2 dependent on resource in the absence of the species 1 and the industrialization pressure.

Existence of $E_6(\tilde{B}, 0, 0, \tilde{I})$:

Here $\tilde{B}$ and $\tilde{I}$ are the positive solutions of the following algebraic equations:

$$r_0 B = r(I) K(I), \tag{4.15}$$

$$\beta_2 B = \beta_0 + \beta_1 I. \tag{4.16}$$

It is easy to check that the two isoclines (4.15) and (4.16) intersect at a unique point $(\tilde{B}, \tilde{I})$ iff

$$\beta_2 K_0 \ > \ \beta_0. \tag{4.17}$$

The inequality (4.17) gives the necessary and sufficient condition for the survival of the resource dependent industrialization in absence of the competing species.

Existence of $E_7(0, N_{1p}, N_{2p}, 0)$:

Here

$$N_{1p} = \frac{K_1 r_{20}(r_{10} - \alpha_{21} K_2)}{r_{10} r_{20} - \alpha_{12}\alpha_{21} K_1 K_2} \tag{4.18}$$

$$and \qquad N_{2p} = \frac{K_2 r_{10}(r_{20} - \alpha_{12} K_1)}{r_{10} r_{20} - \alpha_{12}\alpha_{21} K_1 K_2}. \tag{4.19}$$

The necessary and sufficient conditions for the survival of the two competing species are

$$r_{10} > \alpha_{21} K_2 \ and \ r_{20} > \alpha_{12} K_1. \tag{4.20}$$

It may be noted here that the two competing species will survive even if both the inequalities are reversed in Eq. (4.20).

Existence of $E_8(B_q, N_{1q}, N_{2q}, 0)$:

Here B_q, N_{1q} and N_{2q} are the positive solutions of the following algebraic equations:

$$r_0 B = (r_0 - \delta_1 N_1 - \delta_2 N_2) K_0, \tag{4.21}$$

$$N_1 = \frac{K_1\{r_1(B) r_{20} - r_2(B)\alpha_{21} K_2\}}{r_{10} r_{20} - \alpha_{12}\alpha_{21} K_1 K_2} = f_1(B), \ (say) \tag{4.22}$$

$$N_2 = \frac{K_2\{r_2(B) r_{10} - r_1(B)\alpha_{12} K_1\}}{r_{10} r_{20} - \alpha_{12}\alpha_{21} K_1 K_2} = f_2(B), \ (say) \tag{4.23}$$

Substituting the values of N_1 and N_2 in Eq. (4.21) we get

$$r_0 B = \{r_0 - \delta_1 f_1(B) - \delta_2 f_2(B)\} K_0, \tag{4.24}$$

Taking

$$F(B) = r_0 B - \{r_0 - \delta_1 f_1(B) - \delta_2 f_2(B)\} K_0, \tag{4.25}$$

we note that

$$F(0) = -\{r_0 - \delta_1 f_1(0) - \delta_2 f_2(0)\}K_0 < 0,$$
$$F(K_0) = \{\delta_1 f_1(K_0) + \delta_2 f_2(K_0)\}K_0 > 0.$$

Thus there exists a B_q in the interval $0 < B_q < K_0$ such that $F(B_q) = 0$.

For B_q to be unique we must have

$$F'(B) = r_0 + \{\delta_1 f_1'(B) + \delta_2 f_2'(B)\}K_0 > 0. \tag{4.26}$$

Thus, knowing the value of B_q, the values of N_{1q} and N_{2q} can then be computed from Eq. (4.22) and (4.23) respectively. It may be noted here that for N_1 and N_2 to be positive either

$$r_1(B)r_{20} > r_2(B)\alpha_{21}K_2,\ r_2(B)r_{10} > r_1(B)\alpha_{12}K_1 \tag{4.27}$$

or

$$r_1(B)r_{20} < r_2(B)\alpha_{21}K_2,\ r_2(B)r_{10} < r_1(B)\alpha_{12}K_1 \tag{4.28}$$

must be satisfied.

Thus E_8 exists if condition (4.26) and either (4.27) or (4.28) hold.

Existence of $E_9(B_r, N_{1r}, 0, I_r)$:

Here B_r, N_{1r} and I_r are the positive solutions of the system of algebraic equations:

$$r_0 B = \{r(I) - \delta_1 N_1\}K(I), \tag{4.29}$$
$$N_1 = \frac{K_1 r_1(B)}{r_{10}} = g_1(B),\ (say) \tag{4.30}$$
$$I = \frac{-\beta_0 + \beta_2 B}{\beta_1} = h_1(B).\ (say) \tag{4.31}$$

As in the existence of E_8, it can be shown that E_9 exists iff

$$r_0 - \frac{\partial K}{\partial I}h_1'(B)\{r(h_1(B)) - \delta_1 g_1(B)\} - K(h_1(B))\{\frac{\partial r}{\partial I}h_1'(B) - \delta_1 g_1'(B)\} > 0. \tag{4.32}$$

Existence of $E_{10}(B_s, 0, N_{2s}, I_s)$:

Here B_s, N_{1s} and I_s are the positive solutions of the system of algebraic equations:

$$r_0 B = r(I) - \delta_2 N_2 K(I), \tag{4.33}$$

$$N_2 = \frac{K_2 r_2(B)}{r_{20}} = g_2(B), \ (say) \tag{4.34}$$

$$I = \frac{-\beta_0 + \beta_2 B}{\beta_1} = h_2(B). \ (say) \tag{4.35}$$

As in the existence of E_9, it can be shown that E_{10} exists iff

$$r_0 - \frac{\partial K}{\partial I} h_2'(B)\{r(h_2(B)) - \delta_2 g_2(B)\} - K(h_2(B))\{\frac{\partial r}{\partial I} h_2'(B) - \delta_2 g_2'(B)\} > 0. \tag{4.36}$$

Existence of $E^*(B^*, N_1^*, N_2^*, I^*)$:

Here B^*, N_1^*, N_2^* and I^* are the positive solutions of the following algebraic equations:

$$r_0 B = (r(I) - \delta_1 N_1 - \delta_2 N_2) K(I), \tag{4.37}$$

$$N_1 = \frac{K_1\{r_1(B) r_{20} - r_2(B)\alpha_{21} K_2\}}{r_{10} r_{20} - \alpha_{12}\alpha_{21} K_1 K_2} = f(B), \ (say) \tag{4.38}$$

$$N_2 = \frac{K_2\{r_2(B) r_{10} - r_1(B)\alpha_{12} K_1\}}{r_{10} r_{20} - \alpha_{12}\alpha_{21} K_1 K_2} = g(B), \ (say) \tag{4.39}$$

$$I = \frac{-\beta_0 + \beta_2 B}{\beta_1} = h(B). \ (say) \tag{4.40}$$

It can be checked that E^* exists iff

$$r_0 - \frac{\partial K}{\partial I} h'(B)\{r(h(B)) - \delta_1 f(B) - \delta_2 g(B)\} - K(h(B))\{\frac{\partial r}{\partial I} h'(B) - \delta_1 f'(B) - \delta_2 g'(B)\} > 0. \tag{4.41}$$

and any one of the conditions (4.27) and (4.28) are satisfied.

To study the local stability behaviour of equilibria, we first compute the variational matrices (Freedman, 1987b) corresponding to these equilibria. From these matrices we conclude the following.

E_0 is a saddle point whose stable manifold is locally in the I-direction and unstable manifold locally in the $B - N_1 - N_2$ space. E_1 is also a saddle point with stable manifold locally in the B-direction and unstable manifold locally in the $N_1 - N_2 - I$ space. E_2 is also a saddle point with stable manifold locally in the $N_1 - I$ plane and unstable manifold locally in the $B - N_2$ plane (here $r_{20} - \alpha_{12}K_1$ is taken to be positive). E_3 is also a saddle point with stable manifold locally in the $N_2 - I$ plane and unstable manifold locally in the $B - N_1$ plane (here $r_{10} - \alpha_{21}K_2$ is taken to be positive). E_4 is also a saddle point with stable manifold locally in the $B - N_1$ plane and unstable manifold locally in the $N_2 - I$ plane. E_5 is also a saddle point with stable manifold locally in the $B - N_2$ plane and with unstable manifold locally in the $N_1 - I$ plane. E_6 is also a saddle point with stable manifold locally in the $B - I$ plane and with unstable manifold locally in the $N_1 - N_2$ plane. E_7 is a saddle point with unstable manifold locally in the B-direction and with stable manifold locally in the $N_1 - N_2 - I$ space. E_8 is locally unstable in the I-direction, E_9 is locally unstable in the N_2-direction and E_{10} is locally unstable in the N_1-direction.

In the following theorem we show that E^* is locally asymptotically stable.

Theorem 4.3.1 *Let the following inequality holds*

$$\{c_1\alpha_{21} + c_2\alpha_{12}\}^2 < c_1 c_2 \frac{r_{10}r_{20}}{K_1K_2}, \tag{4.42}$$

where

$$c_1 = \frac{\delta_1}{r_1'(B^*)}, \tag{4.43}$$

$$c_2 = \frac{\delta_2}{r_2'(B^*)}. \tag{4.44}$$

Then E^* is locally asymptotically stable.

Proof: By using the transformations

$$B = B^* + b,\ N_1 = N_1^* + n_1,\ N_2 = N_2^* + n_2,\ I = I^* + i$$

we first linearize the system (4.8). Then we consider the following positive definite function in the linearized form of model (4.8),

$$V = \frac{1}{2}\{\frac{b^2}{B^*} + c_1\frac{n_1^2}{N_1^*} + c_2\frac{n_2^2}{N_2^*} + c_3\frac{i^2}{I^*}\}, \tag{4.45}$$

where c_1, c_2 are given by Eqs. (4.43) and (4.44) and

$$c_3 = -\frac{1}{\beta_2}\{r'(I^*) + \frac{r_0 B^*}{K^2(I^*)}K'(I^*)\} > 0.$$

It can easily be verified that the derivative of V with respect t along the solutions of model (4.8) is negative definite under condition (4.42), proving the theorem.

In order to show that E^* is globally asymptotically stable, we need the following lemma which establishes a region of attraction for system (4.8). The proof of this lemma is easy and hence is omitted.

Lemma 4.3.1 *The set*

$$\Omega = \{(B, N_1, N_2, I) : 0 \leq B \leq K_0,\ 0 \leq N_1 \leq \frac{K_1 r_1(K_0)}{r_{10}},$$

$$0 \leq N_2 \leq \frac{K_2 r_2(K_0)}{r_{20}}\ 0 \leq I \leq \frac{-\beta_0 + \beta_2 K_0}{\beta_1}\}$$

attracts all solutions initiating in the positive orthant.

In the following theorem global stability behaviour of E^* is studied.

Theorem 4.3.2 *In addition to assumptions (4.3)-(4.5) let $r(I)$, $K(I)$, $r_1(B)$ and $r_2(B)$ satisfy the following conditions in Ω*

$$0 \leq -r'(I) \leq \rho_0,\ 0 \leq r_1'(B) \leq \rho_1,\ 0 \leq r_2'(B) \leq \rho_2,\ 0 \leq -K'(I) \leq \rho_3$$
$$\text{and } K_m \leq K(I) \leq K_0, \tag{4.46}$$

for some positive constants ρ_0, ρ_1, ρ_2, ρ_3 and K_m. If the following inequalities hold:

$$\{\rho_0 + \frac{r_0 K_0 \rho_3}{K_m^2} + \beta_2\}^2 < \frac{4}{3}\beta_1 \frac{r_0}{K(I^*)}, \tag{4.47}$$

$$\{\rho_2 \delta_1 \alpha_{21} + \rho_1 \delta_2 \alpha_{12}\}^2 < \frac{r_{10} r_{20}}{K_1 K_2}\rho_1 \rho_2 \delta_1 \delta_2, \tag{4.48}$$

then E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: We consider the following positive definite function around E^*,

$$\begin{aligned} V_1(B, N_1, N_2, I) &= B - B^* - B^* \, ln(\frac{B}{B^*}) + a_1(N_1 - N_1^* - N_1^* \, ln(\frac{N_1}{N_1^*}) \\ &\quad + a_2(N_2 - N_2^* - N_2^* \, ln(\frac{N_2}{N_2^*}) + I - I^* - I^* \, ln(\frac{I}{I^*}). \end{aligned} \quad (4.49)$$

where a_1 and a_2 are positive constants to be chosen suitably.

Differentiating V_1 with respect to t along the solutions of system (4.8), we get

$$\begin{aligned} \frac{dV_1}{dt} &= (B - B^*)[r(I) - \frac{r_0 B}{K(I)} - \delta_1 N_1 - \delta_2 N_2] \\ &\quad + a_1(N_1 - N_1^*)[r_1(B) - \frac{r_{10} N_1}{K_1} - \alpha_{21} N_2] \\ &\quad + a_2(N_2 - N_2^*)[r_2(B) - \frac{r_{20} N_2}{K_2} - \alpha_{12} N_1] \\ &\quad + (I - I^*)[\beta_0 - \beta_1 I + \beta_2 B]. \end{aligned}$$

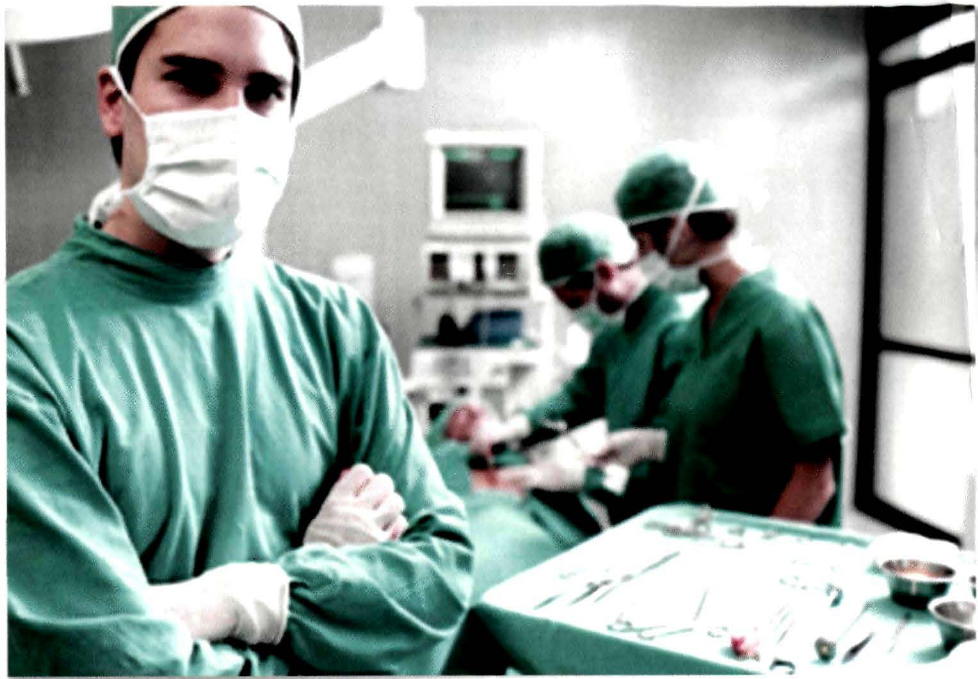
Using (4.37)-(4.40), a little algebraic manipulation yields

$$\begin{aligned} \frac{dV_1}{dt} &= -\frac{r_0}{K(I^*)}(B - B^*)^2 - a_1 \frac{r_{10}}{K_1}(N_1 - N_1^*)^2 \\ &\quad - a_2 \frac{r_{20}}{K_2}(N_2 - N_2^*)^2 - \beta_1 (I - I^*)^2 \\ &\quad + [a_1 \xi_1(B) - \delta_1)](B - B^*)(N_1 - N_1^*) \\ &\quad + [a_2 \xi_2(B) - \delta_2)](B - B^*)(N_2 - N_2^*) \\ &\quad + [\eta_1(I) - r_0 B \eta_2(I) + \beta_2](B - B^*)(I - I^*) \\ &\quad + [a_1 \alpha_{21} + a_2 \alpha_{12}](N_1 - N_1^*)(N_2 - N_2^*), \end{aligned} \quad (4.50)$$

where

$$\eta_1(I) = \begin{cases} \frac{r(I) - r(I^*)}{I - I^*}, & I \neq I^* \\ r'(I), & I = I^* \end{cases},$$

$$\eta_2(I) = \begin{cases} \{\frac{1}{K(I)} - \frac{1}{K(I^*)}\}/(I - I^*), & I \neq I^* \\ -\frac{1}{K^2(I)} K'(I), & I = I^* \end{cases},$$



NH Narayana
Superspeciality Hospital
Unit of Narayana Health

Short Stay Surgery

Minimal Incision • Better Precision
Faster Recovery • Short Hospital Stay

Whole spectrum of Minimal Invasive Surgery under One Roof
Conventional Laparoscopy Surgery • Mini Laparoscopy Surgery (suture less) • Single Incision Surgery • Reduced Port Surgery

MAKING MINIMALLY INVASIVE SURGERY EVEN LESS INVASIVE

Why is it preferred?

Minimally invasive surgery and hence short stay ensures smaller incision, greater precision, faster recovery since it lessens the post-operative complications.

Benefits of Short Stay Surgery

It brings down post-surgery complications like chances of chest and breathing problems, infections, disfigurement of the abdominal wall and incisional hernia • It reduces blood loss and trauma • It allows quicker recovery • The patient can get back home the same day or the next morning. It is a nearly scar-less procedure and yields excellent cosmetic result

Short Stay Surgery (Laparoscopy)

The advancement in technology have made recovery faster. The stay in the hospital has been reduced to 24-72 hours. Your surgery can occur in the morning and you can recover in the comfort and convenience of your own home later the same day. The goal of our team of professionals is for you to have and excellent experience with the best possible outcome which is why our facility will be the first choice if, in the future, you find yourself in need of hospital services.

Services

- Laparoscopic Upper GI Surgery
- Laparoscopic Hepatobilliary Surgery
- Pancreatic Surgery
- Laparoscopic Colorectal Surgery
- Laparoscopic Bariatric & Metabolic Surgery

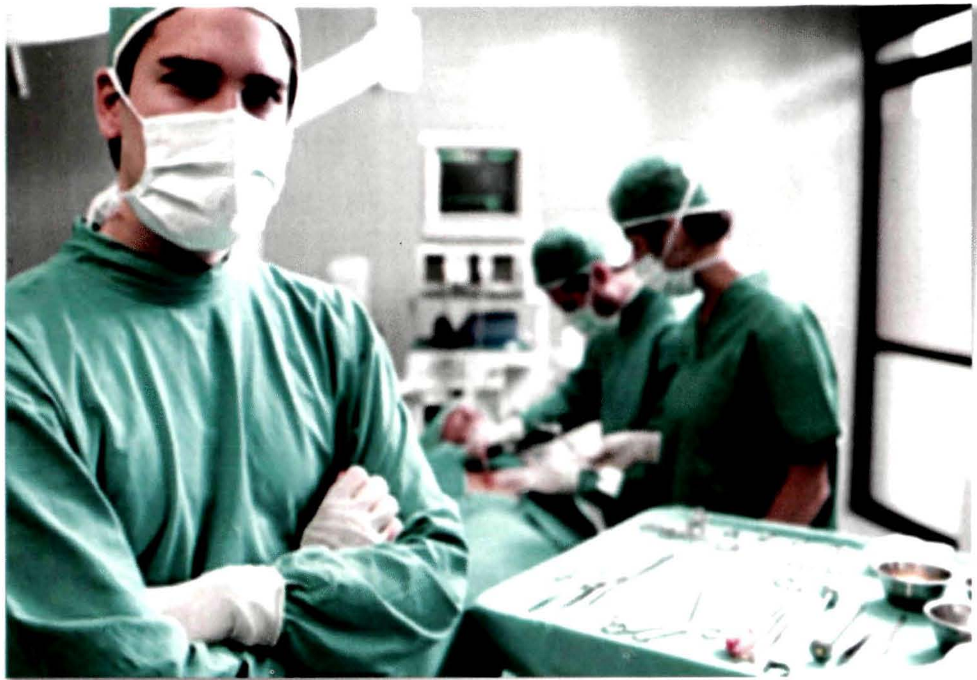
Infrastructure

- State-of-the-art dedicated Laparoscopic Surgical suite
- High definition Laparoscopic Equipments
- Full time team of Laparoscopic & GI surgeons, Gastroenterologist & Dietetics

Narayana Superspeciality Hospital
Near Tularam Bafna Civil Hospital, Amingoan, Guwahati – 781031, ASSAM

NH Helpline
88 11 88 88 88

What I've don-



NH Narayana
Superspeciality Hospital
Unit of Narayana Health

ক্ষন্তেকীয়া ভৰ্তি শল্য চিকিৎসা

নূন্যতম ছেদন • সঠিক নিৰূপণ
দ্ৰুত আৰোগ্য • কমদিন চিকিৎসালয়ত ভৰ্তিকৰণ

একে ঠাইতে হোৱা মিনিমেল ইনভেচিভ চাৰ্জাৰীৰ সুবিধা
কনভেনশ্বনেল লেপ্ৰস্ক'পিক চাৰ্জাৰী • মিনি লেপ্ৰস্ক'পিক চাৰ্জাৰী
• চিংগল ইনচিশ্বন চাৰ্জাৰী • ৰিডিউচ পৰ্ট চাৰ্জাৰী

অগ্ৰাধিকাৰ কিয় দিয়া হয়?

মিনিমেলি ইনভেছিভ চাৰ্জাৰীৰ ফলত নূন্যতম কৰ্তন আৰু নিভূলৰ্তাই অস্ত্ৰোপচাৰৰ পিছত দ্ৰুত আৰোগ্যতা আনে।

অস্ত্ৰোপচাৰৰ পিছত নূন্যতম সময় চিকিৎসাধীন হোৱাৰ সুবিধা

ই অস্ত্ৰোপচাৰৰ পিছৰ সমস্যাসমূহ যেনে বুকু আৰু শ্বাস-প্ৰশ্বাসৰ সমস্যা, ইনফেকশ্বন, এবড্‌মিনেল ৱালৰ বিকৃতি আৰু ইনচিশ্বনেল হাৰ্নিয়াৰ সম্ভাৱনা হ্ৰাস কৰে • ৰক্তক্ষৰণ আৰু ট্ৰ'মা হ্ৰাস কৰে • তাৎক্ষণিক আৰোগ্যতা প্ৰদান কৰে • ৰোগী একেদিনাই নাইবা পিছদিনা ৰাতিপুৱাই ঘৰলৈ যাব পাৰে। সামান্য ক্ষতচিহ্ন যুক্ত পদ্ধতিৰে দাগবিহীন ফলাফল লাভ কৰে।

ক্ষন্তেকীয়া ভৰ্তি শল্য চিকিৎসাই (লেপ্ৰস্ক'পিক)

অত্যাধুনিক কলাকৌশলে দ্ৰুত আৰোগ্যতাত সহায় কৰিছে। লগতে চিকিৎসালয়ত থকাটো 24-72 ঘণ্টালৈ কমাই আনিছে, আপোনাৰ অস্ত্ৰোপচাৰ যদি ৰাতিপুৱা কৰা হয় তেতিয়া একে দিনাই পিছবেলা আপুনি সুকলমে আৰু সুবিধাজনক ভাবে ঘৰলৈ যাব পাৰিব। আপোনাক উৎকৃষ্ট অভিজ্ঞতাৰ লগতে সৰ্ব্বোত্তম ফলাফল দিয়াতোৰেই আমাৰ দলৰ কৰ্মকৰ্তা সকলৰ একমাত্ৰ লক্ষ্য যাতে ভবিষ্যতে যদি কেতিয়াবা চিকিৎসালয়ৰ প্ৰয়োজন হয় তেতিয়া আপোনাৰ প্ৰথম পচন্দ হয় আমাৰ চিকিৎসালয়ৰ সুবিধাখিনি।

মিনিম্যালী ইনভেছিভ চাৰ্জাৰী এতিয়া অধিক ক্ষুদ্ৰতৰ

সুবিধা সমূহ	আন্তঃগাঁথনি
লেপ্ৰস্ক'পিক আপাৰ জি.আই. (GI) চাৰ্জাৰী	বিশেষ সৰ্মপিত অস্ত্ৰোপচাৰ কক্ষ
লেপ্ৰস্ক'পিক হেপাট'বিলিয়াৰী চাৰ্জাৰী	উচ্চমান বিশিষ্ট লেপ্ৰস্ক'পিক সৰঞ্জাম
পেংক্ৰিয়াটিক চাৰ্জাৰী	পূৰ্ণকালীন উচ্চমান বিশিষ্ট লেপ্ৰস্ক'পিক আৰু জি.আই. (GI) চাৰ্জন, গেষ্ট্ৰ'এণ্ট্ৰলজি আৰু ডায়টেটিক্স
লেপ্ৰস্ক'পিক ক'ল'ৰেক্টেল চাৰ্জাৰী	
লেপ্ৰস্ক'পিক বেৰিয়াট্ৰিক আৰু মেটাবলিক চাৰ্জাৰী	

NH Helpline
88 11 88 88 88

নাৰায়না চুপাৰস্পেচিয়েলিটি হস্পিতাল

তোলাৰাম বাফনা অসামৰিক চিকিৎসালয় চৌহদ, আমিনগাওঁ, গুৱাহাটী - 781031, অসম

$$\xi_1(B) = \begin{cases} \frac{r_1(B)-r_1(B^*)}{B-B^*}, & B \neq B^* \\ \\ r_1'(B), & B = B^* \end{cases},$$

$$\xi_2(B) = \begin{cases} \frac{r_2(B)-r_2(B^*)}{B-B^*}, & B \neq B^* \\ \\ r_2'(B), & B = B^* \end{cases}.$$

From (4.46) and the mean value theorem, we note that

$$|\eta_1(I)| \leq \rho_0, \ |\eta_2(I)| \leq \frac{\rho_3}{K_m^2}, \ |\xi_1(B)| \leq \rho_1 \ and \ |\xi_2(B)| \leq \rho_2.$$

Now $\frac{dV_1}{dt}$ can further be written as

$$\begin{aligned} \frac{dV_1}{dt} &= -\frac{1}{2}a_{11}(B-B^*)^2 + a_{12}(B-B^*)(N_1-N_1^*) - \frac{1}{2}a_{22}(N_1-N_1^*)^2 \\ &\quad -\frac{1}{2}a_{11}(B-B^*)^2 + a_{13}(B-B^*)(N_2-N_2^*) - \frac{1}{2}a_{33}(N_2-N_2^*)^2 \\ &\quad -\frac{1}{2}a_{11}(B-B^*)^2 + a_{14}(B-B^*)(I-I^*) - \frac{1}{2}a_{44}(I-I^*)^2 \\ &\quad -\frac{1}{2}a_{22}(N_1-N_1^*)^2 + a_{23}(N_1-N_1^*)(N_2-N_2^*) - \frac{1}{2}a_{33}(N_2-N_2^*)^2, \end{aligned}$$

where

$$\begin{aligned} &a_{11} = \frac{2}{3}\frac{r_0}{K(I^*)}, \ a_{22} = a_1\frac{r_{10}}{K_1}, \ a_{33} = a_2\frac{r_{20}}{K_2}, \ a_{44} = 2\beta_1, \\ &a_{12} = a_1\xi_1(B) - \delta_1, \ a_{13} = a_2\xi_2(B) - \delta_2, \\ &a_{14} = \eta_1(I) - r_0 B\eta_2(I) + \beta_2, \ a_{23} = -(a_1\alpha_{21} + a_2\alpha_{12}). \end{aligned}$$

Sufficient conditions for $\frac{dV_1}{dt}$ to be negative definite are that the following conditions hold:

$$a_{12}^2 < a_{11}a_{22}, \tag{4.51}$$

$$a_{13}^2 < a_{11}a_{33}, \tag{4.52}$$

$$a_{14}^2 < a_{11}a_{44}, \tag{4.53}$$

$$a_{23}^2 < a_{22}a_{33}. \tag{4.54}$$

By choosing $a_1 = \frac{\delta_1}{\rho_1}$ and $a_2 = \frac{\delta_2}{\rho_2}$, we note that inequalities (4.51) and (4.52) are automatically satisfied. Further we note that (4.47) $\Rightarrow$ (4.53) and (4.48) $\Rightarrow$ (4.54).

Thus, V_1 is a Liapunov function with respect to E^*, whose domain contains the region Ω, proving the theorem.

The above theorems imply that the resource biomass settles down to its equilibrium level, the magnitude of which decreases as the equilibrium levels of competing species and the industrialization pressure increase, and even may tend to zero if these factors increase unabatedly. It may be noted here that if the interference coefficients α_{12} and α_{21} are zero, then inequalities (4.42) and (4.48) are automatically satisfied. This implies that if there is no interference between the two species, then the stability of the system increases.

Case II: When the competing species depend wholly on the resource.

In this case, $r_1(B)$ satisfies condition (4.6). It may be noted that there exist nine non-negative equilibria, namely, $E_0(0,0,0,0)$, $E_1(K_0,0,0,0)$, $E_2(\bar{B},\bar{N}_1,0,0)$, $E_3(\hat{B},0,\hat{N}_2,0)$, $E_4(\tilde{B},0,0,\tilde{I})$, $E_5(B_p,N_{1p},0,I_p)$, $E_6(B_q,0,N_{2q},I_q)$, $E_7(B_r,N_{1r},N_{2r},0)$ and $E^*(B^*,N_1^*,N_2^*,I^*)$.

The existence of the equilibria can be checked in a similar way as in case I. Further, the stability behaviour of the equilibria are similar to the corresponding equilibria of case I.

Case III: When the competing species are predating on the resource.

In this case, $r_1(B)$ satisfies condition (4.7). It can be checked that there exist nine equilibria which are similar to those obtained in case II. Further, the existence and the stability behaviour of the equilibria are similar to the corresponding equilibria of case I.

In the above three cases it has been noted that the equilibrium level of the resource biomass is minimum in case I, and is maximum in case III.

4.4 Model With Diffusion

In this section we consider the complete model (4.1)-(4.2) and we state the main results of this section in the form of the following theorem.

Theorem 4.4.1 *(i) If the equilibrium E^* of system (4.8) is globally asymptotically stable, then the corresponding uniform steady state of the initial-boundary value problems (4.1)-(4.2) must also be globally asymptotically stable.*

(ii) If the equilibrium E^* of system (4.8) is unstable, even then the uniform steady state of the initial-boundary value problems (4.1)-(4.2) can be made stable by increasing diffusion coefficients to sufficiently large values.

Proof: Let us consider the following positive definite function

$$V_2(B(t), N_1(t), N_2(t), I(t)) = \int\int_D V_1(B, N_1, N_2, I)dA, \tag{4.55}$$

where V_1 is given in equation (4.49).

We have,

$$\begin{aligned} \frac{dV_2}{dt} &= \int\int_D \{\frac{\partial V_1}{\partial B}\frac{\partial B}{\partial t} + \frac{\partial V_1}{\partial N_1}\frac{\partial N_1}{\partial t} + \frac{\partial V_1}{\partial N_2}\frac{\partial N_2}{\partial t} + \frac{\partial V_1}{\partial I}\frac{\partial I}{\partial t}\}dA \\ &= I_1 + I_2, \end{aligned} \tag{4.56}$$

where

$$I_1 = \int\int_D \frac{dV_1}{dt}dA,$$
$$I_2 = \int\int_D \{D_1\frac{\partial V_1}{\partial B}\nabla^2 B + D_2\frac{\partial V_1}{\partial N_1}\nabla^2 N_1 + D_3\frac{\partial V_1}{\partial N_2}\nabla^2 N_2 + D_4\frac{\partial V_1}{\partial I}\nabla^2 I\}dA.$$

We note the following properties of V_1, namely,

$$\left.\frac{\partial V_1}{\partial B}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial N_1}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial N_2}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial I}\right]_{\partial D} = 0$$

and for all points of D,

$$\frac{\partial^2 V_1}{\partial B\partial N_1} = \frac{\partial^2 V_1}{\partial B\partial N_2} = \frac{\partial^2 V_1}{\partial B\partial I} = \frac{\partial^2 V_1}{\partial N_1\partial N_2} = \frac{\partial^2 V_1}{\partial N_1\partial I} = \frac{\partial^2 V_1}{\partial N_2\partial I} = 0,$$

$$\frac{\partial^2 V_1}{\partial B^2} > 0, \ \frac{\partial^2 V_1}{\partial N_1^2} > 0, \ \frac{\partial^2 V_1}{\partial N_2^2} > 0 \ and \ \frac{\partial^2 V_1}{\partial I^2} > 0.$$

We now consider I_2 and determine the sign of each term. Analysing in a similar fashion as done in chapter 2, we get

$$\begin{aligned}
\int\int_D \{\frac{\partial V_1}{\partial B}\nabla^2 B\} \, dA &= -\int\int_D (\frac{\partial^2 V_1}{\partial B^2})\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\} \, dA \leq 0, \\
\int\int_D \{\frac{\partial V_1}{\partial N_1}\nabla^2 N_1\} \, dA &= -\int\int_D (\frac{\partial^2 V_1}{\partial N_1^2})\{(\frac{\partial N_1}{\partial x})^2 + (\frac{\partial N_1}{\partial y})^2\} \, dA \leq 0, \\
\int\int_D \{\frac{\partial V_1}{\partial N_2}\nabla^2 N_2\} \, dA &= -\int\int_D (\frac{\partial^2 V_1}{\partial N_2^2})\{(\frac{\partial N_2}{\partial x})^2 + (\frac{\partial N_2}{\partial y})^2\} \, dA \leq 0, \\
\int\int_D \{\frac{\partial V_1}{\partial I}\nabla^2 I\} \, dA &= -\int\int_D (\frac{\partial^2 V_1}{\partial I^2})\{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2\} \, dA \leq 0.
\end{aligned} \tag{4.57}$$

Hence, $I_2 \leq 0$.

Thus, we note that if $I_1 \leq 0$, i.e., if E^* is globally asymptotically stable in the absence of diffusion, then the uniform steady state of the initial-boundary value problems (4.1)-(4.2) also must be globally asymptotically stable. This proves the first part of Theorem 4.4.1.

We further note that if $\frac{dV_1}{dt} > 0$, i.e., if $I_1 > 0$, then E^* will be unstable in the absence of diffusion. But Eqs. (4.56) and (4.57) show that by increasing diffusion coefficients D_i sufficiently large, $\frac{dV_2}{dt}$ can be made negative even if $I_1 > 0$. This proves the second part of Theorem 4.4.1.

The above theorem shows that the stability in the diffusive system is more plausible than that of the no diffusion case.

We shall explain the above theorem for a rectangular habitat D defined by

$$D = \{(x, y) : \ 0 \leq x \leq a, \ 0 \leq y \leq b\} \tag{4.58}$$

in the form of the following theorem.

Theorem 4.4.2 *In addition to assumptions (4.3)-(4.5), let $r(I)$, $K(I)$, $r_1(B)$ and $r_2(B)$ satisfy the inequalities in (4.46). If the following inequalities hold:*

$$\begin{aligned}
\{\rho_0 + \frac{r_0 K_0 \rho_3}{K_m^2} + \beta_2\}^2 &< \frac{4}{3}\beta_1\{\frac{r_0}{K(I^*)} + \frac{D_1 B^* \pi^2(a^2+b^2)}{a^2 b^2 K_0^2}\} \times \\
&\quad \{1 + \frac{D_4 I^* \beta_1 \pi^2 (a^2+b^2)}{a^2 b^2 (\beta_2 K_0 - \beta_0)^2}\}, \qquad (4.59) \\
\{\rho_2 \delta_1 \alpha_{21} + \rho_1 \delta_2 \alpha_{12}\}^2 &< \{\frac{r_{10}}{K_1} + \frac{D_2 N_1^* r_{10}^2 \pi^2 (a^2+b^2)}{a^2 b^2 K_1^2 r_1^2(K_0)}\} \times \\
&\quad \{\frac{r_{20}}{K_2} + \frac{D_3 N_2^* r_{20}^2 \pi^2 (a^2+b^2)}{a^2 b^2 K_2^2 r_2^2(K_0)}\}\rho_1 \rho_2 \delta_1 \delta_2, \qquad (4.60)
\end{aligned}$$

then E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Let us consider the rectangular region D given by Eq. (4.58). In this case I_2 which is defined in Eq. (4.56), can be written as

$$\begin{aligned}
I_2 &= -D_1 \int\int_D (\frac{\partial^2 V_1}{\partial B^2})\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\}\, dA \\
&\quad - D_2 \int\int_D (\frac{\partial^2 V_1}{\partial N_1^2})\{(\frac{\partial N_1}{\partial x})^2 + (\frac{\partial N_1}{\partial y})^2\} \\
&\quad - D_3 \int\int_D (\frac{\partial^2 V_1}{\partial N_2^2})\{(\frac{\partial N_2}{\partial x})^2 + (\frac{\partial N_2}{\partial y})^2\}\, dA \\
&\quad - D_4 \int\int_D (\frac{\partial^2 V_1}{\partial I^2})\{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2\}\, dA. \qquad (4.61)
\end{aligned}$$

From Eq. (4.49) we get

$$\frac{\partial^2 V_1}{\partial B^2} = \frac{B^*}{B^2},$$
$$\frac{\partial^2 V_1}{\partial N_1^2} = \frac{N_1^*}{N_1^2},$$
$$\frac{\partial^2 V_1}{\partial N_2^2} = \frac{N_2^*}{N_2^2},$$

and

$$\frac{\partial^2 V_1}{\partial I^2} = \frac{I^*}{I^2}.$$

Hence

$$I_2 \le -\frac{D_1 B^*}{K_0^2}\iint_D\{(\frac{\partial B}{\partial x})^2+(\frac{\partial B}{\partial y})^2\}\ dA - \frac{D_2 a_1 N_1^* r_{10}^2}{K_1^2 r_1^2(K_0)}\iint_D\{(\frac{\partial N_1}{\partial x})^2+(\frac{\partial N_1}{\partial y})^2\}\ dA$$

$$-\frac{D_3 a_2 N_2^* r_{20}^2}{K_2^2 r_2^2(K_0)}\iint_D\{(\frac{\partial N_2}{\partial x})^2+(\frac{\partial N_2}{\partial y})^2\}\ dA - \frac{D_4 I^* \beta_1^2}{(\beta_2 K_0-\beta_0)^2}\iint_D\{(\frac{\partial I}{\partial x})^2+(\frac{\partial I}{\partial y})^2\}\ dA.$$

Now

$$\begin{aligned}\iint_D(\frac{\partial B}{\partial x})^2\ dA &= \iint_D\{\frac{\partial(B-B^*)}{\partial x}\}^2\ dA\\ &= \int_0^b\int_0^a\{\frac{\partial(B-B^*)}{\partial x}\}^2\ dxdy\end{aligned}$$

Under an analysis similar to chapter 2 and using the well known inequality (Denn, 1975, pp. 225)

$$\int_0^1(\frac{\partial B}{\partial x})^2\ dx \ \ge\ \pi^2\int_0^1 B^2\ dx,$$

we note that

$$\iint_D(\frac{\partial B}{\partial x})^2\ dA \ \ge\ \frac{\pi^2}{a^2}\iint_D(B-B^*)^2 dA$$

and

$$\iint_D(\frac{\partial B}{\partial y})^2\ dA \ \ge\ \frac{\pi^2}{b^2}\iint_D(B-B^*)^2 dA$$

Thus,

$$\begin{aligned}I_2 \ \le\ & -\frac{D_1 B^*\pi^2(a^2+b^2)}{a^2 b^2 K_0^2}\iint_D(B-B^*)^2\ dA\\ & -\frac{D_2 a_1 N_1^* r_{10}^2}{K_1^2 r_1^2(K_0)}\iint_D(N_1-N_1^*)^2\ dA\\ & -\frac{D_3 a_2 N_2^* r_{20}^2}{K_2^2 r_2^2(K_0)}\iint_D(N_2-N_2^*)^2\ dA\\ & -\frac{D_4 I^*\beta_1^2}{(\beta_2 K_0-\beta_0)^2}\iint_D(I-I^*)^2\ dA \qquad (4.62)\end{aligned}$$

Now from (4.50), (4.56) and (4.62) we get

$$\begin{aligned}\frac{dV_2}{dt} \le \iint_D [& -\{\frac{r_0}{K(I^*)} + \frac{D_1 B^* \pi^2 (a^2+b^2)}{a^2 b^2 K_0^2}\}(B-B^*)^2 \\ & -a_1\{\frac{r_{10}}{K_1} + \frac{D_2 a_1 N_1^* r_{10}^2}{K_1^2 r_1^2(K_0)}\}(N_1 - N_1^*)^2 \\ & -a_2\{\frac{r_{20}}{K_2} + \frac{D_3 a_2 N_2^* r_{20}^2}{K_2^2 r_2^2(K_0)}\}(N_2 - N_2^*)^2 \\ & -\beta_1\{1 + \frac{D_4 I^* \beta_1^2}{(\beta_2 K_0 - \beta_0)^2}\}(I - I^*)^2 \\ & +\{a_1\xi_1(B) - \delta_1\}(B-B^*)(N_1 - N_1^*) \\ & +\{a_2\xi_2(B) - \delta_2\}(B-B^*)(N_2 - N_2^*) \\ & +\{\eta_1(I) - r_0 B \eta_2(I) + \beta_2\}(B-B^*)(I-I^*) \\ & +\{a_1\alpha_{21} + a_2\alpha_{12}\}(N_1 - N_1^*)(N_2 - N_2^*)]\, dA,\end{aligned}$$

where $\eta_1(I)$, $\eta_2(I)$, $\xi_1(B)$ *and* $\xi_2(B)$ are defined in Eq. (4.50).

Now $\frac{dV_2}{dt}$ can be written as the sum of the quadratics

$$\begin{aligned}\frac{dV_2}{dt} \le \iint_D [& -\frac{1}{2}b_{11}(B-B^*)^2 + b_{12}(B-B^*)(N_1-N_1^*) - \frac{1}{2}b_{22}(N_1 - N_1^*)^2 \\ & -\frac{1}{2}b_{11}(B-B^*)^2 + b_{13}(B-B^*)(N_2-N_2^*) - \frac{1}{2}b_{33}(N_2 - N_2^*)^2 \\ & -\frac{1}{2}b_{11}(B-B^*)^2 + b_{14}(B-B^*)(I-I^*) - \frac{1}{2}b_{44}(I - I^*)^2 \\ & -\frac{1}{2}b_{22}(N_1-N_1^*)^2 + b_{23}(N_1-N_1^*)(N_2-N_2^*) - \frac{1}{2}b_{33}(N_2 - N_2^*)^2] dA,\end{aligned}$$

where

$$b_{11} = \frac{r_0}{K(I^*)} + \frac{D_1 B^* \pi^2 (a^2+b^2)}{a^2 b^2 K_0^2},\ b_{22} = a_1\{\frac{r_{10}}{K_1} + \frac{D_2 a_1 N_1^* r_{10}^2}{K_1^2 r_1^2(K_0)}\},$$

$$b_{33} = a_2\{\frac{r_{20}}{K_2} + \frac{D_3 a_2 N_2^* r_{20}^2}{K_2^2 r_2^2(K_0)}\},\ b_{44} = \beta_1\{1 + \frac{D_4 I^* \beta_1^2}{(\beta_2 K_0 - \beta_0)^2}\},$$

$$b_{12} = a_1\xi_1(B) - \delta_1,\ b_{13} = a_2\xi_2(B) - \delta_2,$$

$$b_{14} = \eta_1(I) - r_0 B \eta_2(I) + \beta_2,\ b_{23} = -(a_1\alpha_{21} + a_2\alpha_{12})$$

Sufficient conditions for $\frac{dV_2}{dt}$ to be negative definite are that the following conditions hold:

$$b_{12}^2 < b_{11}b_{22}, \quad (4.63)$$

$$b_{13}^2 < b_{11}b_{33}, \quad (4.64)$$

$$b_{14}^2 < b_{11}b_{41}, \quad (4.65)$$

$$b_{23}^2 < b_{22}b_{33}. \quad (4.66)$$

By choosing $a_1 = \frac{\delta_1}{\rho_1}$ and $a_2 = \frac{\delta_2}{\rho_2}$, we note that inequalities (4.63) and (4.64) are automatically satisfied. Further we note that (4.59) $\Rightarrow$ (4.65) and (4.60) $\Rightarrow$ (4.66). Thus V_2 is a Liapunov function with respect to E^*, whose domain contains the region Ω, proving the theorem.

4.5 Conservation Model

It has been noted that uncontrolled growth of industrialization may lead to extinction of forestry resources. Therefore, some kind of efforts must be adopted to conserve the resource biomass. In this section a mathematical model is proposed and analysed to conserve the forestry resources and by controlling the undesired level of industrialization by some mechanism. It is assumed that the effort applied to conserve the resource is proportional to the depleted level of resource biomass from its carrying capacity, and effort applied to control industrialization pressure is proportional to its undesired level. Following Shukla et al. (1989), Dubey (1997a) and Shukla and Dubey (1997) differential equations governing the system may be written as

$$\begin{aligned}
\frac{\partial B}{\partial t} &= r(I)B - \frac{r_0 B^2}{K(I)} - \delta_1 BN_1 - \delta_2 BN_2 + \theta_1 F_1 + D_1 \nabla^2 B,\\
\frac{\partial N_1}{\partial t} &= r_1(B)N_1 - \frac{r_{10}N_1^2}{K_1} - \alpha_{21}N_1N_2 + D_2\nabla^2 N_1,\\
\frac{\partial N_2}{\partial t} &= r_2(B)N_2 - \frac{r_{20}N_2^2}{K_2} - \alpha_{12}N_1N_2 + D_3\nabla^2 N_2,\\
\frac{\partial I}{\partial t} &= -\beta_0 I - \beta_1 I^2 + \beta_2 IB - \theta_2 F_2 I + D_4 \nabla^2 I, \qquad (4.67)\\
\frac{\partial F_1}{\partial t} &= \mu_1(1 - \frac{B}{K_0}) - \nu_1 F_1,\\
\frac{\partial F_2}{\partial t} &= \mu_2(I - I_c)H(I - I_c) - \nu_2 F_2.
\end{aligned}$$

We impose the following initial and boundary conditions on the system (4.67):

$$\begin{aligned}
&B(x,y,0) = \phi(x,y) \geq 0,\ N_1(x,y,0) = \psi(x,y) \geq 0,\\
&N_2(x,y,0) = \xi(x,y) \geq 0,\ I(x,y,0) = \chi(x,y) \geq 0,\\
&F_1(x,y,0) = \chi_1(x,y) \geq 0,\ F_2(x,y,0) = \chi_2(x,y) \geq 0\ (x,y)\epsilon D \qquad (4.68)\\
&\frac{\partial B}{\partial n} = \frac{\partial N_1}{\partial n} = \frac{\partial N_2}{\partial n} = \frac{\partial I}{\partial n} = 0,\ (x,y)\ \epsilon\ \partial D, t \geq 0,
\end{aligned}$$

where n is the unit outward normal to ∂D.

In model (4.67), $F_1(x,y,t)$ is the density of effort applied to conserve the resource biomass and $F_2(x,y,t)$ the density of effort applied to control the undesired level of industrialization pressure. θ_1 is the growth rate coefficient of the resource biomass due to effort F_1 and θ_2 is the depletion rate coefficient of $I(x,y,t)$ due to effort F_2. μ_1 and μ_2 are the growth rate coefficients of F_1 and F_2 respectively and ν_1 and ν_2 are their respective depreciation rate coefficients. I_c is the critical level of the industrialization pressure which is assumed to be harmless to the resource biomass. $H(t)$ denotes the unit step function which takes into account the case when $I \leq I_c$. We shall analyse the conservation model (4.67) only for the case when the rate of introduction of pollutant into the environment is constant.

4.6 Conservation Model Without Diffusion

In this case we take $D_1 = D_2 = D_3 = D_4 = 0$ in the model (4.67). Then the model (4.67) has only one interior equilibrium $\bar{E}(\bar{B}, \bar{N}_1, \bar{N}_2, \bar{I}, \bar{F}_1, \bar{F}_2)$, where $\bar{B}$ $\bar{N}_1$, $\bar{N}_2$, $\bar{I}$, $\bar{F}_1$ and $\bar{F}_2$ are the positive solutions of the following algebraic equations:

$$r_0 B = \{r(f_3(B)) - \delta_1 f_1(B) - \delta_2 f_2(B) + \frac{\theta_1\mu_1}{\nu_1 B}(1 - \frac{B}{K_0})\}K(I), \tag{4.69}$$

$$N_1 = \frac{K_1\{r_1(B)r_{20} - \alpha_{21}r_2(B)K_2\}}{r_{10}r_{20} - \alpha_{21}\alpha_{12}K_1K_2} = f_1(B), (say) \tag{4.70}$$

$$N_2 = \frac{K_2\{r_2(B)r_{10} - \alpha_{12}r_1(B)K_1\}}{r_{10}r_{20} - \alpha_{21}\alpha_{12}K_1K_2} = f_1(B), (say) \tag{4.71}$$

$$I = \frac{(\beta_2 B - \beta_0)\nu_2 + \theta_2\mu_2 I_c}{\beta_1\nu_2 + \theta_2\mu_2} = f_3(B), (say) \tag{4.72}$$

$$F_1 = \frac{\mu_1}{\nu_1}(1 - \frac{B}{K_0}), \tag{4.73}$$

$$F_2 = \frac{\mu_2}{\nu_2}(I - I_c)H(I - I_c) = \begin{cases} \{\frac{\mu_2}{\nu_2}(I - I_c), & I > I_c \\ 0, & I \leq I_c \end{cases}. \tag{4.74}$$

It may be noted here that for F_1 to be positive we must have

$$K_0 > B.$$

It is easy to check that $\bar{E}$ exists, provided the following inequality holds at $\bar{E}$,

$$\begin{aligned} r_0 &- \{r(f_3(B) - \delta_1 f_1(B) - \delta_2 f_2(B) + \frac{\theta_1\mu_1}{\nu_1 B}(1 - \frac{B}{K_0})\}K'(I)f_3'(B) - \{r'(I)f_3'(B) \\ &- \delta_1 f_1'(B) - \delta_2 f_2'(B) - \frac{\theta_1\mu_1}{\nu_1 B^2}\}K(f_3(B)) > 0. \end{aligned} \tag{4.75}$$

In the following theorem it is shown that $\bar{E}$ is locally asymptotically stable, the proof of which is similar to Theorem 4.3.1 and hence is omitted.

Theorem 4.6.1 *Let the following inequality holds:*

$$\{\delta_1\alpha_{21}r_2'(\bar{B}) + \delta_2\alpha_{12}r_1'(\bar{B})\}^2 < \frac{\delta_1\delta_2 r_{10}r_{20}r_1'(\bar{B})r_2'(\bar{B})}{K_1K_2}. \tag{4.76}$$

Then equilibrium $\bar{E}$ is locally asymptotically stable.

In the following lemma, a region of attraction for system (4.67) without diffusion is established. The proof of this lemma is similar to Lemma 4.3.1 and hence is omitted.

Lemma 4.6.1 *The set*

$$\Omega_2 = \{(B, N_1, N_2, I, F_1, F_2) \; : \; 0 \leq B \leq K_a, \; 0 \leq N_1 \leq \frac{K_1 r_1(K_a)}{r_{10}}, \; 0 \leq N_2 \leq \frac{K_2 r_2(K_a)}{r_{20}}, \; 0 \leq I \leq \frac{\beta_0 K_a - \beta_0}{\beta_1}\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$K_a = \frac{K_0}{2}\{1 + \sqrt{1 + \frac{4\theta_1\mu_1}{\nu_1 K_0 r_0}}\}$$

The following theorem gives criteria for $\bar{E}$ to be globally asymptotically stable, whose proof is similar to Theorem 4.3.2 and hence is omitted.

Theorem 4.6.2 *In addition to assumptions (4.3)-(4.5) let $r(I)$, $K(I)$, $r_1(B)$ and $r_2(B)$ satisfy the following conditions in Ω_2*

$$0 \leq -r'(I) \leq \bar{\rho}_0, \; 0 \leq r_1'(B) \leq \bar{\rho}_1, \; 0 \leq r_2'(B) \leq \bar{\rho}_2, \; 0 \leq -K'(I) \leq \bar{\rho}_3 \text{ and } \bar{K}_m \leq K(I) \leq K_0, \tag{4.77}$$

for some positive constants $\bar{\rho}_0$, $\bar{\rho}_1$, $\bar{\rho}_2$, $\bar{\rho}_3$ and $\bar{K}_m$. Then if the following inequalities hold:

$$\{\bar{\rho}_0 + \frac{r_0 K_a \bar{\rho}_3}{\bar{K}_m^2} + \beta_2\}^2 < \frac{1}{2}\beta_1 \frac{r_0}{K(\bar{I})}, \tag{4.78}$$

$$\{\delta_1 \alpha_{21} \bar{\rho}_2 + \delta_2 \alpha_{12} \bar{\rho}_1\}^2 < \frac{\delta_1 \delta_2 \bar{\rho}_1 \bar{\rho}_2 r_{10} r_{20}}{K_1 K_2}, \tag{4.79}$$

then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Theorems 4.6.1 and 4.6.2 show that if suitable efforts are made to conserve the resource biomass and to control undesired level of industrialization pressure, an appropriate level of the resource biomass density can be maintained.

4.7 Conservation Model With Diffusion

We now consider the case when $D_i > 0 (i = 1, 2, 3)$ in model (4.67). We shall show that the uniform steady state $B(x, y, t) = B^*, T(x, y, t) = T^*, U(x, y, t) = U^*, W(x, y, t) = W^*, F_1(x, y, t) = F_1^*$ and $F_2(x, y, t) = F_2^*$ is globally asymptotically stable. For this, we consider the following positive definite function

$$V_3(B(t), T(t), U(t), W(t), F_1(t), F_2(t)) = \int\int_D V_2(B, T, U, W, F_1, F_2)\, dA,$$

where

$$\begin{aligned} V_2(B, N_1, N_2, I, F_1, F_2) &= B - B^* - B^* \ln \frac{B}{B^*} + c_1(N_1 - N_1^* - N_1^* \ln \frac{N_1}{N_1^*}) \\ &+ c_2(N_2 - N_2^* - N_2^* \ln \frac{N_2}{N_2^*}) + c_3(I - I^* - I^* \ln \frac{I}{I^*}) \\ &+ \frac{c_4}{2}(F_1 - F_1^*)^2 + \frac{c_5}{2}(F_2 - F_2^*)^2, \end{aligned}$$

where $c_i s$ are positive constants to be chosen suitably.

Then as earlier, it can be checked that if $\frac{dV_2}{dt} < 0$, then $\frac{dV_3}{dt} < 0$. This implies that if E^* is globally asymptotically stable for system (4.67) without diffusion, then the corresponding uniform steady state of system (4.67)-(4.68) is also globally asymptotically stable with respect to solutions such that $\phi(x, y) > 0, \psi(x, y) > 0, \xi(x, y) > 0, \zeta(x, y) > 0, \zeta_1(x, y) > 0, \zeta_2(x, y) > 0,\ (x, y) \in D$.

4.8 Numerical Examples

In this section we present numerical examples to illustrate the applicability of the results obtained. We take the following form of the functions $r(I)$, $K(I)$, $r_1(B)$ and $r_2(B)$ in model (4.8):

$$\begin{aligned} r(I) &= r_0 - r_1 I, \\ K(I) &= K_0 - q_1 I, \\ r_1(B) &= g_{10} + g_{11} B, \qquad (4.80) \\ r_2(B) &= g_{20} + g_{21} B, \end{aligned}$$

where the coefficients are assumed to be positive.

We choose the following values of the parameters in model (4.8) and in Eq. (4.80):

$$\begin{aligned} &r_0 = 15.0,\ r_1 = 0.01,\ K_0 = 50.0, \\ &q_1 = 0.02,\ g_{11} = 0.5,\ g_{21} = 0.48, \\ &\delta_1 = 0.3,\ \delta_2 = 0.4,\ r_{10} = 7.0, \\ &K_1 = 6.0,\ \alpha_{21} = 0.2,\ r_{20} = 10.0,\ K_2 = 8.0, \\ &\alpha_{12} = 0.1,\ \beta_0 = 1.0,\ \beta_1 = 4.0,\ \beta_2 = 0.45. \end{aligned} \tag{4.81}$$

Example 1. In this example we have considered the case when the two species partially depend on the resource. We take $r_1(0) = g_{10} = 7.0$ and $r_2(0) = g_{20} = 10.0$. It can be checked that under the above set of parameters, conditions for the existence of interior equilibrium $E_{11}^*(B_{11}^*, N_{11}^*, N_{21}^*, I_{11}^*)$ are satisfied and E_{11}^* is given by

$$B_{11}^* = 19.05048,\ N_{11}^* = 11.69945,\ N_{21}^* = 14.37943,\ I_{11}^* = 1.89318. \tag{4.82}$$

Again with the set of parameters given in Eq. (4.81) it can be verified that condition (4.42) in Theorem 4.3.1 is satisfied which shows that E_{11}^* is locally asymptotically stable.

By choosing $K_m = 20.0$ in Theorem 4.3.2, it can also be checked that conditions (4.46) and (4.47) are satisfied which shows that E_{11}^* is globally asymptotically stable.

Example 2. In this example we consider the case when the two species wholly depend on resource. We take $r_1(0) = g_{10} = 0.0$ and $r_2(0) = g_{20} = 0.0$. It can be checked that under the same set of parameters given in Eq. (4.81) the interior equilibrium $E_{12}^*(B_{12}^*, N_{12}^*, N_{22}^*, I_{12}^*)$ exists and is given by

$$B_{12}^* = 27.09852,\ N_{12}^* = 9.96648,\ N_{21}^* = 9.60851,\ I_{12}^* = 2.79858. \tag{4.83}$$

It can be seen that conditions corresponding to (4.46) and (4.47) for equilibrium E_{12}^* to be globally asymptotically stable are also satisfied.

Example 3. In this example we assume that the two species are predating on the resource. We take $r_1(0) = g_{10} = -7.0$ and $r_2(0) = g_{20} = -10.0$. With the same set of parameters given in Eq. (4.81) it can be checked that the interior equilibrium $E^*_{13}(B^*_{13}, N^*_{13}, N^*_{23}, I^*_{13})$ exists and is given by

$$B^*_{13} = 35.14338, \ N^*_{13} = 8.23234, \ N^*_{23} = 4.83647, \ I^*_{13} = 3.70363. \qquad (4.84)$$

It can be verified that E^*_{13} is also globally asymptotically stable.

From Eqs. (4.82), (4.83) and (4.84) it may be noted that $B^*_{11} < B^*_{12} < B^*_{13}$, $N^*_{11} > N^*_{12} > N^*_{13}$, $N^*_{21} > N^*_{22} > N^*_{23}$ and $I^*_{11} < I^*_{12} < I^*_{13}$ as expected.

Example 4 In addition to the values of parameters given in (4.81), we choose the following values of parameters in model (4.67) with no diffusion:

$$\theta_1 = 13.0, \ \theta_2 = 0.02, \ \mu_1 = 16.0, \ \nu_1 = 0.03,$$
$$\mu_2 = 18.0, \ \nu_2 = 0.04, \ T_c = 0.12. \qquad (4.85)$$

Then it can be checked that condition (4.75) for the existence of the interior equilibrium $\bar{E}$ is satisfied, and $\bar{E}$ is given by

$$\bar{B} \approx 45.27252, \ \bar{N}_1 \approx 21.34357, \ \bar{N}_2 \approx 23.67716, \ \bar{I} \approx 1.57328,$$
$$\bar{F}_1 \approx 50.42648, \ \bar{F}_2 \approx 653.97580. \qquad (4.86)$$

It can easily be verified that condition (4.76) in Theorem 4.6.1 is satisfied which shows that $\bar{E}$ is locally asymptotically stable.

Further, by choosing $\bar{K}_m = 50.0$ in Theorem 4.6.2, it can be checked that conditions (4.78)-(4.79) are satisfied. This shows that $\bar{E}$ is globally asymptotically stable.

By comparing equilibrium levels E^*_{11} and $\bar{E}$ in Eqs. (4.82) and (3.66) respectively, we note that due to efforts F_1 and F_2, the equilibrium level of the resource biomass has increased whereas equilibrium level of the industrialization pressure has decreased.

4.9 Conclusions

In this chapter, a mathematical model has been proposed and analysed to study the survival of two biological species competing for a single resource under industrialization pressure with and without diffusion. The competing species are assumed to be either partially dependent, wholly dependent or predating on the resource. In the partially dependent case criteria for survival and extinction of competing species and industrialization pressure have been derived. It has been shown that the resource biomass settles down to its equilibrium level, the magnitude of which depends upon the equilibrium levels of the competing species and the industrialization pressure. This magnitude decreases as the densities of the competing species and pressure due to industrialization increase and may driven to extinction if these factors increase without control. It has also been noted that the competing species may coexist even in the absence of the resource biomass in the partially dependent case, whereas in the wholly dependent case the two species will die out in the absence of the resource biomass. In the case when the competing species are predating on the resource, similar results have been found. It has also been found that if the interference coefficient measuring the damage effect of each species on the other is zero (i.e., $\alpha_{12} = \alpha_{21} = 0$), then stability of the system increases. It has been noted that the damage of the resource biomass density is maximum in partially dependent case, and is minimum in the predation case. This has also been established by numerical examples in section 4.8.

A model to study the effect of diffusion on the system under consideration has also been proposed. By analysing the diffusion model it has been shown that stability of the system with diffusion is more plausible than that of without diffusion. It has also been shown that an unstable steady state in the absence of diffusion can be made stable by increasing diffusion coefficients sufficiently large. This implies that solutions approach to the uniform steady state more rapidly as the diffusion coefficients increase.

A model to conserve the resource biomass and to control the undesired level of industrialization pressure is proposed and analysed. It has been noted that if suitable efforts are made, a desired level of resource biomass can be maintained.

Chapter 5

MODELLING THE INTERACTION OF TWO BIOLOGICAL SPECIES IN A POLLUTED ENVIRONMENT

5.1 Introduction

A large amount of pollutants and contaminants released from various industries, motor vehicles and other manmade projects enter into the environment affecting human population and other biological species seriously. In recent years some investigations have been carried out to study the effect of pollution on a single-species population (De Luna and Hallam, 1987; Dubey, 1997a; Freedman and Shukla, 1991; Hallam et al., 1983; Hallam and De Luna, 1984; Hallam and Ma, 1986; Shukla and Dubey, 1996a). In particular, Hallam et al. (1983b) studied the effect of toxicant present in the environment on a single-species population by assuming that its growth rate density decreases linearly with concentration of toxicant but the corresponding carrying capacity does not depend upon the concentration of toxicant present in the environment. Considering this aspect Freedman and Shukla (1991) studied the effect of toxicant on a single species and predator-prey system by taking into account the introduction of toxicant from an external source. Shukla and Dubey (1996a) studied the simultaneous effect of two toxicants, one being more toxic than the other, on a biological species. Dubey (1997a) proposed a model to study the depletion and conservation of forestry resources in a polluted environment.

We know that species do not exist alone in nature. They interact with other species in their surrounding for their survival. So it is of more biological significance to study two-species systems exposed to a pollutant. In recent decades some investigations have been made to study the system of two biological species in a polluted environment (Ma and Hallam, 1987; Huaping and Ma, 1991; Chattopadhyay, 1996; Shukla and Dubey, 1997). In particular, Ma and Hallam (1987) studied two-dimensional nonautonomous Lotka-Volterra models by the average method and obtained sufficient conditions for persistence and extinction of the populations. Huaping and Ma (1991) investigated the effects of toxicants on naturally stable two-species communities and derived persistence-extinction criteria for each population. But in modelling the system they assumed that the individuals of the two species have identical organismal toxicant concentration, which need not be true always in nature. Chattopadhyay (1996) studied the effect of toxic substances on a two-species competitive system. He assumed that each of the competing species produces a substance toxic to the other, but only when the other is present. Shukla and Dubey (1997) studied the effects of population and pollution on the depletion and conservation of forestry resources. It may be pointed out here that the recycle effect of toxicant and the effect of diffusion on the stability of the equilibrium state of the system do not appear in the above literature.

In view of the above, in this chapter we propose a mathematical model to study the effect of environmental pollution on two interacting biological species having different organismal pollutant concentration. Three types of interaction between the two species have been considered, namely, competition, cooperation and prey-predator. The effect of diffusion on the stability of the system is also studied. In the absence of diffusion our model is more general than Huaping and Ma (1991). In the presence of diffusion our results agree with those in Hastings (1978a), Shukla and Verma (1981), Shukla and Shukla (1982), Shukla et al. (1989) and Freedman and Shukla (1989), Dubey and Das (1999). In this chapter, we have also included numerical examples to illustrate the applicability of the results obtained.

5.2 Mathematical Model

Consider a polluted environment where two biological species are interacting with each other in a closed region D with smooth boundary ∂D. The variables of the model are $x_1 = x_1(x,y,t)$ and $x_2 = x_2(x,y,t)$, the densities of the species 1 and 2 respectively; $T = T(x,y,t)$, the concentration of pollutant present in the environment; $U_1 = U_1(x,y,t)$ and $U_2 = U_2(x,y,t)$, the concentration of pollutant in the species 1 and 2 respectively at coordinates $(x,y) \in D$ and time $t \geq 0$. In modelling the system we assume that carrying capacities of the species are constants. Then following Huaping and Ma (1991) and Dubey (1997a) the Lotka-Volterra model of two species with pollutant effect and diffusion can be written as,

$$\begin{aligned}
\frac{\partial x_1}{\partial t} &= r_{10}x_1 - r_{11}x_1U_1 - a_{11}x_1^2 - a_{12}x_1x_2 + D_1\nabla^2 x_1,\\
\frac{\partial x_2}{\partial t} &= r_{20}x_2 - r_{21}x_2U_2 - a_{21}x_1x_2 - a_{22}x_2^2 + D_2\nabla^2 x_2,\\
\frac{\partial T}{\partial t} &= Q(t) - \delta_0 T + \theta_1\delta_1U_1 + \theta_2\delta_2U_2 - \lambda_1x_1T - \lambda_2x_2T + D_3\nabla^2 T, \qquad (5.1)\\
\frac{\partial U_1}{\partial t} &= -\delta_1U_1 + \theta_0\delta_0T + \lambda_1x_1T + \beta_1x_1,\\
\frac{\partial U_2}{\partial t} &= -\delta_2U_2 + \theta_0'\delta_0T + \lambda_2x_2T + \beta_2x_2,\\
&0 \leq \theta_0 + \theta_0' \leq 1,\ 0 \leq \theta_1, \theta_2 \leq 1.
\end{aligned}$$

We impose the following initial and boundary conditions on the system (5.1):

$$
\begin{aligned}
&x_1(x,y,0) = \phi(x,y) \geq 0,\ x_2(x,y,0) = \psi(x,y) \geq 0,\\
&T(x,y,0) = \xi(x,y) \geq 0,\ U_1(x,y,0) = \chi(x,y) \geq 0,\\
&U_2(x,y,0) = \eta(x,y) \geq 0\ (x,y)\epsilon D,\\
&\frac{\partial x_1}{\partial n} = \frac{\partial x_2}{\partial n} = \frac{\partial T}{\partial n} = 0,\ (x,y)\ \epsilon\ \partial D, t \geq 0,
\end{aligned}
\tag{5.2}
$$

where n is the unit outward normal to ∂D.

In model (5.1), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian diffusion operator. $D_i (i = 1, 2, 3)$ are the diffusion rate coefficients of $x_1(x, y, t)$, $x_2(x, y, t)$ and $T(x, y, t)$ respectively in D. r_{i0}, r_{i1} and a_{ij} (i,j=1,2) in the first two equations of model (5.1) are constants. r_{i0} is the intrinsic growth rate of the species i in the absence of pollutant; r_{i1} the depletion rate coefficient of species i due to organismal pollutant concentration. a_{12} and a_{21} are the interspecific interference coefficients and a_{11}, a_{22} are intraspecific interference coefficients of the species 1 and 2 respectively. Q(t) represents the rate of introduction of pollutant into the environment beyond the initial concentration, which is assumed to be positive, zero or periodic. It is assumed that the pollutant in the environment is washed out or broken down with rate δ_0, and fractions θ_0 and θ_0' of it may again reenter into the species 1 and 2 respectively with the uptake of pollutant. λ_1 and λ_2 are the depletion rate coefficients of the pollutant in the environment due to its intake by species 1 and 2 respectively. δ_1 and δ_2 are natural depletion rate coefficients of U_1 and U_2 respectively due to ingestion and depuration of pollutant and fractions θ_1 and θ_2 of these may again reenter into the environment. β_1 and β_2 are the net uptake of pollutant from resource by species 1 and 2 respectively.

It is assumed that the parameters δ_0, δ_1 and δ_2 are strictly positive and θ_1, θ_2, λ_1, λ_2, β_1 and β_2 are nonnegative constants.

The following three cases will be dealt with:

(i) Competition $(r_{10} > 0, r_{20} > 0, a_{12} > 0\ and\ a_{21} > 0)$

(ii) Cooperation $(r_{10} > 0, r_{20} > 0, a_{12} < 0\ and\ a_{21} < 0)$

(iii) Prey-predator $(r_{10} > 0, r_{20} < 0, a_{12} > 0\ and\ a_{21} < 0)$, assuming x_1 as prey and x_2

as predator.

5.3 Competition Model Without Diffusion

We first analyse model (5.1) without diffusion (i.e., $D_1 = D_2 = D_3 = 0$). In such a case model (5.1) reduces to

$$\begin{aligned}
\frac{dx_1}{dt} &= r_{10}x_1 - r_{11}x_1U_1 - a_{11}x_1^2 - a_{12}x_1x_2,\\
\frac{dx_2}{dt} &= r_{20}x_2 - r_{21}x_2U_2 - a_{21}x_1x_2 - a_{22}x_2^2,\\
\frac{dT}{dt} &= Q(t) - \delta_0 T + \theta_1\delta_1 U_1 + \theta_2\delta_2 U_2 - \lambda_1 x_1 T - \lambda_2 x_2 T, \qquad (5.3)\\
\frac{dU_1}{dt} &= -\delta_1 U_1 + \theta_0\delta_0 T + \lambda_1 x_1 T + \beta_1 x_1,\\
\frac{dU_2}{dt} &= -\delta_2 U_2 + \theta_0'\delta_0 T + \lambda_2 x_2 T + \beta_2 x_2,\\
&0 \le \theta_0 + \theta_0' \le 1,\ 0 \le \theta_1, \theta_2 \le 1,\\
&x_i(0) \ge 0,\ T(0) \ge 0,\ U_i(0) \ge 0, i = 1, 2.
\end{aligned}$$

It can be seen that in the case of constant introduction ($Q(t) = Q_0 > 0$) of pollutant into the environment, model (5.3) has four nonnegative equilibria, namely, $E_0(0,0,\frac{Q_0}{\delta_0(1-\theta_0\theta_1-\theta_0'\theta_2)}, \frac{\theta_0 Q_0}{\delta_1(1-\theta_0\theta_1-\theta_0'\theta_2)}, \frac{\theta_0' Q_0}{\delta_2(1-\theta_0\theta_1-\theta_0'\theta_2)})$, $\tilde{E}(\tilde{x}_1, 0, \tilde{T}, \tilde{U}_1, \tilde{U}_2)$, $\hat{E}(0, \hat{x}_2, \hat{T}, \hat{U}_1, \hat{U}_2)$ and $\bar{E}(\bar{x}_1, \bar{x}_2, \bar{T}, \bar{U}_1, \bar{U}_2)$. The equilibrium E_0 exists if

$$1 - \theta_0\theta_1 - \theta_0'\theta_2 > 0. \qquad (5.4)$$

We shall show the existence of other three equilibria as follows.

Existence of $\tilde{E}(\tilde{x}_1, 0, \tilde{T}, \tilde{U}_1, \tilde{U}_2)$:

Here $\tilde{x}_1$, $\tilde{T}$, $\tilde{U}_1$ and $\tilde{U}_2$ are the positive solutions of the following algebraic equations:

$$\begin{aligned}
a_{11}x_1 &= r_{10} - r_{11}U_1,\\
\delta_0 T + \lambda_1 x_1 T &= Q_0 + \theta_1\delta_1 U_1 + \theta_2\delta_2 U_2,\\
\delta_1 U_1 &= \theta_0\delta_0 T + \lambda_1 x_1 T + \beta_1 x_1,\\
\delta_2 U_2 &= \theta_0'\delta_0 T.
\end{aligned}$$

A little algebraic manipulation yields

$$\begin{aligned}
a_{11}x_1 &= r_{10} - r_{11}g(x_1), \\
T &= \frac{Q_0 + \theta_1\beta_1 x_1}{\delta_0(1-\theta_0\theta_1-\theta_0'\theta_2) + \lambda_1(1-\theta_1)x_1} = f(x_1), \ (say) \\
U_1 &= \frac{\theta_0\delta_0 f(x_1) + \lambda_1 x_1 f(x_1) + \beta_1 x_1}{\delta_1} = g(x_1), \ (say) \\
U_2 &= \frac{\theta_0'\delta_0 f(x_1)}{\delta_2} = h(x_1). \ (say)
\end{aligned}$$

Taking,

$$F(x_1) = a_{11}x_1 - r_{10} + r_{11}g(x_1)$$

we note that $F(0) < 0$ if

$$r_{11}\theta_0 Q_0 < r_{10}\delta_1(1-\theta_0\theta_1-\theta_0'\theta_2), \tag{5.5}$$

and $F(\frac{r_{10}}{a_{11}}) > 0$, showing the existence of $\tilde{x}_1$ in the interval $0 < \tilde{x}_1 < \frac{r_{10}}{a_{11}}$. For $\tilde{x}_1$ to be unique the following condition must be satisfied at $\tilde{E}$,

$$Q_0\lambda_1(1-\theta_1) < \theta_1\beta_1\delta_0(1-\theta_0\theta_1-\theta_0'\theta_2). \tag{5.6}$$

Thus from the above analysis we note that the equilibrium $\tilde{E}$ exists under conditions (5.5) and (5.6).

Existence of $\hat{E}(0, \hat{x}_2, \hat{T}, \hat{U}_1, \hat{U}_2)$:

As in the existence of $\tilde{E}$, it can be seen that the equilibrium $\hat{E}$ exists if the following inequalities hold:

$$r_{21}\theta_0' Q_0 < r_{20}\delta_2(1-\theta_0\theta_1-\theta_0'\theta_2), \tag{5.7}$$

$$Q_0\lambda_2(1-\theta_2) < \theta_2\beta_2\delta_0(1-\theta_0\theta_1-\theta_0'\theta_2). \tag{5.8}$$

Existence of $\bar{E}(\bar{x}_1, \bar{x}_2, \bar{T}, \bar{U}_1, \bar{U}_2)$:

Here, $\bar{x}_1$, $\bar{x}_2$, $\bar{T}$, $\bar{U}_1$, and $\bar{U}_2$ are the positive solutions of the following system of algebraic equations:

$$a_{11}x_1 + a_{12}x_2 + r_{11}g(x_1, x_2) = r_{10},$$
$$a_{21}x_1 + a_{22}x_2 + r_{21}h(x_1, x_2) = r_{20},$$
$$T = f(x_1, x_2),$$
$$U_1 = g(x_1, x_2),$$
$$U_2 = h(x_1, x_2),$$

where

$$f(x_1, x_2) = \frac{Q_0 + \theta_1\beta_1 x_1 + \theta_2\beta_2 x_2}{\delta_0(1 - \theta_0\theta_1 - \theta_0'\theta_2) + \lambda_1(1-\theta_1)x_1 + \lambda_2(1-\theta_2)x_2},$$
$$g(x_1, x_2) = \frac{\theta_0\delta_0 f(x_1, x_2) + \lambda_1 x_1 f(x_1, x_2) + \beta_1 x_1}{\delta_1},$$
$$h(x_1, x_2) = \frac{\theta_0'\delta_0 f(x_1, x_2) + \lambda_2 x_2 f(x_1, x_2) + \beta_2 x_2}{\delta_2}.$$

It can be checked that $\bar{E}$ exists if in addition to conditions (5.5) and (5.7), the following conditions hold:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} > \frac{-b' + \sqrt{b'^2 - 4a'c'}}{2a'}, \tag{5.9}$$
$$\frac{-B + \sqrt{B^2 - 4AC}}{2A} > \frac{-B' + \sqrt{B'^2 - 4A'C'}}{2A'}, \tag{5.10}$$
$$\frac{a_{12} + r_{11}\frac{\partial g}{\partial x_2}}{a_{11} + r_{11}\frac{\partial g}{\partial x_1}} > 0, \tag{5.11}$$
$$\frac{a_{22} + r_{21}\frac{\partial h}{\partial x_2}}{a_{21} + r_{21}\frac{\partial h}{\partial x_1}} > 0, \tag{5.12}$$

where

$$a = \lambda_1\{a_{11}\delta_1(1-\theta_1) + r_{11}\beta_1\},$$
$$b = a_{11}\delta_0\delta_1(1 - \theta_0\theta_1 - \theta_0'\theta_2) + r_{11}\delta_0\beta_1(1 - \theta_0'\theta_2) + r_{11}\lambda_1 Q_0 - \lambda_1\delta_1 r_{10}(1-\theta_1),$$
$$c = r_{11}\theta_0\delta_0 Q_0 - \delta_0\delta_1 r_{10}(1 - \theta_0\theta_1 - \theta_0'\theta_2),$$
$$a' = \lambda_1 a_{21}\delta_2(1-\theta_1),$$
$$b' = a_{21}\delta_0\delta_2(1 - \theta_0\theta_1 - \theta_0'\theta_2) + r_{21}\theta_0'\delta_0\theta_1\beta_1 - \lambda_1\delta_2 r_{20}(1-\theta_1),$$
$$c' = r_{21}\theta_0'\delta_0 Q_0 - \delta_0\delta_2 r_{20}(1 - \theta_0\theta_1 - \theta_0'\theta_2),$$

$$A = \lambda_2\{a_{22}\delta_2(1-\theta_2) + r_{21}\beta_2\},$$

$$B = a_{22}\delta_0\delta_2(1-\theta_0\theta_1-\theta_0'\theta_2) + r_{21}\delta_0\beta_2(1-\theta_0\theta_1) + r_{21}\lambda_2 Q_0 - \lambda_2\delta_2 r_{20}(1-\theta_2),$$

$$C = r_{21}\theta_0'\delta_0 Q_0 - \delta_0\delta_2 r_{20}(1-\theta_0\theta_1-\theta_0'\theta_2),$$

$$A' = \lambda_2 a_{12}\delta_1(1-\theta_2),$$

$$B' = a_{12}\delta_0\delta_1(1-\theta_0\theta_1-\theta_0'\theta_2) + r_{11}\theta_0\delta_0\theta_2\beta_2 - \lambda_2\delta_1 r_{10}(1-\theta_2),$$

$$C' = r_{11}\theta_0\delta_0 Q_0 - \delta_0\delta_1 r_{10}(1-\theta_0\theta_1-\theta_0'\theta_2).$$

It may be noted here that $\bar{E}$ exists even when the inequalities (5.9) and (5.10) are reversed.

To study the local stability behaviour of the equilibria, we first compute the variational matrices corresponding to each equilibrium point. From these matrices we conclude the following:

E_0 is a saddle point with unstable manifold locally in the $x_1 - x_2$ plane and stable manifold locally in the $T - U_1 - U_2$ space. $\tilde{E}$ and $\hat{E}$ are locally unstable in the x_2 and x_1 directions respectively.

In the following theorem we have shown that $\bar{E}$ is locally asymptotically stable.

Theorem 5.3.1 *Let the following inequalities hold:*

$$(a_{12}+a_{21})^2 < \frac{4}{9}a_{11}a_{22}, \tag{5.13}$$

$$\{c_1'\theta_1\delta_1 + c_2'(\theta_0\delta_0+\lambda_1\bar{x}_1)\}^2 < \frac{1}{2}c_1'c_2'\delta_1(\delta_0+\lambda_1\bar{x}_1+\lambda_2\bar{x}_2), \tag{5.14}$$

$$\{c_1'\theta_2\delta_2 + c_3'(\theta_0'\delta_0+\lambda_2\bar{x}_2)\}^2 < \frac{1}{2}c_1'c_3'\delta_2(\delta_0+\lambda_1\bar{x}_1+\lambda_2\bar{x}_2), \tag{5.15}$$

where

$$c_1' = min\{\frac{a_{11}(\delta_0+\lambda_1\bar{x}_1+\lambda_2\bar{x}_2)}{4\lambda_1^2\bar{T}^2}, \frac{a_{22}(\delta_0+\lambda_1\bar{x}_1+\lambda_2\bar{x}_2)}{4\lambda_2^2\bar{T}^2}\},$$

$$c_2' = \frac{r_{11}}{\lambda_1\bar{T}+\beta_1},$$

$$c_3' = \frac{r_{21}}{\lambda_2\bar{T}+\beta_2}.$$

Then $\bar{E}$ is locally asymptotically stable.

Proof: By using the transformations

$$x_1 = X_1 + \bar{x}_1,\ x_2 = X_2 + \bar{x}_2,\ T = \tau + \bar{T},\ U_1 = u_1 + \bar{U}_1,\ U_2 = u_2 + \bar{U}_2,$$

we first linearize system (5.3). Then taking the following positive definite function,

$$V = \frac{1}{2}\{\frac{X_1^2}{\bar{x}_1} + \frac{X_2^2}{\bar{x}_2} + c_1'\tau^2 + c_2'u_1^2 + c_3'u_2^2\}, \tag{5.16}$$

it can be seen that the derivative of V with respect t is negative definite under conditions (5.13)-(5.15), proving the theorem.

To show that $\bar{E}$ is globally asymptotically stable, we need the following lemma which establishes a region of attraction for system (5.3). The proof of this lemma is easy and hence is omitted.

Lemma 5.3.1 *The set*

$$\Omega_1 = \{(x_1, x_2, T, U_1, U_2) : 0 \leq x_1 \leq \frac{r_{10}}{a_{11}},\ 0 \leq x_2 \leq \frac{r_{20}}{a_{22}},\ 0 \leq T + U_1 + U_2 \leq L_1\}$$

attracts all solutions initiating in the positive orthant, where

$$L_1 = \frac{1}{\delta}\{Q_0 + \beta_1\frac{r_{10}}{a_{11}} + \beta_2\frac{r_{20}}{a_{22}}\},$$
$$\delta = min\{\delta_0(1 - \theta_0 - \theta_0'), \delta_1(1 - \theta_1), \delta_2(1 - \theta_2)\}.$$

In the following theorem global stability behaviour of $\bar{E}$ is studied.

Theorem 5.3.2 *Let the following inequalities hold:*

$$(a_{12} + a_{21})^2 < \frac{4}{9}a_{11}a_{22}, \tag{5.17}$$
$$\{c_1\theta_1\delta_1 + c_2(\theta_0\delta_0 + \lambda_1\bar{x}_1)\}^2 < \frac{1}{2}c_1c_2\delta_1(\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2), \tag{5.18}$$
$$\{c_1\theta_2\delta_2 + c_3(\theta_0'\delta_0 + \lambda_2\bar{x}_2)\}^2 < \frac{1}{2}c_1c_3\delta_2(\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2), \tag{5.19}$$

where

$$\begin{aligned} c_1 &= min\{\frac{a_{11}(\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2)}{4\lambda_1^2L_1^2}, \frac{a_{22}(\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2)}{4\lambda_2^2L_1^2}\}, \\ c_2 &= \frac{r_{11}}{\lambda_1L_1 + \beta_1}, \\ c_3 &= \frac{r_{21}}{\lambda_2L_1 + \beta_2}. \end{aligned} \tag{5.20}$$

Then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Consider the following positive definite function around $\bar{E}$,

$$\begin{aligned} V_1(x_1,x_2,T,U_1,U_2) &= x_1 - \bar{x}_1 - \bar{x}_1\, ln(\frac{x_1}{\bar{x}_1})x_2 - \bar{x}_2 - \bar{x}_2\, ln(\frac{x_2}{\bar{x}_2}) + \frac{c_1}{2}(T-\bar{T})^2 \\ &\quad + \frac{c_2}{2}(U_1-\bar{U}_1)^2 + \frac{c_3}{2}(U_2-\bar{U}_2)^2. \end{aligned} \tag{5.21}$$

where c_1, c_2 and c_3 are positive constants as defined in (5.20).

Differentiating V_1 with respect to t along the solutions of system (5.3), we get

$$\begin{aligned} \frac{dV_1}{dt} &= (x_1-\bar{x}_1)[r_{10} - r_{11}U_1 - a_{11}x_1 - a_{12}x_2] \\ &\quad +(x_2-\bar{x}_2)[r_{20} - r_{21}U_2 - a_{21}x_1 - a_{22}x_2] \\ &\quad +c_1(T-\bar{T})[Q_0 - \delta_0 T + \theta_1\delta_1 U_1 + \theta_2\delta_2 U_2 - \lambda_1 x_1 T - \lambda_2 x_2 T] \\ &\quad +c_3(U_1-\bar{U}_1)[-\delta_1 U_1 + \theta_0\delta_0 T + \lambda_1 x_1 T + \beta_1 x_1] \\ &\quad +c_3(U_2-\bar{U}_2)[-\delta_2 U_2 + \theta_0'\delta_0 T + \lambda_2 x_2 T + \beta_2 x_2]. \end{aligned} \tag{5.22}$$

After some algebraic manipulations, Eq. (5.22) can be written as

$$\begin{aligned} \frac{dV_1}{dt} &= -\frac{1}{2}A_{11}(x_1-\bar{x}_1)^2 + A_{12}(x_1-\bar{x}_1)(x_2-\bar{x}_2) - \frac{1}{2}A_{22}(x_2-\bar{x}_2)^2 \\ &\quad -\frac{1}{2}A_{11}(x_1-\bar{x}_1)^2 + A_{13}(x_1-\bar{x}_1)(T-\bar{T}) - \frac{1}{2}A_{33}(T-\bar{T})^2 \\ &\quad -\frac{1}{2}A_{11}(x_1-\bar{x}_1)^2 + A_{14}(x_1-\bar{x}_1)(U_1-\bar{U}_1) - \frac{1}{2}A_{44}(U_1-\bar{U}_1)^2 \\ &\quad -\frac{1}{2}A_{22}(x_2-\bar{x}_2)^2 + A_{23}(x_2-\bar{x}_2)(T-\bar{T}) - \frac{1}{2}A_{33}(T-\bar{T})^2 \\ &\quad -\frac{1}{2}A_{22}(x_2-\bar{x}_2)^2 + A_{25}(x_2-\bar{x}_2)(U_2-\bar{U}_2) - \frac{1}{2}A_{55}(U_2-\bar{U}_2)^2 \\ &\quad -\frac{1}{2}A_{33}(T-\bar{T})^2 + A_{34}(T-\bar{T})(U_1-\bar{U}_1) - \frac{1}{2}A_{44}(U_1-\bar{U}_1)^2 \\ &\quad -\frac{1}{2}A_{33}(T-\bar{T})^2 + A_{35}(T-\bar{T})(U_2-\bar{U}_2) - \frac{1}{2}A_{55}(U-\bar{U}_2)^2. \end{aligned}$$

where

$$A_{11} = \frac{2}{3}a_{11},\ A_{22} = \frac{2}{3}a_{22},\ A_{33} = \frac{1}{2}c_1(\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2),$$
$$A_{44} = c_2\delta_1,\ A_{55} = c_3\delta_2,\ A_{12} = -(a_{12}+a_{21}),\ A_{13} = -c_1\delta_1 T,$$

$$A_{14} = c_2(\lambda_1 T + \beta_1) - r_{11}, \; A_{23} = -c_2\lambda_2 T, \; A_{25} = c_3(\lambda_2 T + \beta_2) - r_{21},$$

$$A_{34} = c_1\theta_1\delta_1 + c_2(\theta_0\delta_0 + \lambda_1\bar{x}_1), \; A_{35} = c_1\theta_2\delta_2 + c_3(\theta_0'\delta_0 + \lambda_2\bar{x}_2).$$

Sufficient conditions for $\frac{dV_1}{dt}$ to be negative definite are that the following conditions hold:

$$A_{12}^2 < A_{11}A_{22}, \tag{5.23}$$

$$A_{13}^2 < A_{11}A_{33}, \tag{5.24}$$

$$A_{14}^2 < A_{11}A_{44}, \tag{5.25}$$

$$A_{23}^2 < A_{22}A_{33}, \tag{5.26}$$

$$A_{25}^2 < A_{22}A_{55}, \tag{5.27}$$

$$A_{34}^2 < A_{33}A_{44}, \tag{5.28}$$

$$A_{35}^2 < A_{33}A_{55}. \tag{5.29}$$

Under the suitable choice of constants c_1, c_2 and c_3 as defined in Eq. (5.20). We note that inequalities (5.24)-(5.27) are automatically satisfied and (5.17) $\Rightarrow$ (5.23), (5.18) $\Rightarrow$ (5.28), (5.19) $\Rightarrow$ (5.29). Thus V_1 is a Liapunov function with respect to $\bar{E}$, whose domain contains the region Ω_1, proving the theorem.

Remark 1 In the case of instantaneous introduction of pollutant (i.e., $Q_0 = 0$) into the environment, it can be verified that there are four nonnegative equilibria, namely, $E_0(0,0,0,0,0)$, $\tilde{E}(\tilde{x}_1, 0, \tilde{T}, \tilde{U}_1, \tilde{U}_2)$, $\hat{E}(0, \hat{x}_2, \hat{T}, \hat{U}_1, \hat{U}_2)$ and $\bar{E}(\bar{x}_1, \bar{x}_2, \bar{T}, \bar{U}_1, \bar{U}_2)$. E_0 exists obviously and the existence of the remaining three equilibria can be seen in the similar fashion as discussed earlier. In particular, it may be noted that in this case inequlities (5.5)-(5.8) are satisfied. Further the stability behaviour of the equilibria is similar to the corresponding equilibria as given in the case of constant introduction of pollutant into the environment. It has been noted here that equilibrium levels of the competing species in the case of constant introduction of pollutant into the environment is lower than the case of instantaneous introduction, keeping other parameters and functions same in the model.

Remark 2 When $Q(t) = Q_0 + \epsilon\phi(t)$, $\phi(t+\omega) = \phi(t)$, i.e., in the case of periodic emission of pollutant into the environment, it can be verified that the results corresponding to

Theorem 3.4.1 and 3.4.2 in chapter 3 remain valid. In particular, it has been found that a small periodic influx of pollutant into the environment induces a periodic behaviour in the system.

5.4 Competition Model With Diffusion

In this section we consider the complete model (5.1)-(5.2) with $Q(t) = Q_0 > 0$ and we state the main results of this section in the form of the following theorem.

Theorem 5.4.1 *(i) If the equilibrium $\bar{E}$ of system (5.3) is globally asymptotically stable, then the corresponding uniform steady state of the initial-boundary value problems (5.1)-(5.2) must also be globally asymptotically stable.*

(ii) If the equilibrium $\bar{E}$ of system (5.3) is unstable, even then the unifoi in steady state of the initial-boundary value problems (5.1)-(5.2) can be made stable ny increasing diffusion coefficients to sufficiently large values.

Proof: Consider the following positive definite function

$$V_2(x_1(t), x_2(t), T(t), U_1(t), U_2(t)) = \int\int_D V_1(x_1, x_2, T, U_1, U_2)dA, \tag{5.30}$$

where V_1 is defined by Equation (5.21).

We have,

$$\begin{aligned} \frac{dV_2}{dt} &= \int\int_D \{\frac{\partial V_1}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial V_1}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial V_1}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial V_1}{\partial U_1}\frac{\partial U_1}{\partial t} + \frac{\partial V_1}{\partial U_2}\frac{\partial U_2}{\partial t}\}dA \\ &= I_1 + I_2, \end{aligned} \tag{5.31}$$

where

$$I_1 = \int\int_D \frac{dV_1}{dt}dA$$

$$\text{and} \quad I_2 = \int\int_D \{D_1\frac{\partial V_1}{\partial x_1}\nabla^2 x_1 + D_2\frac{\partial V_1}{\partial x_2}\nabla^2 x_2 + D_3\frac{\partial V_1}{\partial T}\nabla^2 T\}dA.$$

We note the following properties of V_1, namely,

$$\left.\frac{\partial V_1}{\partial x_1}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial x_2}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial T}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial U_1}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial U_2}\right]_{\partial D} = 0$$

and for all points of D,

$$\frac{\partial^2 V_1}{\partial x_1 \partial x_2} = \frac{\partial^2 V_1}{\partial x_1 \partial T} = \frac{\partial^2 V_1}{\partial x_1 \partial U_1} = \frac{\partial^2 V_1}{\partial x_1 \partial U_2} = \frac{\partial^2 V_1}{\partial x_2 \partial T} = \frac{\partial^2 V_1}{\partial x_2 \partial U_1}$$
$$= \frac{\partial^2 V_1}{\partial x_2 \partial U_2} = \frac{\partial^2 V_1}{\partial T \partial U_1} = \frac{\partial^2 V_1}{\partial T \partial U_2} = \frac{\partial^2 V_1}{\partial U_1 \partial U_2} = 0 \; and$$
$$\frac{\partial^2 V_1}{\partial x_1^2} > 0, \; \frac{\partial^2 V_1}{\partial x_2^2} > 0, \; \frac{\partial^2 V_1}{\partial T^2} > 0, \; \frac{\partial^2 V_1}{\partial U_1^2} > 0, \; \frac{\partial^2 V_1}{\partial U_2^2} > 0.$$

Under an analysis similar to chapter 2, we note that

$$\int\int_D \frac{\partial V_1}{\partial x_1} \nabla^2 x_1 dA = -\int\int_D \frac{\partial^2 V_1}{\partial x_1^2} \{(\frac{\partial x_1}{\partial x})^2 + (\frac{\partial x_1}{\partial y})^2\} dA \leq 0, \quad (5.32)$$
$$\int\int_D \frac{\partial V_1}{\partial x_2} \nabla^2 x_2 dA = -\int\int_D \frac{\partial^2 V_1}{\partial x_2^2} \{(\frac{\partial x_2}{\partial x})^2 + (\frac{\partial x_2}{\partial y})^2\} dA \leq 0, \quad (5.33)$$
$$\int\int_D \frac{\partial V_1}{\partial T} \nabla^2 T dA = -\int\int_D \frac{\partial^2 V_1}{\partial T^2} \{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\} dA \leq 0. \quad (5.34)$$

This shows that $I_2 \leq 0$.

Thus, we note that if $I_1 \leq 0$, i.e., if $\bar{E}$ is globally asymptotically stable in the absence of diffusion, then the uniform steady state of the initial-boundary value problems (5.1)-(5.2) also must be globally asymptotically stable. This proves the first part of theorem.

We further note that if $\frac{dV_1}{dt} > 0$, i.e., if $I_1 > 0$, then $\bar{E}$ may be unstable in the absence of diffusion. But Eqs. (5.31) and (8.25)-(5.34) show that by increasing diffusion coefficients D_i to sufficiently large values, $\frac{dV_2}{dt}$ can be made negative even if $I_1 > 0$. This proves the second part of the theorem.

The above theorem shows that the stability in the diffusive system is more plausible than that of the no diffusion case.

We shall explain the above theorem for a rectangular habitat D defined by

$$D = \{(x, y) : \; 0 \leq x \leq a, \; 0 \leq y \leq b\} \quad (5.35)$$

in the form of the following theorem.

Theorem 5.4.2 *Let the following inequalities hold:*

$$(a_{12}+a_{21})^2 < \frac{4}{9}\{a_{11}+\frac{D_1\bar{x}_1 a_{11}^2\pi^2(a^2+b^2)}{r_{10}^2 a^2 b^2}\}\{a_{22}+\frac{D_2\bar{x}_2 a_{22}^2\pi^2(a^2+b^2)}{r_{20}^2 a^2 b^2}\}, \quad (5.36)$$

$$\{c_1\theta_1\delta_1+c_2(\theta_0\delta_0+\lambda_1\bar{x}_1)\}^2 < \frac{1}{2}c_1c_2\delta_1\{\delta_0+\lambda_1\bar{x}_1+\lambda_2\bar{x}_2+\frac{D_3\pi^2(a^2+b^2)}{a^2b^2}\}, \quad (5.37)$$

$$\{c_1\theta_2\delta_2+c_3(\theta_0'\delta_0+\lambda_2\bar{x}_2)\}^2 < \frac{1}{2}c_1c_3\delta_2\{\delta_0+\lambda_1\bar{x}_1+\lambda_2\bar{x}_2+\frac{D_3\pi^2(a^2+b^2)}{a^2b^2}\}, \quad (5.38)$$

then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive octant.

Proof: Let us consider the rectangular region D given by Eq. (5.35). In this case I_2, which is defined in Eq. (5.31), can be written as

$$\begin{aligned} I_2 &= -D_1\int\int_D(\frac{\partial^2 V_1}{\partial x_1^2})\{(\frac{\partial x_1}{\partial x})^2+(\frac{\partial x_1}{\partial y})^2\}\,dA - D_2\int\int_D(\frac{\partial^2 V_1}{\partial x_2^2})\{(\frac{\partial x_2}{\partial x})^2+(\frac{\partial x_2}{\partial y})^2\} \\ &\quad -D_3\int\int_D(\frac{\partial^2 V_1}{\partial T^2})\{(\frac{\partial T}{\partial x})^2+(\frac{\partial T}{\partial y})^2\}\,dA. \end{aligned} \quad (5.39)$$

From Eq. (5.21) we get

$$\frac{\partial^2 V_1}{\partial x_1^2}=\frac{\bar{x}_1}{x_1^2},\ \frac{\partial^2 V_1}{\partial x_2^2}=\frac{\bar{x}_2}{x_2^2}\ and\ \frac{\partial^2 V_1}{\partial T^2}=c_1.$$

Hence

$$\begin{aligned} I_2 &\le -\frac{D_1\bar{x}_1 a_{11}^2}{r_{10}^2}\int\int_D\{(\frac{\partial x_1}{\partial x})^2+(\frac{\partial x_1}{\partial y})^2\}dA - \frac{D_2\bar{x}_2 a_{22}^2}{r_{20}^2}\int\int_D\{(\frac{\partial x_2}{\partial x})^2+(\frac{\partial x_2}{\partial y})^2\}dA \\ &\quad -D_3c_1\int\int_D\{(\frac{\partial T}{\partial x})^2+(\frac{\partial T}{\partial y})^2\}dA. \end{aligned}$$

Now

$$\begin{aligned} \int\int_D(\frac{\partial x_1}{\partial x})^2\,dA &= \int\int_D\{\frac{\partial(x_1-\bar{x}_1)}{\partial x}\}^2\,dA \\ &= \int_0^b\int_0^a\{\frac{\partial(x_1-\bar{x}_1)}{\partial x}\}^2\,dxdy \end{aligned}$$

Substituting $z=\frac{x}{a}$, it can be seen under similar analysis to chapter 2 that

$$\int\int_D(\frac{\partial x_1}{\partial x})^2\,dA \ge \frac{\pi^2}{a^2}\int\int_D(x_1-\bar{x}_1)^2dA$$

and

$$\int\int_D(\frac{\partial x_1}{\partial y})^2\,dA \ge \frac{\pi^2}{b^2}\int\int_D(x_1-\bar{x}_1)^2dA.$$

Thus,

$$
\begin{aligned}
I_2 \leq & -\frac{D_1\bar{x}_1 a_{11}^2\pi^2(a^2+b^2)}{r_{10}^2 a^2 b^2}\iint_D (x_1-\bar{x}_1)^2\,dA \\
& -\frac{D_2\bar{x}_2 a_{22}^2\pi^2(a^2+b^2)}{r_{20}^2 a^2 b^2}\iint_D (x_2-\bar{x}_2)^2\,dA \\
& -\frac{D_3 c_1\pi^2(a^2+b^2)}{a^2 b^2}\iint_D (T-\bar{T})^2\,dA.
\end{aligned}
$$

Now from (5.22) and (5.31) $\frac{dV_2}{dt}$ can be written as the sum of the quadratics

$$
\begin{aligned}
\frac{dV_2}{dt} \leq & \iint_D [-\frac{1}{2}B_{11}(x_1-\bar{x}_1)^2 + B_{12}(x_1-\bar{x}_1)(x_2-\bar{x}_2) - \frac{1}{2}B_{22}(x_2-\bar{x}_2)^2 \\
& -\frac{1}{2}B_{11}(x_1-\bar{x}_1)^2 + B_{13}(x_1-\bar{x}_1)(T-\bar{T}) - \frac{1}{2}B_{33}(T-\bar{T})^2 \\
& -\frac{1}{2}B_{11}(x_1-\bar{x}_1)^2 + B_{14}(x_1-\bar{x}_1)(U_1-\bar{U}_1) - \frac{1}{2}B_{44}(U_1-\bar{U}_1)^2 \\
& -\frac{1}{2}B_{22}(x_2-\bar{x}_2)^2 + B_{23}(x_2-\bar{x}_2)(T-\bar{T}) - \frac{1}{2}B_{33}(T-\bar{T})^2 \\
& -\frac{1}{2}B_{22}(x_2-\bar{x}_2)^2 + B_{25}(x_2-\bar{x}_2)(U_2-\bar{U}_2) - \frac{1}{2}B_{55}(U_2-\bar{U}_2)^2 \\
& -\frac{1}{2}B_{33}(T-\bar{T})^2 + B_{34}(T-\bar{T})(U_1-\bar{U}_1) - \frac{1}{2}B_{44}(U_1-\bar{U}_1)^2 \\
& -\frac{1}{2}B_{33}(T-\bar{T})^2 + B_{35}(T-\bar{T})(U_2-\bar{U}_2) - \frac{1}{2}B_{55}(U_2-\bar{U}_2)^2]dA,
\end{aligned}
$$

where

$$
\begin{aligned}
& B_{11} = \frac{2}{3}\{a_{11} + \frac{D_1\bar{x}_1 a_{11}^2\pi^2(a^2+b^2)}{r_{10}^2 a^2 b^2}\},\ B_{22} = \frac{2}{3}\{a_{22} + \frac{D_2\bar{x}_2 a_{22}^2\pi^2(a^2+b^2)}{r_{20}^2 a^2 b^2}\}, \\
& B_{33} = \frac{1}{2}c_1\{\delta_0 + \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2 + \frac{D_3\pi^2(a^2+b^2)}{a^2b^2}\}, \\
& B_{44} = c_2\delta_1,\ B_{55} = c_3\delta_2,\ B_{12} = -(a_{12}+a_{21}),\ B_{13} = -c_1\delta_1 T, \\
& B_{14} = c_2(\lambda_1 T + \beta_1) - r_{11},\ B_{23} = -c_2\lambda_2 T,\ B_{25} = c_3(\lambda_2 T + \beta_2) - r_{21}, \\
& B_{34} = c_1\theta_1\delta_1 + c_2(\theta_0\delta_0 + \lambda_1\bar{x}_1),\ B_{35} = c_1\theta_2\delta_2 + c_3(\theta_0'\delta_0 + \lambda_2\bar{x}_2).
\end{aligned}
$$

Sufficient conditions for $\frac{dV_2}{dt}$ to be negative definite are that the following conditions hold:

$$B_{12}^2 < B_{11}B_{22}, \tag{5.40}$$

$$B_{13}^2 < B_{11}B_{33}, \tag{5.41}$$

$$B_{14}^2 < B_{11}B_{44}, \tag{5.42}$$

$$
\begin{aligned}
B_{23}^2 &< B_{22}B_{33}, & (5.43)\\
B_{25}^2 &< B_{22}B_{55}, & (5.44)\\
B_{34}^2 &< B_{33}B_{44}, & (5.45)\\
B_{35}^2 &< B_{33}B_{55}. & (5.46)
\end{aligned}
$$

We note that inequalities (5.41)-(5.44) are satisfied automatically and (5.36) $\Rightarrow$ (5.40), (5.37) $\Rightarrow$ (5.45), and (5.38) $\Rightarrow$ (5.46). Thus V_2 is a Liapunov function with respect to $\bar{E}$, whose domain contains the region Ω_1, proving the theorem.

It may be noted here that inequalities (5.36)-(5.38) will be satisfied if we increase D_1, D_2, and D_3 to sufficiently large values. This implies that for a given rectangular region, by increasing diffusion coefficients sufficiently large, an unstable steady state in the absence of diffusion can be made stable. Thus, we conclude that in the presence of diffusion the competing species converge towards their respective carrying capacities faster than the case of no diffusion.

5.5 Cooperation Model

In this case we have $r_{10} > 0$, $r_{20} > 0$, $a_{12} < 0$ and $a_{21} < 0$. There exist four nonnegative equilibria, namely, $E_0(0, 0, \frac{Q_0}{\delta_0(1-\theta_0\theta_1-\theta_0'\theta_2)}, \frac{\theta_0 Q_0}{\delta_1(1-\theta_0\theta_1-\theta_0'\theta_2)}, \frac{\theta_0' Q_0}{\delta_2(1-\theta_0\theta_1-\theta_0'\theta_2)})$, $\tilde{E}_c(\tilde{x}_{1c}, 0, \tilde{T}_c, \tilde{U}_{1c}, \tilde{U}_{2c})$, $\hat{E}_c(0, \hat{x}_{2c}, \hat{T}_c, \hat{U}_{1c}, \hat{U}_{2c})$ and $\bar{E}_c(\bar{x}_{1c}, \bar{x}_{2c}, \bar{T}_c, \bar{U}_{1c}, \bar{U}_{2c})$.

E_0 exists if $1 - \theta_0\theta_1 - \theta_0'\theta_2 > 0$. Existence of $\tilde{E}_c$, $\hat{E}_c$ and $\bar{E}_c$ can be checked as in competition model in section 5.3.

Local stability behaviour of E_0, $\tilde{E}_c$ and $\hat{E}_c$ are similar to the corresponding equilibria of section 5.3. The following theorem shows the local stability character of $\bar{E}_c$, the proof of which is similar to Theorem 5.3.1 and hence is omitted.

Theorem 5.5.1 *Let the following inequalities hold:*

$$(a_{12} + a_{21})^2 < \frac{4}{9} a_{11} a_{22}, \tag{5.47}$$

$$\{k_1' \theta_1 \delta_1 + k_2' (\theta_0 \delta_0 + \lambda_1 \bar{x}_{1c})\}^2 < \frac{1}{2} k_1' k_2' \delta_1 (\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}), \tag{5.48}$$

$$\{k_1' \theta_2 \delta_2 + k_3' (\theta_0' \delta_0 + \lambda_2 \bar{x}_{2c})\}^2 < \frac{1}{2} k_1' k_3' \delta_2 (\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}), \tag{5.49}$$

where

$$k_1' = min\{\frac{a_{11}(\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c})}{4\lambda_1^2 \bar{T}_c^2}, \frac{a_{22}(\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c})}{4\lambda_2^2 \bar{T}_c^2}\},$$

$$k_2' = \frac{r_{11}}{\lambda_1 \bar{T}_c + \beta_1},$$

$$k_3' = \frac{r_{21}}{\lambda_2 \bar{T}_c + \beta_2},$$

then $\bar{E}_c$ is locally asymptotically stable.

In order to show the global stability of $\bar{E}_c$, we need the following lemma whose proof is easy and hence is omitted.

Lemma 5.5.1 *The set*

$$\Omega_2 = \{(x_1, x_2, T, U_1, U_2) \ : \ 0 \le x_1 \le x_{1\infty} < \infty, \ 0 \le x_2 \le x_{2\infty} < \infty,$$
$$0 \le T + U_1 + U_2 \le L_2\}$$

attracts all solutions initiating in the positive orthant, where

$$L_2 = \frac{1}{\delta}\{Q_0 + \beta_1 x_{1\infty} + \beta_2 x_{2\infty}\}$$
$$\delta = min\{\delta_0(1 - \theta_0 - \theta_0'), \delta_1(1 - \theta_1), \delta_2(1 - \theta_2)\}.$$

The following theorem shows the global stability of $\bar{E}_c$ whose proof is similar to Theorem 5.3.2 and hence is omitted.

Theorem 5.5.2 *Let the following inequalities hold:*

$$(a_{12} + a_{21})^2 < \frac{4}{9} a_{11} a_{22}, \tag{5.50}$$

$$\{k_1 \theta_1 \delta_1 + k_2 (\theta_0 \delta_0 + \lambda_1 \bar{x}_{1c})\}^2 < \frac{1}{2} k_1 k_2 \delta_1 (\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}), \tag{5.51}$$

$$\{k_1 \theta_2 \delta_2 + k_3 (\theta_0' \delta_0 + \lambda_2 \bar{x}_{2c})\}^2 < \frac{1}{2} k_1 k_3 \delta_2 (\delta_0 + \lambda_1 \bar{x}_{1c} + \lambda_2 \bar{x}_{2c}), \tag{5.52}$$

where

$$k_1 = min\{\frac{a_{11}(\delta_0 + \lambda_1\bar{x}_{1c} + \lambda_2\bar{x}_{2c})}{4\lambda_1^2 L_2^2}, \frac{a_{22}(\delta_0 + \lambda_1\bar{x}_{1c} + \lambda_2\bar{x}_{2c})}{4\lambda_2^2 L_2^2}\},$$

$$k_2 = \frac{r_{11}}{\lambda_1 L_2 + \beta_1},$$

$$k_3 = \frac{r_{21}}{\lambda_2 L_2 + \beta_2}.$$

Then $\bar{E}_c$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

It may be noted here that conditions in Theorem 5.5.1 are similar to Theorem 5.3.1, and conditions in Theorem 5.5.2 are similar to Theorem 5.3.2 where the equilibrium $\bar{E}$ has been replaced by $\bar{E}_c$.

Remark 3 Effect of diffusion in the case of cooperation model can be studied in a similar way as that of competition model given in section 5.4. It may be noted here that the results of Theorem 5.4.1 are also valid in the case of cooperation.

5.6 Prey-Predator Model

We consider x_1 and x_2 to be prey and predator respectively. Then in this case we have $r_{10} > 0$, $r_{20} < 0$, $a_{12} > 0$ and $a_{21} < 0$.

We take, $a_{21} = -b_{21}$, and $r_{20} = -r'_{20}$, where $b_{21} > 0$, $r'_{20} > 0$.

In this case there exist three nonnegative equilibria, namely, $E_0(0,0,\frac{Q_0}{\delta_0(1-\theta_0\theta_1-\theta'_0\theta_2)}, \frac{\theta_0 Q_0}{\delta_1(1-\theta_0\theta_1-\theta'_0\theta_2)}, \frac{\theta'_0 Q_0}{\delta_2(1-\theta_0\theta_1-\theta'_0\theta_2)})$, $\tilde{E}_p(\tilde{x}_{1p}, 0, \tilde{T}_p, \tilde{U}_{1p}, \tilde{U}_{2p})$, and $\bar{E}_p(\bar{x}_{1p}, \bar{x}_{2p}, \bar{T}_p, \bar{U}_{1p}, \bar{U}_{2p})$.

E_0 exists if $1 - \theta_0\theta_1 - \theta'_0\theta_2 > 0$. Existence of $\tilde{E}_p$, and $\bar{E}_p$ can be established in a similar way as in the competition model.

Local stability behaviour of E_0, and $\tilde{E}_p$ can also be studied in a similar way as in the competition model.

The following theorem shows that $\bar{E}_p$ is locally asymptotically stable. The proof of this theorem is similar to Theorem 5.3.1 and hence is omitted.

Theorem 5.6.1 *Let the following inequalities hold:*

$$\{\bar{k}_1\theta_1\delta_1 + \bar{k}_2(\theta_0\delta_0 + \lambda_1\bar{x}_{1p})\}^2 < \frac{1}{2}\bar{k}_1\bar{k}_2\delta_1(\delta_0 + \lambda_1\bar{x}_{1p} + \lambda_2\bar{x}_{2p}), \quad (5.53)$$
$$\{\bar{k}_1\theta_2\delta_2 + \bar{k}_3(\theta_0'\delta_0 + \lambda_2\bar{x}_{2p})\}^2 < \frac{1}{2}\bar{k}_1\bar{k}_3\delta_2(\delta_0 + \lambda_1\bar{x}_{1p} + \lambda_2\bar{x}_{2p}), \quad (5.54)$$

where

$$\bar{k}_1 = min\{\frac{a_{11}(\delta_0 + \lambda_1\bar{x}_{1p} + \lambda_2\bar{x}_{2p})}{4\lambda_1^2\bar{T}_p^2}, \frac{a_{22}a_{12}(\delta_0 + \lambda_1\bar{x}_{1p} + \lambda_2\bar{x}_{2p})}{4\lambda_2^2 b_{21}\bar{T}_p^2}\},$$
$$\bar{k}_2 = \frac{r_{11}}{\lambda_1\bar{T}_p + \beta_1},$$
$$\bar{k}_3 = \frac{a_{12}}{b_{21}}\frac{r_{21}}{\lambda_2\bar{T}_p + \beta_2}.$$

Then $\bar{E}_p$ is locally asymptotically stable.

In order to show the global stability of $\bar{E}_p$, we need the following lemma whose proof is easy and hence is omitted.

Lemma 5.6.1 *The set*

$$\Omega_3 = \{(x_1, x_2, T, U_1, U_2) : 0 \le x_1 \le \frac{r_{10}}{a_{11}},\ 0 \le x_2 \le \frac{r_{10}b_{21}}{a_{11}a_{22}},\ 0 \le T + U_1 + U_2 \le L_3\}$$

attracts all solutions initiating in the positive orthant, where

$$L_3 = \frac{1}{\delta}\frac{r_{10}}{a_{11}}\{\beta_1 + \frac{b_{21}}{a_{22}}\beta_2\},$$
$$\delta = min\{\delta_0(1 - \theta_0 - \theta_0'), \delta_1(1 - \theta_1), \delta_2(1 - \theta_2)\}.$$

In the following theorem we are able to write down conditions for $\bar{E}_p$ to be globally asymptotically stable. The proof of this theorem is similar to to Theorem 5.3.2 and hence is omitted.

Theorem 5.6.2 *Let the following inequalities hold:*

$$\{\hat{k}_1\theta_1\delta_1 + \hat{k}_2(\theta_0\delta_0 + \lambda_1\bar{x}_{1p})\}^2 < \frac{1}{2}\hat{k}_1\hat{k}_2\delta_1(\delta_0 + \lambda_1\bar{x}_{1p} + \lambda_2\bar{x}_{2p}), \quad (5.55)$$

$$\{\hat{k}_1\theta_2\delta_2 + \hat{k}_3(\theta_0'\delta_0 + \lambda_2\bar{x}_{2p})\}^2 < \frac{1}{2}\hat{k}_1\hat{k}_3\delta_2(\delta_0 + \lambda_1\bar{x}_{1p} + \lambda_2\bar{x}_{2p}), \quad (5.56)$$

where

$$\hat{k}_1 = min\{\frac{a_{11}(\delta_0 + \lambda_1\bar{x}_{1p} + \lambda_2\bar{x}_{2p})}{4\lambda_1^2 L_3^2}, \frac{a_{22}a_{12}(\delta_0 + \lambda_1\bar{x}_{1p} + \lambda_2\bar{x}_{2p})}{4\lambda_2^2 b_{21} L_3^2}\},$$

$$\hat{k}_2 = \frac{r_{11}}{\lambda_1 L_3 + \beta_1},$$

$$\hat{k}_3 = \frac{a_{12}r_{21}}{b_{21}(\lambda_2 L_3 + \beta_2)}.$$

Then $\bar{E}_p$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Remark 4 Effect of diffusion in the case of prey-predator model is found to be similar to the competition model given in section 5.3. In particular, the results of Theorem 5.4.1 remain valid in this case.

5.7 Conservation Model

In the previous sections it has been noted that if the environmental pollution increases without control, then the survival (growth and existence) of the two interacting biological species may be threatened. Therefore, some kind of efforts must be adopted to control the undesired level of the pollutant present in the environment so that the survival of the species may be ensured. Keeping this in mind, in this section a mathematical model is proposed and analysed to control the undesired level of the pollutant present in the environment. It is assumed that the effort applied to control pollutant is proportional to its undesired level. Following Shukla et al. (1989), Dubey (1997a) and Shukla and Dubey (1997) differential equations governing the system may be written as

$$
\begin{aligned}
\frac{\partial x_1}{\partial t} &= r_{10}x_1 - r_{11}x_1U_1 - a_{11}x_1^2 - a_{12}x_1x_2 + D_1\nabla^2 x_1, \\
\frac{\partial x_2}{\partial t} &= r_{20}x_2 - r_{21}x_2U_2 - a_{21}x_1x_2 - a_{22}x_2^2 + D_2\nabla^2 x_2, \\
\frac{\partial T}{\partial t} &= Q(t) - \delta_0 T + \theta_1\delta_1 U_1 + \theta_2\delta_2 U_2 - \lambda_1 x_1 T - \lambda_2 x_2 T \\
&\quad -\alpha F + D_3\nabla^2 T, \qquad (5.57) \\
\frac{\partial U_1}{\partial t} &= -\delta_1 U_1 + \theta_0\delta_0 T + \lambda_1 x_1 T + \beta_1 x_1, \\
\frac{\partial U_2}{\partial t} &= -\delta_2 U_2 + \theta_0'\delta_0 T + \lambda_2 x_2 T + \beta_2 x_2, \\
\frac{\partial F}{\partial t} &= \mu(T - T_c)H(T - T_c) - \nu F, \\
& 0 \le \theta_0 + \theta_0' \le 1,\ 0 \le \theta_1, \theta_2 \le 1.
\end{aligned}
$$

We impose the following initial and boundary conditions on the system (5.57):

$$
\begin{aligned}
& x_1(x,y,0) = \phi(x,y) \ge 0,\ x_2(x,y,0) = \psi(x,y) \ge 0, \\
& T(x,y,0) = \xi(x,y) \ge 0,\ U_1(x,y,0) = \zeta_1(x,y) \ge 0, \\
& U_2(x,y,0) = \zeta_2(x,y) \ge 0,\ F(x,y,0) = \chi(x,y) \ge 0\ (x,y)\epsilon D \qquad (5.58) \\
& \frac{\partial x_1}{\partial n} = \frac{\partial x_2}{\partial n} = \frac{\partial T}{\partial n} = 0,\ (x,y)\ \epsilon\ \partial D, t \ge 0,
\end{aligned}
$$

where n is the unit outward normal to ∂D.

In model (5.57), $F(x, y; t)$ is the density of effort applied to control the undesired level of environmental pollutant. α is the depletion rate coefficient of $T(x, y, t)$ due to the effort F. μ is the growth rate coefficients of F and ν is its depreciation rate coefficient. T_c is the critical level of the environmental pollutants, which is assumed to be harmless to the species. $H(t)$ denotes the unit step function which takes into account the case when $T \le T_c$.

We shall analyse the conservation model (5.57) assuming that the interaction between the two species is of competition type and the introduction of pollutant into the environment being constant, i.e., $Q(t) = Q_0 > 0$.

5.8 Conservation Model Without Diffusion

In this case we take $D_1 = D_2 = D_3 = 0$ in the model (5.57). Then the model (5.57) has only interior equilibrium $E^*(x_1^*, x_2^*, T^*, U_1^*, U_2^*, F^*)$. Existence of E^* can be shown in a similar fashion as $\bar{E}$. In the following theorem it is shown that E^* is locally asymptotically stable, the proof of which is similar to Theorem 5.3.1 and hence is omitted.

Theorem 5.8.1 *Let the following inequalities hold:*

$$(a_{12} + a_{21})^2 < \frac{4}{9} a_{11} a_{22}, \tag{5.59}$$

$$\{d_1' \theta_1 \delta_1 + d_2'(\theta_0 \delta_0 + \lambda_1 x_1^*)\}^2 < \frac{2}{5} d_1' d_2' \delta_1 (\delta_0 + \lambda_1 x_1^* + \lambda_2 x_2^*), \tag{5.60}$$

$$\{d_1' \theta_2 \delta_2 + d_3'(\theta_0' \delta_0 + \lambda_2 x_2^*)\}^2 < \frac{2}{5} d_1' d_3' \delta_2 (\delta_0 + \lambda_1 x_1^* + \lambda_2 x_2^*), \tag{5.61}$$

where

$$d_1' = min\{\frac{a_{11}(\delta_0 + \lambda_1 x_1^* + \lambda_2 x_2^*)}{5(\lambda_1 T^*)^2}, \frac{a_{22}(\delta_0 + \lambda_1 x_1^* + \lambda_2 x_2^*)}{5(\lambda_2 T^*)^2}\},$$
$$d_2' = \frac{r_{11}}{\lambda_1 T^* + \beta_1},$$
$$d_3' = \frac{r_{21}}{\lambda_2 T^* + \beta_2}.$$

Then equilibrium E^* is locally asymptotically stable.

In the following lemma, a region of attraction for system (5.57) without diffusion is established. The proof of this lemma is similar to Lemma 5.3.1 and hence is omitted.

Lemma 5.8.1 *The set*

$$\Omega_4 = \{(x_1, x_2, T, U_1, U_2) : 0 \le x_1 \le \frac{r_{10}}{a_{11}}, 0 \le x_2 \le \frac{r_{20}}{a_{22}}, 0 \le T + U_1 + U_2 \le L_1, 0 \le F \le \frac{\mu}{\nu} L_1\}$$

attracts all solutions initiating in the positive orthant, where

$$L_1 = \frac{1}{\delta}\{Q_0 + \beta_1 \frac{r_{10}}{a_{11}} + \beta_2 \frac{r_{20}}{a_{22}}\},$$
$$\delta = min\{\delta_0(1 - \theta_0 - \theta_0'), \delta_1(1 - \theta_1), \delta_2(1 - \theta_2)\}.$$

The following theorem gives criteria for E^* to be globally asymptotically stable, whose proof is similar to Theorem 5.3.2 and hence is omitted.

Theorem 5.8.2 *Let the following inequalities hold:*

$$
\begin{aligned}
(a_{12}+a_{21})^2 &< \frac{4}{9}a_{11}a_{22}, &(5.62)\\
\{d_1\theta_1\delta_1 + d_2(\theta_0\delta_0 + \lambda_1 x_1^*)\}^2 &< \frac{2}{5}d_1 d_2 \delta_1(\delta_0 + \lambda_1 x_1^* + \lambda_2 x_2^*), &(5.63)\\
\{d_1\theta_2\delta_2 + d_3(\theta_0'\delta_0 + \lambda_2 x_2^*)\}^2 &< \frac{2}{5}d_1 d_3 \delta_2(\delta_0 + \lambda_1 x_1^* + \lambda_2 x_2^*), &(5.64)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= min\{\frac{a_{11}(\delta_0 + \lambda_1 x_1^* + \lambda_2 x_2^*)}{5(\lambda_1 L_1)^2}, \frac{a_{22}(\delta_0 + \lambda_1 x_1^* + \lambda_2 x_2^*)}{5(\lambda_2 L_1)^2}\},\\
d_2 &= \frac{r_{11}}{\lambda_1 L_1 + \beta_1},\\
d_3 &= \frac{r_{21}}{\lambda_2 L_1 + \beta_2}.
\end{aligned}
$$

Then equilibrium E^* is globally asymptotically stable.

Theorems 5.8.1 and 5.8.2 show that if suitable effort is made to control the undesired level of environmental pollutants, then the survival of the two competing species may be ensured.

5.9 Conservation Model With Diffusion

We now consider the case when $D_i > 0 (i = 1, 2, 3)$ in model (5.57). We shall show that the uniform steady state $x_1(x, y, t) = x_1^*, x_2(x, y, t) = x_2^*, T(x, y, t) = T^*, U_1(x, y, t) = U_1^*, U_2(x, y, t) = U_2^*$ and $F(x, y, t) = F^*$ is globally asymptotically stable. For this, we consider the following positive definite function

$$V_3(x_1(t), x_2(t), T(t), U_1(t), U_2(t), F(t)) = \int\int_D V_2(x_1, x_2, T, U_1, U_2, F)\, dA,$$

where

$$
\begin{aligned}
V_2(x_1, x_2, T, U_1, U_2, F) &= x_1 - x_1^* - x_1^* \ln\frac{x_1}{x_1^*} + c_1(x_2 - x_2^* - x_2^*\ln\frac{x_2}{x_2^*}) + \frac{c_2}{2}(T - T^*)^2\\
&\quad + \frac{c_{c_3}}{2}(U_1 - U_1^*)^2 + \frac{c_4}{2}(U_2 - U_2^*)^2 + \frac{c_5}{2}(F - F^*)^2.
\end{aligned}
$$

The constants $c_i s$ are to be chosen suitably.

Then as earlier, it can be checked that if $\frac{dV_2}{dt} < 0$, then $\frac{dV_3}{dt} < 0$. This implies that if E^* is globally asymptotically stable for system (4.67) without diffusion, then the corresponding uniform steady state of system (4.67)-(4.68) is also globally asymptotically stable with respect to solutions such that $\phi(x,y) > 0, \psi(x,y) > 0, \xi(x,y) > 0, \zeta(x,y) > 0, \zeta_1(x,y) > 0, \zeta_2(x,y) > 0, \ (x,y) \epsilon D$.

5.10 Numerical Examples

In this section we present numerical examples to explain the applicability of the results discussed in sections 5.3, 5.5, 5.6 and 5.8. We choose the following values of the parameters in model (5.3).

$$\begin{aligned} &r_{11} = 0.05,\ r_{21} = 0.04,\ a_{11} = 0.22,\ a_{22} = 0.26, \\ &Q_0 = 15.0,\ \delta_0 = 6.7,\ \delta_1 = 15.5,\ \delta_2 = 10.4; \\ &\theta_1 = 0.02,\ \theta_2 = 0.03,\ \theta_0 = 0.01,\ \theta_0' = 0.04, \\ &\lambda_1 = 0.06,\ \lambda_2 = 0.09,\ \beta_1 = 0.25,\ and\ \beta_2 = 0.3. \end{aligned} \tag{5.65}$$

Example 1 In this example, we consider the case when the two species are competing with each other. In addition to the values of the parameters given in Eq. (5.65), we choose the following parameters in model (5.3):

$$r_{10} = 5.0,\ r_{20} = 3.0,\ a_{12} = 0.07\ and\ a_{21} = 0.08.$$

With the above values of the parameters, it can be checked that the interior equilibrium $\bar{E}$ exists, and is given by,

$$\bar{x}_1 = 21.01420,\ \bar{x}_2 = 5.03089,\ \bar{T} = 1.81106,\ \bar{U}_1 = 0.49409,\ \bar{U}_2 = 0.27064. \tag{5.66}$$

It can also be checked that conditions (5.13)-(5.15) in Theorem 5.3.1 are satisfied which shows that $\bar{E}$ is locally asymptotically stable.

Further, we note that conditions (5.17)-(5.19) in Theorem 5.3.2 are also satisfied which shows that $\bar{E}$ is globally asymptotically stable.

Example 2 Here we consider the case when the two species are cooperating with each other. In addition to the values of the parameters given in Eq. (5.65), we choose the following parameters in model (5.3):

$$r_{10} = 5.0, \; r_{20} = 3.0, \; a_{12} = -0.07 \; and \; a_{21} = -0.08.$$

With the above values of the parameters, it can be verified that the interior equilibrium $\bar{E}_c$ exists, and is given by,

$$\bar{x}_{1c} = 29.05284, \; \bar{x}_{2c} = 20.34072, \; \bar{T}_c = 1.50652, \; \bar{U}_{1c} = 0.64453, \; \bar{U}_{2c} = 0.89076. \quad (5.67)$$

It can also be verified that conditions (5.47)-(5.49) in Theorem 5.5.1 are satisfied, showing the local stability character of $\bar{E}_c$.

Further, it is easy to verify that conditions (5.50)-(5.52) in Theorem 5.5.2 are satisfied, showing the global stability character of $\bar{E}_c$.

Example 3 In this example, we consider the case when x_2 is predating on x_1. In addition to the values of the parameters given in Eq. (5.65), we choose the following parameters in model (5.3):

$$r_{10} = 5.0, \; r_{20} = -0.5, \; a_{12} = 0.2 \; and \; a_{21} = -0.1.$$

With the above values of the parameters, it can be verified that the interior equilibrium $\bar{E}_p$ exists, and is given by,

$$\bar{x}_{1p} = 18.09059, \; \bar{x}_{2p} = 4.99304, \; \bar{T}_p = 1.84798, \; \bar{U}_{1p} = 0.42918, \; \bar{U}_{2p} = 0.27150. \quad (5.68)$$

It can also be verified that conditions (5.53) and (5.54) in Theorem 5.6.1 are satisfied. This shows that $\bar{E}_p$ is locally asymptotically stable.

Further, it can also be checked that conditions (5.55) and (5.56) in Theorem 5.6.2 are satisfied. This shows that $\bar{E}_p$ is globally asymptotically stable.

Example 4 Here we present a numerical example for the model with conservation. In this example we consider the case when the two species are competing with each other. In addition to the values of the parameters given in Eq. (5.65), we choose the following parameters in model (5.57) without diffusion:

$$r_{10} = 5.0,\ r_{20} = 3.0,\ a_{12} = 0.07,\ a_{21} = 0.08,$$

$$\alpha = 22.0,\ \mu = 20.0,\ \nu = 0.01,\ T_c = 0.15.$$

With the above values of the parameters, it can be checked that the interior equilibrium $\bar{E}$ exists, and is given by,

$$\bar{x}_1 = 21.04420,\ \bar{x}_2 = 5.03897,\ \bar{T} = 0.15032,\ \bar{U}_1 = 0.35232,\ \bar{U}_2 = 0.15578. \tag{5.69}$$

It can also be checked that conditions (5.59)-(5.61) in Theorem 5.8.1 are satisfied which shows that $\bar{E}$ is locally asymptotically stable.

Further, we note that conditions (5.62)-(5.64) in Theorem 5.8.2 are also satisfied which shows that $\bar{E}$ is globally asymptotically stable.

5.11 Conclusions

In this chapter, a mathematical model has been proposed and analysed to study the survival of two interacting species in a polluted environment, the mode of interaction being competition, cooperation and predation. The model has been analysed with and without diffusion. When there is no diffusion it has been shown that in the case of constant introduction of pollutant into the environment the competing species settle down to their respective equilibrium levels, the magnitude of which depends upon the equilibrium levels of washout and uptake rates of pollutant. It has also been noted that if the concentration of pollutant increase unabatedly, then the survival of the species would be threatened. In the case of instantaneous introduction of pollutant into the environment, it has been found that the competing species again settle down to their respective equilibrium levels whose magnitude is higher than the case of constant introduction of pollutant into the environment. In the case of periodic emission of pollutant into the environment, it has been found that a periodic influx of pollutant with small amplitude causes a periodic behaviour in the system.

The effect of diffusion on the interior equilibrium state of the system has also been investigated. It has been shown that if the positive equilibrium of the system without diffusion is globally asymptotically stable, then the corresponding uniform steady state of the system with diffusion is also globally asymptotically stable. It has further been noted that if the positive equilibrium of the system with no diffusion is unstable, then the corresponding uniform steady state of the system with diffusion can be made stable by increasing diffusion coefficients appropriately. From the proof of Theorem 5.4.1, it should be noted that $\frac{dV_2}{dt}$ contains some extra negative terms implying that the global stability is more feasible in the case of diffusion than the case of no diffusion. In case of cooperation and prey-predator, similar results have been found. In each case, a numerical example has been given to illustrate the results obtained. A model to control the undesired level of the pollutant present in the environment has also been proposed. By analysing this model it has been shown that if suitable efforts are made to control the undesired level of the environmental pollutant, the survival of the species may be ensured.

Chapter 6

MODELS FOR EFFECTS OF INDUSTRIALIZATION AND POLLUTION ON RESOURCES IN A DIFFUSIVE SYSTEM

6.1 Introduction

A rapid pace of industrialization and its by-products has started changing the environment by emanating hazardous waste discharge and poisonous gas fumes and smokes into the environment (Nelson, 1970; Patin, 1982). All these by-products adversely affect the ecosystems- water, air, vegetation, forestry resources and other forms of life. It is therefore absolutely essential to study the effects of industrialization and pollution on forestry resources.

In recent decades, some investigations have been made to study the effects of pollutants on various ecosystems utilizing mathematical models (Hallam and Clark, 1982; Hallam et al., 1983; Hallam and De Luna, 1984; De Luna and Hallam, 1987; Freedman and Shukla, 1991; Huaping and Ma, 1991). As pointed out in the previous chapter, the above studies include the effects of pollutants on a single or two species community. Shukla et al. (1989) proposed and analysed a mathematical model to assess the effects of industrialization on the degradation of forestry biomass together with a reforestation effort. Dubey (1997a) studied the effects of toxicant on depletion and conservation of forestry resources. Shukla and Dubey (1997) also proposed and analysed a mathematical model to study the effects of population and pollution on resources. But in the above studies effects of industrialization and pollution on the biological species in a diffusive system do not appear.

In this chapter we, therefore, consider a dynamical model to study the effects of pollutant emitted by industries on biological species such as plant/tree population in a forest stand. It is assumed that the pollutant is emitted into the environment with a rate which is dependent on the industrialization and is depleted by some natural degradation factors. The model is analysed in two cases, namely, without diffusion and with diffusion. In the analysis of the model, the rate of introduction of pollutant is assumed to be (i) industrialization dependent, (ii) constant, (iii) instantaneous, and (iv) periodic.

6.2 Mathematical Model

Consider a biological species such as plant/tree population in a forest stand (i.e. forestry resource biomass) affected by the pollutant emitted into the environment by different types of industrial processes in a single closed region D with smooth boundary ∂D. It is assumed that the growth rate of the species decreases with the uptake of pollutant by the species and the corresponding carrying capacity decreases with the increase in the density of industrialization as well as the environmental concentration of pollutant. The density of industrialization is assumed to be wholly dependent upon the resource and the interaction is prey-predator type. Following Freedman and Shukla (1991), Huaping and Ma (1991) and Dubey (1997a) the dynamics of the system may be governed by the following differential equations:

$$\begin{aligned}
\frac{\partial B}{\partial t} &= r(U)B - \frac{r_0 B^2}{K(I,T)} - \alpha_1 IB + D_1\nabla^2 B,\\
\frac{\partial I}{\partial t} &= -\gamma_0 I - \gamma_1 I^2 + \alpha_2 IB + D_2\nabla^2 I,\\
\frac{\partial T}{\partial t} &= Q - \delta_0 T - \alpha BT + \theta_1\delta_1 U + \pi\nu BU + D_3\nabla^2 T,\\
\frac{\partial U}{\partial t} &= \beta B + \theta_0\delta_0 T + \alpha BT - \delta_1 U - \nu BU,\\
0 &\le \theta_0, \theta_1, \pi \le 1.
\end{aligned} \tag{6.1}$$

The above model needs to be analysed with following initial and boundary conditions:

$$\begin{aligned}
&B(x,y,0) = \phi(x,y) \ge 0,\ I(x,y,0) = \psi(x,y) \ge 0,\ T(x,y,0) = \xi(x,y) \ge 0,\\
&U(x,y,0) = \zeta(x,y) \ge 0,\ (x,y)\varepsilon D,\\
&\frac{\partial B}{\partial n} = \frac{\partial I}{\partial n}, \frac{\partial T}{\partial n} = 0,\ (x,y)\ \varepsilon\ \partial D,\ t \ge 0,
\end{aligned} \tag{6.2}$$

where n is the unit outward normal to ∂D.

In model (6.1), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian diffusion operator. $B(x,y,t)$ is the forestry resource biomass, $I(x,y,t)$ the industrialization pressure, $T(x,y,t)$ the concentration of the pollutant present in the environment and $U(x,y,t)$ the uptake concentration of pollutant by the resource biomass at coordinates $(x,y)\varepsilon D$ and time $t \ge 0$. D_1, D_2 and D_2 are the diffusion rate coefficients of $B(x,y,t)$, $I(x,y,t)$ and $T(x,y,t)$ respectively in D. β is the net uptake of pollutant by the resource biomass. δ_0 is the depletion rate of pollutant from the environment due to various processes including biological transformation, chemical hydrolysis, volatilization or microbial degradation, and a fraction θ_0 of it may again reenter into the resource biomass with the uptake of pollutant. δ_1 is the natural depletion rate coefficient of U due to ingestion and depuration of pollutant, and a fraction θ_1 of it may again reenter into the environment. α is the depletion rate coefficient of T due to its uptake by the resource biomass. ν denotes the depletion rate coefficient of U due to resource biomass and a fraction π of it reentering into the environment. α_1 is the depletion rate coefficient of resource biomass due to industrialization. α_2 is the growth rate coefficient of industrialization due to resource biomass. γ_0 is the natural depletion rate coefficient of the industrialization and γ_1 is its intraspecific interference coefficient. The parameters α, α_1, α_2, δ_0, δ_1, β, γ_0, γ_1 and ν are assumed to be positive constants.

In model (6.1), Q represents the rate of introduction of pollutant into the environment which may be industrialization dependent, constant, zero or periodic.

The function r(U) denotes the specific growth rate of resource biomass which decreases as U increases, i.e.,

$$r(0) = r_0 > 0, r'(U) < 0 \ for \ U > 0. \tag{6.3}$$

The function $K(I, T)$ is the maximum density of resource biomass which the environment can support and it also decrease as I and T increase, i.e.,

$$K(0,0) = K_0 > 0, \ \frac{\partial K}{\partial I} < 0, \ \frac{\partial K}{\partial T} < 0 \ for \ I > 0, T > 0. \tag{6.4}$$

6.3 Model Without Diffusion

In this section we analyse model (6.1) without diffusion (i.e., $D_1 = D_2 = D_3 = 0$) for different values of Q, namely, when Q is industrialization dependent, constant, zero or periodic. In such a case, model (6.1) reduces to

$$\begin{aligned}
\frac{dB}{dt} &= r(U)B - \frac{r_0 B^2}{K(I,T)} - \alpha_1 IB, \\
\frac{dI}{dt} &= -\gamma_0 I - \gamma_1 I^2 + \alpha_2 IB, \\
\frac{dT}{dt} &= Q - \delta_0 T - \alpha BT + \theta_1 \delta_1 U + \pi \nu BU, \\
\frac{dU}{dt} &= \beta B + \theta_0 \delta_0 T + \alpha BT - \delta_1 U - \nu BU, \\
& B(0) \geq 0, \ I(0) \geq 0, \ T(0) \geq 0, \ U(0) \geq 0.
\end{aligned} \tag{6.5}$$

Case I: Q=Q(I) and it satisfies the following property:

$$Q(0) > 0, Q'(I) > 0 \ for \ I \geq 0. \tag{6.6}$$

In this case, model (6.5) has three nonnegative equilibria, namely, $E_0(0,0,\frac{Q(0)}{\delta_0(1-\theta_0\theta_1)}, \frac{\theta_0 Q(0)}{\delta_1(1-\theta_0\theta_1)})$, $\tilde{E}(\tilde{B}, 0, \tilde{T}, \tilde{U})$ and $\bar{E}(\bar{B}, \bar{I}, \bar{T}, \bar{U})$. The equilibrium E_0 obviously exists, and we shall show the existence of $\tilde{E}$ and $\bar{E}$ as follows.

Existence of $\tilde{E}(\tilde{B}, 0, \tilde{T}, \tilde{U})$:

Here $\tilde{B}$, $\tilde{T}$ and $\tilde{U}$ are the positive solutions of the following algebraic equations:

$$\begin{aligned} r_0 B &= r(g(B))K(0, f(B)), && (6.7)\\ T &= \frac{(\delta_1 + \nu B)Q(0) + (\theta_1\delta_1 + \pi\nu B)\beta B}{\delta_0\delta_1(1-\theta_0\theta_1) + \delta_0\nu B(1-\theta_0\pi) + \delta_1\alpha B(1-\theta_1) + \alpha\nu B^2(1-\pi)} \\ &= f(B), (say) && (6.8)\\ U &= \frac{\beta B}{\delta_1 + \nu B} + \frac{\theta_0\delta_0 + \alpha B}{\delta_1 + \nu B} f(B) = g(B).(say) && (6.9) \end{aligned}$$

Taking

$$F(B) = r_0 B - r(g(B))K(0, f(B)),$$

we note that $F(0) < 0$ and $F(K_0) > 0$, showing the existence of $\tilde{B}$ in the interval $0 < \tilde{B} < K_0$. For $\tilde{B}$ to be unique the following condition must be satisfied at $\tilde{E}$,

$$r_0 - \frac{\partial r}{\partial U} g'(B)K(0, f(B)) - r(g(B))\frac{\partial K}{\partial T} f'(B) > 0. \tag{6.10}$$

By knowing the value of $\tilde{B}$, the values of $\tilde{T}$ and $\tilde{U}$ can then be computed from (6.8) and (6.9) respectively.

Existence of $\bar{E}(\bar{B}, \bar{I}, \bar{T}, \bar{U})$:

Here $\bar{B}$, $\bar{I}$, $\bar{T}$ and $\bar{U}$ are the positive solutions of the following algebraic equations:

$$\begin{aligned} r_0 B &= K(h_1(B), h_2(B))\{r(h_3(B)) - \alpha_1 h_1(B)\}, && (6.11)\\ I &= \frac{\alpha_2 B - \gamma_0}{\gamma_1} = h_1(B), \ (say) && (6.12)\\ T &= \frac{(\theta_1\delta_1 + \pi\nu B)\beta B + Q(h_1(B))(\delta_1 + \nu B)}{\delta_0\delta_1(1-\theta_0\theta_1) + \delta_0\nu B(1-\theta_0\pi) + \delta_1\alpha B(1-\theta_1) + \alpha\nu B^2(1-\pi)} \\ &= h_2(B), \ (say) && (6.13)\\ U &= \frac{\beta B}{\delta_1 + \nu B} + \frac{\theta_0\delta_0 + \alpha B}{\delta_1 + \nu B} h_2(B) = h_3(B). \ (say) && (6.14) \end{aligned}$$

As in the existence of $\tilde{E}$, it is easy to check that $\bar{E}$ exists, provided the following inequality holds at $\bar{E}$:

$$\begin{aligned} r_0 &- \{r(h_3(B)) - \alpha_1 h_1(B)\}\{\frac{\partial K}{\partial I} h_1'(B) + \frac{\partial K}{\partial T} h_2'(B)\} \\ &- K(h_1(B), h_2(B))\{\frac{\partial r}{\partial U} h_3'(B) - \alpha_1 h_1'(B)\} > 0. \end{aligned} \tag{6.15}$$

By computing the variational matrices corresponding to each equilibrium, it can be easily checked that E_0 is a saddle point with unstable manifold locally in the B direction and stable manifold locally in the $I-T-U$ space. $\tilde{E}$ is unstable in the I direction.

In the following theorem it is shown that $\bar{E}$ is locally asymptotically stable.

Theorem 6.3.1 *Let the following inequalities hold*

$$\{\frac{r_0\bar{B}}{K^2(\bar{I},\bar{T})}\frac{\partial K}{\partial I}(\bar{I},\bar{T})+\alpha_1+\alpha_2\}^2 < \frac{2}{3}\frac{r_0\gamma_1}{K(\bar{I},\bar{T})}, \tag{6.16}$$

$$\{\frac{r_0\bar{B}}{K^2(\bar{I},\bar{T})}\frac{\partial K}{\partial T}(\bar{I},\bar{T})+\alpha\bar{T}+\pi\nu\bar{U}\}^2 < \frac{4}{9}\frac{r_0(\delta_0+\alpha\bar{B})}{K(\bar{I},\bar{T})}, \tag{6.17}$$

$$\{r'(\bar{U})+\beta+\alpha\bar{T}+\nu\bar{U}\}^2 < \frac{2}{3}\frac{r_0(\delta_1+\nu\bar{B})}{K(\bar{I},\bar{T})}, \tag{6.18}$$

$$\{Q'(\bar{I})\}^2 < \frac{2}{3}\gamma_1(\delta_0+\alpha\bar{B}), \tag{6.19}$$

$$\{\theta_0\delta_0+\theta_1\delta_1+(\alpha+\pi\nu)\bar{B}\}^2 < \frac{2}{3}(\delta_0+\alpha\bar{B})(\delta_1+\nu\bar{B}). \tag{6.20}$$

Then the equilibrium $\bar{E}$ is locally asymptotically stable.

Proof: We first linearize system (6.5) around the positive equilibrium $\bar{E}$ by taking the transformations $B=\bar{B}+b$, $I=\bar{I}+i$, $T=\bar{T}+\tau$, $U=\bar{U}+u$. Then using the following positive definite function in the linearized system of model (6.5),

$$V=\frac{1}{2}\{\frac{b^2}{\bar{B}}+\frac{i^2}{\bar{I}}+\tau^2+u^2\},$$

it can easily be checked that the derivative of V with respect to t is negative definite under conditions (6.16)-(6.20), proving the theorem.

In order to investigate the global stability behaviour of $\bar{E}$, we first state the following lemma, which establishes a region of attraction for system (6.5). The proof of this lemma is easy, and hence is omitted.

Lemma 6.3.1 *The set*

$$\Omega_1=\{(B,I,T,U):0\le B\le K_0,\ 0\le I\le I_s, 0\le T+U\le L_s\}$$

attracts all solutions initiating in the interior of the positive orthant, where

$$I_s = \frac{-\gamma_0 + \alpha_2 K_0}{\gamma_1}, \ K_0 \ > \ \frac{\gamma_0}{\alpha_2},$$
$$L_s = \frac{Q(I_s) + \beta K_0}{\delta}, \ \delta = min\{\delta_0(1-\theta_0), \delta_1(1-\theta_1)\}.$$

In the following theorem global stability behaviour of $\bar{E}$ is studied.

Theorem 6.3.2 *In addition to the assumptions (6.3) and (6.4), let $r(U)$ and $K(I,T)$ satisfy the following conditions in Ω_1,*

$$K_m \leq K(I,T) \leq K_0, \ 0 \leq -r'(U) \leq \rho_1, 0 \leq \ Q'(I) \ \leq \rho_2,$$
$$0 \leq -\frac{\partial K}{\partial I} \leq k_1, \ 0 \leq -\frac{\partial K}{\partial T} \leq k_2, \tag{6.21}$$

for some positive constants K_m, ρ_1, ρ_2, k_1, and k_2. Then if the following inequalities hold,

$$\{\frac{r_0 k_1 K_0}{K_m^2} + \alpha_1 + \alpha_2\}^2 \ < \ \frac{2}{3}\frac{r_0\gamma_1}{K(\bar{I},\bar{T})}, \tag{6.22}$$
$$\{\frac{r_0 k_2 K_0}{K_m^2} + (\alpha + \pi\nu)L_s\}^2 \ < \ \frac{4}{9}\frac{r_0(\delta_0 + \alpha\bar{B})}{K(\bar{I},\bar{T})}, \tag{6.23}$$
$$\{\rho_1 + \beta + (\alpha + \nu)L_s\}^2 \ < \ \frac{2}{3}\frac{r_0(\delta_1 + \nu\bar{B})}{K(\bar{I},\bar{T})}, \tag{6.24}$$
$$\{\rho_2\}^2 < \frac{2}{3}(\delta_0 + \alpha\bar{B})\gamma_1, \tag{6.25}$$
$$\{\theta_0\delta_0 + \theta_1\delta_1 + (\alpha + \pi\nu)\bar{B}\}^2 \ < \ \frac{2}{3}(\delta_0 + \alpha\bar{B})(\delta_1 + \nu\bar{B}), \tag{6.26}$$

the equilibrium $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Consider the following positive definite function around $\bar{E}$,

$$V_1(B,I,T,U,) = B - \bar{B} - \bar{B}\ ln\frac{B}{\bar{B}} + I - \bar{I} - \bar{I}\ ln\frac{I}{\bar{I}} + \frac{1}{2}(T-\bar{T})^2 + \frac{1}{2}(U-\bar{U})^2. \tag{6.27}$$

Now differentiating V_1 with respect to t along the solutions of (6.5), a little algebraic manipulation yields

$$
\begin{aligned}
\frac{dV_1}{dt} &= -\frac{r_0}{K(\bar{I},\bar{T})}(B-\bar{B})^2 - \gamma_1(I-\bar{I})^2 - \{\delta_0+\alpha\bar{B}\}(T-\bar{T})^2 \\
&\quad -\{\delta_1+\nu\bar{B}\}(U-\bar{U})^2 + \{-r_0B\xi_1(I,T)-\alpha_1+\alpha_2\}(B-\bar{B})(I-\bar{I}) \\
&\quad +\{-r_0B\xi_2(\bar{I},T)-\alpha T+\pi\nu U\}(B-\bar{B})(T-\bar{T}) \\
&\quad +\{\eta_1(U)+\beta+\alpha T-\nu U\}(B-\bar{B})(U-\bar{U}) + \eta_2(I)(I-\bar{I})(T-\bar{T}) \\
&\quad +\{\theta_0\delta_0+\theta_1\delta_1+(\alpha+\pi\nu)\bar{B}\}(T-\bar{T})(U-\bar{U}), \qquad (6.28)
\end{aligned}
$$

where

$$
\xi_1(I,T) = \begin{cases} \{\frac{1}{K(I,T)} - \frac{1}{K(\bar{I},T)}\}/(I-\bar{I}), & I \neq \bar{I} \\ -\frac{1}{K^2(\bar{I},T)}\frac{\partial K}{\partial I}(\bar{I},T), & I = \bar{I} \end{cases},
$$

$$
\xi_2(\bar{I},T) = \begin{cases} \{\frac{1}{K(\bar{I},T)} - \frac{1}{K(\bar{I},\bar{T})}\}/(T-\bar{T}), & T \neq \bar{T} \\ -\frac{1}{K^2(\bar{I},\bar{T})}\frac{\partial K}{\partial T}(\bar{I},\bar{T}), & T = \bar{T} \end{cases},
$$

$$
\eta_1(U) = \begin{cases} \frac{r(U)-r(\bar{U})}{U-\bar{U}}, & U \neq \bar{U} \\ r'(U), & U = \bar{U} \end{cases},
$$

$$
\eta_2(I) = \begin{cases} \frac{Q(I)-Q(\bar{I})}{I-\bar{I}}, & I \neq \bar{I} \\ Q'(\bar{I}), & I = \bar{I} \end{cases}.
$$

From (6.21) and the mean value theorem, we note that

$$
|\eta_1(U)| \leq \rho_1,\ |\eta_2(I)| \leq \rho_2,\ |\xi_1(I,T)| \leq \frac{k_1}{K_m^2} \ \textit{and}\ |\xi_2(\bar{I},T)| \leq \frac{k_2}{K_m^2}. \qquad (6.29)
$$

Now Eq. (6.28) can be rewritten as the sum of the quadratics

$$
\begin{aligned}
\frac{dV_1}{dt} &= -\frac{1}{2}a_{11}(B-\bar{B})^2 + a_{12}(B-\bar{B})(I-\bar{I}) - \frac{1}{2}a_{22}(I-\bar{I})^2 \\
&\quad -\frac{1}{2}a_{11}(B-\bar{B})^2 + a_{13}(B-\bar{B})(T-\bar{T}) - \frac{1}{2}a_{33}(T-\bar{T})^2 \\
&\quad -\frac{1}{2}a_{11}(B-\bar{B})^2 + a_{14}(B-\bar{B})(U-\bar{U}) - \frac{1}{2}a_{44}(U-\bar{U})^2 \\
&\quad -\frac{1}{2}a_{22}(I-\bar{I})^2 + a_{23}(I-\bar{I})(T-\bar{T}) - \frac{1}{2}a_{33}(T-\bar{T})^2 \\
&\quad -\frac{1}{2}a_{33}(T-\bar{T})^2 + a_{34}(T-\bar{T})(U-\bar{U}) - \frac{1}{2}a_{44}(U-\bar{U})^2,
\end{aligned}
$$

where

$$
\begin{aligned}
&a_{11} = \frac{2}{3}\frac{r_0}{K(\bar{I},\bar{T})},\ a_{22} = \gamma_1,\ a_{33} = \frac{2}{3}(\delta_0 + \alpha\bar{B}),\ a_{44} = \delta_1 + \nu\bar{B}, \\
&a_{12} = -r_0 B\xi_1(I,T) - \alpha_1 + \alpha_2,\ a_{13} = -r_0 B\xi_2(\bar{I},T) - \alpha T - \pi\nu U, \\
&a_{14} = \eta_1(U) + \beta + \alpha T - \nu U,\ a_{23} = \eta_2(I),\ a_{34} = \theta_0\delta_0 + \theta_1\delta_1 + (\alpha + \pi\nu)\bar{B}.
\end{aligned}
$$

Sufficient conditions for $\frac{dV_1}{dt}$ to be negative definite are that the following conditions hold:

$$a_{12}^2 < a_{11}a_{22}, \tag{6.30}$$

$$a_{13}^2 < a_{11}a_{33}, \tag{6.31}$$

$$a_{14}^2 < a_{11}a_{44}, \tag{6.32}$$

$$a_{23}^2 < a_{22}a_{33}, \tag{6.33}$$

$$a_{34}^2 < a_{33}a_{44}. \tag{6.34}$$

We note that (6.30) ⇒ (6.22), (6.31) ⇒ (6.23), (6.32) ⇒ (6.24), (6.33) ⇒ (6.25), (6.34) ⇒ (6.26). Thus, V_1 is a Liapunov function with respect to $\bar{E}$, whose domain contains the region of attraction Ω_1, proving the theorem.

The above theorem shows that if inequalities (6.22)-(6.26) hold, then the resource biomass settles down to a steady state whose magnitude depends upon the steady state of industrialization, influx and washout rates of the pollutant present in the environment, the influx rate being dependent upon the steady state of industrialization. The magnitude of the resource biomass decreases with the increase in density of industrialization and influx rate of pollutant present in the environment. It may be noted here that equilibrium level of the resource biomass density may doom to extinction if the densities of industrialization and pollution increase unabatedly.

Case II: Constant emission of pollutant into the environment, i.e., $Q = Q_0 > 0$.

In this case the analysis will be similar as that of the case I and the results corresponding to Theorems 6.3.1 and 6.3.2 can be deduced. In particular, it may be noted that condition (6.19) in Theorem 6.3.1 and condition (6.25) in Theorem 6.3.2 are automatically satisfied. In this case the results are found similar to the industrialization dependent case.

Case III: Instantaneous introduction of pollutant into the environment, i.e., $Q = 0$.

In this case the system can be analysed in a similar fashion as that of case I. In particular, it is noted that the pollutant may be washed out completely from the environment, and then the resource biomass density may return back to a lower equilibrium level than its original carrying capacity, the magnitude of which would depend upon the equilibrium level of industrialization. Even in this case the resource biomass density may tend to zero if the industrialization pressure is very high.

By comparing the equilibrium levels of resource biomass density in cases I, II and III, we note that the extinction rate of the resource biomass density is maximum in case I and minimum in case III, keeping other parameters same.

Case IV. Periodic emission of pollutant into the environment, i.e., $Q(t) = Q_0 + \varepsilon\phi(t), \phi(t+\omega) = \phi(t)$.

In this case, it can be checked that the results corresponding to Theorem 3.4.1 and Theorem 3.4.2 in chapter 3 remain valid. In particular, it is found that a small periodic influx of pollutant into the environment causes a periodic behaviour in the system and for small amplitude the stability behaviour of the system is same as that of the constant introduction of pollutant.

6.4 Model With Diffusion

In this section we consider the complete model (6.1)-(6.2) and state the main results in the form of the following theorem.

Theorem 6.4.1 *(i) If the equilibrium $\bar{E}$ of model (6.5) is globally asymptotically stable, then the corresponding uniform steady state of the initial-boundary value problems (6.1)-(6.2) is also globally asymptotically stable.*

(ii) If the equilibrium $\bar{E}$ of the model (6.5) is unstable, even then the uniform steady state of the initial-boundary value problems (6.1)-(6.2) can be made stable by increasing diffusion coefficients appropriately.

Proof: Let us consider the following positive definite function

$$V_2(B(t), I(t), T(t), U(t)) = \int\int_D V_1(B(t), I(t), T(t), U(t))dA, \qquad (6.35)$$

where V_1 is defined in Eq. (6.27). We have,

$$\begin{aligned} \frac{dV_2}{dt} &= \int\int_D (\frac{\partial V_1}{\partial B}\frac{\partial B}{\partial t} + \frac{\partial V_1}{\partial I}\frac{\partial I}{\partial t} + \frac{\partial V_1}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial V_1}{\partial U}\frac{\partial U}{\partial t})dA \\ &= I_1 + I_2. \end{aligned} \qquad (6.36)$$

where

$$I_1 = \int\int_D \frac{dV_1}{dt}dA \text{ and } I_2 = \int\int_D (D_1\frac{\partial V_1}{\partial B}\nabla^2 B + D_2\frac{\partial V_1}{\partial I}\nabla^2 I + D_3\frac{\partial V_1}{\partial T}\nabla^2 T)dA \quad (6.37)$$

We first note that I_1 has the same sign as that of $\frac{dV_1}{dt}$, if $\frac{dV_1}{dt}$ does not change sign in D.

We also note the following properties of V_1, namely,

$$\frac{\partial V_1}{\partial B}\Big]_{\partial D} = \frac{\partial V_1}{\partial I}\Big]_{\partial D} = \frac{\partial V_1}{\partial T}\Big]_{\partial D} = \frac{\partial V_1}{\partial U}\Big]_{\partial D} = 0$$

and for all points of D,

$$\begin{aligned} &\frac{\partial^2 V_1}{\partial B \partial I} = \frac{\partial^2 V_1}{\partial B \partial T} = \frac{\partial^2 V_1}{\partial B \partial U} = \frac{\partial^2 V_1}{\partial I \partial T} = \frac{\partial^2 V_1}{\partial I \partial U} = \frac{\partial^2 V_1}{\partial T \partial U} = 0, \\ &\frac{\partial^2 V_1}{\partial B^2} > 0, \ \frac{\partial^2 V_1}{\partial I^2} > 0, \ \frac{\partial^2 V_1}{\partial T^2} > 0, \ and \ \frac{\partial^2 V_1}{\partial U^2} > 0. \end{aligned}$$

Under an analysis similar to chapter 2, we note that

$$\int\int_D \frac{\partial V_1}{\partial B}\nabla^2 B dA = -\int\int_D \frac{\partial^2 V_1}{\partial B^2}\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\}dA \leq 0, \tag{6.38}$$

$$\int\int_D \frac{\partial V_1}{\partial I}\nabla^2 I dA = -\int\int_D \frac{\partial^2 V_1}{\partial I^2}\{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2\}dA \leq 0, \tag{6.39}$$

$$\int\int_D \frac{\partial V_1}{\partial T}\nabla^2 T dA = -\int\int_D \frac{\partial^2 V_1}{\partial T^2}\{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\}dA \leq 0. \tag{6.40}$$

This shows that

$$I_2 \leq 0. \tag{6.41}$$

The above results shows that if $I_1 \leq 0$, i.e., if $\bar{E}$ is globally asymptotically stable in the absence of diffusion, then the uniform steady state of the initial-boundary value problems (6.1)-(6.2) also must be globally asymptotically stable. This proves the first part of Theorem 6.4.1.

We further note that if $\frac{dV_1}{dt} > 0$, i.e., if $I_1 > 0$, then $\bar{E}$ may be unstable in the absence of diffusion. But, Eqs. (6.36) and (6.41) show that by increasing diffusion coefficients D_1, D_2 and D_3 sufficiently large, $\frac{dV_2}{dt}$ can be made negative even if $I_1 > 0$. This proves the second part of Theorem 6.4.1.

The above theorem implies that diffusion with reservoir boundary conditions may be thought of as stabilizing the system. We shall explain the above theorem for a rectangular habitat D defined by

$$D = \{(x, y) : \ 0 \leq x \leq a, \ 0 \leq y \leq b\} \tag{6.42}$$

in the form of the following theorem.

Theorem 6.4.2 *In addition to assumptions (6.3) and (6.4), let $r(U)$, $K(I,T)$ satisfy the inequalities in (6.21). If the following inequalities hold:*

$$\{\frac{r_0 k_1 K_0}{K_m^2} + \alpha_1 + \alpha_2\}^2 < \frac{2}{3}\{\frac{r_0}{K(\bar{I},\bar{T})} + \frac{D_1 \bar{B}\pi^2(a^2+b^2)}{K_0^2 a^2 b^2}\} \times \{\gamma_1 + \frac{D_2 \bar{I}\pi^2(a^2+b^2)}{I_s^2 a^2 b^2}\}, \quad (6.43)$$

$$\{\frac{r_0 k_2 K_0}{K_m^2} + (\alpha+\pi\nu)L_s\}^2 < \frac{4}{9}\{\frac{r_0}{K(\bar{I},\bar{T})} + \frac{D_1 \bar{B}\pi^2(a^2+b^2)}{K_0^2 a^2 b^2}\} \times \{\delta_0 + \alpha\bar{B} + \frac{D_3\pi^2(a^2+b^2)}{a^2 b^2}\}, \quad (6.44)$$

$$\{\rho_1 + \beta + (\alpha+\nu)L_s\}^2 < \frac{2}{3}\{\frac{r_0}{K(\bar{I},\bar{T})} + \frac{D_1 \bar{B}\pi^2(a^2+b^2)}{K_0^2 a^2 b^2}\} \times (\delta_1 + \nu\bar{B}), \quad (6.45)$$

$$\rho_2^2 < \frac{2}{3}\{\gamma_1 + \frac{D_2 \bar{I}\pi^2(a^2+b^2)}{I_s^2 a^2 b^2}\} \times \{\delta_0 + \alpha\bar{B} + \frac{D_3\pi^2(a^2+b^2)}{a^2 b^2}\}, \quad (6.46)$$

$$\{\theta_0\delta_0 + \theta_1\delta_1 + (\alpha+\pi\nu)\bar{B}\}2 < \frac{2}{3}\{\delta_0 + \alpha\bar{B} + \frac{D_3\pi^2(a^2+b^2)}{a^2 b^2}\} \times (\delta_1 + \nu\bar{B}), \quad (6.47)$$

then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Let us consider the rectangular region D given by Eq. (6.42). In this case I_2, which is defined in Eq. (6.36), can be written as

$$\begin{aligned} I_2 = & -D_1 \int\int_D (\frac{\partial^2 V_1}{\partial B^2})\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\}\, dA \\ & -D_2 \int\int_D (\frac{\partial^2 V_1}{\partial I^2})\{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2\}\, dA \\ & -D_3 \int\int_D (\frac{\partial^2 V_1}{\partial T^2})\{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\}\, dA. \end{aligned} \quad (6.48)$$

From Eq. (6.27) we get

$$\frac{\partial^2 V_1}{\partial B^2} = \frac{\bar{B}}{B^2},$$
$$\frac{\partial^2 V_1}{\partial I^2} = \frac{\bar{I}}{I^2},$$
$$\frac{\partial^2 V_1}{\partial T^2} = 1.$$

Hence

$$I_2 \leq -\frac{D_1\bar{B}}{K_0^2}\int\int_D\{(\frac{\partial B}{\partial x})^2+(\frac{\partial B}{\partial y})^2\}\ dA-\frac{D_2\bar{I}}{I_s}\int\int_D\{(\frac{\partial I}{\partial x})^2+(\frac{\partial I}{\partial y})^2\}\ dA$$
$$-D_3\int\int_D\{(\frac{\partial T}{\partial x})^2+(\frac{\partial T}{\partial y})^2\}\ dA.$$

Now

$$\int\int_D(\frac{\partial B}{\partial x})^2\ dA = \int\int_D\{\frac{\partial(B-B^*)}{\partial x}\}^2\ dA$$
$$= \int_0^b\int_0^a\{\frac{\partial(B-B^*)}{\partial x}\}^2\ dxdy$$

Under an analysis similar to chapter 2 and using the well known inequality (Denn, 1975, pp. 225)

$$\int_0^1(\frac{\partial B}{\partial x})^2\ dx \geq \pi^2\int_0^1 B^2\ dx,$$

we note that

$$\int\int_D(\frac{\partial B}{\partial x})^2\ dA \geq \frac{\pi^2}{a^2}\int\int_D(B-B^*)^2 dA$$

and

$$\int\int_D(\frac{\partial B}{\partial y})^2\ dA \geq \frac{\pi^2}{b^2}\int\int_D(B-B^*)^2 dA$$

Thus,

$$I_2 \leq -\frac{D_1\bar{B}\pi^2(a^2+b^2)}{K_0^2a^2b^2}\int\int_D(B-\bar{B})^2\ dA$$
$$-\frac{D_2\bar{I}\pi^2(a^2+b^2)}{I_s^2a^2b^2}\int\int_D(I-\bar{I})^2\ dA$$
$$-\frac{D_3\pi^2(a^2+b^2)}{a^2b^2}\int\int_D(T-\bar{T})^2\ dA.$$

Now from (6.28) and (6.38)-(6.40) we get

$$\frac{dV_2}{dt} \leq \int\int_D[-\{\frac{r_0}{K(\bar{I},\bar{T})}+\frac{D_1\bar{B}\pi^2(a^2+b^2)}{K_0^2a^2b^2}\}(B-\bar{B})^2-\{\gamma_1+\frac{D_2\bar{I}\pi^2(a^2+b^2)}{I_s^2a^2b^2}\}(I-\bar{I})^2$$
$$-\{\delta_0+\alpha\bar{B}+\frac{D_3\pi^2(a^2+b^2)}{a^2b^2}\}(T-\bar{T})^2-(\delta_1+\nu\bar{B})(U-\bar{U})^2$$
$$+\{-r_0B\xi_1(I,T)-\alpha_1+\alpha_2\}(B-\bar{B})(I-\bar{I})$$
$$+\{-r_0B\xi_2(\bar{I},T)-\alpha T+\pi\nu U\}(B-\bar{B})(T-\bar{T})$$
$$+\{\eta_1(U)+\beta+\alpha T-\nu U\}(B-\bar{B})(U-\bar{U})+\eta_2(I)(I-\bar{I})(T-\bar{T})$$
$$+\{\theta_0\delta_0+\theta_1\delta_1+(\alpha+\pi\nu)\bar{B}\}(T-\bar{T})(U-\bar{U})]\ dA,$$

where $\eta_1(I)$, $\eta_2(I)$, $\xi_1(B)$ *and* $\xi_2(B)$ are defined in Eq. (6.28).

Now $\frac{dV_2}{dt}$ can be written as

$$\begin{aligned}\frac{dV_2}{dt} \leq \int\int_D [&-\frac{1}{2}b_{11}(B-\bar{B})^2 + b_{12}(B-\bar{B})(I-\bar{I}) - \frac{1}{2}b_{22}(I-\bar{I})^2 \\ &-\frac{1}{2}b_{11}(B-\bar{B})^2 + b_{13}(B-\bar{B})(T-\bar{T}) - \frac{1}{2}b_{33}(T-\bar{T})^2 \\ &-\frac{1}{2}b_{11}(B-\bar{B})^2 + b_{14}(B-\bar{B})(U-\bar{U}) - \frac{1}{2}b_{44}(U-\bar{U})^2 \\ &-\frac{1}{2}b_{22}(I-\bar{I})^2 + b_{23}(I-\bar{I})(T-\bar{T}) - \frac{1}{2}b_{33}(T-\bar{T})^2 \\ &-\frac{1}{2}b_{33}(T-\bar{T})^2 + b_{34}(T-\bar{T})(U-\bar{U}) - \frac{1}{2}b_{44}(U-\bar{U})^2]dA,\end{aligned}$$

where

$$\begin{aligned}&b_{11} = \frac{2}{3}\{\frac{r_0}{K(\bar{I},\bar{T})} + \frac{D_1\bar{B}\pi^2(a^2+b^2)}{K_0^2a^2b^2}\},\ b_{22} = \gamma_1 + \frac{D_2\bar{I}\pi^2(a^2+b^2)}{I_s^2a^2b^2},\\ &b_{33} = \frac{2}{3}\{\delta_0 + \alpha\bar{B} + \frac{D_3\pi^2(a^2+b^2)}{a^2b^2}\},\ b_{44} = \delta_1 + \nu\bar{B},\\ &b_{12} = -r_0B\xi_1(I,T) - \alpha_1 + \alpha_2,\ b_{13} = -r_0B\xi_2(\bar{I},T) - \alpha T - \pi\nu U,\\ &b_{14} = \eta_1(U) + \beta + \alpha T - \nu U,\ b_{23} = \eta_2(I),\\ &b_{34} = \theta_0\delta_0 + \theta_1\delta_1 + (\alpha+\pi\nu)\bar{B}.\end{aligned}$$

Sufficient conditions for $\frac{dV_2}{dt}$ to be negative definite are that the following conditions hold:

$$b_{12}^2 < b_{11}b_{22}, \tag{6.49}$$

$$b_{13}^2 < b_{11}b_{33}, \tag{6.50}$$

$$b_{14}^2 < b_{11}b_{44}, \tag{6.51}$$

$$b_{23}^2 < b_{22}b_{33}, \tag{6.52}$$

$$b_{34}^2 < b_{33}b_{44}. \tag{6.53}$$

We note that (6.43) $\Rightarrow$ (6.49), (6.44) $\Rightarrow$ (6.50), (6.45) $\Rightarrow$ (6.51), (6.46) $\Rightarrow$ (6.52), (6.47) $\Rightarrow$ (6.53). Thus V_2 is a Liapunov function with respect to E^*, whose domain contains the region Ω_1, proving the theorem.

From the above theorem we note that if we increase D_1, D_2 and D_3 to sufficiently large values, then inequalities (6.43)-(6.47) may be satisfied. This implies that solutions of the system with diffusion approaches to its equilibrium faster than the case of no diffusion.

6.5 Conservation Model

In the previous section it has been noted that uncontrolled growth of industrialization and pollution may lead to the extinction of forestry resources. Therefore, some kinds of efforts must be adopted to conserver the forestry resources (Munn and Fedorov, 1986; Huttl and Wisniewski, 1987; Lamberson, 1986; Shukla et al., 1989; Reed and Heras, 1992; Dubey, 1997a; Shukla and Dubey, 1997). In this section a mathematical model is proposed to conserve the forestry resources by some efforts like plantation, irrigation, fencing etc. and by controlling the undesired levels of industrialization and pollution by some mechanisms. It is assumed that the effort applied to conserve the resource is proportional to the depleted level of resource biomass from its carrying capacity, and efforts applied to control the industrialization pressure and the concentration of pollutant are proportional to their respective undesired levels. Following Shukla et al. (1989), Dubey (1997a) and Shukla and Dubey (1997), differential equations governing the system may be written as

$$
\begin{aligned}
\frac{\partial B}{\partial t} &= r(U)B - \frac{r_0 B^2}{K(I,T)} - \alpha_1 IB + r_{10}F_1 + D_1\nabla^2 B,\\
\frac{\partial I}{\partial t} &= -\gamma_0 I - \gamma_1 I^2 + \alpha_2 IB - r_{20}F_2 I + D_2\nabla^2 I,\\
\frac{\partial T}{\partial t} &= Q(I) - \delta_0 T - \alpha BT + \theta_1\delta_1 U + \pi\nu BU - r_{30}F_3 + D_3\nabla^2 T,\\
\frac{\partial U}{\partial t} &= \beta B + \theta_0\delta_0 T + \alpha BT - \delta_1 U - \nu BU, \qquad (6.54)\\
\frac{\partial F_1}{\partial t} &= r_1(1 - \frac{B}{K_0}) - \mu_1 F_1,\\
\frac{\partial F_2}{\partial t} &= r_2(I - I_c)H(I - I_c) - \mu_2 F_2,\\
\frac{\partial F_3}{\partial t} &= r_3(T - T_c)H(T - T_c) - \mu_3 F_3,\\
0 &\le \theta_0, \theta_1, \pi \le 1.
\end{aligned}
$$

We analyse the model with the following initial and boundary conditions:

$$
\begin{aligned}
&B(x,y,0) = \phi(x,y) \geq 0, I(x,y,0) = \psi(x,y) \geq 0,\\
&T(x,y,0) = \xi(x,y) \geq 0, U(x,y,0) = \zeta(x,y) \geq 0,\\
&F_1(x,y,0) = \zeta_1(x,y) \geq 0, F_2(x,y,0) = \zeta_2(x,y) \geq 0,\\
&F_3(x,y,0) = \zeta_3(x,y) \geq 0, (x,y)\ \varepsilon\ D \ and\\
&\frac{\partial B}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial T}{\partial n} = 0,\ (x,y)\ \varepsilon\ \partial D,\ t \geq 0,
\end{aligned}
\tag{6.55}
$$

where n is the unit outward normal to ∂D.

In model (6.54), $F_1(x,y,t)$ is the density of effort applied to conserve the resource biomass, $F_2(x,y,t)$ the density of effort applied to control the undesired level of industrialization pressure and $F_3(x,y,t)$ the density of effort applied to control the undesired level of the concentration of pollutant in the environment. $r_{10} > 0$ represents the growth rate coefficient of resource biomass due to effort F_1. $r_{20} > 0$ and $r_{30} > 0$ are depletion rate coefficients of $I(x,y,t)$ and $T(x,y,t)$ due to the efforts F_2 and F_3 respectively. r_1, r_2, r_3 are the growth rate coefficients of F_1, F_2, F_3 respectively and μ_1, μ_2 and μ_3 are their respective depreciation rate coefficients. I_c and T_c are critical levels of industrialization pressure and concentration of pollutant respectively which are assumed to be harmless to the resource. In the last two equations of system (6.54), $H(t)$ denotes the unit step function which takes into account the cases when $I \leq I_c$ and $T \leq T_c$. It may be noted that in the unusual circumstances, even in the face of industrialization, if the forest exceeds its carrying capacity, then $\frac{\partial F_1}{\partial t}$ will be negative, giving a decrease in the effort to conserve the biomass.

We analyse conservation model (6.54) only for the case when rate of introduction of pollutant into the environment is industrialization dependent.

6.6 Conservation Model Without Diffusion

In this section we take, $D_1 = D_2 = D_3 = 0$ in model (6.54). Then the model has only one interior equilibrium, namely, $E^*(B^*, I^*, T^*, U^*, F_1^*, F_2^*, F_3^*)$, where B^*, I^*, T^*, U^*,

F_1^*, F_2^* and F_3^* are positive solutions of the system of algebraic equations given below.

$$
\begin{aligned}
r_0 B &= \{r(U) - \alpha_1 I + \frac{r_{10} r_1}{B\mu_1}(1 - \frac{B}{K_0})\} K(I,T), \\
I &= \frac{(-\gamma_0 + \alpha_2 B)\mu_2 + r_{20} r_2 I_c}{\mu_2 + r_{20} r_0} = h_1(B), (say) \\
T &= \frac{Q(h_1(B))\mu_3(\delta_1 + \nu B) + \mu_3(\theta_1\delta_1 + \pi\nu B)\beta B + r_{30} r_3 T_c(\delta_1 + \nu B)}{\mu_3[\delta_0\delta_1(1-\theta_0\theta_1) + \delta_0\nu B(1 - \pi\theta_0) + \alpha\delta_1 B(1-\theta_1) + \alpha\nu B^2(1-\pi)] + r_{30} r_3(\delta_1 + \nu B)} \\
&= h_2(B), (say) \\
U &= \frac{\beta B + (\theta_0\delta_0 + \alpha B) h_2(B)}{\delta_1 + \nu B} = h_3(B), (say) \\
F_1 &= \frac{r_1}{\mu_1}(1 - \frac{B}{K_0}), \\
F_2 &= \frac{r_2}{\mu_2}(I - I_c) H(I - I_c) = \begin{cases} \{\frac{r_2}{\mu_2}(I - I_c), & I > I_c \\ 0, & I \le I_c \end{cases}, \\
F_3 &= \frac{r_3}{\mu_3}(T - T_c) H(T - T_c) = \begin{cases} \{\frac{r_3}{\mu_3}(T - T_c), & T > T_c \\ 0, & T \le T_c \end{cases}.
\end{aligned}
$$

It may be noted here that for F_1 to be positive, we must have

$$K_0 > B. \tag{6.56}$$

As earlier, it is easy to check that E^* exists if and only if the following inequality holds at E^*,

$$
\begin{aligned}
r_0 &- \{r(h_3(B)) - \alpha_1 h_1(B) + \frac{r_{10} r_1}{B\mu_1}(1 - \frac{B}{K_0})\}\{\frac{\partial K}{\partial I} h_1'(B) + \frac{\partial K}{\partial T} h_2'(B)\} \\
&- \{\frac{\partial r}{\partial U} h_3'(B) - \alpha_1 h_1'(B) - \frac{r_{10} r_1}{\mu B^2 K_0}\} K(h_1(B), h_2(B)) > 0.
\end{aligned} \tag{6.57}
$$

In the following theorem it is shown that E^* is locally asymptotically stable, the proof of which is similar to Theorem 6.3.1 and hence is omitted.

Theorem 6.6.1 *Let the following inequalities hold:*

$$\{\frac{r_0 B^*}{K^2(I^*,T^*)}\frac{\partial K}{\partial I}(I^*,T^*) + \alpha_1 + \alpha_2\}^2 < \frac{2}{3}\frac{r_0\gamma_1}{K(I^*,T^*)}, \tag{6.58}$$

$$\{\frac{r_0 B^*}{K^2(I^*,T^*)}\frac{\partial K}{\partial T}(I^*,T^*) + \alpha T^* + \pi\nu U^*\}^2 < \frac{4}{9}\frac{r_0(\delta_0 + \alpha B^*)}{K(I^*,T^*)}, \tag{6.59}$$

$$\{r'(U^*) + \beta + \alpha T^* + \nu U^*\}^2 < \frac{2}{3}\frac{r_0(\delta_1 + \nu B^*)}{K(I^*,T^*)}, \tag{6.60}$$

$$\{Q'(I^*)\}^2 < \frac{2}{3}\gamma_1(\delta_0 + \alpha B^*), \tag{6.61}$$

$$\{\theta_0\delta_0 + \theta_1\delta_1 + (\alpha + \pi\nu)B^*\}^2 < \frac{2}{3}(\delta_0 + \alpha B^*)(\delta_1 + \nu B^*), \tag{6.62}$$

then E^* is locally asymptotically stable.

In the following lemma a region of attraction for system (6.54) without diffusion is established. The proof of this lemma is similar to Lemma 6.3.1 and hence is omitted.

Lemma 6.6.1 *The set*

$$\begin{aligned}\Omega_2 = \{(B,I,T,U,F_1,F_2,F_3) \ : \ & 0 \le B \le K_a, 0 \le I \le I_a, 0 \le T + U \le L_a,\\ & 0 \le F_1 \le \frac{r_1}{\mu_1}, 0 \le F_2 \le \frac{r_2 I_a}{\mu_2}, 0 \le F_3 \le \frac{r_3 L_a}{\mu_3}\}\end{aligned}$$

attracts all solutions initiating in the positive orthant, where

$$\begin{aligned}K_a &= \frac{K_0}{2}\{1 + \sqrt{1 + \frac{4r_{10}r_1}{\mu_1 K_0 r_0}}\},\\ I_a &= \frac{-\gamma_0 + \alpha_2 K_a}{\gamma_1},\\ L_a &= \frac{1}{\delta}\{Q(I_a) + \beta K_a\}, \ \delta = min\{\delta_0(1-\theta_0), \delta_1(1-\theta_1)\}.\end{aligned}$$

The following theorem gives criteria for E^* to be globally asymptotically stable, whose proof is similar to Theorem 6.3.2 and hence is omitted.

Theorem 6.6.2 *In addition to the assumptions (6.3) and (6.4), let $r(U)$ and $K(I,T)$ satisfy in Ω_2,*

$$\begin{aligned}&K_m^* \le K(I,T) \le K_0, \ 0 \le -\frac{\partial K}{\partial I} \le k_1^*, \ 0 \le -\frac{\partial K}{\partial T} \le k_2^*,\\ &0 \le -r'(U) \le \rho_1^*, 0 \le Q'(I) \le \rho_2^*,\end{aligned} \tag{6.63}$$

for some positive constants K_m^*, k_1^*, k_2^*, ρ_1^* and ρ_2^*. Then if the following inequalities hold:

$$\{\frac{r_0 k_1^* K_a}{K_m^{*2}} + \alpha_1 + \alpha_2\}^2 < \frac{2}{3}\frac{r_0\gamma_1}{K(I^*,T^*)}, \tag{6.64}$$

$$\{\frac{r_0 k_2^* K_a}{K_m^{*2}} + (\alpha + \pi\nu)L_a\}^2 < \frac{4}{9}\frac{r_0(\delta_0 + \alpha B^*)}{K(I^*,T^*)}, \tag{6.65}$$

$$\{\rho_1^* + \beta + (\alpha + \nu)L_a\}^2 < \frac{2}{3}\frac{r_0(\delta_1 + \nu B^*)}{K(I^*,T^*)}, \tag{6.66}$$

$$\{\rho_2^{*2}\}^2 < \frac{2}{3}\gamma_1(\delta_0 + \alpha B^*), \tag{6.67}$$

$$\{\theta_0\delta_0 + \theta_1\delta_1 + (\alpha + \pi\nu)B^*\}^2 < \frac{2}{3}(\delta_0 + \alpha B^*)(\delta_1 + \nu B^*), \tag{6.68}$$

E^* is globally asymptotically stable with respect to all solutions initiating in the positive orthant.

Theorems 6.6.1 and 6.6.2 show that if suitable efforts are made to conserve the resource biomass and to control undesired levels of industrialization and pollution, an appropriate level of the resource biomass density may be maintained.

6.7 Conservation Model With Diffusion

We now consider the case when $D_i > 0 (i = 1,2,3)$ in model (6.54). Then we can show that the uniform steady state $B(x,y,t) = B^*$, $I(x,y,t) = I^*$, $T(x,y,t) = T^*$, $U(x,y,t) = U^*$, $F_1(x,y,t) = F_1^*$, $F_2(x,y,t) = F_2^*$, $F_3(x,y,t) = F_3^*$ is globally asymptotically stable. For this, let us consider the following positive definite function:

$$V_3(B,I,T,U,F_1,F_2,F_3) = \int\int_D V_2(B,I,T,U,F_1,F_2,F_3)dA,$$

where,

$$\begin{aligned} V_2(B,I,T,U,F_1,F_2,F_3) &= B - B^* - B^* ln(\frac{B}{B^*}) + I - I^* - I^* ln(\frac{I}{I^*}) \\ &+\frac{1}{2}(T-T^*)^2 + \frac{1}{2}(U-U^*)^2 + \frac{1}{2}\frac{r_{10}K_0}{r_1 B^*}(F_1 - F_1^*)^2 \\ &+\frac{1}{2}\frac{r_{20}}{r_2}(F_2 - F_2^*)^2 + \frac{1}{2}\frac{r_{30}}{r_3}(F_3 - F_3^*)^2. \end{aligned}$$

Then as earlier, it can be shown that

$$\frac{dV_3}{dt} = \int\int_D \frac{dV_2}{dt}(B,I,T,U,F_1,F_2,F_3)dA - D_1\int\int_D \frac{\partial^2 V_2}{\partial B^2}\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\}dA$$
$$-D_2\int\int_D \frac{\partial^2 V_2}{\partial I^2}\{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2\}dA - D_3\int\int_D \frac{\partial^2 V_2}{\partial T^2}\{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\}dA.$$

This shows that if $\frac{dV_2}{dt} < 0$, then $\frac{dV_3}{dt} < 0$. This implies that if E^* is globally asymptotically stable for system (6.54) without diffusion, then the corresponding uniform steady state of system (6.54)-(6.55) is also globally asymptotically stable with respect to solutions such that $\phi(x,y) > 0$, $\psi(x,y) > 0$, $\xi(x,y) > 0$, $\zeta(x,y) > 0$, $\zeta_1(x,y) > 0$, $\zeta_2(x,y) > 0$, $\zeta_3(x,y) > 0$, $(x,y)\varepsilon D$.

We also note that if $\frac{dV_2}{dt} > 0$, then $\frac{dV_3}{dt}$ can be made negative by increasing D_1, D_2, D_3 to sufficiently large values. This implies that if system (6.54) without diffusion is unstable, even then the corresponding uniform steady state of system (6.54)-(6.55) can be made stable. We also note that $\frac{dV_3}{dt}$ contains some extra negative terms implying that the global stability in this case is more plausible than the case of no diffusion. This shows that solutions approach E^* more rapidly as the diffusion coefficients D_1, D_2 and D_3 increase. So, with diffusion the biomass will converge towards its carrying capacity at a faster rate than with no diffusion.

6.8 Numerical Examples

Example 1 Here a numerical example is presented to illustrate the results obtained in section 6.3. We consider the following particular form of the functions in model (6.5).

$$r(U) = r_0 - b_1 U,$$
$$K(I,T) = K_0 - K_1 I - K_2 T, \quad (6.69)$$
$$Q(I) = q_0 + q_1 I.$$

Now choose the following set of values of the parameters in Eq. (6.69) and in model (6.5).

$$r_0 = 20.0,\ b_1 = 0.02,\ K_0 = 100.0,\ K_1 = 0.8,\ K_2 = 0.9,\ q_0 = 10.0,$$
$$q_1 = 0.7,\ \alpha_1 = 0.25,\ \alpha_2 = 0.15,\ \gamma_0 = 0.29,\ \gamma_1 = 6.5,\ \delta_0 = 7.0, \tag{6.70}$$
$$\delta_1 = 6.0,\ \alpha = 0.04,\ \theta_1 = 0.05,\ \pi = 0.01,\ \nu = 0.1,\ \beta = 0.064,\ \theta_0 = 0.03.$$

With the above values of the parameters, it can be checked that the condition (6.15) for the existence of the interior equilibrium $\bar{E}$ is satisfied and $\bar{E}$ is given by

$$\bar{B} = 94.63914,\ \bar{I} = 2.13936,\ \bar{T} = 1.09066,\ \bar{U} = 0.67348. \tag{6.71}$$

It can also be checked that conditions (6.16)-(6.20) in Theorem 6.3.1 are satisfied which shows that $\bar{E}$ is locally asymptotically stable.

By choosing $K_m = 60.0$ in Theorem 6.3.2 it can also be verified that conditions (6.22)-(6.26) are satisfied which shows that $\bar{E}$ is globally asymptotically stable.

Example 2 Here we present a numerical example to illustrate the results obtained in section 6.6. In addition to the values of parameters given in (6.70), we choose the following values of parameters in model (6.54) with no diffusion:

$$r_{10} = 3.0,\ r_{20} = 5.0,\ r_{30} = 2.0,\ r_1 = 3.5,\ r_2 = 4.0,\ r_3 = 4.5,$$
$$\mu_1 = 0.12,\ \mu_2 = 0.8,\ \mu_3 = 0.75,\ I_c = 0.07,\ T_c = 0.09. \tag{6.72}$$

Then it can be checked that condition (6.57) for the existence of the interior equilibrium E^* is satisfied, and E^* is given by

$$B^* = 99.26837,\ I^* = 0.12976,\ T^* = 0.49550,\ U^* = 0.52896,$$
$$F_1^* = 0.21339,\ F_2^* = 0.29882\ \textit{and}\ F_3 = 2.43301. \tag{6.73}$$

It can easily be verified that conditions (6.58)-(6.62) in Theorem 6.6.1 are satisfied which shows that E^* is locally asymptotically stable.

Further, by choosing $K_m^* = 60.0$ in Theorem 6.6.2, it can be checked that conditions (6.64)-(6.68) are satisfied. This shows that E^* is globally asymptotically stable.

By comparing equilibrium levels $\bar{E}$ and E^* in Eqs. (6.71) and (6.73) we note that due to efforts F_1, F_2 and F_3, the equilibrium level of the resource biomass has increased whereas equilibrium levels of the industrialization pressure, concentration of pollutant in the environment and in the resource biomass have decreased.

6.9 Conclusions

In this chapter, a mathematical model has been proposed and analysed to study the effects of industrialization and pollution on forestry resources with diffusion. The rate of introduction of pollutant into the environment is considered to be industrialization dependent, constant, zero or periodic. The model has been analysed with and without diffusion.

When there is no diffusion in the system, it has been shown that in the case of industrialization dependent introduction of pollutant into the environment the resource biomass settles down to its equilibrium level whose magnitude depends upon the equilibrium level of industrialization, influx and washout rates of pollutant present in the environment. The magnitude of the resource biomass density decreases as the density of industrialization and influx rate of pollutant increase, and even it may tend to zero if these factors increase without control. In the case of constant spill of pollutant into the environment and without diffusion in the system the results are found similar to the industrialization dependent case. Without diffusion and in the case of instantaneous introduction of pollutant into the environment it has been noted that the pollutant may be washed out completely and the resource biomass may settle down to a lower equilibrium level than its original carrying capacity whose magnitude depends only upon the equilibrium level of industrialization pressure. Even in this case the resource biomass may vanish if industrialization pressure increases unabatedly. In the case of periodic emission of pollutant into the environment it has been found that a small periodic influx of pollutant causes a periodic behaviour in the system and the stability behaviour of the system is same as that of the constant introduction of pollutant.

A mathematical model to conserve the resource biomass by plantation, irrigation, fencing, fertilization etc., and to control the undesired levels of industrialization pressure and concentration of pollutant in the environment by some mechanisms has also been proposed. By analysing this model it has been shown that if suitable efforts are made, an appropriate level of resource biomass density can be maintained.

In the diffusive system with reservoir boundary conditions a complete analysis has been carried out for the model. It has been shown that if the positive equilibrium of the system without diffusion is globally asymptotically stable, then the corresponding uniform steady state of the system with diffusion is also globally asymptotically stable. It has been noted that there are cases where the positive equilibrium of the system with no diffusion is unstable, but the corresponding uniform steady state of the system with diffusion can be made stable by increasing diffusion coefficients appropriately. It has also been noted that the global stability is more plausible in the diffusive system than the case with no diffusion, that is, with diffusion the resource biomass density converges towards its carrying capacity at a faster rate than the case with no diffusion. Thus, it has been concluded that the solutions of the system with diffusion converge towards its equilibrium faster than the case of no diffusion.

Chapter 7

TIME DELAY MODEL FOR DEPLETION OF FORESTRY RESOURCES AND THEIR CONSERVATION

7.1 Introduction

Environmental pollution is one of the challenges that mankind is facing as a result of industrialization. Main gaseous pollutants from various industrial units are sulphur dioxide, nitrogen oxides, carbon monoxide, hydrocarbons, fluorine, fly ash, etc. These pollutants affect the ecosystem in general and plants in particular (Gordon and Gorham, 1963; Rao and Rao, 1989). Automobiles constitute another major source of air-borne pollutants in the majority of cities of industrialized countries. The main pollutants which automobiles emit are carbon monoxide, nitrogen oxides, unburned hydrocarbons, smoke and particulate matter. In developing and under developed countries vehicles are poorly maintained and as a result, cause more air pollution. In addition to the pollutants emitted in the gaseous form, solid and liquid pollutants are also coming out from industries. Many of the trees stop bearing fruits due to high level of air pollution. Plants do not blossom and even if they do the flowers are very small and rickety. Air pollution has already led to the disappearance of much of the vegetation including trees. Therefore, it is absolutely essential to study the effect of pollutants on forestry resource biomass.

In recent decades some investigations have been made to study the effect of pollution on a single biological species (Hallam et. al., 1983; Hallam and De Luna, 1984; Hallam and Ma, 1986; De Luna and Hallam, 1987; Freedman and Shukla, 1991; Shukla and Dubey, 1996a; Dubey, 1997a). As pointed out in the previous chapter the above studies have been conducted to study the effect of pollutant on a single or two species communities. In all the above investigations it has been assumed that as soon as the pollutant enters into the body of the species it starts affecting the species without any delay. But there are many other substances emitted by different industries which do not harm the species directly but after some metabolic change these substances get converted to toxic substances which affect the species (MacDonald, 1977). Some other pollutants go on accumulating in the body of the species until their concentrations do not cross the threshold value for affecting the species. This introduces a delay in the system, which was not considered in the earlier investigations. In this chapter, therefore, we have proposed and analysed a mathematical model where time-delay factor has been considered.

7.2 The Model

Consider a forestry resource which is being degraded due to environmental pollution. It is assumed that the dynamics of the forest biomass is governed by nonlinear logistic type equations. It is also assumed that environmental pollutant does not affect the forestry biomass directly, but the pollutant after entering into the biomass gets converted to a substance which is toxic to the resource biomass and consequently the growth rate of the biomass decreases. This conversion causes a time delay in the depletion of forestry resource. Then the dynamics of the system may be governed by following system of autonomous differential equations:

$$\begin{aligned}
\frac{\partial B}{\partial t} &= r(W)B - \frac{r_0 B^2}{K(T)} + D_1 \nabla^2 B, \\
\frac{\partial T}{\partial t} &= Q(t) - \delta_0 T - \alpha_1 BT + D_2 \nabla^2 T, \\
\frac{\partial U}{\partial t} &= -\delta_1 U + \alpha_1 BT, \\
\frac{\partial W}{\partial t} &= \alpha U - \alpha_0 W.
\end{aligned} \tag{7.1}$$

We impose the following initial and boundary conditions on the system:

$$\begin{aligned}
& B(x,y,0) = \phi(x,y) \geq 0, \; T(x,y,0) = \psi(x,y) \geq 0, \\
& U(x,y,0) = \xi(x,y) \geq 0, \; W(x,y,0) = \zeta(x,y) \geq 0, \; (x,y) \, \varepsilon D, \\
& \frac{\partial B}{\partial n} = \frac{\partial T}{\partial n} = 0, \; (x,y) \varepsilon \partial D, \; t \geq 0,
\end{aligned} \tag{7.2}$$

where n is the unit outward normal to ∂D.

In model (7.1), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian diffusion operator. $B(x,y,t)$ is the density of the forest biomass, $T(x,y,t)$ the concentration of environmental pollutants, $U(x,y,t)$ the uptake concentration of pollutant from the environment, $W(x,y,t)$ the concentration of the toxic substance which has been formed by the conversion of $U(x,y,t)$ due to some metabolic changes at coordinates $(x,y)\varepsilon D$ and time $t \geq 0$. D_1 and D_2 are the diffusion rate coefficients of $B(x,y,t)$ and $T(x,y,t)$ respectively in D. $Q(t)$ is the rate of introduction of pollutant into the environment beyond initial concentration. δ_0 is the natural depletion rate coefficient of environmental pollutant. α_1 is the depletion rate coefficient of environmental pollutant due to the resource biomass. δ_1 is the natural depletion rate coefficient of U. α is the growth rate coefficient of $W(x,y,t)$ which is assumed to be proportional to the concentration of $U(x,y,t)$ and α_0 is the natural depletion rate coefficient of $W(x,y,t)$.

In model (7.1), the function $r(W)$ is the specific growth rate of the forest biomass which decreases as W increases, i.e.,

$$r(0) = r_0 > 0 \; and \; r'(W) < 0 \; for \; W \geq 0. \tag{7.3}$$

The function $K(T)$ is the carrying capacity of the forest biomass which satisfies the following properties:

$$K(0) = K_0 > 0, \text{ and } K'(T) < 0 \text{ for } T \geq 0,$$
$$\text{and there exists a } T = T_a \text{ such that } K(T_a) = 0. \tag{7.4}$$

The model is analysed for three different values of Q(t), namely, positive constant, zero or periodic. The model has also been analysed with and without diffusion.

7.3 Model Without Diffusion

In this section we analyse model (7.1) when $D_1 = D_2 = 0$. Then the model reduces to

$$\begin{aligned}
\frac{dB}{dt} &= r(W)B - \frac{r_0 B^2}{K(T)}, \\
\frac{dT}{dt} &= Q(t) - \delta_0 T - \alpha_1 BT, \\
\frac{dU}{dt} &= -\delta_1 U + \alpha_1 BT, \\
\frac{dW}{dt} &= \alpha U - \alpha_0 W, \\
B(0) &\geq 0,\ T(0) \geq 0,\ U(0) \geq 0,\ W(0) \geq 0.
\end{aligned} \tag{7.5}$$

Case I: Constant introduction of pollutant, i.e., $Q(t) = Q_0 > 0$.

In this case it can be checked that there exist two nonnegative equilibria, namely,

$$E_1(0, \frac{Q_0}{\delta_0}, 0, 0) \text{ and } \bar{E}(\bar{B}, \bar{T}, \bar{U}, \bar{W}),$$

where $\bar{B}$, $\bar{T}$, $\bar{U}$ and $\bar{W}$ are the positive solutions of the following algebraic equations:

$$\begin{aligned}
r_0 B &= r(W)K(T), \\
T &= \frac{Q_0}{\delta_0 + \alpha_1 B} = f_1(B), \ (say) \\
U &= \frac{\alpha_1}{\delta_1} B f_1(B) = f_2(B), \ (say) \\
W &= \frac{\alpha}{\alpha_0} f_2(B) = f_3(B). \ (say)
\end{aligned}$$

It can be checked that there exists a unique $\bar{B}$ in the interval $0 < \bar{B} < K_0$, provided the following inequalities hold at $\bar{E}$:

$$r_0 - \frac{dr}{dW} f_3' K(f_1(B)) - r(f_3(B)) \frac{dK}{dT} f_1'(B) > 0, \tag{7.6}$$

$$Q_0 < \delta_0 T_a. \tag{7.7}$$

By computing the variational matrix corresponding to the equilibrium E_1, it can be checked that E_1 is a saddle point with unstable manifold locally in the B direction and with stable manifold in the $T - U - W$ space.

In the following theorem, it is shown that $\bar{E}$ is locally asymptotically stable.

Theorem 7.3.1 *Let the following inequalities hold:*

$$\left(\frac{r_0 \bar{B}}{K^2(\bar{T})} K'(\bar{T}) + \alpha_1 \bar{T}\right)^2 < \frac{2}{3} \frac{r_0}{K(\bar{T})} (\delta_0 + \alpha_1 \bar{B}), \tag{7.8}$$

$$c_2 \alpha^2 < \frac{2}{3} c_1 \alpha_0 \delta_1, \tag{7.9}$$

where

$$c_1 = min\left\{\frac{1}{3} \frac{\delta_1}{(\alpha_1 \bar{T})^2} \frac{r_0}{K(\bar{T})}, \ \frac{1}{3} \frac{\delta_1}{(\alpha_1 \bar{B})^2} (\delta_0 + \alpha_1 \bar{B})\right\} > 0,$$

$$c_2 = 2 \frac{(r'(\bar{W}))^2}{\alpha_0 r_0} K(\bar{T}) > 0.$$

Then the equilibrium $\bar{E}$ is locally asymptotically stable.

Proof: We first linearlze the system (7.5) around the equilibrium $\bar{E}$ by using the following transformations:

$$B = \bar{B} + b, \ T = \bar{T} + \tau, \ U = \bar{U} + u, \ W = \bar{W} + w.$$

Then in the linearized model of (7.5), taking the following positive definite function,

$$V(b, \tau, u, w) = \frac{1}{2}\left\{\frac{b^2}{\bar{B}} + \tau^2 + c_1 u^2 + c_2 w^2\right\}$$

it can be checked that the derivative of V with respect to t is negative definite under conditions (7.8)-(7.9), proving the theorem.

Remark 1 In Theorem 7.3.1 it may be noted that condition (7.9) will be satisfied for $\alpha = 0$. This shows that stability of $\bar{E}$ is more plausible in the absence of W. In the following theorem it is shown that the equilibrium $\bar{E}$ is globally asymptotically stable. To prove this theorem, we need the following lemma which establishes a region of attraction for system (7.5). The proof of this lemma is easy and hence is omitted.

Lemma 7.3.1 *The set*

$$\Omega_1 = \{(B,T,U,W) : 0 \le B \le K_0,\ 0 \le T+U+W \le \frac{Q_0}{\delta}\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$\delta_1 > \alpha \text{ and } \delta = min\{\delta_0, \delta_1 - \alpha, \alpha_0\}.$$

Theorem 7.3.2 *In addition to the assumptions (7.3)-(7.4), let $r(W)$ and $K(T)$ satisfy in Ω_1,*

$$0 \le -r'(W) \le \rho,\ K_m \le K(T) \le K_0 \text{ and } 0 \le -K'(T) \le k, \tag{7.10}$$

for some positive constants ρ, K_m and k. Let the following inequalities hold:

$$\{\frac{r_0 K_0 k}{K_m^2} + \alpha_1 \frac{Q_0}{\delta}\}^2 < \frac{2}{3}\frac{r_0}{K(\bar{T})}(\delta_0 + \alpha_1 \bar{B}), \tag{7.11}$$

$$c_2 \alpha^2 < \frac{2}{3} c_1 \alpha_0 \delta_1. \tag{7.12}$$

Then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the positive orthant, where

$$c_1 = min\{\frac{1}{3}\frac{r_0 \delta_1 \delta^2}{(\alpha_1 Q_0)^2 K(\bar{T})},\ \frac{1}{3}\frac{\delta_1}{(\alpha_1 \bar{B})^2}(\delta_0 + \alpha_1 \bar{B})\}, \tag{7.13}$$

$$c_2 = 2\frac{\rho^2}{r_0 \alpha_0} K(\bar{T}). \tag{7.14}$$

Proof: Consider the following positive definite function around $\bar{E}$,

$$V_1(B,T,U,W) = B - \bar{B} - \bar{B}\, ln\frac{B}{\bar{B}} + \frac{1}{2}(T-\bar{T})^2 + \frac{c_1}{2}(U-\bar{U})^2 + \frac{c_2}{2}(W-\bar{W})^2. \tag{7.15}$$

Now differentiating V_1 with respect to t along the solutions of (7.5), we get

$$\begin{aligned}\frac{dV_1}{dt} &= -\frac{r_0}{K(\bar{T})}(B-\bar{B})^2 - (\delta_0+\alpha_1\bar{B})(T-\bar{T})^2 - c_1\delta_1(U-\bar{U})^2 \\ &-c_2\alpha_0(W-\bar{W})^2 - (r_0B\xi(T)+\alpha_1T)(B-\bar{B})(T-\bar{T}) \\ &+c_1\alpha_1\bar{B}(T-\bar{T})(U-\bar{U}) + c_2\alpha(U-\bar{U})(W-\bar{W}) \\ &+\eta(W)(B-\bar{B})(W-\bar{W}) + c_1\alpha_1T(B-\bar{B})(U-\bar{U}), \end{aligned} \tag{7.16}$$

where

$$\eta(W) = \begin{cases} \frac{r(W)-r(\bar{W})}{W-\bar{W}}, & W \neq \bar{W} \\ \\ r'(\bar{W}), & W = \bar{W} \end{cases}$$

$$\xi(T) = \begin{cases} \left(\frac{1}{K(T)} - \frac{1}{K(\bar{T})}\right)/(T-\bar{T}), & T \neq \bar{T} \\ \\ -\frac{K'(\bar{T})}{K^2(\bar{T})}, & T = \bar{T} \end{cases}$$

From (7.10) and mean value theorem, we note that

$$|\eta(W)| \leq \rho \; and \; |\xi(T)| \leq \frac{k}{K_m^2}.$$

Now Eq. (7.16) can be rewritten as the sum of the quadratics

$$\begin{aligned}\frac{dV_1}{dt} &= -\frac{1}{2}a_{11}(B-\bar{B})^2 + a_{12}(B-\bar{B})(T-\bar{T}) - \frac{1}{2}a_{22}(T-\bar{T})^2 \\ &-\frac{1}{2}a_{11}(B-\bar{B})^2 + a_{13}(B-\bar{B})(U-\bar{U}) - \frac{1}{2}a_{33}(U-\bar{U})^2 \\ &-\frac{1}{2}a_{11}(B-\bar{B})^2 + a_{14}(B-\bar{B})(W-\bar{W}) - \frac{1}{2}a_{44}(W-\bar{W})^2 \\ &-\frac{1}{2}a_{22}(T-\bar{T})^2 + a_{23}(T-\bar{T})(U-\bar{U}) - \frac{1}{2}a_{33}(U-\bar{U})^2 \\ &-\frac{1}{2}a_{33}(U-\bar{U})^2 + a_{34}(U-\bar{U})(W-\bar{W}) - \frac{1}{2}a_{44}(W-\bar{W})^2, \end{aligned}$$

where

$$a_{11} = \frac{2}{3}\frac{r_0}{K(\bar{T})}, \; a_{22} = \delta_0 + \alpha_1\bar{B}, \; a_{33} = \frac{2}{3}c_1\delta_1,$$

$$a_{44} = c_2\alpha_0, \; a_{12} = -(r_0B\xi(T)+\alpha_1T), \; a_{13} = c_1\alpha_1T,$$

$$a_{14} = \eta(W), \; a_{23} = c_1\alpha_1\bar{B}, \; a_{34} = c_2\alpha.$$

Sufficient conditions for $\frac{dV_1}{dt}$ to be negative definite are that the following conditions hold:

$$a_{12}^2 < a_{11}a_{22}, \quad (7.17)$$

$$a_{13}^2 < a_{11}a_{33}, \quad (7.18)$$

$$a_{14}^2 < a_{11}a_{44}, \quad (7.19)$$

$$a_{23}^2 < a_{22}a_{33}, \quad (7.20)$$

$$a_{34}^2 < a_{33}a_{44}. \quad (7.21)$$

From (7.13) and (7.14) we note that the constants c_1 and c_2 are such that inequalities (7.18)-(7.20) are satisfied automatically. We also note that (7.11) $\Rightarrow$ (7.17) and (7.12) $\Rightarrow$ (7.21). Hence V_1 is a Liapunov function with respect to $\bar{E}$, whose domain contains the region of attraction Ω_1, proving the theorem.

It is interesting to note here that after linearizing the conditions (7.17) and (7.18) we get conditions (7.8) and (7.9) respectively as expected.

The above analysis shows that in the case of constant introduction of pollutant into the environment, the resource biomass settles down to its equilibrium level, whose magnitude depends upon the rate of formation of chemical toxicant in the resource biomass and upon the environmental concentration of pollutant. It may be pointed out here that if the time delay in the formation of the chemical toxicant is large, then the over all effect on decreasing the resource biomass density may be reduced

Case II: Instantaneous introduction of pollutant, i.e., $Q(t) = 0$

In this case there exists two nonnegative equilibria, namely, $E_0(0,0,0,0)$ and $E_1(K_0,0,0,0)$. By computing the variational matrix corresponding to each equilibria it can be checked that E_0 is a saddle point with unstable manifold locally along B direction and stable manifold locally in the $T-U-W$ space. E_1 is locally asymptotically stable. In the following theorem we have shown that E_1 is globally asymptotically stable.

Theorem 7.3.3 *If $B(0) > 0$, then E_1 is globally asymptotically stable with respect to the nonnegative orthant*

Proof: We have

$$\frac{dB}{dt} = r(W)B - \frac{r_0 B^2}{K(T)} \le r_0 B(1 - \frac{B}{K_0})$$

Hence

$$\lim_{n \to \infty} B(t) \le K_0.$$

Now

$$\frac{dT}{dt} + \frac{dU}{dt} + \frac{dW}{dt} = -\delta_0 T - (\delta_1 - \alpha)U - \alpha_0 W \le -\delta(T + U + W)$$

where $\delta = min\{\delta_0,\ \delta_1 - \alpha,\ \alpha_0\}$ and $\delta_1 > \alpha$. Then

$$T(t) + U(t) + W(t) \le \{T(0) + U(0) + W(0)\}e^{-\delta t}$$

and hence the system is dissipative.

From the above analysis it follows that

$$\lim_{n \to \infty} T(t) = \lim_{n \to \infty} U(t) = \lim_{n \to \infty} W(t) = 0.$$

In the limit $B(t)$ is given by the solutions of $\frac{dB}{dt} = r_0 B(1 - \frac{B}{K_0})$. Since $B(0) > 0$, the theorem follows.

The above theorem shows that if the concentration of environmental pollution is not sufficient to destroy the resource biomass, eventually the pollutant will be removed and the resource would recover to its original level.

Case III. Periodic introduction of pollutant into the environment, i.e., $Q(t) = Q_0 + \varepsilon\phi(t), \phi(t + \omega) = \phi(t)$.

In this case it can be checked that the results corresponding to Theorem (3.4.1) and Theorem (3.4.2) in chapter 3 remain valid. In particular, it is found that a small periodic influx of toxicant causes a periodic behaviour in the system.

7.4 Model With Diffusion

In this section we consider the complete model (7.1)-(7.2) and state the main results in the form of the following theorem.

Theorem 7.4.1 *(i) If the equilibrium $\bar{E}$ of model (7.5) is globally asymptotically stable, then the corresponding uniform steady state of the initial-boundary value problems (7.1)-(7.2) is also globally asymptotically stable.*

(ii) If the equilibrium $\bar{E}$ of model (7.5) is unstable even then the uniform steady state of the initial-boundary value problems (7.1)-(7.2) can be made stable by increasing diffusion coefficients to sufficiently large values.

Proof: Let us consider the following positive definite function

$$V_2(B(t), T(t), U(t), W(t)) = \int\int_D V_1(B, T, U, W)dA,$$

where V_1 is given in Eq. (7.15).

We have,

$$\begin{aligned} \frac{dV_2}{dt} &= \int\int_D (\frac{\partial V_1}{\partial B}\frac{\partial B}{\partial t} + \frac{\partial V_1}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial V_1}{\partial U}\frac{\partial U}{\partial t} + \frac{\partial V_1}{\partial W}\frac{\partial W}{\partial t})dA \\ &= I_1 + I_2, \end{aligned} \tag{7.22}$$

where

$$I_1 = \int\int_D \frac{dV_1}{dt}dA,$$
$$I_2 = \int\int_D (D_1\frac{\partial V_1}{\partial B}\nabla^2 B + D_2\frac{\partial V_1}{\partial T}\nabla^2 T)dA.$$

We note the following properties of V_1, namely,

$$\left.\frac{\partial V_1}{\partial B}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial T}\right]_{\partial D} = 0$$

and for all points of D,

$$\frac{\partial^2 V_1}{\partial B\partial T} = \frac{\partial^2 V_1}{\partial B\partial U} = \frac{\partial^2 V_1}{\partial B\partial W} = \frac{\partial^2 V_1}{\partial T\partial U} = \frac{\partial^2 V_1}{\partial T\partial W} = \frac{\partial^2 V_1}{\partial U\partial W} = 0,$$
$$\frac{\partial^2 V_1}{\partial B^2} > 0, \ \frac{\partial^2 V_1}{\partial T^2} > 0, \ \frac{\partial^2 V_1}{\partial U^2} > 0, \ and \ \frac{\partial^2 V_1}{\partial W^2} > 0.$$

Under an analysis similar to chapter 2, we note that

$$\int\int_D \frac{\partial V_1}{\partial B}\nabla^2 B dA = -\int\int_D \frac{\partial^2 V_1}{\partial B^2}\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\}dA \leq 0, \tag{7.23}$$
$$\int\int_D \frac{\partial V_1}{\partial T}\nabla^2 T dA = -\int\int_D \frac{\partial^2 V_1}{\partial T^2}\{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\}dA \leq 0. \tag{7.24}$$

This shows that

$$I_2 \leq 0. \tag{7.25}$$

The above results imply that if $I_1 \leq 0$, i.e., if $\bar{E}$ is globally asymptotically stable in the absence of diffusion, then the uniform steady state of the initial-boundary value problems (7.1)-(7.2) also must be globally asymptotically stable. This proves the first part of Theorem 7.4.1.

We further note that if $\frac{dV_1}{dt} > 0$, i.e., if $I_1 > 0$, then $\bar{E}$ may be unstable in the absence of diffusion. But, Eqs. (7.22) and (7.25) show that by increasing diffusion coefficients D_1 and D_2 sufficiently large, $\frac{dV_2}{dt}$ can be made negative even if $I_1 > 0$. This proves the second part of Theorem 7.4.1.

The above theorem implies that diffusion with reservoir boundary conditions stabilizes a system which is otherwise unstable.

We shall explain the above theorem for a rectangular habitat D defined by

$$D = \{(x, y) : \ 0 \leq x \leq a, \ 0 \leq y \leq b\} \tag{7.26}$$

in the form of the following theorem.

Theorem 7.4.2 *In addition to assumptions (7.3) and (7.4), let $r(W)$, $K(T)$, satisfy the inequalities in (7.10). If the following inequalities hold:*

$$\begin{aligned} \{\frac{r_0 K_0 k}{K_m^2} + \frac{\alpha_1 Q_0}{\delta}\}^2 &< \frac{2}{3}\{\frac{r_0}{K(\bar{T})} + \frac{D_1 \bar{B} \pi^2 (a^2+b^2)}{K_0^2 a^2 b^2}\} \times \\ &\quad \{\delta_0 + \alpha \bar{B} + \frac{D_2 \pi^2 (a^2+b^2)}{a^2 b^2}\}, \end{aligned} \tag{7.27}$$

$$\rho^2 < \frac{2}{3} c_2 \alpha_0 \{\frac{r_0}{K(\bar{T})} + \frac{D_1 \bar{B} \pi^2 (a^2+b^2)}{K_0^2 a^2 b^2}\}, \tag{7.28}$$

where

$$\begin{aligned} c_1 &= min\{\frac{\delta_1 \delta^2}{3\alpha_1^2 Q_0^2}(\frac{r_0}{K(\bar{T})} + \frac{D_1 \bar{B} \pi^2 (a^2+b^2)}{K_0^2 a^2 b^2}), \\ &\quad \frac{\delta_1}{3\alpha_1^2 \bar{B}^2}(\delta_0 + \alpha_1 \bar{B} + \frac{D_2 \pi^2 (a^2+b^2)}{a^2 b^2})\} \end{aligned} \tag{7.29}$$

$$c_2 = \frac{c_1 \alpha_0 \delta_1}{3\alpha^2}. \tag{7.30}$$

Then the uniform steady state of the initial-boundary value problems (7.1)-(7.2) is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Let us consider the rectangular region D given by Eq. (7.26). In this case I_2, which is defined in Theorem 7.4.1, can be written as

$$I_2 = -D_1 \int\int_D (\frac{\partial^2 V_1}{\partial B^2})\{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\}\ dA - D_2 \int\int_D (\frac{\partial^2 V_1}{\partial T^2})\{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\}. \quad (7.31)$$

From Eq. (7.15) we get

$$\frac{\partial^2 V_1}{\partial B^2} = \frac{\bar{B}}{B^2} \ and \ \frac{\partial^2 V_1}{\partial T^2} = 1.$$

Hence

$$I_2 \leq -\frac{D_1 \bar{B}}{K_0^2} \int\int_D \{(\frac{\partial B}{\partial x})^2 + (\frac{\partial B}{\partial y})^2\}\ dA - D_2 \int\int_D \{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\}\ dA.$$

Now

$$\begin{aligned}\int\int_D (\frac{\partial B}{\partial x})^2\ dA &= \int\int_D \{\frac{\partial (B-\bar{B})}{\partial x}\}^2\ dA \\ &= \int_0^b \int_0^a \{\frac{\partial (B-\bar{B})}{\partial x}\}^2\ dxdy\end{aligned}$$

Letting $z = \frac{x}{a}$, it can be seen under an analysis similar to chapter 2 that

$$\int\int_D (\frac{\partial B}{\partial x})^2\ dA \geq \frac{\pi^2}{a^2} \int\int_D (B-\bar{B})^2\ dA$$

and

$$\int\int_D (\frac{\partial B}{\partial y})^2\ dA \ \geq \ \frac{\pi^2}{b^2} \int\int_D (B-\bar{B})^2\ dA$$

Thus,

$$I_2 \leq \ -\frac{D_1 \bar{B} \pi^2 (a^2+b^2)}{K_0^2 a^2 b^2} \int\int_D (B-\bar{B})^2 dA - \frac{D_2 \pi^2 (a^2+b^2)}{a^2 b^2} \int\int_D (T-\bar{T})^2\ dA.$$

Now from (7.16) and (7.22) we get

$$\begin{aligned}\frac{dV_2}{dt} \leq & \int\int_D [-\{\frac{r_0}{K(\bar{T})} + \frac{D_1\bar{B}\pi^2(a^2+b^2)}{K_0^2a^2b^2}\}(B-\bar{B})^2 \\ & -\{\delta_0 + \alpha_1\bar{B} + \frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\}(T-\bar{T})^2 - c_1\delta_1(U-\bar{U})^2 \\ & -c_2\alpha_0(W-\bar{W})^2 - \{r_0B\xi(T) + \alpha_1 T\}(B-\bar{B})(T-\bar{T}) \\ & +c_1\alpha_1\bar{B}(T-\bar{T})(U-\bar{U}) + c_2\alpha(U-\bar{U})(W-\bar{W}) \\ & +\eta(W)(B-\bar{B})(W-\bar{W}) + c_1\alpha_1 T(B-\bar{B})(U-\bar{U}), \qquad (7.32)\end{aligned}$$

where $\xi(T)$ and $\eta(W)$ are defined in Eq. (7.16).

Now Eq. (7.32) can be written as the sum of the quadratics

$$\begin{aligned}\frac{dV_2}{dt} \leq & \int\int_D \{-\frac{1}{2}b_{11}(B-\bar{B})^2 + b_{12}(B-\bar{B})(T-\bar{T}) - \frac{1}{2}b_{22}(T-\bar{T})^2 \\ & -\frac{1}{2}b_{11}(B-\bar{B})^2 + b_{13}(B-\bar{B})(U-\bar{U}) - \frac{1}{2}b_{33}(U-\bar{U})^2 \\ & -\frac{1}{2}b_{11}(B-\bar{B})^2 + b_{14}(B-\bar{B})(W-\bar{W}) - \frac{1}{2}b_{44}(W-\bar{W})^2 \\ & -\frac{1}{2}b_{22}(T-\bar{T})^2 + b_{23}(T-\bar{T})(U-\bar{U}) - \frac{1}{2}b_{33}(U-\bar{U})^2 \\ & -\frac{1}{2}b_{33}(U-\bar{U})^2 + b_{34}(U-\bar{U})(W-\bar{W}) - \frac{1}{2}b_{44}(W-\bar{W})^2\}dA\end{aligned}$$

where

$$\begin{aligned} & b_{11} = \frac{2}{3}\{\frac{r_0}{K(\bar{T})} + \frac{D_1\bar{B}\pi^2(a^2+b^2)}{K_0^2a^2b^2}\}, b_{22} = \delta_0 + \alpha_1\bar{B} + \frac{D_2\pi^2(a^2+b^2)}{a^2b^2}, \\ & b_{33} = \frac{2}{3}c_1\delta_1,\ b_{44} = c_2\alpha_0,\ b_{12} = -(r_0B\xi(T) + \alpha_1 T), \\ & b_{13} = c_1\alpha_1 T,\ b_{14} = \eta(W),\ b_{23} = c_1\alpha_1\bar{B}, b_{34} = c_2\alpha.\end{aligned}$$

Sufficient conditions for $\frac{dV_2}{dt}$ to be negative definite are that the following conditions hold:

$$
\begin{aligned}
b_{12}^2 &< b_{11}b_{22}, &\quad (7.33)\\
b_{13}^2 &< b_{11}b_{33}, &\quad (7.34)\\
b_{14}^2 &< b_{11}b_{44}, &\quad (7.35)\\
b_{23}^2 &< b_{22}b_{33}, &\quad (7.36)\\
b_{34}^2 &< b_{33}b_{44}. &\quad (7.37)
\end{aligned}
$$

We note that inequalities (7.34), (7.36) and (7.37) are automatically satisfied for the value of c_1 and c_2 given in (7.29) and (7.30) respectively. We further note (7.27) $\Rightarrow$ (7.33), (7.28) $\Rightarrow$ (7.35). Hence V_2 is a Liapunov function with respect to $\bar{E}$ whose domain contains the region of attraction Ω_1, proving the theorem.

From the above theorem we note that inequalities (7.27)-(7.28) may be satisfied by increasing D_1 and D_2 to sufficiently large values. This implies that in the case of diffusion stability is more plausible than the case of no diffusion. Thus, in the case of diffusion the population converges towards its carrying capacity faster than the case of no diffusion, and hence the survival of the population may be ensured.

7.5 Conservation Model

In the previous section it has been noted that uncontrolled level of environmental pollution may lead to the extinction of forestry resources. Therefore, some kind of efforts must be adopted to conserve the forestry resources and to control the emission of pollutant into the environment (Peterson et al., 1984; Huttl and Wisniewski, 1987; Shukla et al., 1989; Shukla and Dubey, 1997). In this section a mathematical model is proposed and analysed to conserve the forestry resources and to control the undesired level of environmental pollution by some mechanisms. It is assumed that the effort applied to conserve the forest biomass is proportional to the depleted level of forest biomass from its carrying capacity and the effort applied to control the concentration of pollutant is proportional to the undesired level of pollutant. The dynamics of the system is assumed to be governed by the following differential equations:

$$\begin{aligned}
\frac{\partial B}{\partial t} &= r(W) - \frac{r_0 B^2}{K(T)} + r_1 F_1 + D_1 \nabla^2 B, \\
\frac{\partial T}{\partial t} &= Q(t) - \delta_0 T - \alpha_1 BT - r_2 F_2 + D_2 \nabla^2 T, \\
\frac{\partial U}{\partial t} &= -\delta_1 U + \alpha_1 BT, \\
\frac{\partial W}{\partial t} &= \alpha U - \alpha_0 W, \\
\frac{\partial F_1}{\partial t} &= \mu_1 (1 - \frac{B}{K_0}) - \mu_0 F_1, \\
\frac{\partial F_2}{\partial t} &= \nu_1 (T - T_c) H(T - T_c) - \nu_0 F_2.
\end{aligned} \tag{7.38}$$

We impose the following initial and boundary conditions on system (7.38):

$$\begin{aligned}
& B(x,y,0) = \phi(x,y) \geq 0, T(x,y,0) = \psi(x,y) \geq 0, \\
& U(x,y,0) = \xi(x,y) \geq 0, W(x,y,0) = \zeta(x,y) \geq 0, \\
& F_1(x,y,0) = \zeta_1(x,y) \geq 0, F_2(x,y,0) = \zeta_2(x,y) \geq 0, \ (x,y) \in D \\
& \frac{\partial B}{\partial n} = \frac{\partial T}{\partial n} = 0, \ (x,y) \in \partial D, \ t \geq 0.
\end{aligned} \tag{7.39}$$

In model (7.38), $F_1(x,y,t)$ is the density of effort applied to conserve the resource biomass and $F_2(x,y,t)$ the density of effort applied to control the undesired level of the concentration of pollutant in the environment. $r_1 > 0$ represents the growth rate coefficient of resource biomass due to effort F_1. $r_2 > 0$ is depletion rate coefficient of $T(x,y,t)$ due to the effort F_2. μ_1 and ν_1 are the growth rate coefficients of F_1 and F_2 respectively and μ_0 and ν_0 are their respective depreciation rate coefficients. T_c is the critical level of the concentration of pollutant which is assumed to be harmless to the resource. In the last equation of system (7.38), $H(t)$ denotes the unit step function which takes into account the case $T \leq T_c$. It is interesting to note that in the unusual circumstances, if the forest exceeds its carrying capacity, then $\frac{\partial F_1}{\partial t}$ will be negative, giving a decrease in the effort to conserve the biomass.

We analyse the conservation model (7.38) only for the case when rate of introduction of pollutant into the environment is constant.

7.6 Conservation Model Without Diffusion

In this section we take, $D_1 = D_2 = 0$ in model (7.38). Then model (7.38) has only one interior equilibrium, namely, $E^*(B^*, T^*, U^*, W^*, F_1^*, F_2^*)$, where $B^*, T^*, U^*, W^*, F_1^*$ and F_2^* are the positive solutions of the system of algebraic equations given below.

$$r_0 B = \{r(W) + \frac{r_1\mu_1}{B\mu_0}(1 - \frac{B}{K_0})\}K(T),$$

$$T = \frac{\nu_0 Q_0 + r_2\nu_1 T_c}{\nu_0(\delta_0 + \alpha_1 B) + r_2\nu_1} = g_1(B), \ (say)$$

$$U = \frac{\alpha_1}{\delta_1} B g_1(B) = g_2(B), \ (say)$$

$$W = \frac{\alpha}{\alpha_0} g_2(B) = g_3(B), \ (say)$$

$$F_1 = \frac{\mu_1}{\mu_0}(1 - \frac{B}{K_0}),$$

$$F_2 = \frac{\nu_1}{\nu_0}(T - T_c)H(T - T_c) = \begin{cases} \frac{\nu_1}{\nu_0}(T - T_c), & T > T_c \\ 0, & T \leq T_c \end{cases}.$$

It may be noted here that for F_1 to be positive, we must have

$$B < K_0.$$

As earlier, it is easy to check that E^* exists provided the following inequality holds at E^*,

$$r_0 - \{\frac{\partial r}{\partial W} g_3'(B) - \frac{r_1\mu_1}{B^2\mu_0}\}K(g_1(B)) - \{r(g_3(B)) + \frac{r_1\mu_1}{B\mu_0}(1 - \frac{B}{K_0})\}\frac{\partial K}{\partial T} g_1'(B) > 0. \tag{7.40}$$

In the following theorem it is shown that E^* is locally asymptotically stable, the proof of which is similar to Theorem 7.3.1 and hence is omitted.

Theorem 7.6.1 *Let the following inequalities hold:*

$$\{\frac{r_0 B^*}{K^2(T^*)} K'(T^*) + \alpha_1 T^*\}^2 < \frac{1}{3}\{\frac{r_0}{K(T^*)} + \frac{r_1 F_1^*}{B^{*2}}\}(\delta_0 + \alpha_1 B^*), \tag{7.41}$$

$$c_2\alpha^2 < \frac{2}{3} c_1 \alpha_0 \delta_1, \tag{7.42}$$

where

$$c_1 = min\{\frac{1}{4}\frac{\delta_1}{(\alpha_1 T^*)^2}(\frac{r_0}{K(T^*)} + r_1\frac{F_1^*}{B^{*2}}), \frac{1}{3}\frac{\delta_1}{(\alpha_1 B^*)^2}(\delta_0 + \alpha_1 B^*)\},$$
$$c_2 = \frac{3}{\alpha_0}\frac{(r'(W^*))^2}{\frac{r_0}{K(T^*)} + r_1\frac{F_1^*}{B^{*2}}}.$$

Then E^* is locally asymptotically stable.

In the following lemma a region of attraction for system (7.38) without diffusion is established. The proof of this lemma is easy and hence is omitted.

Lemma 7.6.1 *The set*

$$\Omega_2 = \{(B, T, U, W, F_1, F_2) \ : \ 0 \le B \le K_c, 0 \le T + U + W \le \frac{Q_0}{\delta}, 0 \le F_1 \le \frac{\mu_1}{\mu_0}, \quad 0 \le F_2 \le \frac{\nu_1 Q_0}{\nu_0 \delta})\}$$

attracts all solutions initiating in the positive orthant. where

$$K_c = \frac{K_0}{2}\{1 + \sqrt{1 + \frac{4r_1\mu_1}{r_0\mu_0 K_0}}\}, \ \delta = min\{\delta_0, \delta_1 - \alpha, \alpha_0\} \ and \ \delta_1 > \alpha.$$

The following theorem gives criteria for global stability of E^*, whose proof is similar to Theorem 7.3.2 and hence is omitted.

Theorem 7.6.2 *In addition to the assumptions (7.3 and (7.4), let $r(W)$ and $K(T)$ satisfy in Ω_2,*

$$0 \le -r'(W) \le \rho^*, \ K_m^* \le K(T) \le K_0 \ and \ 0 \le -K'(T) \le k^*, \tag{7.43}$$

for some positive constants ρ^*, K_m^* and k^*. Let the following inequalities hold:

$$\{\frac{r_0 K_c k^*}{K_m^{*2}} + \alpha_1\frac{Q_0}{\delta}\}^2 < \frac{1}{3}\frac{r_0}{K(T^*)}(\delta_0 + \alpha_1 B^*), \tag{7.44}$$
$$c_2\alpha^2 < \frac{2}{3}c_1\alpha_0\delta_1, \tag{7.45}$$

where

$$c_1 = min\{\frac{1}{4}\frac{\delta_1\delta^2}{(\alpha_1 Q_0)^2}\frac{r_0}{K(T^*)}, \frac{1}{3}\frac{\delta_1}{(\alpha_1 B^*)^2}(\delta_0 + \alpha_1 B^*)\},$$
$$c_2 = \frac{3}{\alpha_0}\frac{K(T^*)}{r_0}\rho^{*2}.$$

Then E^* is globally asymptotically stable with respect to all solutions initiating in the positive orthant.

Theorems 7.6.1 and 7.6.2 show that if suitable efforts are made to conserve the forest biomass and to control the undesired level of the concentration of environmental pollutant, an appropriate level of the resource biomass may be maintained.

7.7 Conservation Model With Diffusion

We now consider the case when $D_i > 0 (i = 1, 2, 3)$ in model (7.38). We shall show that the uniform steady state $B(x,y,t) = B^*, T(x,y,t) = T^*, U(x,y,t) = U^*, W(x,y,t) = W^*, F_1(x,y,t) = F_1^*$ and $F_2(x,y,t) = F_2^*$ is globally asymptotically stable. For this, we consider the following positive definite function

$$V_3(B(t),T(t),U(t),W(t),F_1(t),F_2(t)) = \int\int_D V_2(B,T,U,W,F_1,F_2)\, dA,$$

where

$$\begin{aligned} V_2(B,T,U,W,F_1,F_2) &= B - B^* - B^* \ln \frac{B}{B^*} + \frac{1}{2}(T-T^*)^2 + \frac{c_1}{2}(U-U^*)^2 \\ &+ \frac{c_2}{2}(W-W^*)^2 + \frac{c_3}{2}(F_1-F_1^*)^2 + \frac{c_4}{2}(F_2-F_2^*)^2 \end{aligned}$$

and the $c_i s$ are positive constants to be chosen suitably.

Then as earlier, it can be checked that if $\frac{dV_2}{dt} < 0$, then $\frac{dV_3}{dt} < 0$. This implies that if E^* is globally asymptotically stable for system (7.38) without diffusion, then the corresponding uniform steady state of system (7.38)-(7.39) is also globally asymptotically stable with respect to solutions such that $\phi(x,y) > 0, \psi(x,y) > 0, \xi(x,y) > 0, \zeta(x,y) > 0, \zeta_1(x,y) > 0, \zeta_2(x,y) > 0,\ (x,y) \in D$.

7.8 Numerical Examples

Example 1 Here a numerical example is presented to illustrate the results obtained in section 7.3. We consider the following particular form of the functions in model (7.5).

$$r(W) = r_0 - r_{10}W,$$
$$K(T) = K_0 - K_1 T. \tag{7.46}$$

Now choose the following set of values of the parameters in Eq. (7.46) and in model (7.5).

$$r_0 = 10.00,\ r_{10} = 0.08,\ K_0 = 30.00,$$
$$K_1 = 0.09,\ Q_0 = 15.00,\ \delta_0 = 8.00,\ \alpha_1 = 0.04, \tag{7.47}$$
$$\delta_1 = 7.50,\ \alpha = 1.50,\ \alpha_0 = 0.80.$$

With the above values of the parameters, it can be checked that the conditions (7.6) and (7.7) for the existence of the interior equilibrium $\bar{E}$ are satisfied and $\bar{E}$ is given by

$$\bar{B} = 29.73716,\ \bar{T} = 1.63230,\ \bar{U} = 0.25888,\ \bar{W} = 0.48540. \tag{7.48}$$

It can also be checked that conditions (7.8)-(7.9) in Theorem 7.3.1 are satisfied which shows that $\bar{E}$ is locally asymptotically stable.

By choosing $K_m = 20.0$ in Theorem 7.3.2 it can also be verified that conditions (7.11)-(7.12) are satisfied which shows that $\bar{E}$ is globally asymptotically stable.

Example 2 Now to illustrate the results obtained in section 7.6 we present a numerical example. In addition to the values of parameters given in (7.47), we choose the following values of parameters in model (7.38) with no diffusion:

$$r_1 = 0.30,\ r_2 = 0.07,\ \mu_1 = 10.00,\ \mu_0 = 0.05,$$
$$\nu_1 = 11.0,\ \nu_0 = 0.06,\ T_c = 0.12. \tag{7.49}$$

Then it can be checked that condition (7.40) for the existence of the interior equilibrium E^* is satisfied, and E^* is given by

$$B^* = 29.89891,\ T^* = 0.75082,\ U^* = 0.11973,\ W^* = 0.22449,$$
$$F_1^* = 0.67393,\ F_2^* = 115.65010. \tag{7.50}$$

It can easily be verified that conditions (7.41)-(7.42) in Theorem 7.6.1 are satisfied which shows that E^* is locally asymptotically stable.

Further, by choosing $K_m^* = 20.0$ in Theorem 7.6.2, it can be checked that conditions (7.44)-(7.45) are satisfied. This shows that E^* is globally asymptotically stable.

By comparing equilibrium levels $\bar{E}$ and E^* in Eqs. (7.48) and (7.50) we note that due to efforts F_1 and F_2, the equilibrium level of the resource biomass has increased whereas equilibrium level of the concentration of pollutant in the environment and in the resource biomass have decreased.

7.9 Conclusions

In this chapter, a mathematical model has been proposed and analysed to study the effect of environmental pollution on forestry resource biomass with time-delay. The model has been analysed with and without diffusion. When there is no diffusion it has been shown that in the case of constant introduction of pollutant into the environment the resource biomass settles down to its equilibrium level, the magnitude of which depends upon the washout and uptake rates of pollutant. It has further been noted that if the concentration of pollutant increases unabatedly, the survival of the species would be threatened. In our model (7.1), the concentration of the environmental pollutant T does not affect the growth of the resource biomass directly. This pollutant when uptaken by the species is being converted into some other chemical toxicant due to some metabolic changes, which affects the growth rate of the biomass. The effect of time delay due to the formation of the chemical toxicants on decreasing the equilibrium level of resource biomass is determined by the rate of formation of the chemical toxicant and depletion of the resource biomass. If the delay in formation of the toxicant is large, then this may help in reducing over all effect of the pollutant provided other parameters remain same. In the case of instantaneous introduction of pollutant into the environment, it has been found that perhaps the concentration of pollutant was not enough to deplete the resource biomass and hence the pollutant will be washed out completely and the resource biomass would recover at its original carrying capacity. It has also been noted that a small periodic introduction of pollutant into the environment induces a periodic behaviour in the system.

By analysing the conservation model it has been shown that if suitable efforts are made to conserve the resource biomass and to control the undesired level of pollutant in the environment, then the desired level of resource biomass may be maintained. The effect of diffusion on the interior equilibrium state of the system has also been investigated. It has been shown that if the positive equilibrium of the system without diffusion is globally asymptotically stable, then the corresponding uniform steady state of the system with diffusion is also globally asymptotically stable. It has further been noted that if the positive equilibrium of the system with no diffusion is unstable, then the corresponding uniform steady state of the system with diffusion can be made stable by increasing diffusion coefficients appropriately. This shows that the global stability is more plausible in the case of diffusion than the case of no diffusion. Thus we conclude that in the case of diffusion solutions approach to its equilibrium levels faster than the case of no diffusion.

Chapter 8

MODELLING THE EFFECT OF POLLUTANTS FORMED BY PRECURSORS IN THE ATMOSPHERE ON POPULATION

8.1 Introduction

With the rapid pace of industrialization, urbanization, deforestation etc. our environment is getting polluted day by day. The effects of pollution caused by various human factors on structure and functions of ecosystems have been studied by several researchers (Woodwell, 1970; Smith, 1981; McLaughlin, 1985; Hari et al., 1986; Woodman and Cowling, 1987; Schulze, 1989). In recent decades some investigations have been made to study the effect of pollution on a single biological species (Hallam et al., 1983; Hallam and De Luna. 1984; Hallam and Ma, 1986; De Luna and Hallam, 1987; Freedman and Shukla, 1991; Shukla and Dubey, 1996a; Dubey, 1997a; Shukla and

Dubey, 1997). As pointed out in the previous chapters, in the above investigations it is assumed that the pollutant enters into the environment by some manmade projects which may be population (industrialization) dependent, constant, zero or periodic. In the above studies the effect of pollutant lag has not been considered. In this regard, Rescigno (1977) studied the effect of a precursor pollutant on a single species, but he did not consider the rate of uptake concentration of the pollutant on the growth of the species. Further, in the above works the effects of diffusion has not been considered. Keeping the above in view, in this chapter we propose and analyse a mathematical model to study the effect of a precursor pollutant, which is formed by various human activities in the atmosphere, on population where the effect of uptake concentration, diffusion and conservation are considered.

8.2 The Model

We consider an environment which is polluted by various population activities. It is assumed that the population is affected by the pollutant formed in the atmosphere by its precursor. Let $P(x, y, t)$ be the population density, $Q(x, y, t)$ the concentration of the precursor pollutant emitted by various activities of the population, $T(x, y, t)$ the concentration of the pollutant formed by Q in the atmosphere and $U(x, y, t)$ the uptake concentration of pollutant by the population at coordinates $(x, y)\epsilon D$ and time $t \geq 0$. It is also assumed that the larger the population, the faster the precursor is produced. It is further assumed that the larger the precursor, the faster the pollutant is produced. Then the system may be governed by the following set of differential equations:

$$\begin{aligned}
\frac{\partial P}{\partial t} &= r(U)P - \frac{r_0 P^2}{K(T)} + D_1 \nabla^2 P, \\
\frac{\partial Q}{\partial t} &= \gamma P - \gamma_0 Q, \\
\frac{\partial T}{\partial t} &= hQ - h_0 T + \theta_1 \delta_1 U - \alpha PT + D_2 \nabla^2 T, \qquad (8.1) \\
\frac{\partial U}{\partial t} &= -\delta_1 U + \theta_0 h_0 T + \alpha PT, \\
0 &\leq \theta_0, \; \theta_1 \leq 1.
\end{aligned}$$

We analyse the system 8.1 with the following initial and boundary conditions:

$$\begin{aligned}
&P(x,y,0)=\phi(x,y)\geq 0, Q(x,y,0)=\psi(x,y)\geq 0,\\
&T(x,y,0)=\xi(x,y)\geq 0, U(x,y,0)=\zeta(x,y)\geq 0,\ (x,y)\ \varepsilon\ D \qquad (8.2)\\
&\frac{\partial P}{\partial n}=\frac{\partial T}{\partial n}=0,\ (x,y)\ \varepsilon\ \partial D,\ t\geq 0,
\end{aligned}$$

where n is the unit outward normal to ∂D.

In model (8.1), $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplacian diffusion operator. D_1 and D_2 are the diffusion rate coefficients of $P(x,y,t)$ and $T(x,y,t)$ respectively in D. γ is the growth rate of Q due to the population P, γ_0 the natural depletion rate coefficient of Q. h is the growth rate coefficient of T due to Q, h_0 the natural depletion rate coefficient of T, a fraction θ_0 of which goes inside the body of the population. α is the depletion rate coefficient of T due to P. δ_1 is the natural depletion rate coefficient of U, a fraction θ_1 of which reenters into the environment.

In model (8.1), the function $r(U)$ is the specific growth rate of the population which decreases as U increases, i.e.,

$$r(0)=r_0 \ and\ r'(U)<0\ for\ U\geq 0. \qquad (8.3)$$

The function $K(T)$ is the carrying capacity of the population which satisfies the following properties:

$$\begin{aligned}
&K(0)=K_0,\ and\ K'(T)<0\ for\ T\geq 0,\\
&and\ there\ exists\ a\ T=T_a\ such\ that\ K(T_a)=0. \qquad (8.4)
\end{aligned}$$

The model is analysed with and without diffusion.

8.3 Model Without Diffusion

In this section we take $D_1 = D_2 = 0$ in model (8.1). Then model (8.1) reduces to

$$\begin{aligned}
\frac{dP}{dt} &= r(U)P - \frac{r_0 P^2}{K(T)}, \\
\frac{dQ}{dt} &= \gamma P - \gamma_0 Q, \\
\frac{dT}{dt} &= hQ - h_0 T + \theta_1 \delta_1 U - \alpha PT, \\
\frac{dU}{dt} &= -\delta_1 U + \theta_0 h_0 T + \alpha PT, \\
P(0) &\geq 0, \ Q(0) \geq 0, \ T(0) \geq 0, \ U(0) \geq 0.
\end{aligned} \tag{8.5}$$

It can be checked that there exist two nonnegative equilibria, namely,

$$E_0(0,0,0,0) \ and \ \bar{E}(\bar{P},\bar{Q},\bar{T},\bar{U}),$$

where $\bar{P}$, $\bar{Q}$, $\bar{T}$ *and* $\bar{U}$ are the positive solutions of the following algebraic equations:

$$\begin{aligned}
r_0 P &= r(U)K(T), \\
Q &= \frac{\gamma}{\gamma_0} P, \\
T &= \frac{hQ}{h_0(1-\theta_0\theta_1) \ + \ \alpha(1-\theta_1)P} = f(P), \ say \\
U &= \frac{1}{\delta_1}(\theta_0 h_0 f(P) + \alpha P f(P)) = g(P). \ say
\end{aligned}$$

It can be verified that the equilibrium $\bar{E}$ exists if the following inequality holds at $\bar{E}$:

$$r_0 - r'(U)g'(P)K(f(P)) - r(g(P))K'(T)f'(P) > 0. \tag{8.6}$$

By computing the variational matrix corresponding to the equilibrium $\bar{E}$, it can be checked that E_0 is a saddle point with unstable manifold locally in the P direction and with stable manifold locally in the $Q-T-U$ space.

In the following theorem, it is shown that $\bar{E}$ is locally asymptotically stable.

Theorem 8.3.1 *Let the following inequalities hold:*

$$\{\frac{r_0\bar{P}}{K^2(\bar{T})}K'(\bar{T}) + \alpha\bar{T}\}^2 \ < \ \frac{4}{9}\frac{r_0}{K(\bar{T})}(h_0 + \alpha\bar{P}), \tag{8.7}$$

$$\{\theta_1\delta_1 + c_2(\theta_0 h_0 + \alpha\bar{P})\}^2 < \frac{2}{3}c_2\delta_1(h_0 + \alpha\bar{P}), \tag{8.8}$$

$$h^2 < \frac{2}{3}c_1\gamma_0(h_0 + \alpha\bar{B}), \tag{8.9}$$

where

$$c_1 = \frac{1}{3}\frac{r_0\gamma_0}{\gamma^2 K(\bar{T})} \text{ and } c_2 = -\frac{r'(\bar{U})}{\alpha\bar{T}}. \tag{8.10}$$

Then the equilibrium $\bar{E}$ is locally asymptotically stable.

Proof: By taking the transformations

$$P = \bar{P} + p,\ Q = \bar{Q} + q,\ T = \bar{T} + \tau,\ U = \bar{U} + u,$$

we first linearize model (8.5). Then we consider the following positive definite function in the linearized form of model (8.5):

$$V(p, q, \tau, u) = \frac{1}{2}\{\frac{p^2}{\bar{P}} + c_1 q^2 + \tau^2 + c_2 u^2\}, \tag{8.11}$$

where c_1 and c_2 are positive constants given by (8.10). It can be checked that the derivative of V with respect to t is negative definite under the conditions (8.7)-(8.9), proving the theorem.

In the following theorem it is shown that the equilibrium $\bar{E}$ is globally asymptotically stable. To prove this theorem, we need the following lemma which establishes a region of attraction for system (8.5). The proof of this lemma is easy and hence is omitted.

Lemma 8.3.1 *The set*

$$\Omega_1 = \{(P, Q, T, U) : 0 \le P \le K_0,\ 0 \le Q + T + U \le \frac{\gamma K_0}{\delta}\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$\gamma_0 > h \text{ and } \delta = min\{\gamma_0 - h,\ h_0(1 - \theta_0),\ \delta_1(1 - \theta_1)\}.$$

Theorem 8.3.2 *In addition to the assumptions (8.3) and (8.4), let $r(U)$ and $K(T)$ satisfy in Ω_1,*

$$0 \le -r'(U) \le \rho,\ K_m \le K(T) \le K_0 \text{ and } 0 \le -K'(T) \le k, \tag{8.12}$$

for some positive constants ρ, K_m and k. Let the following inequalities hold:

$$\{\frac{r_0 K_0 k}{K_m^2} + \frac{\alpha\gamma K_0}{\delta}\}^2 < \frac{4}{9}\frac{r_0}{K(\bar{T})}(h_0 + \alpha\bar{P}), \tag{8.13}$$

$$\{\rho + \frac{\alpha\gamma K_0}{\delta}\}^2 < \frac{2}{3}\delta_1 \frac{r_0}{K(\bar{T})}, \tag{8.14}$$

$$h^2 < \frac{2}{3}c_1\gamma_0(h_0 + \alpha\bar{P}), \tag{8.15}$$

$$(\theta_1\delta_1 + \theta_0 h_0 + \alpha\bar{P})^2 < \frac{2}{3}\delta_1(h_0 + \alpha\bar{P}), \tag{8.16}$$

where

$$c_1 = \frac{1}{3}\frac{r_0\gamma_0}{\gamma^2 K(\bar{T})}.$$

Then $\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the positive orthant.

Proof: Consider the following positive definite function around $\bar{E}$,

$$V_1(P,Q,T,U) = P - \bar{P} - \bar{P}\ln\frac{P}{\bar{P}} + \frac{c_1}{2}(Q-\bar{Q})^2 + \frac{1}{2}(T-\bar{T})^2 + \frac{1}{2}(U-\bar{U})^2. \tag{8.17}$$

Now differentiating V_1 with respect to t along the solutions of (8.5), we get

$$\begin{aligned}\frac{dV_1}{dt} &= -\frac{r_0}{K(\bar{T})}(P-\bar{P})^2 - c_1\gamma_0(Q-\bar{Q})^2 - (h_0+\alpha\bar{P})(T-\bar{T})^2\\ &\quad -\delta_1(U-\bar{U})^2 + c_1\gamma(P-\bar{P})(Q-\bar{Q}) - (r_0 P\xi(T) + \alpha T)(P-\bar{P})(T-\bar{T})\\ &\quad +(\eta(U) + \alpha T)(P-\bar{P})(U-\bar{U}) + h(Q-\bar{Q})(T-\bar{T})\\ &\quad +(\theta_1\delta_1 + \theta_0 h_0 + \alpha\bar{P})(T-\bar{T})(U-\bar{U}),\end{aligned} \tag{8.18}$$

where

$$\eta(U) = \begin{cases} \frac{r(U)-r(\bar{U})}{U-\bar{U}}, & U \neq \bar{U} \\ r'(\bar{U}), & U = \bar{U} \end{cases}$$

$$\xi(T) = \begin{cases} (\frac{1}{K(T)} - \frac{1}{K(\bar{T})})/(T-\bar{T}), & T \neq \bar{T} \\ -\frac{K'(\bar{T})}{K^2(\bar{T})}, & T = \bar{T} \end{cases}$$

From (8.12) and the mean value theorem, we note that

$$|\eta(U)| \le \rho \text{ and } |\xi(T)| \le \frac{k}{K_m^2}.$$

Now Eq. (8.18) can be rewritten as the sum of the quadratics

$$\begin{aligned}
\frac{dV_1}{dt} &= -\frac{1}{2}a_{11}(P-\bar{P})^2 + a_{12}(P-\bar{P})(Q-\bar{Q}) - \frac{1}{2}a_{22}(Q-\bar{Q})^2 \\
&\quad -\frac{1}{2}a_{11}(P-\bar{P})^2 + a_{13}(P-\bar{P})(T-\bar{T}) - \frac{1}{2}a_{33}(T-\bar{T})^2 \\
&\quad -\frac{1}{2}a_{11}(P-\bar{P})^2 + a_{14}(P-\bar{P})(U-\bar{U}) - \frac{1}{2}a_{44}(U-\bar{U})^2 \\
&\quad -\frac{1}{2}a_{22}(Q-\bar{Q})^2 + a_{23}(Q-\bar{Q})(T-\bar{T}) - \frac{1}{2}a_{33}(T-\bar{T})^2 \\
&\quad -\frac{1}{2}a_{33}(T-\bar{T})^2 + a_{34}(T-\bar{T})(U-\bar{U}) - \frac{1}{2}a_{44}(U-\bar{U})^2,
\end{aligned}$$

where

$$a_{11} = \frac{2}{3}\frac{r_0}{K(\bar{T})},\ a_{22} = c_1\gamma_0,\ a_{33} = \frac{2}{3}(h_0 + \alpha\bar{P}),\ a_{44} = \delta_1,$$

$$a_{12} = c_1\gamma,\ a_{13} = -(r_0 P\xi(T) + \alpha T),\ a_{14} = \eta(U) + \alpha T,$$

$$a_{23} = h,\ a_{34} = \theta_1 h_1 + \theta_0 h_0 + \alpha\bar{P}.$$

Sufficient conditions for $\frac{dV_1}{dt}$ to be negative definite are that the following conditions hold:

$$a_{12}^2 < a_{11}a_{22}, \tag{8.19}$$

$$a_{13}^2 < a_{11}a_{33}, \tag{8.20}$$

$$a_{14}^2 < a_{11}a_{44}, \tag{8.21}$$

$$a_{23}^2 < a_{22}a_{33}, \tag{8.22}$$

$$a_{34}^2 < a_{33}a_{44}. \tag{8.23}$$

We note that inequality (8.19) is satisfied automatically. We also note that (8.13) $\Rightarrow$ (8.20), (8.14) $\Rightarrow$ (8.21), (8.15) $\Rightarrow$ (8.22) and (8.16) $\Rightarrow$ (8.23). Hence V_1 is a Liapunov function with respect to $\bar{E}$, whose domain contains the region of attraction Ω_1, proving the theorem.

The above theorem implies that the population living in a polluted environment caused by its own pollutant attains an equilibrium level under certain conditions, and the equilibrium level of the precursor pollutant is crucial in affecting the equilibrium level of population which decreases as the equilibrium level of precursor pollutant increases.

8.4 Model With Diffusion

In this section we consider the complete model (8.1)-(8.2) and state the main results in the form of the following theorem.

Theorem 8.4.1 *(i) If the equilibrium $\bar{E}$ of model (8.5) is globally asymptotically stable, then the corresponding uniform steady state of the initial-boundary value problems (8.1)-(8.2) is also globally asymptotically stable.*

(ii) If the equilibrium $\bar{E}$ of model (8.5) is unstable even then the uniform steady state of the initial-boundary value problems (8.1)-(8.2) can be made stable by increasing diffusion coefficients to sufficiently large values.

Proof: Let us consider the following positive definite function

$$V_2(P(t), Q(t), T(t), U(t)) = \int\int_D V_1(P, Q, T, U)dA$$

where V_1 is given by Eq. (8.17).

We have,

$$\begin{aligned} \frac{dV_2}{dt} &= \int\int_D \left(\frac{\partial V_1}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial V_1}{\partial Q}\frac{\partial Q}{\partial t} + \frac{\partial V_1}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial V_1}{\partial U}\frac{\partial U}{\partial t}\right)dA \\ &= I_1 + I_2. \end{aligned} \tag{8.24}$$

where

$$I_1 = \int\int_D \frac{dV_1}{dt}dA \text{ and } I_2 = \int\int_D \left(D_1\frac{\partial V_1}{\partial P}\nabla^2 P + D_2\frac{\partial V_1}{\partial T}\nabla^2 T\right)dA.$$

We note the following properties of V_1, namely,

$$\left.\frac{\partial V_1}{\partial P}\right]_{\partial D} = \left.\frac{\partial V_1}{\partial T}\right]_{\partial D} = 0$$

and for all points of D,

$$\frac{\partial^2 V_1}{\partial P \partial Q} = \frac{\partial^2 V_1}{\partial P \partial T} = \frac{\partial^2 V_1}{\partial P \partial U} = \frac{\partial^2 V_1}{\partial Q \partial T} = \frac{\partial^2 V_1}{\partial Q \partial U} = \frac{\partial^2 V_1}{\partial T \partial U} = 0,$$
$$\frac{\partial^2 V_1}{\partial P^2} > 0, \ \frac{\partial^2 V_1}{\partial Q^2} > 0, \ \frac{\partial^2 V_1}{\partial T^2} > 0, \ and \ \frac{\partial^2 V_1}{\partial U^2} > 0.$$

Under an analysis similar to chapter 2, we note that

$$\int\int_D \frac{\partial V_1}{\partial P} \nabla^2 P dA = -\int\int_D \frac{\partial^2 V_1}{\partial P^2} \{(\frac{\partial P}{\partial x})^2 + (\frac{\partial P}{\partial y})^2\} dA \leq 0, \tag{8.25}$$
$$\int\int_D \frac{\partial V_1}{\partial T} \nabla^2 T dA = -\int\int_D \frac{\partial^2 V_1}{\partial T^2} \{(\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2\} dA \leq 0. \tag{8.26}$$

This shows that

$$I_2 \leq 0. \tag{8.27}$$

Thus we note that if $I_1 \leq 0$, i.e., if $\bar{E}$ is globally asymptotically stable in the absence of diffusion, then the uniform steady state of the initial-boundary value problems (8.1)-(8.2) also must be globally asymptotically stable. This proves the first part of Theorem 8.4.1.

We further note that if $\frac{dV_1}{dt} > 0$, i.e., if $I_1 > 0$, then $\bar{E}$ may become unstable in the absence of diffusion. But, Eqs. (8.24) and (8.27) show that by increasing diffusion coefficients D_1 and D_2 to sufficiently large values, $\frac{dV_2}{dt}$ can be made negative even if $I_1 > 0$. This proves the second part of Theorem 8.4.1.

The above theorem implies that diffusion with reservoir boundary conditions may stabilize a system which is otherwise unstable.

We shall explain the above theorem for a rectangular habitat D defined by

$$D = \{(x, y) : \ 0 \leq x \leq a, \ 0 \leq y \leq b\} \tag{8.28}$$

in the form of the following theorem.

Theorem 8.4.2 *In addition to assumptions (8.3) and (8.4), let $r(U)$, $K(T)$ satisfy the inequalities in (8.12). If the following inequalities hold:*

$$\begin{aligned}
\{\frac{r_0K_0k}{K_m^2}+\frac{\alpha\gamma K_0}{\delta}\}^2 &< \frac{4}{9}\{\frac{r_0}{K(\bar{T})}+\frac{D_1\bar{P}\pi^2(a^2+b^2)}{K_0^2a^2b^2}\}\times \\
&\quad \{h_0+\alpha\bar{P}+\frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\}, \qquad (8.29)\\
\{\rho+\frac{\alpha\gamma K_0}{\delta}\}^2 &< \frac{2}{3}\delta_1\{\frac{r_0}{K(\bar{T})}+\frac{D_1\bar{P}\pi^2(a^2+b^2)}{K_0^2a^2b^2}\}, \qquad (8.30)\\
h^2 &< \frac{2}{3}c_1\gamma_0\{h_0+\alpha\bar{P}+\frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\}, \qquad (8.31)\\
\{\theta_0h_0+\theta_1\delta_1+\alpha\bar{P}\}^2 &< \frac{2}{3}\delta_1\{h_0+\alpha\bar{P}+\frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\}, \qquad (8.32)
\end{aligned}$$

where

$$c_1=\frac{\gamma_0}{3\gamma^2}\{\frac{r_0}{K(\bar{T})}+\frac{D_1\bar{P}\pi^2(a^2+b^2)}{K_0^2a^2b^2}\}. \qquad (8.33)$$

Then the uniform steady state of the initial-boundary value problems (8.1)-(8.2) is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Let us consider the rectangular region D given by Eq. (8.28). In this case I_2, which is defined in Theorem 8.4.1, can be written as

$$I_2=-D_1\int\int_D(\frac{\partial^2V_1}{\partial P^2})\{(\frac{\partial P}{\partial x})^2+(\frac{\partial P}{\partial y})^2\}\,dA-D_2\int\int_D(\frac{\partial^2V_1}{\partial T^2})\{(\frac{\partial T}{\partial x})^2+(\frac{\partial T}{\partial y})^2\}. \quad (8.34)$$

From Eq. (8.17) we get

$$\frac{\partial^2V_1}{\partial P^2}=\frac{\bar{P}}{P^2}\ and\ \frac{\partial^2V_1}{\partial T^2}=1.$$

Hence

$$I_2\le-\frac{D_1\bar{P}}{K_0^2}\int\int_D\{(\frac{\partial P}{\partial x})^2+(\frac{\partial P}{\partial y})^2\}\,dA-D_2\int\int_D\{(\frac{\partial T}{\partial x})^2+(\frac{\partial T}{\partial y})^2\}\,dA.$$

Now

$$\begin{aligned}
\int\int_D(\frac{\partial P}{\partial x})^2\,dA &= \int\int_D\{\frac{\partial(P-\bar{P})}{\partial x}\}^2\,dA\\
&= \int_0^b\int_0^a\{\frac{\partial(P-\bar{P})}{\partial x}\}^2\,dxdy
\end{aligned}$$

Letting $z = \frac{x}{a}$, it can be seen under an analysis similar to chapter II that

$$\int\int_D (\frac{\partial P}{\partial x})^2 \, dA \geq \frac{\pi^2}{a^2} \int\int_D (P - \bar{P})^2 \, dA$$

and

$$\int\int_D (\frac{\partial P}{\partial y})^2 \, dA \geq \frac{\pi^2}{b^2} \int\int_D (P - \bar{P})^2 \, dA$$

Thus,

$$I_2 \leq -\frac{D_1 \bar{P} \pi^2 (a^2 + b^2)}{K_0^2 a^2 b^2} \int\int_D (P - \bar{P})^2 dA - \frac{D_2 \pi^2 (a^2 + b^2)}{a^2 b^2} \int\int_D (T - \bar{T})^2 \, dA.$$

Now from (8.18) and (8.24) we get

$$\begin{aligned}
\frac{dV_2}{dt} \leq & \int\int_D [-\{\frac{r_0}{K(\bar{T})} + \frac{D_1 \bar{P} \pi^2 (a^2 + b^2)}{K_0^2 a^2 b^2}\}(P - \bar{N})^2 - c_1 \gamma_0 (Q - \bar{Q})^2 \\
& -\{h_0 + \alpha \bar{B} + \frac{D_2 \pi^2 (a^2 + b^2)}{a^2 b^2}\}(T - \bar{T})^2 - \delta_1 (U - \bar{U})^2 \\
& + c_1 \gamma (P - \bar{P})(Q - \bar{Q}) - \{r_0 P \xi(T) + \alpha T\}(P - \bar{P})(T - \bar{T}) \\
& + \{\eta(U) + \alpha T\}(P - \bar{P})(U - \bar{U}) \\
& + \{\theta_0 h_0 + \theta_1 \delta_1 + \alpha \bar{B}\}(T - \bar{T})(U - \bar{U})] dA,
\end{aligned}$$

where $\xi(T)$ and $\eta(U)$ are defined in Eq. (8.18).

Now Eq. (8.35) can be written as the sum of the quadratics

$$\begin{aligned}
\frac{dV_2}{dt} \leq & \int\int_D \{-\frac{1}{2} b_{11} (P - \bar{P})^2 + b_{12} (P - \bar{P})(Q - \bar{Q}) - \frac{1}{2} b_{22} (Q - \bar{Q})^2 \\
& -\frac{1}{2} b_{11} (P - \bar{P})^2 + b_{13} (P - \bar{P})(T - \bar{T}) - \frac{1}{2} b_{33} (T - \bar{T})^2 \\
& -\frac{1}{2} b_{11} (P - \bar{P})^2 + b_{14} (P - \bar{P})(U - \bar{U}) - \frac{1}{2} b_{44} (U - \bar{U})^2 \\
& -\frac{1}{2} b_{22} (Q - \bar{Q})^2 + b_{23} (Q - \bar{Q})(T - \bar{T}) - \frac{1}{2} b_{33} (T - \bar{T})^2 \\
& -\frac{1}{2} b_{33} (T - \bar{T})^2 + b_{34} (T - \bar{T})(U - \bar{U}) - \frac{1}{2} b_{44} (U - \bar{U})^2\} dA
\end{aligned}$$

where

$$b_{11} = \frac{2}{3}\{\frac{r_0}{K(\bar{T})} + \frac{D_1 \bar{P} \pi^2 (a^2 + b^2)}{K_0^2 a^2 b^2}\}, b_{22} = c_1 \gamma_0,$$

$$b_{33} = \frac{2}{3}(h_0 + \alpha\bar{P} + \frac{D_2\pi^2(a^2+b^2)}{a^2b^2}\}, \ b_{44} = \delta_1,$$

$$b_{12} = c_1\gamma, \ b_{13} = -(r_0P\xi(T) + \alpha T),$$

$$b_{14} = \eta(U) + \alpha T, \ b_{23} = h, \ b_{34} = \theta_0 h_0 + \theta_1\delta_1 + \alpha\bar{P}.$$

Sufficient conditions for $\frac{dV_2}{dt}$ to be negative definite are that the following conditions hold:

$$b_{12}^2 < b_{11}b_{22}, \tag{8.35}$$

$$b_{13}^2 < b_{11}b_{33}, \tag{8.36}$$

$$b_{14}^2 < b_{11}b_{44}, \tag{8.37}$$

$$b_{23}^2 < b_{22}b_{33}, \tag{8.38}$$

$$b_{34}^2 < b_{33}b_{44}. \tag{8.39}$$

We note that inequality (8.35) is automatically satisfied for the value of c_1 given in (8.33). We further note (8.29) $\Rightarrow$ (8.36), (8.30) $\Rightarrow$ (8.37), (8.31) $\Rightarrow$ (8.38) and (8.32) $\Rightarrow$ (8.39). Hence V_2 is a Liapunov function with respect to $\bar{E}$ whose domain contains the region of attraction Ω_1, proving the theorem.

From the above theorem we note that inequalities (8.29)-(8.32) may be satisfied by increasing D_1 and D_2 to sufficiently large values. This implies that in the case of diffusion stability is more plausible than the case of no diffusion. Thus, in the case of diffusion the population converges towards its carrying capacity faster than the case of no diffusion, and hence the survival of the population may be ensured.

8.5 Conservation Model

In the previous section it has been noted that uncontrolled human activities that are polluting the environment may harm itself considerably. Therefore, some kind of efforts must be adopted to stop further deterioration of the environment. In this section a mathematical model is proposed and analysed to control the undesired level of precursor pollutant by some mechanisms. It is assumed that the effort applied to control the precursor pollutant is proportional to the undesired level of the precursor pollutant. Then the dynamics of the system is assumed to be governed by the system of differential equations given below.

$$\begin{aligned}
\frac{\partial P}{\partial t} &= r(U)P - \frac{r_0 P^2}{K(T)} + D_1 \nabla^2 P, \\
\frac{\partial Q}{\partial t} &= \gamma P - \gamma_0 Q - r_1 F, \\
\frac{\partial T}{\partial t} &= hQ - h_0 T + \theta_1 \delta_1 U - \alpha PT + D_2 \nabla^2 T, \\
\frac{\partial U}{\partial t} &= -\delta_1 U + \theta_0 h_0 T + \alpha PT, \\
\frac{\partial F}{\partial t} &= \mu_1 (Q - Q_c) H(Q - Q_c) - \nu_1 F, \\
0 &\le \theta_0,\ \theta_1 \le 1.
\end{aligned} \tag{8.40}$$

The above model (8.40) is to be analysed with following initial and boundary conditions:

$$\begin{aligned}
&P(x,y,0) = \phi(x,y) \ge 0,\ Q(x,y,0) = \psi(x,y) \ge 0, \\
&T(x,y,0) = \xi(x,y) \ge 0,\ U(x,y,0) = \zeta(x,y) \ge 0, \\
&F(x,y,0) = \zeta_1(x,y) \ge 0,\ (x,y) \in D \\
&\frac{\partial P}{\partial n} = \frac{\partial T}{\partial n} = 0,\ (x,y) \in \partial D,\ t \ge 0,
\end{aligned} \tag{8.41}$$

where n is the unit outward normal to ∂D.

In model (8.40), $F(x,y,t)$ is the density of effort applied to control the undesired level of precursor pollutant formed by the population. $r_1 > 0$ is depletion rate coefficient of $Q(x,y,t)$ due to the effort F. μ_1 is the growth rate coefficient of F and ν_1 its natural depreciation rate coefficient. Q_c is the critical level of precursor pollutant which is assumed to be harmless to the population. In the last equation of system (8.40), $H(t)$ denotes the unit step function which takes into account the case for which $Q \le Q_c$.

8.6 Conservation Model Without Diffusion

In this section we take, $D_1 = D_2 = 0$ in model (5.1). Then model (5.1) has only one interior equilibrium, namely, $E^*(P^*, Q^*, T^*, U^*, F^*)$, where P^*, Q^*, T^*, U^* and F^* are the positive solutions of the system of algebraic equations given below.

$$
\begin{aligned}
r_0 P &= r(U)K(T), \\
Q &= \frac{\gamma\nu_1 P + r_1\mu_1 Q_c}{\nu_1\gamma_0 + r_1\mu_1} = f_1(P), (say) \\
T &= \frac{hf_1(P)}{h_0(1-\theta_0\theta_1) + \alpha(1-\theta_1)P} = f_2(P), (say) \\
U &= \frac{1}{\delta_1}(\theta_0 h_0 + \alpha P)f_2(P) = f_3(P), (say) \\
F &= \begin{cases} 0, & Q \le Q_c \\ \frac{\mu_1}{\nu_1}(Q - Q_c), & Q > Q_c \end{cases}
\end{aligned}
$$

As earlier, it is easy to check that E^* exists if the following inequality holds at E^*,

$$r_0 - r'(U)f_3'(P)K(f_2(P)) - K'(T)f_2'(P)r(f_3(P)) > 0. \tag{8.42}$$

In the following theorem it is shown that E^* is locally asymptotically stable. The proof is similar to Theorem 8.3.1 and hence is omitted.

Theorem 8.6.1 *Let the following inequalities hold:*

$$\{\frac{r_0 P^*}{K^2 T^*}K'(T^*) + \alpha T^*\}^2 < \frac{4}{9}\frac{r_0}{K(T^*)}(h_0 + \alpha P^*), \tag{8.43}$$

$$h^2 < \frac{4}{9}c_1\gamma_0(h_0 + \alpha P^*), \tag{8.44}$$

$$\{\theta_1\delta_1 + c_2(\theta_0 h_0 + \alpha P^*)\}^2 < \frac{2}{3}c_2\delta_1(h_0 + \alpha P^*), \tag{8.45}$$

where

$$c_1 = \frac{r_0\gamma_0}{3\gamma^2 K(T^*)} \text{ and } c_2 = -\frac{r'(U^*)}{\alpha T^*}.$$

Then E^* is locally asymptotically stable.

In the following lemma a region of attraction for system (8.40) without diffusion is established. The proof of this lemma is easy and hence is omitted.

Lemma 8.6.1 *The set*

$$\Omega_2 = \{(P,Q,T,U,F) : 0 \le P \le K_0,\ 0 \le Q + T + U \le \frac{\gamma K_0}{\delta}, 0 \le F \le \frac{\mu_1}{\nu_1}\frac{\gamma K_0}{\delta}\}$$

attracts all solutions initiating in the positive orthant, where

$$\gamma_0 > h \text{ and } \delta = min\{\gamma_0 - h,\ h_0(1-\theta_0),\ \delta_1(1-\theta_1)\}.$$

The following theorem gives criteria for global stability of E^*, whose proof is similar to Theorem 8.6.2 and hence is omitted.

Theorem 8.6.2 *In addition to the assumptions (8.3) and (8.4), let $r(U)$ and $K(T)$ satisfy in Ω_2,*

$$0 \le -r'(U) \le \rho^*,\ K_m^* \le K(T) \le K_0 \text{ and } 0 \le -K'(T) \le k^*, \tag{8.46}$$

for some positive constants ρ^*, K_m^* and k^*. Let the following inequalities hold:

$$\{\frac{r_0 K_0 k^*}{K_m^{*2}} + \frac{\alpha\gamma K_0}{\delta}\}^2 < \frac{4}{9}\frac{r_0}{K(T^*)}(h_0 + \alpha P^*), \tag{8.47}$$

$$\{\rho^* + \frac{\alpha\gamma K_0}{\delta}\}^2 < \frac{2}{3}\delta_1\frac{r_0}{K(T^*)}, \tag{8.48}$$

$$h^2 < \frac{4}{9}c_1\gamma_0(h_0 + \alpha P^*), \tag{8.49}$$

$$(\theta_1\delta_1 + \theta_0 h_0 + \alpha P^*)^2 < \frac{2}{3}\delta_1(h_0 + \alpha P^*), \tag{8.50}$$

where

$$c_1 = \frac{r_0\gamma_0}{3\gamma^2 K(T^*)}.$$

Then E^* is globally asymptotically stable with respect to all solutions initiating in the positive orthant.

Theorems 8.6.2 and 8.6.2 show that if suitable efforts are made to control the undesired level of precursor pollutant formed by the activities of populations in the environment, the population density may be maintained at a desired level under certain conditions.

8.7 Conservation Model With Diffusion

We now consider the case when $D_i > 0$ $(i = 1, 2)$ in model (8.40). Under an analysis similar to section 8.4 of this chapter, it can be established that if the interior equilibrium E^* of model (8.40) with no diffusion is globally asymptotically stable, then the corresponding uniform steady state of system (8.40)-(8.41) is also globally asymptotically stable with respect to solutions such that $\phi(x,y) > 0$, $\psi(x,y) > 0$, $\xi(x,y) > 0$, $\zeta(x,y) > 0$, $\zeta_1(x,y) > 0$, $(x,y) \in D$.

Further, it should be noted if the system (8.40) with no diffusion is unstable even then the corresponding uniform steady state of system (8.40)-(8.41) can be made stable by increasing diffusion coefficients to sufficiently large values.

Thus, we conclude that diffusion in our model plays the general role of stabilizing the system.

8.8 Numerical Examples

Example 1 Here we present a numerical example to illustrate the results obtained in section 8.3. We consider the following particular form of the functions in model (8.5).

$$\begin{aligned} r(U) &= r_0 - r_{10}U, \\ K(T) &= K_0 - K_1 T. \end{aligned} \tag{8.51}$$

Now choose the following set of values of the parameters in Eq. (8.51) and in model (8.5).

$$\begin{aligned} &r_0 = 20.0,\ r_{10} = 0.07,\ K_0 = 60.0,\ K_1 = 0.08,\ \gamma = 0.05, \\ &\gamma_0 = 0.04,\ h = 0.30,\ h_0 = 0.20,\ \delta_1 = 7.0,\ \theta_0 = 0.01, \\ &\theta_1 = 0.02,\ \alpha_0 = 0.06. \end{aligned} \tag{8.52}$$

With the above values of the parameters, it can be checked that the condition (8.6) for the existence of the interior equilibrium $\bar{E}$ is satisfied and $\bar{E}$ is given by

$$\bar{P} = 58.88342,\ \bar{Q} = 73.60427,\ \bar{T} = 6.02934,\ \bar{U} = 3.04482. \tag{8.53}$$

It can also be checked that conditions (8.7)-(8.9) in Theorem 8.3.1 are satisfied which shows that $\bar{E}$ is locally asymptotically stable.

By choosing $K_m = 50.0$ in Theorem 8.3.2 it can also be verified that conditions (8.13)-(8.16) are satisfied which shows that $\bar{E}$ is globally asymptotically stable.

Example 2 Now to illustrate the results obtained in section 8.6, we present a numerical example. In model (8.40) without diffusion we consider the same particular form of functions as given in (8.51). Now in addition to the values of parameters given in (8.52), we choose the following values of parameters in model (8.40) with no diffusion:

$$r_1 = 0.09,\ \mu_1 = 12.0,\ \nu_1 = 0.09,\ Q_c = 0.14. \tag{8.54}$$

Then it can be checked that condition (8.42) for the existence of the interior equilibrium E^* is satisfied, and E^* is given by

$$\begin{aligned} &P^* = 59.99146,\ Q^* = 0.38868,\ T^* = 0.03128,\ U^* = 0.01609, \\ &F^* = 33.15734. \end{aligned} \tag{8.55}$$

It can easily be verified that conditions (8.43)-(8.45) in Theorem 8.6.1 are satisfied which shows that E^* is locally asymptotically stable.

Further, by choosing $K_m^* = 50.0$ in Theorem 8.6.2, it can be checked that conditions (8.47)-(8.50) are satisfied. This shows that E^* is globally asymptotically stable.

By comparing equilibrium levels $\bar{E}$ and E^* in Eqs. (8.53) and (8.55) we note that due to effort F, the equilibrium level of the population has increased whereas equilibrium level of the concentration of precursor pollutant, concentration of pollutant in the environment and in the population have decreased.

8.9 Conclusions

In this chapter, a mathematical model is proposed and analysed to study the effect of a pollutant on a population which is living in a an environment polluted by its own activities. It has been assumed that the pollutant enters into the environment not directly by the population but by a precursor produced by the population itself. It has been further assumed that the larger the population, the faster the precursor is produced, and the larger the precursor, the faster the pollutant is produced. The model has been studied with and without diffusion. In the case of no diffusion it has been shown that population density settles down to its equilibrium level, the magnitude of which depends upon the equilibrium levels of emission and washout rates of pollutant as well as on the rate of precursor formation and its depletion. It has been noted that the rate of precursor formation is crucial in effecting the population. It has further been noted that if the concentration of pollutant increase unabatedly, the survival of the population would be threatened.

The effect of diffusion on the interior equilibrium state of the system has also been investigated. It has been shown that if the positive equilibrium of the system without diffusion is globally asymptotically stable, then the corresponding uniform steady state of the system with diffusion is also globally asymptotically stable. It has further been noted that if the positive equilibrium of the system with no diffusion is unstable, then the corresponding uniform steady state of the system with diffusion can be made stable by increasing diffusion coefficients appropriately. Thus, it has been concluded that the global stability is more plausible in the case of diffusion than the case of no diffusion.

In case of conservation model it has been shown that if the rate of formation of the precursor pollutant is controlled by some external means, its effect on the population can be minimised.

Bibliography

Anderson N. and Arthur A. M., Analytical boundary functions for diffusion problems with Michaelis-Menten kinetics, *Bull. Math. Biol.* 47(1) (1985) pp. 145-153.

Angulo J. and Linares F., Global existence of solutions of a nonlinear dispersive model, *J. Math. Anal. and Appl.* **195(3)** (1995) pp. 797-808.

Anon, *Desertification and its control* (ICAR, New Delhi, 1977a).

Anon (Ed.), *Desertification, its Causes and Consequences* (Pregman Press, Oxford, 1977b).

Armstrong R. A. and McGhehee R., Coexistence of species competing for shared resources, *Theor. Popul. Biol.* 9 (1976) pp. 317-328.

Armstrong R. A. and McGhehee R., Competitive exclusion *Am. Nat.* **115** (1980) pp. 151-170.

Banerjee U. K. and Banerjee S., North Western Himalayas: Impact of human activities on its Ecosystem, *Advances in Forestry Research in India*, XVI (1997) pp. 53-62.

Bergerud A. T., Butler H. E. and Miller D. R., Antipredator tactics of calving caribou: dispersion in mountains, *Cand. J. Zool.* 62 (1984) pp. 1566-1575.

Bergerud A. T. and Page R. E., Displacement and dispersion of parturient caribou at calving as antipredators tactics, *Can. J. Zool.* 65 (1987) pp. 1597-1606.

Biswas M. R. and Biswas A. K. (Eds.), *Desertification Case Studies* (Pergamon Press, Oxford, 1986).

Borsillino A. and Torre V., Limits to growth from Volterra Theory of population, *Kybernetik* **16** (1974) pp. 113-118.

Brauer F., Stability of some population models with Delay, *Math. Biosci.* **33** (1978) pp. 345-358.

Brown L. R., World population growth soil erosion and food security, *Science*, **214** (1981) pp. 1087-1095.

Brown L. R. and Wolf E. C., Soil Erosion: Quiet Crisis in the World Economy, Worldwatch Paper 60, 1984.

Burton T. A., *Volterra Integral and Differential Equations*, (Academic Press, New York, 1983).

Butler G. J., Hsu S. B. and Waltman P., Coexistence of competing predators in a chemostat, *J. Math. Biol.* **17** (1983) pp. 133-151.

Cantrell R. S. and Cosner C., On the steady-state problem for the Volterra-Lotka competition model with diffusion, *Houston J. Math.* **13** (1987) pp. 337-352.

Cantrell R. S. and Cosner C., Diffusive logistic equations with indefinite weights: population models in disrupted environments, *Proc. of the Royal Society of Edinburgh* **112A** (1989) pp. 293-318.

Casten R. G. and Holland C. J., Instability results for reaction-diffusion equations with Neumann boundary conditions, *J. Diff. Eq.* **27** (1978) pp. 266-273.

Caswell H., A simulation study of a time lag population model, *J. Theor. Biol.* **34** (1972) pp. 419-439.

Cavani M. and Avis R., Wave Train Solutions for General Reaction-Diffusion Systems, *DEDS* **3(3)** (1995) pp. 225-234.

Chattopadhyay J., Effect of toxic substances on a two-species competitive system, *Ecol. Model.* 84 (1996) pp. 287-289.

Chattopadhyay J., Sarkar A. K. and Tapaswi P. K., Effect of cross-diffusion on a diffusive prey-predator system: A nonlinear analysis, *J. Biol. Sys.* 4(2) (1996) pp. 159-169.

Chewning W., Migratory effects in predator-prey models, *Math. Biosci.* 23 (1975) pp. 253-262.

Chow P. L. and Tam W. C., Periodic and travelling wave solutions to Volterra-Lotka equations with diffusion, *Bull. Math. Biol.* 38 (1976) pp. 643-658.

Cohen D. S. and Murray J. D., A generalized diffusion model for growth and dispersal in a population, *J. Math. Biol.* 12 (1981) pp. 237-249.

Comins H. N. and Blatt D. W. E., Prey-predator models in spatially heterogeneous environments, *J. Theor. Biol.* 48 (1974) pp. 75-83.

Cosner C. and Laser A. C., Stable coexistence states in the Volterra-Lotka competition model with diffusion, *SIAM J. Appl. Math.* 44 (1984) pp. 1112-1132.

Cushing J. M., Predator Prey interactions with time delays, *J. Math. Biol.* 3 (1976) pp. 369-380.

De Luna J. T. and Hallam T. G., Effects of toxicants on populations: a qualitative approach IV. Resource-Consumer-Toxicant models, *Ecol. Model.* 35 (1987) pp. 249-273.

Crank J., *The Mathematics of Diffusion*, (Clarendon Press, Oxford, 1975).

Das D. C., Soil conservation practices and erosion control in India - A case study, *FAO Soils Bulletin* 33 (1977) pp. 11-50.

Davis D. R., Sulphur dioxide fumigation of soybeans: Effect on yield, *J. Air Pollution Control Assoc.*, 22 (1972) pp. 12-17.

Deininger R. A., *Models for Environmental Pollution Control*, (Ann Arbor Science Publishers Inc., Michigan, 1974).

De Jong G., A model of competition for food. 1. Frequency dependent viabilities, *Am. Nat.* 110 (1976) pp. 1013-1037.

De Luna J. T. and Hallam T. G., Effects of toxicants on populations: a qualitative approach IV. Resource-Consumer-Toxicant models, *Ecol. Model.* 35 (1987) pp. 249-273.

Denn M. M., *Stability of Reaction and Transport Processes* (Prentice-Hall, Englewood Cliffs, N. J., 1972).

Detwyler T. R. (Ed.), *Man's Impact on Environment* (McGraw Hill, New York, 1971).

Driver R. D., Existence theory for a delay-differential system, *Contr. Diff. Eqs.*, 1 (1961) pp. 317-366.

Driver R. D., Existence and stability of solutions of a delay-differential system, *Arch. Rational Mech. Anal.* 10 (1962) pp. 401-426.

Dubey B., Modelling the effect of toxicant on forestry resources, *Ind. J. Pure and Appl. Math.* 28 (1997a) pp. 1-12.

Dubey B., Modelling the depletion and conservation of resources: Effects of two interacting populations, *Ecol. Model.* 101 (1997b) pp. 123-136.

Dubey B., Time delay model for degradation of forestry resources, *Ind. J. Ecol.* 24(1) (1997c) pp. 10-16.

Dubey B. and Das B., Models for the Survival of species Dependent on Resource in Industrial Environments, *J. Math. Anal. and Appl.* 231 (1999) pp. 374-396.

Erbe L. H., Freedman H. I. and Rao V. S. H., Three Species Food-chain Models with a Mutual Interference and Time Delays, *J. Math. Biol.* 80 (1986) pp. 57-80.

Fife P. C., Mathematical aspects of reacting and diffusion-systems, Lecture notes in Biomathematics, 28, Springer-Verlag, Heidelberg, 1979.

Forrester J. W., *World Dynamics* (Cambridge, Mass, 1971).

Freedman H. I., Graphical Stability, Enrichment and Pest Control by a Natural Enemy, *Math. Biosci.* 31 (1976) pp. 207-225.

Freedman H. I., Stability Analysis of a Predator-Prey System with Mutual Interference and Density Dependent Death Rates, *Bull. Math. Biol.* 41 (1979) pp. 67-78.

Freedman H. I., Single species migration in two habitats: Persistence and extinction, *Math. Model.* 8 (1987a) pp. 778-780.

Freedman H. I., *Deterministic mathematical models in population ecology* (HIFR Consulting Ltd., Edmonton, 1987b).

Freedman H. I. and Rao V. S. H., The Trade off Between Mutual Interference and Time Lags in a Predator-Prey Systems, *Bull. Math. Biol.* 45 (1983) pp. 991-1004.

Freedman H. I. and Gopalsamy K., Global stability in time-delayed single-species dynamics, *Bull. Math. Biol.* 48(5/6) (1986) pp. 485-492.

Freedman H. I., Rai B. and Waltman P., Mathematical models of population interactions with dispersal II: Differential survival in a change of habitat, *J. Math. Anal. Appl.* 115(1) (1986).

Freedman H. I. and Shukla J. B., Models for the effect of toxicant in single-species and predator-prey systems, *J. Math. Biol.* 30 (1991) pp. 15-30.

Freedman H. I. and Shukla J. B., The effect of a predator resource on a Diffusive Predator-Prey System, *Nat. Res. Model.* 3(3) (1989) pp. 359-383.

Freedman H. I., Shukla J. B. and Takeuchi Y., Population diffusion in a two patch environment, *Math. Biosci.* 95 (1989) pp. 111-123.

Freedman H. I. and Waltman P., Mathematical models of population interactions with dispersal I: Stability of two habitats with and without a predator, *SIAM J. Appl. Math.* 32(3) (1977) pp. 631-648.

Freedman H. I. and Wu J., Steady-state analysis in a model for population diffusion in multi-patch environment, *Nonlinear analysis, Theory, Methods and Applications* 18(6) (1992) pp. 517-542.

Frevert R. K., Schwab G. O., Edminster T. M. and Barnes K. K., *Soil and Water Conservation Engineering*, (John Wiley Sons, Inc., New York, 1962).

Gadgil M., Social Restraint on Resource Utilization: The Indian Experience, 1985.

Gadgil M., Diversity: Cultural and Biological, *Trend in Ecology and Evolution* 2(12) (1987) pp. 369-373.

Gadgil M. and Chandran M. D.S., Environmental Impact of Forest based Industries on the Evergreen Forests of Uttara Kannada District: A case study, Dept. of Ecology and Environment, Govt. of Karnataka, 1989.

Gadgil M. and Prasad S. N., Vanishing bamboo stocks, *Commerce* 136(3497) (1978) pp. 1-5.

Gadgil M., Prasad S. N. and Ali R., Froest management in India: A critical review, *Social Action*, 33 (1983) pp. 127-155.

Gatto M. and Rinaldi S., Stability analysis of predator-prey models via the Liapunov method, *Bull. Math. Biol.* 39 (1977) pp. 339-347.

Ghosh R. C. and Lohani D. N., Plantation Forestry - Its Implication in Indian Economy, Proc. 7th World for Cong. Buenos Aires, Argentina, 1972.

Gilpin M. E., A Liapunov function for competition communities, *J. Theor. Biol.* 44 (1974) pp. 35-48.

Goh B. S., *Management and Analysis of Biological Populations*, (Elsevier Scientific Publishing Company, New York, 1980).

Goh B. S., Global stability in two species interactions, *J. Math. Biol.* 3 (1976) pp. 313-318.

Goh B. S., Global stability in many species systems, *Am. Nat.* **111** (1977) pp. 135-143.

Goh B. S., Global stability in a class of prey-predator models, *Bull. Math. Biol.* 40 (1978) pp. 525-533.

Gomatam J., A New Model for Interacting Populations 1, *Bull. Math. Biol.* 36 (1974) pp. 347-353.

Gopalsamy K., Competition, dispersion and coexistence, *Math. Biosci.*, **33** (1977) pp. 25-34.

Gopalsamy K., Optimal stabilization and harvesting in logistic population models, *Ecol. Model.*, **11** (1980) pp. 67-69.

Gopalsamy K., Limit cycles in periodically perturbed population systems *Bull. Math. Biol.*, **43**(4) (1981) pp. 463-485.

Gopalsamy K., Convergence in a resource based competition system, *Bull. Math. Biol.*, 48 (1986) pp. 681-699.

Gopalsamy K., *Stability and oscillations in Delay Differential Equations of population Dynamics* (Kluwer Academic Publishers, 1992).

Gopalsamy K. and Aggarwalla B. D., Recurrence in two species competition, *Ecol. Model.* 9 (1980) pp. 153-163.

Gordon A. G. and Gorham E., Ecological aspects of air pollution from iron sintering plant at Wawa, *Cand. J. Bot.* 41 (1963) pp. 1063-1078.

Gurtin M. E. and MacCamy R. C., On the diffusion of biological populations, *Math. Biosci.* 33 (1977) pp. 35-49.

Gurney W. S. C. and Nisbet R. M., The regulation of inhomogeneous populations, *J. Theor. Biol.* 52 (1975) pp. 441-457.

Gyori I. and Ladas G., *Oscillation Theory of Delay Differential Equations with Applications* (Clarendu Press, Oxford, 1991).

Hadeler K. P., van der Heiden U., Rothe F., Nonhomogeneous spatial distributions of populations, *J. Math. Biol.* 1 (1974) pp. 165-176.

Hadeler K. P. and Rothe. F., Travelling fronts in non-linear diffusion equations, *J. Math. Biol.* 2 (1975) pp. 251-264.

Haigh M. J., Deforestation and Disasters in Northern India, Land use policy (1984) pp. 187-198.

Hale J. K., Asymptotic behavior of the solutions of differential-difference equations, *RIAS Tech. Rept.* 61-10, 1961.

Hale J. K., Functional-differential equations with parameters, *Contr. Diff. Eqs.* 1(4) (1962) pp. 401-410.

Hale J. K., Linear functional-differential equations with constant coefficients, *Contr. Diff. Eqs.* 2 (1963) pp. 291-317.

Hale J. K., Averaging methods for differential equations with retarded arguments and small parameters, Tech. Rept. No. 64-1, Center for Dynamic Systems, Div. App. Math., Brown Univ., Providence, R. I., 1964a.

Hale J. K., Periodic and almost periodic solutions of functional-differential equations, *Arch. Rational Mech. Anal.* 15(4) (1964b) pp. 289-304.

Hallam T. G. and Clark C. E., Nonautonomous logistic equations as models of populations in a deteriorating environment, *J. Theor. Biol.* 93 (1982) pp. 303-311.

Hallam T. G., Clark C. E. and Jordan G. S., Effects of toxicants on populations: a qualitative approach II. First order kinetics, *J. Math. Biol.* 18 (1983a) pp. 25-37.

Hallam T. G., Clark C. E. and Lassiter R. R., Effects of toxicants on populations: a qualitative approach I. Equilibrium environmental exposure, *Ecol. Model.* 18 (1983b) pp. 291-304.

Hallam T. G. and De Luna J. T., Effects of toxicants on populations: a qualitative approach III. Environmental and food chain pathways, *J. Theor. Biol.* 109 (1984) pp. 411-429.

Hallam T. G. and Ma Z., Persistence in population models with demographic fluctuations, *J. Math. Biol.* 24 (1986) pp. 327-339.

Hallam T. G., Svoboda L. J. and Gard T. C., Persistence and extinction in three species Lotka-Volterra competitive systems, *Math. Biosci.* 46 (1979) pp. 117-124.

Harada K. and Fukao T., Coexistence of Competing Species Over a Linear Habitat of Finite Length, *Math. Biosci.* 38 (1978) pp. 279-291.

Hari P, Raunemaa T. and Hautojarvi A., The effect on forest growth of air pollution from energy production, *Atom. Environ.* 20(1) (1986) pp. 129-137.

Harrison G. W., Global stability of predator-prey interactions, *J. Math. Biol.* 8 (1979) pp. 159-171.

Hastings A., Global stability in Lotka-Volterra systems with Diffusion, *J. Math. Biol.* 6 (1978a) pp. 163-168.

Hastings A., Global stability of two species systems, *J. Math. Biol.* 5 (1978b) pp. 399-403.

Hastings A., Dynamics of a single species in a spatially varying environment: The stabilizing role of high dispersal rates, *J. Math. Biol.* 16 (1982) pp. 49-55.

Holling C. S., The functional response of predators to prey density and its role in mimicry and population regulation, *Mem. Entomol. Soc. Can.* 45 (1965) pp. 1-60.

Hsu S. B., Limiting behavior of competing species, *SIAM J. Appl. Math.* 34 (1978a) pp. 760-763.

Hsu S. B., On global stability of predator-prey systems, *Math. Biosci.* 39 (1978b) pp. 1-10.

Hsu S. B., On a resource based ecological competition model with interference, *J. Math. Biol.* 12 (1981a) pp. 45-52.

Hsu S. B., Predator mediated coexistence and extinction, *Math. Biosci.* 54 (1981b) pp. 231-248.

Hsu S. B. and Huang T. W., Global stability for a class of predator-prey systems, *SIAM J. Appl. Math.* 55(3) (1995) pp. 763-783.

Hsu S. B. and Hubbell S. P., Two predators competing for two prey species: An analysis of MacArthur's model, *Math. Biosci.* 47 (1979) pp. 143-171.

Huaping L. and Ma Z., The threshold of survival for system of two species in a polluted environment, *J. Math. Biol.* 30 (1991) pp. 49-61.

Huffaker C. B., Experimental studies on predation; dispersion factors and prey predator oscillations, *Hilgardia 27* (1958) pp. 305-329.

Huffaker C. B., Experimental studies on predation; Complex dispersion and levels of food in an Acarsian predator-prey interactions, *Hilgardia 37* (1963) pp. 305-309.

Huttl R. F. and Wisniewski J., Fertilization as a tool to mitigate forest decline associated with nutrient deficiencies, *Water, Air and Soil Pollution* 33 (1987) pp. 265-276.

Jensen A. L., Simple models for exploitative and interference competition, *Ecol. Model.* 35 (1987) pp. 113-121.

Jensen A. L. and Marshall J. S., Application of a surplus production model to assess environmental impacts on exploited populations of Daphina pluex in the laboratory, *Environmental pollution* A28 (1982) pp. 273-280.

Jorne J., The diffusive Lotka-Volterra oscillating systems, *J. Theor. Biol.* 65 (1977) pp. 133-139.

Jorne J. and Carmi S., Liapunov stability of the diffusive Lotka-Volterra equations, *Math. Biosci.* 37 (1977) pp. 51-61.

Karamchandani K. P., Environmental management: Forestry, Reading in Environmental Management edited by Rama D. V., Bharadwaj J. and Vidaken V., UNNAPDI, Bangkok (1980).

Khoshoo T. N., *Environmental Properties in India and Sustainable Development* (ISCA, New Delhi, 1986).

Kormondy E. J., *Concepts of Ecology*, (Third Edition, Prentice Hall of India Pvt. Ltd., New Delhi, 1986).

Krasovskii N. N., On periodical solution of differential equations involving a time lag, *Dokl. Akad. Nauk USSR* 114 (1957) pp. 252-255.

Lakshmikantham V., Lyapunov function and a basic inequality in delay-differential equations, *Arch. Rational Mech. Anal.* 10 (1962) pp. 305-310.

Lakshmikantham V., Functional-differential systems and extensions of Lyapunov's method, *F. Math. Anal. Appl.* 8(3) (1964) pp. 392-405.

Lakshmikantham V. and Rao M. R. M., Stability in Variation for Non-linear Integro-Differential Equations, *Applicable Analysis* 24 (1987) pp. 165-173.

Lamberson R. H., The conservation and maintenance of valuable resources: Optimal expenditure strategies, Second Autumn Course on Mathematical Ecology, ICTP, Trieste, Italy, 1986.

Lancaster P. L. and Tismanetsky M., *The theory of Matrices, Second Edition,* (Academic Press, New York, 1985).

Landahl H. D., A note on the population growth under random dispersal, *Bull. Math. Biophysics* 21 (1959) pp. 153-159.

Larson W. E., Pierce F. J. and Dowdy R. H., The Threat of Soil Erosion to Long-Term Crop Production, *Science* 219 (1983) pp. 458-465.

La Salle J. and Lefschetz S., *Stability by Liapunov's Direct Method with Applications* (Academic Press, New York, London, 1961).

Leung A., Conditions for Global Stability Concerning a Prey-Predator Model with Delay Effects, *SIAM J. Appl. Math.*, 36 (1979) pp. 281-286.

Leung A. W. and Zhou Z., Global stability for large systems of Volterra-Lotka type integro-differential population delay equations, *Nonlinear Anal. Theor. Method and Appl.* 12 (1988) pp. 495-505.

Levins R. and Culver D. Regional coexistence of species and competition between rare species, *Proc. Nat. Acad. Sci. USA* 68 (1971) pp. 1246-1248.

Levin S. A., Dispersion and population interactions, *Am.Nat.* 108 (1974) pp. 207-228.

Levin S. A., Population Dynamic Models in Heterogeneous environments, *Annu. Rev. Ecol. Syst.* 7 (1976) pp. 287-310.

Ma Z. and Hallam T. G., Effects of parameter fluctuations on community survival, *Math. Biosci.* 86 (1987) pp. 35-49.

Martino J. P., *Technological Forecasting for Decision Making,* (American Elsevier, New York, 1983).

Maugh T. H., Restoring damaged lakes, *Science*, 203 (1979) pp.425-427.

May R. M., Stability in multispecies community models, *Math. Biosci.* 12 (1971) pp. 59-79.

May R. M., *Stability and Complexity in Model Ecosystems*, (Princeton, Princeton U. P., 1973).

May R. M., Ecosystem patterns in randomly fluctuating environments. In *Progress in Theoretical Biology*, ed. Rosen R., Snell F., pp. 1-50, Academic P., New York, 1974).

Maynard Smith J., *Mathematical ideas in Biology*, (Cambridge University Press, 1968).

Maynard Smith J., *Models in Ecology* (Cambridge, Cambridge U. P., 1974).

MacDonald N., Time delay in prey-predator models, *Math. Biosci.* 28 (1976) pp. 321-330.

MacDonald N., Time Delay in Prey-Predator Models, *Math. Biosci.* 33 (1977) pp. 227-234.

MacDonald N., *Time lags in Biological Models, Lecture Notes in Biomathematics Vol. 27* (Springer-Verlag, Berlin, 1978).

McLaughlin S. B., Effects of air pollution on forests, *J. Air Pollution Control Assoc.* 35 (1985) pp. 512-534.

McMurtrie R., Persistence and stability of single species and prey-predator systems in spatially heterogenous environments, *Math. Biosci.* 39 (1978) pp. 11-51.

Meadows D. H., *The Limits to Growth*, New York, 1972.

Miller, R. K., On Volterra's Population Equations, *SIAM J. Appl. Math.*, 14(3) (1966) pp. 446-452.

Miller R. S., Pattern and process in competition, *Adv. in Ecol. Res.* 4 (1976) pp. 1-74.

Mitra D., Mukherjee D., Roy A. B. and Ray S., Permanent coexistence in a resource-based competition system, *Ecol. Model.* 60 (1992) pp. 77-85.

Mukherjee D. and Roy A. B., Uniform persistence and global stability of two prey-predator pairs linked by competition, *Math. Biosci.* 99 (1990) pp. 31-45.

Munn R. E. and Fedorov V., *The environmental assessment, IIASA, Project Report, vol. 1*, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1986.

Murakami S. and Hamaya Y., Global Attractivity in an Integro Differential Equation with Diffusion, *DEDS* 3(1) (1995) pp. 35-42.

Nallaswamy R. and Shukla J. B., Effects of convective and dispersive migration on the linear stability of a two species system with mutualistic interactions and functional response, *Bull. Math. Biol.* 44(2 (1982a) pp. 271-282.

Nallaswamy R. and Shukla J. B., Effects of dispersal on the stability of a prey-predator system with functional response, *Math. Biosci.* 60 (1982b) pp. 123-132.

Nelson S. A., The problem of oil pollution of the sea, In: *Advances in Marine Biology* pp. 215-306 (Academic Press, London, 1970).

Okubo A., *Diffusion and ecological problem: Mathematical models*, (Springer-Verlag, Berlin Heidelberg, New York, 1980).

Pasquill F., *Atmospheric Diffusion* (Van Nostrand, N. J., Princeton, 1962).

Pathak S., Role of forests in soil conservation with special reference to Ramganga watershed, *Soil Conservation Digest* 2(1) (1974) pp. 44-47.

Parry M. L. and Carter T. R., The assessment of effect of climate variation on agriculture: Aims, methods and summary In: M. L. Parry, T. R. Carter and N. T. Konijn (editors), *The impact of climate variation on agriculture*, (Kluwer Academic Publisher Dordrecht, Netherlands, 1988).

Peterson C. E., Rayon P. J. and Gassel S. P., Response of northwest Douglas-fir stands to urea: Correlations with forest soil properties, *Soil Sci. Soc. of America J.* **48** (1984) pp. 162-169.

Pimental D. et al., Land degradation: Effects on food and energy resources, *Science*, **194** (1976) pp. 149-155.

Patin S. A., *Pollution and the biological resource of the Ocean* (Butterworth Scientific, London, 1982).

Putnam A. R. and Chung-Shih Tang (Eds.), *The Science of Allelopathy*, 1986.

Rao M. R. M., *Ordinary Differential Equations-Theory and Application* (1981), Edward Arnold (Publishers) Ltd., London (U. K.).

Rao M. R. M and Pal V. N., Asymptotic Stability of Grazing Systems with Unbounded Delay, *J. Math. Anal. and Appl.* **163(1)** (1992) pp. 60-72.

Rao M. N. and Rao H. V. N., *Air Pollution*, (Tata McGraw Hill Publishing Co. Ltd., New Delhi, 1989).

Rao M. R. M. and Sivasundaram S., Asymptotic Stability for Equations with Unbounded Delay, *J. Math. Analysis and Applications* **131(11)** (1988) pp. 97-105.

Raichaudhury S. Sinha D. K. and Chattopadhyay J., Effect of time-varying cross-diffusivity in a two-species Lotka-Volterra competitive system, *Ecol. Model.* **92** (1996) pp. 55-64.

Reed W. J. and Heras H. E., The conservation and exploitation of vulnerable resources, *Bull. Math. Biol.* **54(2/3)** (1992) pp. 185-207.

Reish D. J., Gill G. G., Frank G. W., Phillips S. O., Alan J. M., Steven S. R. and Thomas C. G., Marine and estuarine pollution, *Journal WPCF*, 54 (1982) pp. 786-812.

Reish D. J., Gill G. G., Frank G. W., Phillips S. O., Alan J. M., Steven S. R. and Thomas C. G., Marine and estuarine pollution, *Journal WPCF*, 55 (1983) pp. 767-787.

Repetto R. and Holmes T., The Role of Population in Resource Depletion in Developing Countries, *Population and Development Review* 9 (1983) pp. 609-632.

Rescigno A., The Struggle For Life-II. Three Competitors, *Bull. Math. Biophy.* 30 (1968) pp. 291-298.

Rescigno A., The Struggle For Life-V. One species living in a limited environment, *Bull. Math. Biol.* 39 (1977) pp. 479-485.

Rescigno A. and Richardson I. W., The Struggle for Life-I. Two Species, *Bull. Math. Biophys.* 29 (1967) pp. 377-388.

Rice E. L., *Allelopathy*, 2nd ed., 1984.

Roff D. A., The analysis of a population model demonstrating the importance of dispersal in a heterogeneous environment, *Oecologia* 15 (1974) pp. 259-275.

Rosen G., Global Theorems for species distributions governed by reaction-diffusion equations, *J. Chem. Phys.* 61(9) (1974) pp. 3676-3679.

Rosen G., Solutions to systems of nonlinear reaction-diffusion equations, *Bull. Math. Biol.* 37 (1975) pp. 277-289.

Rosen R., *Dynamic system theory in biology, Vol. 1* (Wiley-Interscience, New York, 1970).

Rosen G., Effects of diffusion on the stability of the equilibrium in multi-species ecological systems, *Bull. Math. Biol.* 39 (1977) pp. 373-383.

Sahani D. V., Air Pollution, *Advances in Forestry Research in India*, XVIII (1998) pp. 54-63.

Sanchetz D. A., *Ordinary Differential Equations and Stability Theory: An Introduction*, (Freeman, San Francisco, 1968).

Scorer R., *Air Pollution*, (Pergamon Press, Oxford, 1968).

Schulze E. D., Air pollution and forest decline in a Spruce (Picea abies) forest, *Science* 224 (1989) pp. 776-783.

Segel L. A. and Jackson J. L., Dissipative structure: an explanation and an ecological example, *J. Theor. Biol.* 37 (1972) pp. 545-559.

Shukla J. B. and Dubey B., Simultaneous effect of two toxicants on biological species: a mathematical model, *J. Biol. Syst.*4(1) (1996a) pp. 109-130.

Shukla J. B. and Dubey B., Effect of changing habitat on species: Application to Keoladeo National Park, India, *Ecol. Model.* 86 (1996b) pp. 91-99.

Shukla J. B., Dubey, B. and Freedman H. I., Effect of changing habitat on survival of species, *Ecol. Model.* 87(1-3) (1996) pp. 205-216.

Shukla J. B. and Dubey B., Modelling the depletion and conservation of forestry resources: effects of population and pollution, *J. Math. Biol.* 36 (1997) pp. 71-94.

Shukla J. B., Hallam T. G. and Capasso V (Eds.), *Mathematical Modelling of Environmental and Ecological Systems* (Elsevier Sci. Publi. Amsterdam, 1987).

Shukla J. B., Pal. V. N., Mishra O. P., Agarwal M. and Shukla A., Effects of Population and Industrialization on the Degradation of Biomass and its Regeneration by Afforestation: A Mathematical Model, *Journal of Biomath*, 3(1) (1988) pp. 1-9.

Shukla J. B., Freedman H. I., Pal V. N., Misra O. P., Agarwal M. and Shukla A., Degradation and subsequent regeneration of forestry resource: A mathematical model, *Ecol. Model.* 44 (1989) pp. 219-229.

Shukla V. P. and Shukla J. B., Multispecies food webs with diffusion, *J. Math. Biol.* **13** (1982) pp. 339-344.

Shukla V. P., Shukla J. B. and Das P. C., Environmental effects on the linear stability of a three species food chain model, *Math. Biosci.* **57** (1981) pp. 35-58.

Shukla J. B. and Verma S., Effects of convective and dispersive interactions on the stability of two species, *Bull. Math. Biol.* **43(5)** (1981) pp. 593-610.

Singh K. D., Forest Resource Assessment 1990, Tropical Countries, Food and Agriculture Organization, the United Nations, Rome, *FAO Forestry Paper No.* 112, 1993.

Skellam J. G., Random dispersal in theoretical populations, *Biometrika* **38** (1951) pp. 196-218.

Smith R. L., *The Ecology of Man: An Ecosystem Approach* (Harper and Row, New York, 1972).

Smith W. H., *Air Pollution and Forests*, (Springer-Verlag, New York, 1981).

Stepan G., Great Delay in predator-prey model, *Nonlinear Anal. Theory Method and Appl.* **10(9)** (1986) pp. 913-929.

Stern A. C., *Air Pollution*, (Academic Press, New York, 1968).

Sutton O. G., *Micrometeorology*, (McGraw-Hill, New York, 1953).

Takeuchi Y., Global stability in generalized Lotka-Volterra diffusion systems, *J. Math. Anal. Appl.* **116(1)** (1986a).

Takeuchi Y., Diffusion effect on stability of Lotka-Volterra models, *Bull. Math. Biol.* **48(5/6)** (1986b) pp. 585-601.

Takeuchi Y., Adachi N. and Tokumaru H., The stability of generalized Volterra equations, *J. Math. Anal. Appl.* **62** (1978) pp. 453-473.

Tansky M., Switching effect in prey-predator systems, *J. Theor. Biol.* 70 (1978) pp. 263-271.

Timm U. and Okubo A., Diffusion-driven instability in a predator-prey system with time varying diffusivities, *J. Math. Biol.* 30 (1992) pp. 307-320.

Thompson A. C. (Ed.), *The chemistry of Allelopathy: Biochemical Interactions among Plants*, ACS Symp. Ser. 268, 1985.

Treshow M., The impact of air pollutants on plant populations, *Phytopathology*, 58 (1968) pp. 1108-1113.

Vandermeer J. H., On the regional stabilization of locally unstable predator-prey relationships, *J. Theor. Biol.* 41 (1973) pp. 161-170.

Veeman T. S., Land degradation, Conservation reserves and Rationalizing excess Agricultural production: an economic prospective, Preprint for Agricultural Institute of Canada National Conference, August 21-24, 1988, Calgary Canada.

Verma S., Effects of convective and dispersive migration on stability of interactive species system in heterogeneous habitats, *Ph. D. thesis*, Department of Mathematics, Indian Institute of Technology, Kanpur, India (1980).

Volterra V., *Theory of Functionals and Integral and Integro-Differential Equations*, (Dover, New York, 1959).

Waller G. R. (Ed.), *Allelochemicals: Role in Agriculture and Forestry*, ACS Symp. Ser. 330, 1987.

Wang L. and Zhan Yi, Global Stability of Volterra-Lotka Systems with Delay, *DEDS* 3(2) (1995) pp. 205-216.

Wangersky P. J. and Cunningham W. J., Time lag in prey-predator population models, *Ecology* 38 (1957) pp. 136-139.

Waring R. H. and Schiessinger W. H., *Froest Ecosystems: Concepts and Management* (Academic Press, New York, 1985).

Wilson E. O., Threats of biodiversity, In the Scientific American Special Issue (Managing Planet Earth), September, pp. 108-117, New York, 1989.

Wollkind D. J., Collins J. B. and Barba M. C. B., Diffusive instabilities in one-dimensional temperature-dependent model system for a mite predator-prey interaction on fruit trees: Dispersal mobility and aggregative preytaxis effects, *J. Math. Biol.* **29** pp. 339-362.

Woodman J. N. and Cowling E. B., Airborne chemical and forest health, *Environ. Sci. Technol.* **21** (1987) pp. 120-126.

Woodwell G. M., Effects of pollution on the structure and physiology of ecosystems, *Science* **168**(3930) (1970) pp. 429-431.