# SOME ASPECTS OF DISCRETE PROBABILITY DISTRIBUTIONS

A Thesis submitted
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by

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*Dedicated to the enduring memory of my beloved father...*

# CERTIFICATE

This is to certify that the thesis entitled "Some aspects of discrete probability distributions" submitted by Mr. Subrata Chakraborty to the Tezpur University, Tezpur, is a record of bonafide research work carried out by him under my guidance and supervision in the Department of Mathematical Sciences, Tezpur University, Tezpur. This thesis, is in my opinion, worthy for the degree of Doctor of philosophy in accordance with the regulation of this University. The results presented in this thesis have not been submitted to any other University or Institution for the award of any other degree or diploma.

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# Preface

One of the important direction of research in the theory of discrete probability distribution is to develop/obtain large class/family of probability distributions and investigate their various properties, problem of estimation, data fitting, etc. This thesis deals with the study of various distributional properties, estimation of parameters, and fitting of real life data to some new classes of discrete probability distributions. The thesis has been organised into six chapters as described below.

In chapter one an introduction of the work is presented. Chapter two deals with various properties, estimation and data fitting problems of a class of weighted quasi binomial distributions. Chapter three describes two discrete probability models developed using urn models with different predetermined strategies. Some of their important properties are studied. Chapter four deals with a class of $\alpha$-modified binomial and some related distributions. In chapter five various properties of class of weighted generalized Poisson distributions are presented. In the sixth chapter a class of generalized multivariate generalized Poisson distributions is proposed, some properties of new distributions are studied. Finally, some important results on the Abel's generalizations of the binomial identities and exponential sums related to our works are presented in two appendices.

Subrata Chakraborty

## LIST OF TABLES

# Contents

# Glossary

$a^{(k)} \quad = a(a-1)\cdots(a-\overline{k-1})$

$a^{[k]} \quad = a(a+1)\cdots(a+\overline{k-1})$

$a^{(k,s)} \quad = a(a-s)\cdots(a-\overline{k-1}s)$

$a^{[k,s]} \quad = a(a+s)\cdots(a+\overline{k-1}s)$

$\binom{a}{k}_s \quad = \frac{a^{(k,s)}}{k!}$

$\left\langle{a \atop k}\right\rangle_s \quad = \frac{a^{[k,s]}}{k!}$

$\binom{a}{k} \quad = \frac{a^{(k)}}{k!}$

$\left\langle{a \atop k}\right\rangle \quad = \frac{a^{[k]}}{k!}$

$\binom{n}{\boldsymbol{k}} \quad = \frac{n!}{k_1!,\ldots,k_p!}$, where $\boldsymbol{k} = (k_1, \ldots, k_p)$ and $\sum_{i=1}^{p} k_i = n$

$S(n,i)$ Stirling number of the second kind

$S_n^{(i)}$ Stirling number of the first kind

$\Gamma(x)$ gamma function $= \int_0^\infty e^{-z} z^{n-1} dz, \;\; n > 0$

$| D |$ determinant of the matrix $D$

$D^{-1}$ Inverse of the matrix $D$, $| D | \neq 0$

$\Delta f(x) \quad = f(x+1) - f(x)$

$\Delta^n f(x) \quad = \Delta[\Delta^{n-1} f(x)], \;\; n = 2, 3, \ldots$

$\Delta^n 0^i \quad = \Delta^n x^i \,|_{x=0}$

$f_1(x;\theta) \underset{\theta}{\wedge} f_2(\theta;\beta)$ distribution of $f_1(x;\theta)$ compounded (by $\theta$) with distribution $f_2(\theta;\beta)$

$f_1(.) \vee f_2(.)$ generalized $f_1(.)$ distribution generalized by $f_2(.)$

$D_k \quad = (-1+\alpha)^k$

$\alpha^k \quad \equiv \alpha_k = k!$

$\alpha^k(j) \quad \equiv \alpha_k(j) = (\underbrace{\alpha + \cdots + \alpha}_{j\ terms})^k = \binom{k+j-1}{k}\ k!$

$\beta'_k(x) \quad \equiv \beta'^k(x) = k!\ (x+kz)$

$\gamma'_k(x) \quad \equiv \gamma'^k(x) = (kz)\ k!\ (x+kz)$

$\psi'_k(x) \quad \equiv \psi'^k(x) = (kz)\ (kz)\ k!\ (x+kz)$

$\beta'^k(x;2) \quad = [\beta'(x) + \beta'(x)]^k$

$I_p(x, n-x+1) \quad = \sum_{j=x}^{n} \binom{n}{j} p^j (1-p)^{n-j}$

$I_d(x) \quad = \left(\frac{x}{2}\right)^d \sum_{j=0}^{\infty} \frac{(\frac{1}{4}x^2)^j}{j!\Gamma(d+j+1)}$

$\delta_{li} \quad = \begin{cases} 0, & \text{if } i = l \\ 1, & \text{if } i \neq l \end{cases}$

$K(a;s;z) \quad = \sum_{i \geq 0} \frac{1}{i!} e^{-iz} (a+iz)^{i+s}$

$B_n(p,q;s,t;\phi) \quad = \sum_{k=0}^{n} \binom{n}{k} (p+k\phi)^{k+s} (q+\overline{n-k}\phi)^{n-k+t}$

$B_n(p_1,\ldots,p_m;s_1,\ldots,s_m,;\phi) \quad = \sum_{\boldsymbol{k}} \binom{n}{\boldsymbol{k}} \prod_{i=1}^m (p+k\phi)^{k_i+s_i}$, where $\sum_{i=1}^{p} k_i = n$

$M(b_0,b_1,\ldots,b_n;s_0,s_1,\ldots,s_n;z_0,z_1,\ldots,z_n) \quad = \sum_{x_1 \geq 0} \cdots \sum_{x_n \geq 0} \sum_{j \geq 0}^{\min\{x_1,..,x_n\}} (b_0+jz_0)^{j+s_0} \frac{e^{-jz_0}}{j!} \prod_{i=1}^n \left[ \frac{(b_i+\overline{x_i-j}z_i)^{x_i-j+s_i} e^{-\overline{x_i-j}z_i}}{(x_i-j)!} \right]$

$B'_{x_1}(b_0,b_1;s_0,s_1;z_0,z_1) \quad = \sum_{j=0}^{x_1} \binom{x_1}{j} (b_0+jz_0)^{j+s_0} (b_1+\overline{x_1-j}z_1)^{x_1-j+s_1} e^{-\overline{z_0-z_1 j}}$

$b(x;n,p) \quad = \binom{n}{x} p^x q^{n-x}$

$B(x;n,p) \quad = (q+p)^n b(x;n,p)$

$i(j)k \quad \equiv i, i+j, i+2j, \ldots, k$

$[a(j,m)]^k \quad = \binom{mk+j-1}{j-1}\ k!$

# Notations

$\Pr(X = k)$ *probability that the rv $X$ takes value $k$*

$\Pr(k_1, k_2)$ $\Pr(X_1 = k_1, X_2 = k_2)$

$E(X)$ expectation of $X$

$V(X)$ variance of $X$

$\text{Cov}(X_1, X_2)$ covariance between $X_1$ and $X_2$

$\mu'_{(r)}$ descending $r$th factorial moment

$\mu'_{(r_1,\ldots,r_n)} = E[X_1^{(r_1)}, \ldots, X_n^{(r_n)}]$

$\mu'_r$ $r$th moment about the origin

$\mu_r$ $r$th central moment

$\bar{x}$ obsereved sample mean

$m_2$ observed sample variance

$p_k$ probability that a rv takes the value $k$

$M_r{}'(.)$ $r$th moment about origin

$G(s)$ pgf

$g(s)$ pgf

$E_p(X)$ $E(X)$ when $X \sim$ Poisson

$E_{mp}(X)$ $E(X)$ when $X \sim$ modified Poisson

$E_\nu[X]$ $E(X)$ when $X \sim$ class of $\alpha$-modified binomial distribution for given $\nu$

${}_xM_k$ incomplete moment of order $k$ on the right about origin

${}^xM^k$ incomplete moment of order $k$ on the left about origin

$\delta$ mean deviation about mean $= \mathrm{E}[\mid X - \mathrm{E}(X) \mid]$

$\beta_1$ $\frac{\mu_3^2}{\mu_2^3}$

$\beta_2$ $\frac{\mu_4}{\mu_2^2}$

$k_r$ $r$th cumulant

$\mathrm{E}\{X_1 \mid X_2\}$ conditional expectation of $X_1$ on $X_2$

$m'_{10}$ sample mean of $X_1$

$m'_{01}$ sample mean of $X_2$

$m_{11}$ sample covariance $(X_1, X_2)$

$m_{20}$ sample variance of $X_1$

$m_{02}$ sample variance of $X_2$

$\chi^2$ chi-square staistic

$f_0$ observed sample frequency of zeros

$f_1$ observed sample frequency of ones

$f_{00}$ observed sample frequency of double zeros

$N$ total frequency

$\widehat{\theta}$ estimated value of the parameter $\theta$

# Chapter 1

# Introduction

Discrete probability distributions arise whenever we are dealing with a problem of random nature involving counting. It is one of the most important basic field of study in the theory of statistics. Discrete probability distributions are applied in many areas of application. One of the most important area of research in the theory of discrete probability distribution is to obtain general frame works or probabilistic models which can serve as a parent model for a large and varied collection of discrete probability distributions. Such classifications into broad classes are often useful in understanding large number of known distributions and their inter relations and helps in unification of results, construction of models, derivation of general method of analysis. Quasi binomial distributions, $\alpha$-modified binomial distributions, generalized Poisson distributions, generalizations of Polya-Eggenberger distributions using urn models with predetermined strategies and multivariate Poisson distributions are some of the most important discrete probability distributions that have drawn attention of many research workers.

## 1.1 Previous works

### 1.1.1 Quasi binomial distributions

Consul [11] first introduced the notion of urn model with predetermined strategy with a two urn model and developed quasi binomial distribution (QBD) with probability function (pf)

$$Pr(X=k)=\binom{n}{k}p(p+k\phi)^{k-1}(1-p-k\phi)^{n-k},\ \ k=0(1)n,\ -p/n\leq\phi\leq(1-p)/n \tag{1.1.1}$$

using sampling with replacement schemes. He gave justification and mentioned applications of these distributions in various fields. Consul and Mittal [20] defined QBD type II using four urn model with a predetermined strategy as

$$Pr(X=k)=\binom{n}{k}\frac{p(1-p-n\phi)}{(1-n\phi)}(p+k\phi)^{k-1}(1-p-k\phi)^{n-k-1},\ \ k=0(1)n;-p/n\leq\phi\leq(1-p)/n \tag{1.1.2}$$

and indicated large number of possible applications. Berg and Mutafchiev [5] have shown applications of some QBD and modified QBDs in random mapping problems. Consul [15] studied properties of (1.1.1) with deduction of moments, inverse moments, maximum likelihood estimation and data fitting. Using Abel's generalization of binomial identities (Riordan [62]), Das [28] proposed a class of QBD with pf

$$Pr(X=k)=\binom{n}{k}\frac{(p+k\phi)^{k+s}(1-p-k\phi)^{n-k+t}}{B_n(p,q;s,t;\phi)},\ \ k=0(1)n \tag{1.1.3}$$

where $s$ and $t$ are integers, $p+q+n\phi=1$ and

$$B_n(p,q;s,t;\phi)=\sum_{k=0}^{n}\binom{n}{k}(p+k\phi)^{k+s}(1-p-k\phi)^{n-k+t} \tag{1.1.4}$$

### 1.1.2 Generalization of Polya-Eggenberger distributions

Urn models were used to define Polya-Eggenberger (PE) distribution (Eggenberger and Polya [30]) with pf

$$Pr(X=k)=\frac{\langle{a \atop k}\rangle_s \langle{b \atop n-k}\rangle_s}{\langle{a+b \atop n}\rangle_s}, \quad k=0(1)n \tag{1.1.5}$$

and Inverse Polya-Eggenberger (IPE) (Johnson and Kotz [48], pp.229-332) distributions having the pf

$$Pr(X=k)=\binom{n+k-1}{k}\frac{a^{[k,s]}\,b^{[n,s]}}{(a+b)^{[n+k,s]}}, k=0,1,\ldots \tag{1.1.6}$$

Later, many modifications and generalizations of these two models were suggested by different authors (Johnson and Kotz [49]). Consul [11], Consul and Mittal [20] discussed urn models with pre-determined strategies. Using different urn models with different pre-determined strategies Janardan ([43], [44] and [46]) obtained quasi Polya distributions (QED) as

$$Pr(X=k)=\frac{a}{a+kz}\frac{\langle{a+kz \atop k}\rangle_s \langle{b+(n-k)z \atop n-k}\rangle_s}{\langle{a+b+nz \atop n}\rangle_s}, \quad k=0(1)n, \tag{1.1.7}$$

where $a^{[k,s]}=a(a+s)\cdots(a+(k-1)s)$, $\langle{a \atop k}\rangle=\frac{a^{[k,s]}}{k!}$

quasi inverse Polya distributions (QIPD) with pf

$$Pr(X=k)=\binom{n+k-1}{k}\frac{a}{a+kz}\frac{(a+kz)^{[k,s]}(b+nz)^{[n,s]}}{\{a+b+(n+k)z\}^{[n+k,s]}}, \quad k=0,1,\ldots \tag{1.1.8}$$

generalized Markov-Polya distributions (GMPD) having the pf

$$Pr(X=k)=\frac{a}{a+kz}\frac{b}{b+(n-k)z}\frac{a+b+nz}{a+b}\frac{\langle{a+kz \atop k}\rangle_s \langle{b+(n-k)z \atop n-k}\rangle_s}{\langle{a+b+nz \atop n}\rangle_s}, \quad k=0(1)n \tag{1.1.9}$$

and generalized inverse Markov-Polya distributions (GIMPD) as

$$Pr(X=k) = \binom{n+k-1}{k}\frac{a}{a+kz}\frac{b}{b+nz}\frac{a+b+(n+k)z}{a+b}\frac{(a+kz)^{[k,s]}(b+nz)^{[n,s]}}{\{a+b+(n+k)z\}^{[n+k,s]}}, \quad k=0,1,\ldots \tag{1.1.10}$$

Sen and Mishra [63] used combinatorial method to obtain generalized Polya-Eggenberger model (GPE) unifying PE and IPE with the pf

$$Pr(X=k) = \frac{n}{n+(\mu+1)k}\binom{n+(\mu+1)}{k}\frac{a^{[k,s]}b^{[n+\mu k,s]}}{(a+b)^{[n+(\mu+1)k,s]}}, \quad k=0,1,\ldots,\min(n,a), \text{ when } s=-1. \tag{1.1.11}$$

### 1.1.3 $\alpha$-modified binomial and Poisson distributions

Berg and Jaworski [4] introduced $\alpha$-modified binomial distribution with pf

$$Pr(X=k) = \binom{n}{k}\frac{(p+\alpha\phi)^k q^{n-k}}{(1+\alpha\phi)^n}, \quad k=0(1)n,\ p>0, p+\phi\geq 0 \tag{1.1.12}$$

They derived E$[(n-X)^{(\nu)}]$ and also a weighted form of the pf (1.1.12) as

$$Pr(X=k) = \binom{n}{k}\{q+(n-k)\phi\}(p+\alpha\phi)^k q^{n-k-1}, \qquad k=0(1)n,\ p+\phi\geq 0 \text{ and } \phi\leq q/n \tag{1.1.13}$$

As a limiting form of the pf (1.1.12) and the pf (1.1.13), they obtained one and two parameter $\alpha$-modified Poisson distributions with pf

$$Pr(X=k) = \frac{\lambda^k}{k!}D_k(1-\lambda)\exp(\lambda),\ k=0,1,\ldots;\ 0<\lambda<1 \tag{1.1.14}$$

and

$$Pr(X=k) = \frac{(\lambda+\alpha\psi)^k}{k!}(1-\psi)\exp(-\lambda), \qquad k=0,1,\ldots;\ \lambda>0,\ \lambda+\psi\geq 0 \text{ and } |\psi|<1 \tag{1.1.15}$$

respectively, where

$$D_k = (-1+\alpha)^k$$
$$\alpha_k \equiv \alpha^k = k!$$

They discussed how these distributions arise in connection with random mapping model (Jaworski [47]) and its inference.

Berg and Mutafchiev [5] obtained the probability distribution

$$Pr(X=k)=\frac{(1-\lambda)^2}{k!}\{\lambda_{11}+\lambda_{12}+\lambda(\alpha+k)\}^k\exp\{-(\lambda_{11}+\lambda_{12}+\lambda k)\},\ \ k=0,1,\dots \quad (1.1.16)$$

as a sum of two weighted Lagrangian Poisson distribution (Consul and Jain [18], Berg [3]). Berg and Nowicki [6] introduced two classes of distributions with pf

$$Pr(X=k)=\frac{\lambda^k}{k!}[\alpha(m)+n]^k(1-\lambda)^m\exp(-n\lambda),\qquad k=0,1,\dots \quad (1.1.17)$$

$$m,n=0,1,\dots;\ \text{not both equal to zero};\ 0<\lambda<1$$

$$Pr(X=k)=\frac{(\lambda\exp(-\lambda))^k}{k!}[\alpha(m-1)+n+k]^k(1-\lambda)^m\exp(-n\lambda),\ \ k=0,1,\dots \quad (1.1.18)$$

$$m,n=0,1,\dots;\ \text{not both equal to zero};\ 0<\lambda<1$$

generated respectively by a power series and modified power series expansion and studied various properties, where

$$\alpha^k(m)=\binom{k+m-1}{m-1}k!$$

### 1.1.4 Generalized Poisson Distributions

Discrete probability distributions have been used as a modelling tool in various branches of statistical sciences. Poisson distribution is among the most widely used discrete probability distributions and has been receiving very high degree of attention from researchers in discrete distributions. A large volume of works are done to obtained generalizations, modifications of Poisson distributions by different workers.

Consul and Jain ([18], [19]) developed generalized Poisson distribution (GPD) with two parameters having pf

$$Pr(X=k)=\begin{cases}\frac{1}{k!}\,a(a+kz)^{k-1}e^{-(a+kz)} & k=0,1,\dots\\ 0 & \text{for } k>m \text{ when } z<0\end{cases} \tag{1.1.19}$$

and zero otherwise, where $a>0, \max(-1,a/4)\leq z<1$ and $m$ the largest positive integer for which $a+zm>0$ when $z<0$.

Since then, a lot of works have been done on this model (Consul [14]).

Nandi et al. [55] proposed a class of discrete distributions called generalized Poisson distributions with pf

$$Pr(X=k)=\frac{1}{k!}\frac{(a+kz)^{k+s}e^{-kz}}{K(a;s;z)},\quad k=0,1,\dots;a>0,\mid z\mid<1 \tag{1.1.20}$$

by defining a class of exponential sums as

$$K(a;s;z)=\sum_{k\geq 0}\frac{1}{k!}(a+kz)^{k+s}e^{-kz}$$

subject to the simultaneous realization of the constraints $a+kz>0$, for all $k$ and

$$0<\left(z+\frac{a}{k+1}\right)\left(1+\frac{z}{a+kz}\right)^{k+s}e^{-z}<1$$

for all sufficiently large $k$ where $s$ is an integer.

### 1.1.5 Multivariate Poisson distributions

The multinomial distribution (Johnson et al. [50], p.31) with parameters $(n;,p_1,\dots,p_k)$ is defined by probability function

$$Pr\left(\bigcap_{i=1}^{k}N_i=n_i\right)=n!\prod_{i=1}^{k}\frac{{p_i}^{n_i}}{n_i!} \tag{1.1.21}$$

where $n_i\geq 0, \sum\limits_{i=1}^{k}n_i=n, \sum\limits_{i=1}^{k}p_i=1$. If $p_1,p_2,\dots,p_{k-1}\to 0$ hence $p_k\to 1$ as $n\to\infty$ such that $np_i=\lambda_i; i=1,2,\dots,k-1$ then (1.1.21) tends to

$$Pr\left(\bigcap_{i=1}^{k-1}N_i=n_i\right)=\prod_{i=1}^{k-1}\frac{e^{-\lambda_i}\lambda_i^{n_i}}{n_i!} \tag{1.1.22}$$

which is known as the multiple Poisson distribution. (Patil and Bildikar [58], Johnson et al. [50]). The pf (1.1.22) is simply joint distribution of $(k-1)$ independent Poisson $(\lambda_i)$ variates. Consul and Mittal [21] defined $k$- variate quasi Multinomial distribution with parameters $(n; p_1, p_2, \cdots, p_k; \phi)$ with pf

$$Pr\left(\bigcap_{i=1}^{k} N_i = n_i\right) = \frac{n!(1+n\phi)}{n_1!\, n_2! \,\cdots\, n_k!} \prod_{i=1}^{k} \frac{p_i}{1+n\phi}\left(\frac{p_i + n_i\phi}{1+n\phi}\right)^{n_i - 1} \tag{1.1.23}$$

where $\sum_{i=1}^{k} n_i = n, 0 < p_i < 1, \sum_{i=1}^{k} p_i = 1, \phi \geq 0$. As $n \to \infty$ and $p_1, p_2, \ldots, p_{k-1} \to 0$ hence $p_k \to 1$ such that $np_i = \lambda_i; i = 1, 2, \ldots, k-1$ and $n\phi = \beta; 0 < \lambda_i < \infty$ then (1.1.23) tends to multiple generalized (Lagrangian) Poisson distribution with parameters

$$\frac{\lambda_i}{1+\beta}; \quad i = 1, \ldots, (k-1) \text{ and } \frac{\beta}{1+\beta}$$

with probability function (Consul and Mittal [21])

$$Pr\left(\bigcap_{i=1}^{k-1} N_i = n_i\right) = \prod_{i=1}^{k-1}\left[\frac{1}{n_i!}\left(\frac{\lambda_i}{1+\beta}\right)\left(\frac{\lambda_i + n_i\beta}{1+\beta}\right)^{n_i - 1} e^{-\left(\frac{\lambda_i + n_i\beta}{1+\beta}\right)}\right] \tag{1.1.24}$$

The pf (1.1.24) is the joint distribution of $(k-1)$ independent generalized Poisson (Consul and Jain [18]) with parameters $\frac{\lambda_i}{1+\beta}; \quad i = 1, \ldots, (k-1)$ and $\frac{\beta}{1+\beta}$. Das [28] obtained a class of quasi multinomial distributions with probability function

$$Pr\left(\bigcap_{i=1}^{k} N_i = n_i\right) = \frac{n!}{B_n(p_1, \ldots, p_k; s_1, \ldots, s_k; \phi)} \prod_{i=1}^{k} \frac{(p_i + n_i\phi)^{n_i + s_i}}{n_1! n_2! \cdots n_k!} \tag{1.1.25}$$

where

$$B_n(p_1, \ldots, p_k; s_1, \ldots, s_k; \phi) = \sum \frac{n!}{n_1! n_2! \cdots n_k!} \prod_{i=1}^{k} (p_i + n_i\phi)^{n_i + s_i}$$

wherein the summation is over all non-negative integers $n_i, i = 1(1)k$ such that $\sum_{i=1}^{k} n_i = n$. The pf (1.1.23) is a particular case of the pf (1.1.25) when $s_i = -1; i = 1(1)k$. As a limiting distribution of the pf (1.1.25), a class of multiple generalized Poisson is defined as

$$Pr\left(\bigcap_{i=1}^{k-1} N_i = n_i\right) = \prod_{i=1}^{k-1} \frac{1}{n_i!}(\lambda_i + n_i\beta)^{n_i + s_i} e^{-(\lambda_i + n_i\beta)} \tag{1.1.26}$$

Clearly (1.1.24) is a member of (1.1.26). Like (1.1.24), (1.1.26) too is a joint distribution of $(k-1)$ independent variates belonging to a class of weighted generalized Poisson distribution. Among non-trivial class of Poisson distributions, Holgate [40] [See Johnson et al. [50], p.124] defined a class of bivariate Poisson distributions as a joint distribution of the variables $X_1 = V_0 + V_1$ and $X_2 = V_0 + V_2$ where $V_0, V_1$ and $V_2$ are mutually independent Poisson variates with mean $\lambda_0, \lambda_1$ and $\lambda_2$ respectively. The probability function is given by

$$Pr(X_1 = k_1, X_2 = k_2) = e^{-(\lambda_0+\lambda_1+\lambda_2)} \sum_{i=0}^{\min(k_1,k_2)} \frac{{\lambda_0}^i {\lambda_1}^{k_1-i} {\lambda_2}^{k_2-i}}{i!(k_1-i)!(k_2-i)!}, \quad k_1, k_2 = 0, 1, \ldots \tag{1.1.27}$$

The pf (1.1.27) can be derived as a limiting form of (Hamdan and Al-Bayyati [38])

$$Pr(X_1 = k_1, X_2 = k_2) = \sum_{i=0}^{n} \frac{n!}{i!(k_1-i)!(k_2-i)!(n+k_1-k_2)!} {p_{11}}^i {p_{10}}^{k_1-i} {p_{01}}^{k_2-i} {p_{00}}^{n-k_1-k_2+i} \tag{1.1.28}$$

as $\to \infty$, $p_{11}, p_{01}, p_{10} \to 0$ such that $np_1 \to \lambda_1$, $np_2 \to \lambda_2$, $np_{11} \to \lambda_0$ where $p_{11} + p_{10} + p_{01} + p_{00} = 1$ and $p_{11} + p_{01} = p_2$, $p_{00} + p_{11} = p_1$ (Johnson et al. [50], p.125). The pf (1.1.27) can be generalized by considering the distributions of $V_0, V_1$ and $V_2$ as generalized Poisson (Consul [18]) with parameters $(\lambda_0, \phi_0), (\lambda_1, \phi_1), (\lambda_2, \phi_2)$ respectively (Johnson et al. [50], p.133). The probability function is then given by

$$Pr(k_1, k_2) = \lambda_0 \lambda_1 \lambda_2 e^{-(\lambda_0+\lambda_1+\lambda_2+\phi_1 k_1+\phi_2 k_2)} \sum_{i=0}^{\min(k_1,k_2)} \frac{(\lambda_0 + \phi_0 i)^{i-1} (\lambda_1 + (k_1-i)\phi_1)^{k_1-i-1} (\lambda_2 + (k_2-i)\phi_2)^{k_2-i-1}}{i!(k_1-i)!(k_2-i)!} \tag{1.1.29}$$

for $\phi_0 = \phi_1 = \phi_2 = \phi$ (1.1.29) reduces to

$$Pr(k_1, k_2) = \lambda_0 \lambda_1 \lambda_2 e^{-(\lambda_0+\lambda_1+\lambda_2+\phi k_1+\phi k_2)} \sum_{i=0}^{\min(k_1,k_2)} \frac{(\lambda_0 + \phi i)^{i-1} (\lambda_1 + (k_1-i)\phi)^{k_1-i-1} (\lambda_2 + (k_2-i)\phi)^{k_2-i-1}}{i!(k_1-i)!(k_2-i)!} \tag{1.1.30}$$

Das [28] derived a class of bivariate generalized Poisson distribution by defining a class of bivariate exponential sums, (1.1.29) is a member of the class.

## 1.2 Objective of the thesis

1. Study the class of QBDs (1.1.3) for various distributional properties, inter-relationships, problem of parameter estimations, data fitting in general and properties of some of the members of this class in particular.

2. *Develop two unified probability models using urn models with pre-determined strategies to* unify QED, QIPD and GMPD, GIMPD and study some distributional aspects of these models.

3. Obtain a class of weighted $\alpha$-modified binomial distribution and study its various distributional properties, generalizations and compounding.

4. Study various aspects of the class of GPDs (1.1.20) in general and some of the new distribution belonging to this class in particular.

5. Define a class of generalized multivariate GPD, study its various properties in general and bivariate case in particular.

## 1.3 Organisation of the thesis

An introduction of the work is given in the first chapter.

In chapter two an attempt has been made to develop some new discrete distributions of the QBD type using urn models with pre-determined strategies. It has been shown that these distributions are infact members of a class of weighted QBD (WQBD)s, that can also be derived by Abel's generalization of the binomial identities (Riordan [62]). Then the following properties of this WQBD class in general and some new QBDs in particular have been studied.

1. Derivation of new QBDs using urn models with pre-determined strategies.

2. A class of QBD as a weighted (mixed factorial moment) QBD.

3. Various moment properties of the WQBD class in general and some members in particular.

4. Negative (Inverse) moments for the WQBD class in general and new QBDs in particular.

5. Bound for the mode of the WQBD class.

6. Estimation of the parameters of different QBDs by the method of

   - Zero one class frequencies.
   - Zero class frequency and mean.
   - Moments.
   - Maximum likelihood.

7. Fitting of QBDs to data from various field of applications and comparative performance study by goodness of fit.

8. Zero-truncated WQBDs their first two factorial moments and inverse moments.

9. A class of weighted generalized Poisson distribution as the Limiting distributions of the class of WQBDs.

In the third chapter, first an attempt has been made to develop a unified probability model based on a three urn setup with a predetermined strategy which will generate both QED and QIPD and hence all their particular cases. Some recurrence relations among the moments and probabilities are established. A few limiting distributions are mentioned. Then a generalized probability model (GPM) based on a five urn setup with a predetermined strategy has been discussed. This model generates both GMPD and GIMPD and hence all their particular cases and many more important discrete distributions. Some recurrence relations among moments and probabilities are obtained. Formulae for first and second moments of GMPD are obtained using different approaches. A few limiting distributions of the models are cited. The steps involved in the maximum likelihood estimation of the parameters of QED and GMPD by using numerical methods have also been discussed.

The fourth chapter begins with the study of the general form of (1.1.12). Some useful identities are derived and a class of weighted $\alpha$-modified binomial(w$\alpha$mb) distribution is proposed. Both (1.1.12) and (1.1.13) are seen as particular cases of the proposed class. Some important recurrence relations among and between the different probability functions are established and the pgf of the class is derived. The factorial moments of the w$\alpha$mb class is deduced in general and first few moments for some of the distributions belonging to class of w$\alpha$mb distributions are obtained. A few useful recurrence relations among moments are also obtained. Limiting distributions under different set of conditions are derived. It is has been observed that both (1.1.14) and (1.1.15) occur as particular case of these limiting distributions. Next, some generalized and compound (Johnson and Kotz [48], Johnson et al. [51]) $\alpha$-modified binomial distributions have been discussed. Then, some of the occurrences of the various distributions studied above in different fields of applications such as matching problem, extended matching problems, rumor problem, random mapping (when number of points is fixed as well as stochastic) are mentioned. For each of these the formulas for the first two moments of the variable of interest are provided.

In chapter five it has been first shown how the class of GPD (1.1.20) can be obtained as a class of weighted distribution of the generalized Poisson distribution (Consul and Jain [18]) and then the various properties of the class of weighted GPD (WGPD) in general and two new distributions, GPD II (obtained by taking $s = -2$ in (1.1.20)) and GPD III (derived by putting $s = 0$ in (1.1.20)) in particular are studied. Some new distributions of the class and their basic properties like pgf, mean and variance are mentioned. The following aspects of class of WGPD have been investigated.

1. Some results on moments and inverse moments.

2. Incomplete moments, mean deviation about mean.

3. Probability generating function (pgf).

4. Relationship between the derivative of pgf and first four moments.

5. Distributions of the sum.

6. Distributions of the difference.

7. Distribution of the sum of left truncated variates.

8. Some new distributions and their properties.

9. Characterizations of the class of distributions.

10. Limiting distributions.

In addition we have also studied the following aspects of GPD II and GPD III

1. First four factorial, central moments, cumulants, $\beta_1$, $\beta_2$.

2. Estimation of parameters by the method of

    - Maximum likelihood.
    - Moments.
    - zero class frequency and mean.

3. A relation between moments of GPD I and GPD III

4. Fitting of data sets taken from various areas of application and test of goodness of the fits.

5. Models leading to GPD III

In the last chapter a class of generalized multivariate generalized Poisson distribution has been proposed by defining a class of multivariate exponential sums. Various forms of GMGPD are derived by choosing different values for parameters. Various distributional properties viz. characterization, marginal, conditional distributions, regression function, moment vectors, dispersion matrices, formula for mixed factorial moments and estimation in bivariate cases of some of these distributions are studied.

Finally, two appendices are provided to list the various formulae and results related to the present work.

# Chapter 2

# A class of weighted Quasi Binomial Distributions

## 2.1 Introduction

In this chapter, a class of weighted quasi binomial distributions, the moments, inverse moments, recurrence relations among moments, bounds for mode, truncated distributions, problem of estimation and fitting of data from real life situations using different methods have been studied.

## 2.2 Urn models with predetermined strategies

### 2.2.1 *A two urn model*

Let there be two urns marked I and II, urn I containing $a$ white and urn II $a$ white $b$ black balls. Let $n$ and $z$ be two known positive integers, for given $n$ and $z$, a strategy is determined by choosing an integer $k$ such that $0 \leq k \leq n$ before making two draws from urn I and $n$ draws from urn II under the following rules:

(i) $kz$ black balls will be added to urn I and $kz$ white, $(n-k)z$ black to urn II,

(ii) Two balls are drawn from urn I with replacement, if both the balls are white, $n$ draws are made from urn II with replacement, otherwise no draws are made.

A success is achieved, if exactly $k$ out of $n$ draws are white balls.

Clearly, therefore the probability of success is equal to

$$Pr(X=k) = \left(\frac{a}{a+kz}\right)^2 \binom{n}{k} \left(\frac{a+kz}{a+b+nz}\right)^k \left(\frac{b+(n-k)z}{a+b+nz}\right)^{n-k} \tag{2.2.1}$$

With

$$p = \frac{a}{a+b+nz}, \quad q = \frac{b}{a+b+nz} \text{ and } \phi = \frac{z}{a+b+nz}$$

$$Pr(X=k) = \binom{n}{k} p^2 (p+k\phi)^{k-2} (q+(n-k)\phi)^{n-k} \tag{2.2.2}$$

### 2.2.2 *An model with three urns*

Let there be three urns marked I, II and III. Urn I containing $a$ white, urn II $b$ black and urn III $a$ white, $b$ black balls respectively. For given $n$ and $z$, a strategy $0 \le k \le n$ is chosen before making two draws from urn I, one from urn II and $n$ from urn III under the following conditions:

(i) $kz$ black to urn I, $(n-k)z$ white to urn II and $kz$ white, $(n-k)z$ black balls to urn III will be added,

(ii) Two balls are drawn with replacement from urn I, if both white, then one is drawn from urn II if black, $n$ draws with replacement are made from urn III, otherwise the game is stopped as a failure.

A success is achieved, if out of $n$ draws from urn II exactly $k$ are white.

Then the probability of success is given by

$$Pr(X=k) = \binom{n}{k} p^2 q (p+k\phi)^{k-2} (q+(n-k)\phi)^{n-k-1} \tag{2.2.3}$$

### 2.2.3 *Another three urn model*

Here the same setup as that in the last model has been considered only the drawing pattern is changed as follows.

Two balls are drawn from urn I if both white, then two draws are made from urn II, if both black $n$ draws with replacement are made from urn III, a success is defined in the same manner as earlier. Hence, here

$$Pr(X = k) = \binom{n}{k} p^2 q^2 (p + k\phi)^{k-2}(q + (n-k)\phi)^{n-k-2} \tag{2.2.4}$$

None of these models are proper probability distribution. But using Abel's generalisation of the binomial identities (Riordan [62]), it is possible to find norming constants $c$ in such a way that $\sum_k c\, Pr(X = k) = 1$

All these and many more distributions are infact members of a class of quasi binomial distributions (Das [28]) defined using Abel's formula as

$$p_k = \binom{n}{k} \frac{(p + k\phi)^{k+s}(q + (n-k)\phi)^{n-k+t}}{B_n(p, q; s, t; \phi)} \tag{2.2.5}$$

$$B_n(p, q; s, t; \phi) = \sum_{k=0}^{n} \binom{n}{k} (p + k\phi)^{k+s} (q + (n-k)\phi)^{n-k+t}, \quad \text{(Riordan [62])} \tag{2.2.6}$$

where $p$ and $q$ being the non-negative fractions, $p + q + n\phi = 1$; $-\frac{p}{n} < \phi < \frac{(1-p)}{n}$ and $s, t$ integers. Alternatively, (2.2.5) can also be written as

$$p_k = \binom{n}{k} \frac{(p + k\phi)^{k+s}(1 - p - k\phi)^{n-k+t}}{B_n(p, 1 - p - n\phi; s, t; \phi)} \tag{2.2.7}$$

***Some special cases of (2.2.5)*** : For

i) $s = -1,\ t = 0$ , QBD type I (Consul [11]).

ii) $s = -1,\ t = -1$, QBD type II (Consul and Mittal [20]).

iii) $s = -2,\ t = 0$, QBD type III (Das [28]).

iv) $s = -2,\ t = -1$, QBD type IV (Das [28]).

v) $s = 0,\ t = 0$, QBD type VII (Das [28]), etc.

In fact, the above class can also be derived as a class of weighted QBD I with some reparametrisation as stated below. Here, it should be noted that the weighted probability mass function(pmf)

$Q(x)$ corresponding to the pmf $p(x)$ of the random variable $X$ with weight function $w(x)$ is given by

$$Q(x) = \frac{w(x)p(x)}{\sum_x w(x)p(x)}$$

In case $w(x) = x$, the distribution is said to be size biased distribution (See Johnson et al. [51], p.145).

**Theorem 2.2.1** *If $X \sim$ QBD I (Consul [11]) with $n, p, \phi$, then the weighted distribution of $X$ with weight $w(x) = x^{(s+1)}(n-x)^{(t)}$ is given by*

$$\begin{aligned} p_k &= \frac{\binom{n-t-s-1}{k-s-1}}{B_{n-s-t-1}(p+(s+1)\phi, 1-p-(n-t)\phi; s,t;\phi)} (p+(s+1)\phi+(k-s-1)\phi)^{k-s-1+s} \\ & (1-p-(s+1)\phi-(k-s-1)\phi)^{n-t-s-k+s+t}, \quad k = 0(1)n \end{aligned} \tag{2.2.8}$$

Hence the distribution of $Y = X - s - 1$ is given by

$$Pr(Y=k) = \frac{\binom{m}{k}(p'+k\phi)^{k+s}(1-p'-k\phi)^{m-k+t}}{B_m(p', 1-p'-m\phi; s,t;\phi)}, \quad k = 0(1)m \tag{2.2.9}$$

where $m = n-s-t-1$, $p' = p+(s+1)\phi$ and $-\frac{p'}{m} < \phi < \frac{(1-p')}{m}$, which is the class of QBD given in (2.2.7). Thus, all the distributions belonging to the (2.2.7) can be derived as the weighted distribution of QBD I by choosing appropriate values for the integers $s$ and $t$ in the weight function $w(x)$. Denoting the class (2.2.7) by WQBD$(n; p, \phi; s, t)$, it can be seen that the form of the weighted QBD I (2.2.8) is

$$1 + s + \text{WQBD}(n-s-t-1; p+(s+1)\phi, \phi; s, t).$$

In particular, for

I. $s = 0, t = 0$ we get when $\phi = 0$ the size biased for of binomial distribution (Johnson et al. [51], p.146) as

$$1 + \text{Binomial}(n-1; p)$$

II. $s = 0, t = 0$ the size biased for of QBD I become

$$1 + \text{WQBD}(n-1; p+(s+1)\phi, \phi; 0, 0)$$

## 2.3 Moments

In this section, the Abel's generalisations of binomial identities (see appendix A) and umbrals (Riordan [62]) have been exploited for the derivation of the formulae of moments of the class of weighted quasi binomial distributions in general and various quasi binomial distributions in particular in compact form.

**Theorem 2.3.1** *The r-th order descending factorial moments of the class of* WQBD *is given by (Das [28])*

$$\mu'_{(r)}(n;p,q;s,t;\phi) = \frac{n^{(r)} B_{n-r}(p+r\phi,q;s+r,t;\phi)}{B_n(p,q;s,t;\phi)}, \tag{2.3.1}$$

*where* $n^{(r)} = n(n-1)\cdots(n-r+1)$ *and* $\mu'_{(r)}(n;p,q;s,t;\phi)$ *denotes the r-th order descending factorial moments.*

**Theorem 2.3.2** *The r-th order moments about origin of the class of* WQBD *is given by*

$$\mu'_r(n;p,q;s,t;\phi) = \sum_{j=0}^{r} S(r,j) n^{(j)} \frac{B_{n-j}(p+j\phi,q;s+j,t;\phi)}{B_n(p,q;s,t;\phi)}, \tag{2.3.2}$$

*where* $S(r,j)$ *are the Stirling numbers of second kind (See Riordan [61], Johnson et al. [51]) defined as*

$$\begin{aligned} S(i,j) &= \frac{\Delta^j 0^i}{j!} \quad for\ i \geq j \\ &= 0 \quad for\ i < j. \\ &= 1 \quad for\ j=1\ or\ i=j \end{aligned} \tag{2.3.3}$$

*also* $S(i+1,j) = jS(i,j) + S(i,j-1)$

*Proof.* The $r$-th order moments of about origin is

$$\begin{aligned} \mu'_r(n;p,q;s,t;\phi) &= \mathrm{E}[X^r] = \sum_{j=0}^{r} S(r,j)\mu'_{(j)}(n;p,q;s,t;\phi) \\ &= \sum_{j=0}^{r} S(r,j) n^{(j)} \frac{B_{n-j}(p+j\phi,q;s+j,t;\phi)}{B_n(p,q;s,t;\phi)}. \end{aligned}$$

**Theorem 2.3.3** *The r-th order moments about mean of the class of* WQBD *is given by*

$$\mu_r(n;p,q;s,t;\phi) = \frac{1}{B_n(p,q;s,t;\phi)}\sum_{j=0}^{r}\sum_{\nu=0}^{j}\binom{r}{j}(-1)^{r-j}{\mu'_1}^{r-j}S(j,\nu)\,n^{(\nu)} B_{n-\nu}(p+\nu\phi,q;s+\nu,t;\phi). \quad (2.3.4)$$

*Proof.* The $r$-th order moments about mean is

$$\begin{aligned}\mu_r(n;p_1,p_2;s,t;\phi) &= \mathrm{E}[X-\mu'_1]^r \\ &= (\mu'_1)^r\sum_{j=0}^{r}\binom{r}{j}\frac{(-1)^{r-j}}{(\mu'_1)^j}\mu'_j \\ &= \frac{{\mu'_1}^r}{B_n(p,q;s,t;\phi)}\sum_{j=0}^{r}\sum_{\nu=0}^{j}\binom{r}{j}\frac{(-1)^{r-j}}{{\mu'_1}^j}S(j,\nu)n^{(\nu)} \\ &\quad B_{n-\nu}(p+\nu\phi,q;s+\nu,t;\phi).\end{aligned}$$

The relations (2.3.1), (2.3.2) and (2.3.4) are the general formulae of the moments for the class of quasi binomial distributions. Using different combinations of integer values of $r, s$ and $t$, the different moments of the weighted quasi binomial distributions may be obtained. The above results are used to find moments of some of the QBDs. It may be noted that in all the results above, $p+q+n\phi=1$ that is $q=1-p-n\phi$.

### 2.3.1 First four central moments of some of the WQBDs

#### 2.3.1.1 Moments of the QBD I

$$\mu'_1 = np\sum_{\nu=0}^{n-1}(n-1)^{(\nu)}\phi^\nu \quad (2.3.5)$$

$$\begin{aligned}\mu_2 &= n^{(2)}p\sum_{\nu=0}^{n-2}\sum_{\gamma=0}^{\nu}(n-2)^{(\nu)}\phi^\nu(p+2\phi+\gamma\phi)+np\sum_{\nu=0}^{n-1}(n-1)^{(\nu)}\phi^\nu \\ &- n^2p^2\left\{\sum_{\nu=0}^{n-1}\{(n-1)^{(\nu)}\phi^\nu\}^2+\sum_{i\neq j=0}^{n-1}(n-1)^{(i)}(n-j)^{(j)}\phi^{i+j}\right\}\end{aligned} \quad (2.3.6)$$

It can be verified that this result is equal to that of Consul [15].

$$\mu_3 = (2{\mu'_1}^3-3{\mu'_1}^2+\mu'_1)$$

$$
\begin{aligned}
&+ \quad 3p(1-\mu_1')\sum_{\nu=0}^{n-2}\sum_{\gamma=0}^{\nu} n^{(\nu+2)}\phi^{(\nu)}(p+2\phi+\gamma\phi)\\
&+ \quad p\sum_{\nu=0}^{n-3}\sum_{\gamma=0}^{\nu} n^{(\nu+3)}\phi^{(\nu+1)}\gamma(\nu-\gamma+1)(p+3\phi+\gamma\phi)\\
&+ \quad p\sum_{\nu=0}^{n-3}\sum_{\gamma=0}^{\nu}\sum_{\mu=0}^{\gamma} n^{(\nu+3)}\phi^{\nu}(p+3\phi+\mu\phi)(p+3\phi+(\gamma-\mu)\phi). \qquad (2.3.7)
\end{aligned}
$$

$$
\begin{aligned}
\mu_4 &= p\left[p^{-1}{\mu_1'}^4 + n(1+\alpha\phi)^{n-1}\{-4{\mu_1'}^3+6{\mu_1'}^2-4\mu_1'+1\}\right.\\
&+ \quad n^{(2)}(1+\alpha\phi+\beta'(p+2\phi)\phi)^{n-2}\{6{\mu_1'}^2-12\mu_1'+7\}\\
&+ \quad n^{(3)}[(1+\alpha\phi+\beta'(p+3\phi;2)\phi)^{n-3}+(1+\alpha(2)\phi+\gamma'(p+2\phi)\phi)^{n-3}]\{-4\mu_1'+6\}\\
&+ \quad n^{(4)}[(1+\alpha\phi+\phi\beta'(p+4\phi;3))^n+3(1+\phi\alpha(2)+\phi\beta'(p+4\phi)+\phi\gamma'(p+4\phi))^n\\
&+ \quad (1+\phi\alpha(2)+\phi\beta'(0)+\phi\gamma'(p+4\phi))^n+(1+\phi\alpha(3)+\phi\psi'(p+4\phi))^n]\,] \qquad (2.3.8)
\end{aligned}
$$

#### 2.3.1.2 Moments of the QBD II

$$
\mu_1' = \frac{np}{1-n\phi} \qquad (2.3.9)
$$

$$
\mu_2 = \frac{np}{1-n\phi}\left[\sum_{\nu=0}^{n-2}(n-1)^{(\nu+1)}\phi^{\nu}(p+2\phi+\nu\phi)+\frac{1-n(\phi+p)}{1-n\phi}\right] \qquad (2.3.10)
$$

$$
\begin{aligned}
\mu_3 &= (2{\mu_1'}^3-3{\mu_1'}^2+1)\\
&+ \quad \frac{p}{1-n\phi}\left[3n^{(2)}p(1-\mu_1')\sum_{\nu=0}^{n-2}(n-2)^{\nu}\phi^{(\nu)}(p+2\phi+\gamma\phi)\right.\\
&+ \quad n^{(3)}\{\sum_{\nu=0}^{n-3}\sum_{\gamma=0}^{\nu}(n-3)^{(\nu)}\phi^{(\nu)}(p+3\phi+\gamma\phi)(p+3\phi+(\nu-\gamma)\phi)\\
&+ \quad \left.\sum_{\nu=0}^{n-3}\sum_{\gamma=0}^{\nu}(n-3)^{(\nu)}\phi^{\nu+1}\gamma(p+3\phi+\gamma\phi)\}\right]. \qquad (2.3.11)
\end{aligned}
$$

$$
\begin{aligned}
\mu_4 &= {\mu_1'}^4+\frac{p}{1-n\phi}\left[n(-4{\mu_1'}^3+6{\mu_1'}^2-4\mu_1'+1)\right.\\
&+ \quad n^{(2)}(6{\mu_1'}^2-12\mu_1'+7)(1+\phi\beta'(p+2\phi))^{n-2}+n^{(3)}(-4\mu_1'+6)\\
&\quad \left\{(1+\phi\beta'(p+3\phi;2))^{n-3}+(1+\alpha\phi+\phi\gamma'(p+3\phi))^{n-3}\right\}+n^{(4)}\{B_n(p+4\phi,q;3,0;\phi)\\
&+ \quad n\phi B_{n-1}(p+4\phi,q+\phi;3,0;\phi)\}] \qquad (2.3.12)
\end{aligned}
$$

Further expansion has been avoided as the expression becomes too long.

#### 2.3.1.3 Moments of the QBD III

$$\mu_1' = \frac{np^2}{p+\phi-np\phi}. \tag{2.3.13}$$

$$\mu_2 = \frac{np^2(p+\phi)}{p+\phi-np\phi}\left[\sum_{\nu=0}^{n-2}(n-1)^{(\nu+1)}\phi^{\nu} + \frac{1-np}{p+\phi-np\phi}\right] \tag{2.3.14}$$

$$\begin{aligned}\mu_3 &= (2{\mu_1'}^3 - 3{\mu_1'}^2 + 1) + 3(1-\mu_1')\sum_{\nu=0}^{n-2} n^{\nu+2}\phi^{(\nu)} \\ &+ \sum_{\nu=0}^{n-3}\sum_{\gamma=0}^{\nu} n^{(\nu+3)}\phi^{(\nu)}(p+3\phi+\gamma\phi)\end{aligned} \tag{2.3.15}$$

$$\begin{aligned}\mu_4 &= (-3{\mu_1'}^4 + 6{\mu_1'}^3 - 4{\mu_1'}^2 + \mu_1') + \frac{p^2(p+\phi)}{p+\phi-np\phi}\left[n^{(2)}(6{\mu_1'}^2 - 12\mu_1' + 7)\right. \\ &\sum_{\nu=0}^{n-2}(n-2)^{(\nu)}\phi^{\nu} + n^{(3)}(6-4\mu_1')\sum_{\nu=0}^{n-3}(n-3)^{(\nu)}\phi^{\nu}(p+3\phi+\gamma\phi) \\ &+ n^{(4)}\left\{\sum_{\nu=0}^{n-4}\sum_{\gamma=0}^{\nu}\sum_{\mu=0}^{\nu}(n-4)^{(4)}\phi^{\nu}(p+4\phi+\mu\phi)(p+4\phi+(\gamma-\mu)\phi)\right. \\ &+ \left.\left.\sum_{\nu=0}^{n-4}\sum_{\gamma=0}^{\nu}(n-4)^{(\nu)}\phi^{\nu}\gamma\,\phi\,(p+4\phi+\gamma\phi)\right\}\right]\end{aligned} \tag{2.3.16}$$

#### 2.3.1.4 Moments of the QBD IV

$$\mu_1' = \frac{np^2(1-(n-1)\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-1)\phi)} \tag{2.3.17}$$

$$\begin{aligned}\mu_2 &= \frac{np^2(1-(n-1)\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-1)\phi)}\left(1 - \frac{np^2(1-(n-1)\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-1)\phi)}\right) \\ &+ \frac{n^{(2)}p^2(p+\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-1)\phi)}\end{aligned} \tag{2.3.18}$$

$$\begin{aligned}\mu_3 &= (2{\mu_1'}^3 - 3{\mu_1'}^2 + 1) + \frac{p^2(p+\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-1)\phi)}\{-3(1-\mu_1')n^{(2)} \\ &+ n^{(3)}\sum_{\nu=0}^{n-3}(n-3)^{\nu}\phi^{\nu}(p+3\phi+\nu\phi)\}\end{aligned} \tag{2.3.19}$$

$$\begin{aligned}\mu_4 &= (-3{\mu_1'}^4 + 6{\mu_1'}^3 - 4{\mu_1'}^2 + 1) + \frac{p^2(p+\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-1)\phi)} \\ &\left\{n^{(2)}(6{\mu_1'}^2 - 12\mu_1' + 7) + n^{(3)}(6-4\mu_1')\sum_{\nu=0}^{n-3}(n-3)^{(\nu)}\phi^{\nu}(p+3\phi+\nu\phi)\right. \\ &+ \left. n^{(4)}\left\{\sum_{\nu=0}^{n-4}(n-4)^{(\nu)}\phi^{\nu}(p+4\phi+\nu\phi) + \sum_{\nu=0}^{n-4}\sum_{\gamma=0}^{\nu}(n-4)^{(\nu)}\phi^{\nu+1}\gamma(p+4\phi+\gamma\phi)\right\}\right\}\end{aligned} \tag{2.3.20}$$

#### 2.3.1.5 Moments of the QBD VII

$$\mu_1' = n\left(\sum_{\nu=0}^{n} n^{(\nu)}\phi^{\nu}\right)^{-1}\left[\sum_{\nu=0}^{n-1}\sum_{\gamma=0}^{\nu}(n-1)^{(\nu)}\phi^{\nu}(p+\phi+\gamma\phi)\right] \tag{2.3.21}$$

$$\begin{aligned}\mu_2 &= (\mu_1' - {\mu_1'}^2) + \frac{n^{(2)}}{\sum_{\nu=0}^{n} n^{(\nu)}\phi^{\nu}}\left\{\sum_{\nu=0}^{n-2}\sum_{\gamma=0}^{\nu}\sum_{\mu=0}^{\gamma}(n-2)^{(\nu)}\phi^{\nu}(p+2\phi+\mu\phi)(p+2\phi+(\gamma-\mu)\phi)\right.\\ &+ \left.\sum_{\nu=0}^{n-2}\sum_{\gamma=0}^{\nu}(n-2)^{(\nu)}\phi^{\nu}(\nu-\gamma+1)\gamma\phi(p+2\phi+\gamma\phi)\right\}\end{aligned} \tag{2.3.22}$$

Expressions for $\mu_3$ and $\mu_4$ for this distribution become too messy to present here.

Remark: Formulas for $\alpha, \beta'(.), \gamma'(.), \psi'(.)$ and $B_n(.)$ used in expressions above are provided in appendix A.

### 2.3.2 Recurrence relation of moments

In the following, two recurrence relations for the moments of (2.2.5) have been stated.

1.
$$\mu_r'(n;p,q;s,t;\phi) = \frac{nB_{n-1}(p+\phi,q;s+1,t;\phi)}{B_n(p,q;s,t;\phi)}\sum_{j=0}^{r-1}\binom{r-1}{j}\mu_j'(n-1;p+\phi,q;s+1,t;\phi) \tag{2.3.23}$$

Repeated application of (2.3.23) gives

2.
$$\mu_r'(n;p,q;s,t;\phi) = \frac{n^{(2)}B_{n-2}(p+2\phi,q;s+2,t;\phi)}{B_n(p,q;s,t;\phi)}\sum_{j=0}^{r-1}\sum_{\nu=0}^{j-1}\binom{r-1}{j}\binom{j-1}{\nu}\mu_\nu'(n-2;p+2\phi,q;s+2,t;\phi). \tag{2.3.24}$$

## 2.4 Inverse moments

The importance of inverse or negative moments are well known in the estimation of the parameters of a model and also for testing the efficiency of various estimates. Besides they are equally useful in life testing and in survey sampling, where ratio estimates are being employed. Consul [15] gave a detailed account of negative moments of QBD I. In this section similar properties have been studied for the class of WQBD by deriving general formulas and listing some particular cases.

**Theorem 2.4.1** *If $X \sim$class of* WQBD *(2.2.7), then*

$$\mathrm{E}\left[\frac{X^{(r)}(n-X)^{(u)}}{(p+X\phi)^v(1-p-X\phi)^w}\right] = n^{(r+u)}\frac{B_{n-r-u}(p+r\phi, 1-p-n\phi+u\phi; s+r-v, t+u-w; \phi)}{B_n(p, 1-p-n\phi; s, t; \phi)} \tag{2.4.1}$$

Some important results on negative moments using the above general formula are listed below.

### 2.4.1 QBD I

$$\mathrm{E}(p+X\phi)^{-1} = \frac{1}{p} - \frac{np}{p+\phi} \tag{2.4.2}$$

$$\mathrm{E}X(p+X\phi)^{-1} = \frac{np}{p+\phi} \tag{2.4.3}$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-1} = p\sum_{j=0}^{n-3}\phi^j n^{(j+2)} \tag{2.4.4}$$

$$\begin{aligned}\mathrm{E}(p+X\phi)^{-2} &= p^{-2} - np^{-1}\phi(p+\phi)^{-1} - n\phi(p+\phi)^{-2} \\ &- n^{(2)}\phi^2(p+\phi)^{-1}(p+2\phi)^{-1}\end{aligned} \tag{2.4.5}$$

$$\mathrm{E}X(p+X\phi)^{-2} = np(p+\phi)^{-2} - n^{(2)}p\phi(p+\phi)^{-1}(p+2\phi)^{-1} \tag{2.4.6}$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-2} = \frac{n^{(2)}p}{p+2\phi} \tag{2.4.7}$$

$$\mathrm{E}X^2(p+X\phi)^{-2} = np(p+\phi)^{-2} + n^{(2)}p^2(p+\phi)^{-1}(p+2\phi)^{-1} \tag{2.4.8}$$

$$\mathrm{E}X^{(3)}(p+X\phi)^{-2} = p\sum_{j\geq 0}^{n-3}\phi^j n^{(j+3)} \tag{2.4.9}$$

$$\mathrm{E}X^2(X-1)(p+X\phi)^{-2} = p\sum_{j=0}^{n-3}\phi^j n^{(j+3)} + 2pn^{(2)}(p+2\phi)^{-1} \tag{2.4.10}$$

$$\begin{aligned}\mathrm{E}(p+X\phi)^{-3} &= p^{-3} - n\phi(p+\phi)^{-1}[p^{-2} + p^{-1}(p+\phi)^{-1} + (p+\phi)^{-2}] \\ &+ n^{(2)}\phi^2(p+\phi)^{-1}(p+2\phi)^{-1}[p^{-1} + (p+2\phi)^{-2} + (p+\phi)^{-1} \\ &\quad (p+2\phi)^{-1}] + n^{(3)}\phi^3(p+\phi)^{-1}(p+2\phi)^{-1}(p+3\phi)^{-1}\end{aligned} \tag{2.4.11}$$

$$\begin{aligned}\mathrm{E}X(p+X\phi)^{-3} &= np(p+\phi)^{-3} - n^{(2)}p\phi[(p+\phi)^{-1}(p+2\phi)^{-2} + (p+\phi)^{-2} \\ &\quad (p+2\phi)^{-1}] + n^{(3)}\phi^2 p(p+\phi)^{-1}(p+2\phi)^{-1}(p+3\phi)^{-1}\end{aligned} \tag{2.4.12}$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-3} = n^{(2)}p(p+2\phi)^{-2} - n^{(3)}p\phi(p+2\phi)^{-1}(p+3\phi)^{-1} \tag{2.4.13}$$

$$\mathrm{E}X^{(3)}(p+X\phi)^{-3} = \frac{n^{(3)}p}{p+3\phi} \tag{2.4.14}$$

$$\mathrm{E}(1-p-X\phi)^{-1} = \frac{1-n\phi}{1-p-n\phi} \tag{2.4.15}$$

$$\mathrm{E}X(1-p-X\phi)^{-1} = \frac{np}{1-p-n\phi} \tag{2.4.16}$$

$$\mathrm{E}(1-p-x\phi)^{-2} = \frac{1}{1-p-n\phi}\left[\frac{1-n\phi}{1-p-n\phi} - n\phi\frac{1-(n-1)\phi}{1-p-n\phi+\phi}\right] \tag{2.4.17}$$

$$\mathrm{E}X(1-p-X\phi)^{-2} = \frac{np}{1-p-n\phi}\left[\frac{1}{1-p-n\phi} - \frac{(n-1)\phi}{1-p-(n-1)\phi}\right] \tag{2.4.18}$$

$$\mathrm{E}(n-X)(1-p-X\phi)^{-2} = \frac{n(1-(n-1)\phi)}{1-p-(n-1)\phi} \tag{2.4.19}$$

$$\mathrm{E}(n-X)X(1-p-X\phi)^{-2} = \frac{n^{(2)}p}{1-p-(n-1)\phi} \tag{2.4.20}$$

$$\mathrm{E}(n-X)X^{(2)}(1-p-X\phi)^{-2} = \frac{p}{1-p-(n-1)\phi}\sum_{\nu=0}^{n-3} n^{(\nu+3)}\phi^{\nu}(p+(\nu+2)\phi) \tag{2.4.21}$$

$$\begin{aligned}\mathrm{E}(n-X)(1-p-X\phi)^{-3} &= n(1-p-(n-1)\phi)^{-2}(1-(n-1)\phi) \\ &- n^{(2)}\frac{\phi(1-(n-2)\phi)}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)}\end{aligned} \tag{2.4.22}$$

$$\begin{aligned}\mathrm{E}(n-X)X(1-p-X\phi)^{-3} &= \frac{n^{(2)}p}{1-p-(n-1)\phi}\left[\frac{1}{1-p-(n-1)\phi}\right. \\ &- \left.\frac{(n-2)\phi}{1-p-(n-2)\phi}\right]\end{aligned} \tag{2.4.23}$$

$$\begin{aligned}\mathrm{E}(n-X)X^{(2)}(1-p-X\phi)^{-3} &= \frac{n^{(3)}p}{(1-p-(n-1)\phi)^2}\sum_{\nu=0}^{n-3}(n-3)^{(\nu)}\phi^{(\nu)}(p+2\phi+\nu\phi) \\ &- n^{(4)}p\,\phi\frac{\sum_{\nu=0}^{n-4}(n-4)^{(\nu)}\phi^{\nu}(p+2\phi+\nu\phi)}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)}\end{aligned} \tag{2.4.24}$$

### 2.4.2 QBD II

$$\mathrm{E}(p+X\phi)^{-1} = \frac{1}{p} - n\phi\frac{1-(n-1)\phi}{(1-n\phi)(p+\phi)} \tag{2.4.25}$$

$$\mathrm{E}\frac{X}{(p+X\phi)} = n\frac{p(1-(n-1)\phi)}{(1-n\phi)(p+\phi)} \tag{2.4.26}$$

$$\mathrm{E}\frac{X^{(2)}}{(p+X\phi)} = n^{(2)}\frac{p}{1-n\phi} \tag{2.4.27}$$

$$\begin{aligned}\mathrm{E}(p+X\phi)^{-2} &= \frac{1}{p^2} - \frac{n\phi(2p+\phi)(1-(n-1)\phi)}{p(1-n\phi)(p+\phi)} \\ &+ \frac{n^{(2)}\phi^2(1-(n-2)\phi)}{(1-n\phi)(p+\phi)(p+2\phi)}\end{aligned} \tag{2.4.28}$$

$$\mathrm{E}\frac{X}{(p+X\phi)^2} = \frac{np}{1-n\phi}\left[\frac{1-(n-1)\phi}{(p+\phi)^2} - \frac{\phi(n-1)(1-(n-2)\phi)}{(p+\phi)(p+2\phi)}\right] \quad (2.4.29)$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-2} = \frac{n^{(2)}p(1-(n-2)\phi}{(p+2\phi)(1-n\phi)} \quad (2.4.30)$$

$$\mathrm{E}X^2(p+X\phi)^{-2} = \frac{np(1-(n-1)\phi)}{(1-n\phi)(p+\phi)^2} + \frac{n^{(2)}p^2(1-(n-2)\phi)}{(p+2\phi)(1-n\phi)(p+\phi)} \quad (2.4.31)$$

$$\mathrm{E}X^{(3)}(p+X\phi)^{-2} = \frac{n^{(3)}p}{1-n\phi} \quad (2.4.32)$$

$$\mathrm{E}X^2(X-1)(p+X\phi)^{-2} = \frac{n^{(2)}p}{1-n\phi}\left[(n-2) + \frac{2(1-(n-2)\phi)}{p+2\phi}\right] \quad (2.4.33)$$

$$\begin{aligned}\mathrm{E}X(p+X\phi)^{-3} &= \frac{np}{(1-n\phi)(p+\phi)}\left[1+\frac{1-p-n\phi}{p+\phi}\right] + \frac{n^{(2)}p\phi}{1-n\phi}\\ &\quad \left[1+\frac{(1-p-n\phi)}{(p+2\phi)} + \frac{1}{p+\phi} + \frac{1-p-n\phi}{(p+\phi)(p+2\phi)}\right]\\ &+ \frac{n^{(3)}p\phi^2(1-p-n\phi)}{(1-n\phi)(p+2\phi)(p+3\phi)}\end{aligned} \quad (2.4.34)$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-3} = \frac{p}{1-n\phi}\left[\frac{n^{(2)}(1-(n-2)\phi)}{p+2\phi} - \frac{n^{(3)}\phi(1-(n-3)\phi)}{(p+2\phi)(p+3\phi)}\right] \quad (2.4.35)$$

$$\mathrm{E}X^{(3)}(p+X\phi)^{-3} = \frac{n^{(3)}p(1-(n-3)\phi)}{(1-n\phi)(p+3\phi)} \quad (2.4.36)$$

$$\mathrm{E}(1-p-X\phi)^{-1} = \frac{1}{1-p-n\phi} - \frac{np(1-(n-1)\phi)}{1-n\phi} \quad (2.4.37)$$

$$\mathrm{E}X(1-p-X\phi)^{-1} = \frac{p}{1-n\phi}\left[\frac{n}{1-p-n\phi} - \frac{n^{(2)}\phi}{1-p-(n-1)\phi}\right] \quad (2.4.38)$$

$$\begin{aligned}\mathrm{E}(1-p-x\phi)^{-2} &= \frac{1}{1-n\phi}\left[\frac{1-n\phi}{(1-p-n\phi)^2} - n\phi\frac{(2-2p-(2n-1)\phi)(1-(n-1)\phi)}{(1-p-n\phi)(1-p-(n-1)\phi)^2}\right.\\ &+ \left. n^{(2)}\phi^2\frac{1-(n-2)\phi}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)}\right]\end{aligned} \quad (2.4.39)$$

$$\begin{aligned}\mathrm{E}X(1-p-X\phi)^{-2} &= \frac{np}{1-n\phi}\left[\frac{(1-p-n\phi)^2}{1-p-(n-2)\phi} - \frac{(n-1)(2-2p-(2n-1)\phi)}{(1-p-n\phi)(1-p-(n-1)\phi)^2}\right.\\ &+ \left.\frac{(n-1)^{(2)}\phi^2}{1-p-(n-1)\phi}\right]\end{aligned} \quad (2.4.40)$$

$$\begin{aligned}\mathrm{E}(n-X)(1-p-X\phi)^{-2} &= \frac{n(1-p-n\phi)}{1-n\phi}\left[\frac{1-(n-1)\phi}{1-p-(n-1)\phi}\right.\\ &- \left.\frac{\phi(n-1)(1-(n-2)\phi)}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)}\right]\end{aligned} \quad (2.4.41)$$

$$\begin{aligned}\mathrm{E}(n-X)X(1-p-X\phi)^{-2} &= \frac{n^{(2)}p(1-p-n\phi)}{(1-n\phi)(1-p-(n-1)\phi)}\left[\frac{1}{1-p-(n-1)\phi}\right.\\ &- \left.\frac{\phi(n-2)}{1-p-(n-2)\phi}\right]\end{aligned} \quad (2.4.42)$$

$$\mathrm{E}[\frac{(n-X)X^{(2)}}{(1-p-X\phi)^2}] = \frac{n^{(3)}p(1-p-n\phi)}{1-n\phi}\left[\frac{\sum_{\nu=0}^{n-3}\phi^\nu(n-3)^{(\nu)}(p+2\phi+\nu\phi)}{(1-p-(n-1)\phi)^2}\right.$$
$$\left.-\frac{(n-3)\phi\sum_{\nu=0}^{n-4}\phi^\nu(n-4)^{(\nu)}(p+2\phi+\nu\phi)}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)}\right] \quad (2.4.43)$$

$$\mathrm{E}[\frac{(n-X)}{(1-p-X\phi)^3}] = \frac{n(1-p-n\phi)}{1-n\phi}\left[\frac{1-(n-1)\phi}{(1-p-(n-1)\phi)^3}\right.$$
$$-\frac{(n-1)\phi(2p+3\phi)(1-(n-2)\phi)}{(1-p-(n-1)\phi)^2(1-p-(n-2)\phi)^2}$$
$$\left.+\frac{(n-1)^{(2)}\phi^2(1-(n-3)\phi)}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)(1-p-(n-3)\phi)}\right] \quad (2.4.44)$$

$$\mathrm{E}(n-X)X(1-p-X\phi)^{-3} = \frac{n^{(2)}p(1-p-n\phi)}{(1-n\phi)(1-p-(n-1)\phi)}\left[\frac{1}{(1-p-(n-1)\phi)^2}\right.$$
$$-\frac{(n-2)\phi(2p+3\phi)}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)^2}$$
$$\left.+\frac{(n-2)^{(2)}\phi^2}{(1-p-(n-2)\phi)(1-p-(n-3)\phi)}\right] \quad (2.4.45)$$

### 2.4.3 QBD III

$$\mathrm{E}(p+X\phi)^{-1}) = \frac{\frac{1}{p^2}-\frac{n\phi(2p+\phi)}{p(p+\phi)^2}+\frac{n^{(2)}\phi^2}{(p+\phi)(p+2\phi)}}{\frac{1}{p}-\frac{n\phi}{(p+\phi)}} \quad (2.4.46)$$

$$\mathrm{E}X(p+X\phi)^{-1} = \frac{np^2}{p(1-n\phi)+\phi}\left[\frac{1}{p+\phi}-\frac{(n-1)\phi}{p+2\phi}\right] \quad (2.4.47)$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-1} = \frac{n^{(2)}p^2(p+\phi)}{(p+2\phi)(p(1-n\phi)+\phi)} \quad (2.4.48)$$

$$\mathrm{E}(p+X\phi)^{-2} = \frac{p^2(p+\phi)}{(p+\phi-np\phi)}\left[\frac{1}{p^4}-\frac{n\phi}{p(p+\phi)}\left(\frac{1}{p^2}+\frac{1}{p(p+\phi)}+\frac{1}{(p+\phi)^2}\right.\right.$$
$$-\frac{(n-1)\phi}{(p+2\phi)}\left(\frac{1}{p}+\frac{1}{(p+2\phi)^2}+\frac{1}{(p+\phi)(p+2\phi)}\right)$$
$$\left.\left.-\frac{(n-1)^{(2)}\phi^2}{(p+2\phi)(p+3\phi)}\right)\right] \quad (2.4.49)$$

$$\mathrm{E}X(p+X\phi)^{-2} = \frac{np^2(p+\phi)}{p+.\phi-np\phi}\left[\frac{1}{(p+\phi)^3}-\frac{(n-1)(2p+3\phi)\phi}{(p+\phi)^2(p+2\phi)^2}\right.$$
$$\left.+\frac{(n-1)^{(2)}\phi^2}{(p+\phi)(p+2\phi)(p+3\phi)}\right] \quad (2.4.50)$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-2} = \frac{n^{(2)}p^2(p+\phi)}{(p+\phi-np\phi)}\left[\frac{1}{p+2\phi}-\frac{(n-2)\phi}{p+3\phi}\right] \quad (2.4.51)$$

$$\mathrm{E}X^2(p+X\phi)^{-2} = \frac{np^2(p+\phi)}{p+\phi-np\phi}\left[\frac{(n-1)}{(p+2\phi)^2}\left(1-\frac{\phi(2p+3\phi)}{(p+\phi)^2}\right)\right.$$
$$\left. - \frac{(n-1)^{(2)}\phi}{(p+2\phi)(p+3\phi)}\left(1-\frac{\phi}{p+\phi}\right)+\frac{1}{(p+\phi)^3}\right] \qquad (2.4.52)$$

$$\mathrm{E}X^{(3)}(p+X\phi)^{-2} = \frac{n^{(3)}p^2(p+\phi)}{(p+3\phi)(p+\phi-np\phi)} \qquad (2.4.53)$$

$$\mathrm{E}X^2(X-1)(p+X\phi)^{-2} = \frac{n^{(3)}p^3(p+\phi)}{(p+2\phi)(p+3\phi)(p+\phi-np\phi)}$$
$$+ \frac{2n^{(2)}p^2(p+\phi)}{(p+2\phi)^2(p+\phi-np\phi)} \qquad (2.4.54)$$

$$\mathrm{E}X(p+X\phi)^{-3} = \frac{np^2}{p+\phi-np\phi}\left[\frac{1}{(p+\phi)^{-3}} - \frac{(n-1)\phi}{(p+2\phi)}\left[\frac{1}{(p+\phi)^{-2}}\right.\right.$$
$$\left. + \frac{1}{(p+\phi)(p+2\phi)}+\frac{1}{(p+2\phi)}\right] - \frac{(n-1)^{(2)}\phi^2}{(p+2\phi)(p+3\phi)}$$
$$\left[\frac{1}{(p+\phi)}+\frac{1}{(p+3\phi)^2}+\frac{1}{(p+2\phi)(p+3\phi)}\right]$$
$$\left. + \frac{(n-1)^{(3)}\phi^3}{(p+2\phi)(p+3\phi)(p+4\phi)}\right] \qquad (2.4.55)$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-3} = \frac{n^{(2)}p^2(p+\phi)}{p+\phi-np\phi}\left[\frac{1}{(p+2\phi)^3} - \frac{(n-3)\phi(2p+5\phi)}{(p+2\phi)^2}\right.$$
$$\left. + \frac{(n-2)^{(2)}\phi^2(p+4\phi)}{(p+2\phi)(p+3\phi)}\right] \qquad (2.4.56)$$

$$\mathrm{E}X^{(3)}(p+X\phi)^{-3} = \frac{n^{(3)}p^2(p+\phi)}{(p+\phi-np\phi)(p+3\phi)}\left[\frac{1}{p+3\phi}-\frac{(n-3)\phi}{p+4\phi}\right] \qquad (2.4.57)$$

$$\mathrm{E}(1-p-X\phi)^{-1} = \frac{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi)}{(1-n\phi)(p+\phi-np\phi)} \qquad (2.4.58)$$

$$\mathrm{E}X(1-p-X\phi)^{-1} = \frac{np^2(1-(n-1)\phi)}{(p+\phi-np\phi)(1-p-n\phi)} \qquad (2.4.59)$$

$$\mathrm{E}(1-p-X\phi)^{-2} = \frac{1-p-n\phi}{p+\phi-np\phi}\left[\frac{(1-n\phi)(p+\phi)}{1-p-n\phi}\right.$$
$$- \frac{n\phi(1-(n-1)\phi)}{(1-p-n\phi)(1-p-(n-1)\phi)}\{p(1-p-(n-1)\phi)$$
$$\left. + (1-p-n\phi)(p+\phi)\} + \frac{n^{(2)}\phi^2 p(1-(n-2)\phi)}{1-p-(n-1)\phi}\right] \qquad (2.4.60)$$

$$\mathrm{E}X(1-p-X\phi)^{-2} = \frac{np^2}{(p+\phi-np\phi)(1-p-n\phi)}\left[\frac{1-(n-1)\phi}{1-p-n\phi}\right.$$
$$\left. - \frac{(n-1)\phi(1-(n-2)\phi)}{1-p-(n-1)\phi}\right] \qquad (2.4.61)$$

$$\mathrm{E}(n-X)(1-p-X\phi)^{-2} = n\,\frac{(1-(n-1)\phi)(p+\phi-(n-1)\phi p)}{(1-p-(n-1)\phi)(p+\phi-np\phi)} \qquad (2.4.62)$$

$$\mathrm{E}(n-X)X(1-p-X\phi)^{-2} = n^{(2)}p^2\,\frac{1-(n-2)\phi}{(1-p-(n-1)\phi)(p+\phi-np\phi)} \qquad (2.4.63)$$

$$\mathrm{E}(n-X)X^{(2)}(1-p-X\phi)^{-2} = \frac{n^{(3)}p^2(p+\phi)}{(p+\phi-np\phi)(1-p-(n-1)\phi)} \tag{2.4.64}$$

$$\begin{aligned}\mathrm{E}(n-X)(1-p-X\phi)^{-3} &= \frac{n}{p+\phi-n\phi}\left[\frac{(1-(n-1)\phi)(p+\phi)}{(1-p-(n-1)\phi)^2} - (n-1)\phi(1-(n-2)\phi)\right.\\ &\quad \frac{\{(1-p-(n-1)\phi)(p+\phi)+p(1-p-(n-2)\phi)\}}{(1-p-(n-2)\phi)(1-p-(n-1)\phi)^2}\\ &+ \left.\frac{(n-1)^{(2)}\phi^2 p(1-(n-3)\phi)}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)}\right]\end{aligned} \tag{2.4.65}$$

### 2.4.4 QBD IV

$$\begin{aligned}\mathrm{E}(p+X\phi)^{-1}) &= \left[\frac{1-n\phi}{p} - \frac{n\phi(1-(n-1)\phi)}{p+\phi}\right]^{-1}\left[\frac{1-n\phi}{p^2}\right.\\ &- \left.\frac{n\phi(2p+\phi)(1-(n-1)\phi)}{p(1-p-n\phi)(p+\phi)^2} + \frac{n^{(2)}\phi^2}{(p+\phi)(p+2\phi)}\right]\end{aligned} \tag{2.4.66}$$

$$\begin{aligned}\mathrm{E}X(p+X\phi)^{-1} &= \frac{p}{p+\phi}\left[\frac{1-(n-1)\phi}{p+\phi} - \frac{(n-1)(1-(n-2)\phi)}{p+2\phi}\right]\\ &\quad \left[\frac{1-n\phi}{p} - \frac{n\phi(1-(n-1)\phi)}{p+\phi}\right]^{-1}\end{aligned} \tag{2.4.67}$$

$$\mathrm{E}X^{(2)}(p+X\phi)^{-1} = \frac{\left[n^{(2)}p\frac{(1-(n-2)\phi)}{p+2\phi}\right]}{\left[\frac{1-n\phi}{p} - n\phi\frac{1-(n-1)\phi}{p+\phi}\right]} \tag{2.4.68}$$

$$\begin{aligned}\mathrm{E}X(p+X\phi)^{-2} &= \frac{np^2(p+\phi)}{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi}\left[\frac{1-(n-1)\phi}{(p+\phi)^3}\right.\\ &- \frac{(n-1)\phi(2p+3\phi)(1-(n-2)\phi)}{(p+\phi)^2(p+2\phi)^2}\\ &+ \left.\frac{(n-1)^{(2)}\phi^2(1-(n-3)\phi)}{(p+\phi)(p+2\phi)(p+3\phi)}\right]\end{aligned} \tag{2.4.69}$$

$$\begin{aligned}\mathrm{E}X^{(2)}(p+X\phi)^{-2} &= \frac{n^{(2)}p^2(p+\phi)}{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi)}\left[\frac{1-(n-2)\phi}{(p+2\phi)^2}\right.\\ &- \left.\frac{(n-2)\phi(1-(n-3)\phi)}{(p+2\phi)(p+3\phi)}\right]\end{aligned} \tag{2.4.70}$$

$$\begin{aligned}\mathrm{E}X^2(p+X\phi)^{-2} &= \frac{n^{(2)}p^2(p+\phi)}{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi)}\left[\frac{1-(n-1)\phi}{(p+\phi)^3}\right.\\ &- \frac{(n-1)(1-(n-2)\phi)}{(p+2\phi)^2}(1-\frac{\phi(2p+3\phi)}{(p+\phi)^2})\\ &- \left.\frac{(n-2)^{(2)}\phi(1-(n-3)\phi)}{(p+2\phi)(p+3\phi)}(1-\frac{\phi}{p+\phi})\right]\end{aligned} \tag{2.4.71}$$

$$\mathrm{E}X^{(3)}(p+X\phi)^{-2} = \frac{n(3)p^2(p+\phi)(1-(n-3)\phi)}{(p+3\phi)\{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi\}} \tag{2.4.72}$$

$$\mathrm{E}X^2(X-1)(p+X\phi)^{-2} = \frac{n^{(2)}p^2(p+\phi)}{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi)}$$

$$\left[\frac{2(1-(n-2)\phi)}{(p+2\phi)^2}+\frac{(n-2)p(1-(n-3)\phi)}{(p+2\phi)(p+3\phi)}\right] \quad (2.4.73)$$

$$\begin{aligned} \mathrm{E}X^{(2)}(p+X\phi)^{-3} &= \frac{n^{(2)}p^2(p+\phi)}{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi)}\left[\frac{1-(n-2)\phi}{(p+2\phi)^3}\right. \\ &- \frac{(n-2)\phi(2p+5\phi)(1-(n-3)\phi)}{(p+2\phi)^2(p+3\phi)^2} \\ &+ \left.\frac{(n-2)^{(2)}\phi^2(1-(n-4)\phi)}{(p+2\phi)(p+3\phi)(p+4\phi)}\right] \end{aligned} \quad (2.4.74)$$

$$\begin{aligned} \mathrm{E}X^{(3)}(p+X\phi)^{-3} &= \frac{n^{(3)}p^2(p+\phi)}{(p+\phi)(1-n\phi)-np\phi(1-(n-1)\phi)}\left[\frac{1-(n-3)\phi)}{(p+3\phi)^2}\right. \\ &- \left.\frac{(n-3)\phi(1-(n-4)\phi)}{(p+3\phi)(p+4\phi)}\right] \end{aligned} \quad (2.4.75)$$

$$\begin{aligned} \mathrm{E}(1-p-X\phi)^{-1} &= \frac{1}{1-p-(n-1)\phi)}\left[(1-n\phi)(p+\phi)(1-p-(n-1)\phi)\right. \\ &- n\phi(1-(n-1)\phi)\{(1-p-n\phi)(p+\phi)+p(1-p-(n-1)\phi)\} \\ &+ \left.p(1-p-(n-1)\phi)+n^{(2)}p\phi^2(1-p-n\phi)\right] \\ & \{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi)\}^{-1} \end{aligned} \quad (2.4.76)$$

$$\begin{aligned} \mathrm{E}X(1-p-X\phi)^{-1} &= \frac{np^2}{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi)} \\ & \left[\frac{1-(n-1)\phi}{1-p-n\phi}-\frac{(n-1)(1-(n-2)\phi)}{1-p-(n-1)\phi}\right] \end{aligned} \quad (2.4.77)$$

$$\begin{aligned} \mathrm{E}X(1-p-X\phi)^{-2} &= \frac{np^2}{(1-n\phi)(p+\phi)-np\phi(1-(n-1)\phi)}\left[\frac{1-(n-1)\phi}{(1-p-n\phi)^2}\right. \\ &- \frac{(n-1)\phi(2-2p-(2n-1)\phi)(1-(n-2)\phi)}{(1-p-(n-1)\phi)^2} \\ &+ \left.\frac{(n-2)^{(2)}\phi^2(1-(n-3)\phi)}{(1-p-(n-1)\phi)(1-p-(n-2)\phi)}\right] \end{aligned} \quad (2.4.78)$$

### 2.4.5 Inverse factorial moments

**Theorem 2.4.2** *If $X \sim$ class of* WQBD *(2.2.7), then*

$$\begin{aligned} \mathrm{E}\left[\frac{1}{(X+1)^{[r]}}\right] &= \mathrm{E}\left[\frac{1}{(X+r)^{(r)}}\right] = \frac{B_{n+r}(p-r\phi,1-p-n\phi;s-r,t;\phi)}{(n+1)^{[r]}B_n(p,1-p-n\phi;s,t;\phi)} \\ &- \frac{\sum_{y\geq 0}^{r-1}\binom{n+r}{y}(p-r\phi+y\phi)^{y+(s-r)}(1-p+(r-y)\phi)^{n+r-y+t}}{(n+1)^{[r]}B_n(p,1-p-n\phi;s,t;\phi)} \end{aligned} \quad (2.4.79)$$

Putting $\phi = 0$, we get for binomial distribution (Johnson et al. [51], p.109) with parameters $n, p$

$$\mathrm{E}\left[\frac{1}{(X+r)^{(r)}}\right] = \frac{1 - \sum_{y=0}^{r-1} \binom{n+r}{y} p^y (1-p)^{n+r-y}}{(n+r)^{(r)} p^r} \tag{2.4.80}$$

In particular, the following results can be derived for

### 2.4.5.1 QBD I

$$\mathrm{E}\left[\frac{1}{X+1}\right] = \frac{p}{(n+1)(p-\phi)^2}\left[1 - (n+1)\phi - (1-p+\phi)^{n+1}\right] \tag{2.4.81}$$

$$\begin{aligned} \mathrm{E}\left[\frac{1}{(X+1)(X+2)}\right] &= \frac{p(p-\phi)^{-2}(p-2\phi)^{-3}}{(n+1)(n+2)}\Big[p(p-\phi)^2 - p\phi(n+2)(p-2\phi)(2p-3\phi) \\ &+ (n+1)(n+2)(p-\phi)(p-2\phi)^2 - (p-\phi)^2(1-p+2\phi)^{n+2} \\ &- (n+2)(p-2\phi)^3(1-p+\phi)^{n+1}\Big] \end{aligned} \tag{2.4.82}$$

### 2.4.5.2 QBD II

$$\mathrm{E}\left[\frac{1}{X+1}\right] = \frac{p(1-(n+1)\phi) + \phi(n+1)(1-n\phi)(p-\phi) + p(1-p-n\phi)(1-p+\phi)^n}{(n+1)(1-n\phi)(p-\phi)^2} \tag{2.4.83}$$

$$\begin{aligned} \mathrm{E}\left[\frac{1}{(X+1)(X+2)}\right] &= \frac{1}{(n+1)(n+2)(1-n\phi)(p-\phi)^2(p-2\phi)^3}\Big\{p(p-\phi)(1-(n+2)\phi) \\ &+ np\phi(2p-3\phi)(1-(n+1)\phi)(p-2\phi) + n^{(2)}\phi^2(1-n\phi)(p-2\phi)^2 \\ &- (p-\phi)p(1-p-n\phi)(1-p-2\phi)^{n+1} \\ &+ (n+2)p(1-p-n\phi)(p-2\phi)^3(1-p+\phi)^n(p-\phi)^{-1}\Big\} \end{aligned} \tag{2.4.84}$$

## 2.5 Mode of the class of WQBDs

Consul [15] obtained bounds for the mode of QBD I. Here an attempt has been made to find similar result for the class of WQBDs.

Denoting the mode by $m$, it is observed that $m$ lies between $l$ and $u$ where $l$ is the real positive root of the equation

$$\begin{aligned}\phi^2 m^3 &- [2n-1+2(t-1)]m^2 + [1-p(1+(s+1)\phi-2\phi-2n\phi+n\phi+(t-1)\phi) \\ &- 2\phi+(s+1)\phi(1-n\phi-(t-1)\phi)-2n\phi(1-n\phi-(t-1)\phi)]m \\ &+ (1-p)^2+n(p+(s+1)\phi)(1-p-n\phi-(t-1)\phi) > 0 \qquad (2.5.1)\end{aligned}$$

and

$$u = \frac{(n+1)(p+(s-1)\phi)}{1-(n-(t+s)+2)\phi} \qquad (2.5.2)$$

Proof: Similar to the one provided for QBD I (Consul [15]).

It is seen that for $s=-1, t=0$ the above result reduces to the bound given by Consul [15]. Similar bounds for other distributions belonging to the WQBD class can be easily derived using the above result.

## 2.6 Estimation

Consul [15] discussed MLE (Maximum likelihood estimation) of the parameters of the QBD I for raw as well as for grouped data set and suggested starting values for solving the ML equations numerically. He also provide exact solutions when number of classes are small (two, three and four). Here the problem of estimation of the parameters of QBD I, QBD II, QBD III and QBD IV using different methods of estimation have been discussed. It is assumed, that observed frequency in a random sample of size $N$ are $n_k$, $k=0(1)m$ for different classes, i.e., $\sum_{k=0}^{m} n_k = N$, where $m$ is of course the largest value observed. Here, the parameter $n$ is estimated by $m$, $\bar{x}$ is the sample mean, $f_0$ frequency of zeros and $f_1$ frequency ones. Since the analytical solution of the ML equations are not easy, they are solved numerically using Newton-Rapson method. The second ordered partial derivatives needed for implementing the method have been provided. In solving ML equations numerically by successive approximation, estimates of $p$ and $\phi$ obtained by other methods may be taken as the starting values for $p$ and $\phi$.

### 2.6.1 QBD I

The pf of QBD I with parameters $p, \phi$ is given by

$$p_k = \binom{n}{k} p\,(p + k\phi)^{k-1}(1 - p - k\phi)^{n-k} \tag{2.6.1}$$

I. ***By proportion of zeros and ones***

$$\hat{p} = 1 - \left(\frac{f_0}{N}\right)^{\frac{1}{n}} \tag{2.6.2}$$

$$\hat{\phi} = 1 - \hat{p} - \left(\frac{f_1}{n\hat{p}N}\right)^{\frac{1}{n-1}} \tag{2.6.3}$$

II. ***By proportion of zeros and sample mean***

cOnce the estimate of $p$ is obtained, the estimate of $\phi$ can be obtained by numerically solving the equation

$$np\sum_{i-0}^{n-1}(n-1)^{(i)}\phi^i - \bar{x} = 0 \tag{2.6.4}$$

Equation (2.6.4) can be solved by standard techniques like Newton-Rapson or by employing direct search methods. Initial value may be 0 or the estimate of $\phi$ as given by method of proportion of zeros and ones.

III. ***ML method***

The loglikelihood function is given by

$$\begin{aligned} l = \log L \quad &\propto \quad N \log p + \sum_{k=0}^{n} n_k(k-1)\log(p + k\phi) \\ &+ \quad \sum_{k=0}^{n} n_k(n-k)\log(1 - p - k\phi) \end{aligned} \tag{2.6.5}$$

The two likelihood equations obtained by partially differentiating $l$ w.r.t $p$ and $\phi$ are

$$\frac{\partial l}{\partial p} = \frac{N}{p} + \sum_{k=0}^{n}\frac{n_k(k-1)}{p + k\phi} - \sum_{k=0}^{n}\frac{n_k(n-k)}{1 - p - k\phi} = 0$$

$$\Rightarrow h = 0, \quad \text{say} \tag{2.6.6}$$

$$\frac{\partial l}{\partial \phi} = \sum_{k=0}^{n} \frac{n_k(k-1)k}{p+k\phi} - \sum_{k=0}^{n} \frac{n_k(n-k)k}{1-p-k\phi} = 0$$

$$\Rightarrow g = 0, \quad \text{say} \tag{2.6.7}$$

The partial derivatives of $h$ and $g$ w.r.t $p$ and $\phi$ are

$$\frac{\partial h}{\partial p} = -\frac{N}{p^2} - \sum_{k=0}^{n} \frac{n_k(k-1)}{(p+k\phi)^2} - \sum_{k=0}^{n} \frac{n_k(n-k)}{(1-p-k\phi)^2} \tag{2.6.8}$$

$$\frac{\partial g}{\partial p} = -\sum_{k=0}^{n} \frac{n_k(k-1)k}{(p+k\phi)^2} - \sum_{k=0}^{n} \frac{n_k(n-k)k}{(1-p-k\phi)^2} \tag{2.6.9}$$

$$\frac{\partial g}{\partial \phi} = -\sum_{k=0}^{n} \frac{n_k(k-1)k^2}{(p+k\phi)^2} - \sum_{k=0}^{n} \frac{n_k(n-k)k^2}{(1-p-k\phi)^2} \tag{2.6.10}$$

It may be noted that

$$\frac{\partial h}{\partial \phi} = \frac{\partial g}{\partial p}$$

### 2.6.2 QBD II

The pf of QBD II is given by

$$p_k = \left[\frac{p(1-p-n\phi)}{1-n\phi}\right] \binom{n}{k} (p+k\phi)^{k-1}(1-p-k\phi)^{n-k-1}, \quad 1-n\phi \neq 0 \tag{2.6.11}$$

I. ***By proportion of zeros and ones***

First, the estimate of the parameter $p$ is obtained by numerically solving the following equation

$$\left(1-p-\frac{(1-p)^n-p_0}{n[(1-p)^{n-1}-p_0]}\right)\left[np\left(1-p-\frac{(1-p)^n-p_0}{(1-p)^{n-1}-p_0}\right)\right]^{\frac{1}{n-2}} - \left[p\left(1-\frac{(1-p)^n-p_0}{(1-p)^{n-1}-p_0}\right)\right]^{\frac{1}{n-2}} = 0 \tag{2.6.12}$$

Using direct search method, one can get a root of (2.6.12). Then the estimate of the parameter $\phi$ is obtained from the equation

$$n\phi = \frac{(1-p)^n-p_0}{(1-p)^{n-1}-p_0} \tag{2.6.13}$$

II. ***Sample proportion of zeros and the mean***

The estimate of $p$ is obtained by numerically solving the following equation

$$p(1-p)^{n-1}(n-\bar{x}) - np_0 p = 0 \tag{2.6.14}$$

then substituting the value of $p$ in (2.6.13), $\phi$ can be estimated.

III. ***ML method***

The loglikelihood function is given by

$$\begin{aligned} l = \log L \quad \propto \quad & N \log\left[\frac{p(1-p-n\phi)}{1-n\phi}\right] + \sum_{k=0}^{n} n_k(k-1)\log(p+k\phi) \\ + \quad & \sum_{k=0}^{n} n_k(n-k-1)\log(1-p-k\phi) \end{aligned} \tag{2.6.15}$$

The two likelihood equations obtained by partially differentiating $l$ w.r.t $p$ and $\phi$ are

$$\begin{aligned} \frac{\partial l}{\partial p} &= \frac{N}{p} - \frac{N}{1-p-n\phi} + \sum_{k=0}^{n} \frac{n_k(k-1)}{p+k\phi} - \sum_{k=0}^{n} \frac{n_k(n-k-1)}{1-p-k\phi} = 0 \\ \Rightarrow g &= 0, \text{ say} \end{aligned} \tag{2.6.16}$$

$$\begin{aligned} \frac{\partial l}{\partial \phi} &= -\frac{nNp}{(1-p-n\phi)(1-n\phi)} + \sum_{k=0}^{n} \frac{n_k(k-1)k}{p+k\phi} - \sum_{k=0}^{n} \frac{n_k(n-k-1)k}{1-p-k\phi} = 0 \\ \Rightarrow h &= 0, \text{ say} \end{aligned} \tag{2.6.17}$$

The partial derivatives of $g$ and $h$ w.r.t $p$ and $\phi$ are

$$\frac{\partial g}{\partial p} = -\frac{N}{p^2} - \frac{N}{(1-p-n\phi)^2} - \sum_{k=0}^{n} \frac{n_k(k-1)}{(p+k\phi)^2} - \sum_{k=0}^{n} \frac{n_k(n-k-1)}{(1-p-k\phi)^2} \tag{2.6.18}$$

$$\frac{\partial g}{\partial \phi} = -\frac{nN}{(1-p-n\phi)^2} - \sum_{k=0}^{n} \frac{n_k(k-1)k}{(p+k\phi)^2} - \sum_{k=0}^{n} \frac{n_k(n-k-1)k}{(1-p-k\phi)^2} \tag{2.6.19}$$

$$\frac{\partial h}{\partial \phi} = \frac{n^2Np(2-p-2n\phi)}{(1-p-n\phi)^2(1-np)^2} - \sum_{k=0}^{n} \frac{n_k(k-1)k^2}{(p+k\phi)^2} - \sum_{k=0}^{n} \frac{n_k(n-k-1)k^2}{(1-p-k\phi)^2} \tag{2.6.20}$$

It may be noted that

$$\frac{\partial h}{\partial p} = \frac{\partial g}{\partial \phi}$$

### 2.6.3 QBD III

The pf of QBD III is given by

$$p_k = \frac{p^2(p+\phi)}{p+\phi-np\phi}\binom{n}{k}(p+k\phi)^{k-2}(1-p-k\phi)^{n-k} \tag{2.6.21}$$

I. ***Proportion of zeros and mean***

Here first, the parameter $p$ is estimated numerically by solving the following equation for $p$

$$[p_0 - np_0p - (1-p)^n]\left[\frac{np^2-\bar{x}p}{(1-np)\bar{x}}\right] + p(p_0 - (1-p)^n) = 0 \tag{2.6.22}$$

Then estimate of $\phi$ is obtained from

$$\phi = \frac{np^2-\bar{x}p}{(1-np)\bar{x}}, \quad np \neq 1 \tag{2.6.23}$$

II. ***ML method***

The loglikelihood function is given by

$$\begin{aligned} l = \log L \quad &\propto \quad N\log\left[\frac{p^2(p+\phi)}{p+\phi-np\phi}\right] + \sum_{k=0}^{n} n_k(k-2)\log(p+k\phi) \\ &+ \sum_{k=0}^{n} n_k(n-k)\log(1-p-k\phi) \end{aligned} \tag{2.6.24}$$

The two likelihood equations obtained by partially differentiating $l$ w.r.t $p$ and $\phi$ are

$$\begin{aligned} \frac{\partial l}{\partial p} &= -\frac{N(1-n\phi)}{p+\phi-np\phi} + \frac{N}{p+\phi} + \frac{2N}{p} + \sum_{k=0}^{n}\frac{n_k(k-2)}{p+k\phi} - \sum_{k=0}^{n}\frac{n_k(n-k)}{1-p-k\phi} = 0 \\ \Rightarrow g &= 0, \text{ say} \end{aligned} \tag{2.6.25}$$

$$\begin{aligned} \frac{\partial l}{\partial \phi} &= \frac{nNp^2}{(p+\phi)(p+\phi-np\phi)} + \sum_{k=0}^{n}\frac{n_k(k-2)k}{p+k\phi} - \sum_{k=0}^{n}\frac{n_k(n-k)k}{1-p-k\phi} = 0 \\ \Rightarrow h &= 0, \text{ say} \end{aligned} \tag{2.6.26}$$

The partial derivatives of $h$ and $g$ w.r.t $p$ and $\phi$ are

$$\frac{\partial g}{\partial p} = \frac{N(1-n\phi)^2}{(p+\phi-np\phi)^2} - \frac{N}{(p+\phi)^2} - \frac{2N}{p^2} - \sum_{k=0}^{n}\frac{n_k(k-2)}{(p+k\phi)^2} - \sum_{k=0}^{n}\frac{n_k(n-k)}{(1-p-k\phi)^2} \tag{2.6.27}$$

$$\begin{aligned}\frac{\partial g}{\partial \phi} &= \frac{N(1-n\phi)(1-np)}{(p+\phi-np\phi)^2}+\frac{Nn}{p+\phi-np\phi}-\frac{N}{(p+\phi)^2}-\sum_{k=0}^{n}\frac{n_k(k-2)k}{(p+k\phi)^2}\\ &- \sum_{k=0}^{n}\frac{n_k(n-k)k}{(1-p-k\phi)^2} \qquad (2.6.28)\\ \frac{\partial h}{\partial \phi} &= \frac{N(1-np)^2}{(p+\phi-np\phi)^2}-\frac{N}{(p+\phi)^2}-\sum_{k=0}^{n}\frac{n_k(k-2)k^2}{(p+k\phi)^2}-\sum_{k=0}^{n}\frac{n_k(n-k)k^2}{(1-p-k\phi)^2} \qquad (2.6.29)\end{aligned}$$

It may be noted that

$$\frac{\partial h}{\partial p}=\frac{\partial g}{\partial \phi}$$

### 2.6.4 QBD IV

The pf of QBD IV is given by

$$p_k=\frac{p^2(p+\phi)(1-p-n\phi)}{(p+\phi)(1-n\phi)-np\phi(1-(n-1)\phi)}\binom{n}{k}(p+k\phi)^{k-2}(1-p-k\phi)^{n-k-1} \qquad (2.6.30)$$

I. ***Method of moments***

Here, the moment estimates for the parameters $p,\phi$ are obtained by solving the following equations

$$\phi=\frac{1-r}{n-1}-rp \qquad (2.6.31)$$

$$\begin{aligned}&\left[\frac{1-r}{n-1}-rp\right]^2 n(1-(n-1)p)\bar{x}-\left[\frac{1-r}{n-1}-rp\right]\\ &\{\bar{x}(1-2np)+n^{(2)}p^2\}+[np^2-\bar{x}p]=0\\ &\Rightarrow f=0, \quad \text{say} \qquad (2.6.32)\end{aligned}$$

where $r=\frac{\bar{x}}{\text{2nd order sample moment about the origin}}$

Equation (2.6.32) can be solved using Newton Rapson method. The initial value of $p$ is first obtained by searching in $(0,1)$.

Following result is needed for implementing the routine.

$$\frac{\partial f}{\partial p} = \frac{1}{n-1}[-\bar{x}+2rpn+\bar{x}nr+6rp^2n^2-3rp^2n-n\bar{x}r^2$$

$$
\begin{aligned}
&- \quad 3rn^3p^2 - 2rpn^2 + 3n^3\bar{x}r^2p^2 - 6n^2\bar{x}r^2p^2 \\
&+ \quad 3n\bar{x}r^2p^2 + r\bar{x} + 2\bar{x}n^2r^2p - 2\bar{x}nr^2p]
\end{aligned} \tag{2.6.33}
$$

II. ***ML method***

The loglikelihood function is given by

$$
\begin{aligned}
l = \log L \quad &\propto \quad N \log\left[\frac{p^2(p+\phi)(1-p-n\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-1)\phi)}\right] + \sum_{k=0}^{n} n_k(k-2)\log(p+k\phi) \\
&+ \quad \sum_{k=0}^{n} n_k(n-k-1)\log(1-p-k\phi)
\end{aligned} \tag{2.6.34}
$$

The two likelihood equations are

$$
\begin{aligned}
\frac{\partial l}{\partial p} &= -N\frac{(1-n\phi) - n\phi(1-(n-1)\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-1)\phi)} \\
&+ \quad N\frac{(3p^2+2\phi)(1-p-n\phi) - p^2(p+\phi)}{p^2(p+\phi)(1-p-n\phi)} \\
&+ \quad \sum_{k=0}^{n}\frac{n_k(k-2)}{p+k\phi} - \sum_{k=0}^{n}\frac{n_k(n-k-1)}{1-p-k\phi} = 0 \\
\Rightarrow g &= 0, \quad \text{say}
\end{aligned} \tag{2.6.35}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial \phi} &= -N\frac{(1-n\phi) - n(p+\phi) - np(1-2(n-1)\phi)}{(p+\phi)(1-n\phi) - np\phi(1-(n-10\phi)} \\
&+ \quad N\frac{p^2(1-p-n\phi) - np^2(p+\phi)}{p^2(p+\phi)(1-p-n\phi)} \\
&+ \quad \sum_{k=0}^{n}\frac{n_k(k-2)k}{p+k\phi} - \sum_{k=0}^{n}\frac{n_k(n-k-1)k}{1-p-k\phi} = 0 \\
\Rightarrow h &= 0, \quad \text{say}
\end{aligned} \tag{2.6.36}
$$

The partial derivatives of $g$ and $h$ w.r.t $p$ and $\phi$ are

$$
\begin{aligned}
\frac{\partial g}{\partial p} &= N\frac{(6p^2+2\phi)(1-p-n\phi) - 4p^2 - 2p\phi - 2p(p+\phi)}{p^2(p+\phi)(1-p-n\phi)} \\
&- \quad 2N\frac{(3p^2+2p\phi)(1-p-n\phi) - p^2(p+\phi)}{p^3(p+\phi)(1-p-n\phi)} \\
&- \quad N\frac{(3p^2+2p\phi)(1-p-n\phi) - p^2(p+\phi)}{p^2(p+\phi)^2(1-p-n\phi)} \\
&+ \quad N\frac{(3p^2+2p\phi)(1-p-n\phi) - p^2(p+\phi)}{p^2(p+\phi)(1-p-n\phi)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad N\frac{(1-n\phi-n\phi(1-(n-1)\phi))^2}{((p+\phi)(1-n\phi)-np\phi(1-(n-1)\phi)^2} \\
&- \quad \sum_{k=0}^{n}\frac{n_k(k-2)}{(p+k\phi)^2}-\sum_{k=0}^{n}\frac{n_k(n-k)}{(1-p-k\phi)^2} \qquad (2.6.37) \\
\frac{\partial g}{\partial \phi} &= N\frac{2p(1-p-n\phi)-(3p^2+2p\phi)n-p^2}{p^2(p+\phi)(1-p-n\phi)} \\
&- \quad N\frac{(3p^2+2p\phi)(1-p-n\phi)-p^2(p+\phi)}{p^2(p+\phi)^2(1-p-n\phi)} \\
&+ \quad N\frac{(3p^2+2p\phi)(1-p-n\phi)-p^2(p+\phi)}{p^2(p+\phi)(1-p-n\phi)^2} \\
&+ \quad N\frac{(1-n\phi-n\phi(1-(n-1)\phi))(1-n\phi-n(p+\phi)-np(1-(n-1)\phi)+np\phi(n-1)}{[(p+\phi)(1-n\phi)-np\phi(1-(n-1)\phi)]^2} \\
&+ \quad N\frac{2n(1-\phi(n-1))}{(p+\phi)(1-n\phi)-np\phi(1-(n-1)\phi)} \\
&- \quad \sum_{k=0}^{n}\frac{n_k(k-2)k}{(p+k\phi)^2}-\sum_{k=0}^{n}\frac{n_k(n-k-1)k}{(1-p-k\phi)^2} \qquad (2.6.38) \\
\frac{\partial h}{\partial \phi} &= N\left[\frac{(1-n\phi)-n(p+\phi)-np(1-2(n-1)\phi)}{(p+\phi)(1-n\phi)-np\phi(1-(n-10\phi)}\right]^2 \\
&+ \quad 2nN\frac{1-p(n-1)}{(p+\phi)(1-n\phi)-np\phi(1-(n-1)\phi)} \\
&- \quad 2nN\frac{1}{(p+\phi)(1-p-n\phi)} \\
&- \quad N\frac{p^2(1-p-n\phi)-np^2(p-\phi)}{p^2(p+\phi)^2(1-p-n\phi)} \\
&+ \quad Nn\frac{p^2(1-p-n\phi)-np^2(p+\phi)}{p^2(p+\phi)(1-p-n\phi)^2} \\
&- \quad \sum_{k=0}^{n}\frac{n_k(k-2)k^2}{(p+k\phi)^2}-\sum_{k=0}^{n}\frac{n_k(n-k-1)k^2}{(1-p-k\phi)^2} \qquad (2.6.39)
\end{aligned}
$$

It may be noted that

$$\frac{\partial h}{\partial p}=\frac{\partial g}{\partial \phi}$$

## 2.7 Data fitting

In this section, the fittings of the different QBDs to four sets of data using the method of maximum likelihood have been presented.

Example 1. Here, McGuire, Brindley and Bancroft's data on the Europian Corn borer, used by Shumway and Gurland [65], Crow and Bardwell [27] and Consul [15] is considered.

Table 2.1: Observed and Expected frequencies of Europian Corn borer in 1296 Corn plants

| No. of borers per plant | Observed no. of plants | QBD I | QBD II | QBD III | QBD IV |
|---|---|---|---|---|---|
| 0 | 907 | 906.41 | 906.45 | 905.94 | 906.01 |
| 1 | 275 | 277.40 | 277.26 | 278.84 | 278.63 |
| 2 | 88 | 85.90 | 86.01 | 84.75 | 84.90 |
| 3 | 23 | 22.59 | 22.60 | 22.56 | 22.58 |
| $\geq 4$ | 3 | 3.70 | 3.68 | 3.91 | 3.88 |
| $\hat{p}$ | | 0.0855 | 0.0797 | 0.1281 | 0.1187 |
| $\hat{\phi}$ | | 0.0591 | 0.0557 | 0.0655 | 0.0613 |
| $\chi^2$ | | 0.2124 | 0.0.1975 | 0.3991 | 0.2788 |
| d.f. | | 2 | 2 | 2 | 2 |

As measured by $\chi^2$, all the models give almost equally good fit, but QBD II is better than the rest.

Example 2. Classical data derived from haemacytometer yeast cell counts observed by 'Student' in 400 squares of haemacytometer used by Crow and Bardwell [27], Consul [15] see also Hand et al. [39].

Table 2.2: Distribution of yeast cells per square in a haemacytometer

| No. of cells per square | Observed no. of squares | QBD I | QBD II | QBD III | QBD IV |
|---|---|---|---|---|---|
| 0 | 213 | 215.73 | 215.72 | 215.84 | 215.83 |
| 1 | 128 | 118.28 | 118.29 | 118.31 | 118.31 |
| 2 | 37 | 47.23 | 47.25 | 46.99 | 47.02 |
| 3 | 18 | 14.89 | 14.89 | 14.89 | 14.89 |
| 4 | 3 | 3.43 | 3.42 | 3.50 | 3.49 |
| 5 | 1 | 0.44 | .44 | .47 | .46 |
| $\hat{p}$ | | 0.1162 | 0.1110 | 0.1147 | 0.1400 |
| $\hat{\phi}$ | | 0.0391 | 0.0376 | 0.0419 | 0.0401 |
| $\chi^2$ | | 3.7031 | 3.6996 | 3.6047 | 3.6162 |
| d.f. | | 2 | 2 | 2 | 2 |

Here also, all the models give almost equally good fit, but QBD III is better than the rest.

Example 3. Taken from Ord et al. [57] is based on field data on D. bimaculatus by time of the day.

Table 2.3: Distribution of number of seeds by time of day

| Time | Observed no. seeds | QBD I | QBD II | QBD III | QBD IV |
|---|---|---|---|---|---|
| 0 | 7 | 6.50 | 6.77 | 6.25 | 6.59 |
| 1 | 4 | 5.38 | 4.95 | 5.67 | 5.24 |
| 2 | 5 | 4.55 | 4.28 | 4.73 | 4.39 |
| 3 | 5 | 4.18 | 4.28 | 4.16 | 4.18 |
| 4 | 4 | 4.40 | 4.95 | 4.18 | 4.72 |
| 5 | 7 | 6.99 | 6.77 | 7.02 | 6.88 |
| $\hat{p}$ | | .2729 | .1934 | .3612 | .2734 |
| $\hat{\phi}$ | | .1346 | .1226 | .1219 | .1143 |
| $\chi^2$ | | .6342 | .6225 | .7647 | .6765 |
| d.f. | | 3 | 3 | 3 | 3 |

While all the models are equally good it can be seen that only the QBD II preserves the symmetry of the original data. It should be noted here that QBD II is a symmetric distribution when $\phi = \frac{1-2p}{n}$

Example 4. This data about incidence of flying bombs in an area in south London during world war II is taken from Feller [33] used by Clarke [9].

Table 2.4: Distribution of number of hits per square.

| No. of hits | No. of 1/4 km squares | QBD I | QBD II | QBD III | QBD IV |
|---|---|---|---|---|---|
| 0 | 229 | 231.35 | 231.35 | 231.35 | 231.36 |
| 1 | 211 | 203.60 | 203.59 | 203.66 | 203.65 |
| 2 | 93 | 100.24 | 100.26 | 100.15 | 100.16 |
| 3 | 35 | 33.01 | 33.01 | 32.99 | 33.00 |
| 4 | 7 | 7.04 | 7.04 | 7.07 | 7.07 |
| 5 | 1 | .76 | .76 | .77 | .77 |
| $\hat{p}$ | | .1668 | .1619 | .1847 | .1819 |
| $\hat{\phi}$ | | .0263 | .0257 | .0271 | .0263 |
| $\chi^2$ | | .9409 | .9344 | .9346 | .9156 |
| d.f. | | 2 | 2 | 2 | 2 |

Clearly from the value of the $\chi^2$ and the expected frequencies, all the models are equally good but QBD IV is better than the rest.

## 2.8 Zero truncated WQBDs

Some times in real life situations it may not be possible to count the number of zeros in other words zero is not observable. Such occurrences are common in ecological experiments (Ord et al. [57]. Here, two zero truncated quasi binomial distributions, their factorial and inverse factorial moments have been studied. The pf of zero truncated WQBD class is given by

$$Pr(X=k)=\frac{\binom{n}{k}(p+k\phi)^{k+s}(1-p-k\phi)^{n-k+t}}{B_n(p,1-p-n\phi;s,t;\phi)-p^s(1-p)^{n+t}} \tag{2.8.1}$$

The $r$th factorial moment about origin for (2.8.1) is

$$\begin{aligned} \mathrm{E}[X^{(r)}] &= n^{(r)}\frac{B_{n-r}(p+r\phi,1-p-n\phi;s+r,t;\phi)}{B_n(p,1-p-n\phi;s,t;\phi)-p^s(1-p)^{n+t}} && (2.8.2)\\ &= \frac{B_n(p,1-p-n\phi;s,t;\phi)}{B_n(p,1-p-n\phi;s,t;\phi)-p^s(1-p)^{n+t}}\ \mu_{(r)} && (2.8.3)\end{aligned}$$

where $\mu_{(r)}$ is the corresponding factorial moment of the WQBD class.

The $r$th inverse factorial moment of (2.8.1) is given by

$$\begin{aligned}\mathrm{E}\left[\frac{1}{(X+1)^{[r]}}\right] &= \frac{1}{B_n(p,1-p-n\phi;s,t;\phi)-p^s(1-p)^{n+t}(n+1)^{[r]}}\,[B_{n+r}(p-r\phi,1-p-n\phi;\\ & s-r,t;\phi)-\sum_{y=0}^{r}\binom{n+r}{y}(p-r\phi+y\phi)^{y+s-r}(1-p+(r-y)\phi)^{n+r-y+t}] \quad (2.8.4)\end{aligned}$$

For $\phi=0$ (i.e. for binomial distribution with parameters $n,p$), left hand side of (2.8.4) reduces to

$$\frac{I_p(r+1;n)}{p^r(n+1)^{[r]}(1-(1-p)^n)} \tag{2.8.5}$$

where

$$I_p(x,n-x+1)=\sum_{j=x}^{n}\binom{n}{j}p^j(1-p)^{n-j},\quad \text{(Johnson et al. [51], p.130)}$$

### 2.8.1 Zero truncated QBD I

The pf is

$$p_k=\binom{n}{k}\frac{p(p+k\phi)^{k-1}(1-p-k\phi)^{n-k}}{(1-(1-p)^n)} \tag{2.8.6}$$

It's mean is

$$\mathrm{E}[X] = \frac{np(1+\alpha\phi)^{n-1}}{1-(1-p)^n} \tag{2.8.7}$$

where $\alpha$ is umbral defined earlier and

$$\mathrm{E}[X^{(2)}] = \frac{n^{(2)}p}{1-(1-p)^n} \sum_{\nu=0}^{n-2} \sum_{\gamma=0}^{\nu} (n-2)^{(\nu)} \phi^{\nu} (p + 2\phi + \gamma\phi) \tag{2.8.8}$$

### 2.8.2 Zero truncated QBD II

The pf is

$$p_k = \binom{n}{k} \frac{p(p+k\phi)^{k-1}(1-p-n\phi)(1-p-k\phi)^{n-k-1}}{1-n\phi-(1-p-n\phi)(1-(1-p)^{n-1})} \tag{2.8.9}$$

The mean is

$$\mathrm{E}[X] = \frac{np}{1-n\phi-(1-p-n\phi)(1-(1-p)^{n-1})} \tag{2.8.10}$$

and

$$\mathrm{E}\left[X^{(2)}\right] = \frac{p}{1-n\phi-(1-p-n\phi)(1-(1-p)^{n-1})} \sum_{\nu=0}^{n-2} n^{(\nu+2)} \phi^{\nu} (p + 2\phi + \nu\phi) \tag{2.8.11}$$

## 2.9 Limiting Distribution

**Theorem 2.9.1** *As $n \to \infty$ and $p, \phi \to 0$ such that $np = \lambda, n\phi = \psi$ the class of WQBD (2.2.7) tends to the WGPD class with parameters $(\lambda; s; \psi)$.*

*Proof.* The proof is given in the theorem (5.18.1) in page number 136.

# Chapter 3

# Unification of Probability Models

## 3.1 Introduction

In the first part of this chapter, a three urn setup with a predetermined strategy has been used to combine QED and QIPD to construct a unified probability model (UPM) and hence obtain all their particular cases. Some recurrence relations among the moments and probabilities are established. A few limiting distributions are mentioned. In the second part starting with section §3.3, a five urn setup with a predetermined strategy is used to unify GMPD and GIMPD to obtain a generalized probability model (GPM) which gives all the particular cases of GMPD and GIMPD and some new distributions. For this model also some recurrence relation among moments and probabilities are derived and limiting distributions are mentioned. Steps of ML estimation by numerical method are discussed for both the models.

## 3.2 Model I : *An urn model with three urns*

Consider three urns $A, B$ and $C$. $A$ is empty, $B$ contains $a$ white, $C$ $a$ white and $b$ black balls. For given positive integers $n, \phi, z$ and integer $s$, a strategy is determined by selecting an integer $k \geq 0$. Once this integer is selected, $n$ white, $\phi k$ black balls to $A$, $kz$ black balls to $B$ and $kz$ white,

$(n+(\phi-1)k)z$ black balls to $C$ are added. The constitution of the urns will now be as below:

Table 3.1: Constitution of the urns

| | *Number of balls* | |
|---|---|---|
| *Urn* | white | black |
| $A$ | $n$ | $\phi k$ |
| $B$ | $a$ | $kz$ |
| $C$ | $a+kz$ | $b+(n+(\phi-1)k)z$ |

Now a ball is drawn from $A$, if it is white, a ball is drawn from $B$, if it is white too, then $n+\phi k$ draws are made from $C$ successively one by one with replacement, where after each draw, the ball drawn is replaced with $s$ additional balls of the same colour before the next draw. Success is achieved if exactly $k$ of this $n+\phi k$ balls are white.

Clearly, the probability of success is the joint probability of drawing a white from $A$ in the first trial, a white from $B$ in the second trial and then exactly $k$ white balls in $n+\phi k$ repeated trials from $C$ using sampling scheme stated above.

$$\begin{aligned} Pr(\text{Success} \mid \text{Strategy } k) &= P(k) = \frac{n}{n+\phi k}\frac{a}{a+kz}\binom{n+\phi k}{k} \\ &\quad \frac{(a+kz)^{[k,s]}\,(b+(n+(\phi-1)k)z)^{[n+(\phi-1)k,s]}}{(a+b+(n+\phi k)z)^{[n+\phi k,s]}} \\ &= \frac{n}{n+\phi k}\,\frac{a}{a+kz}\,\frac{\left\langle {a+kz \atop k} \right\rangle_s \left\langle {b+(n+(\phi-1)k)z \atop n+(\phi-1)k} \right\rangle_s}{\left\langle {a+b+(n+\phi k)z \atop n+\phi k} \right\rangle_s}, \end{aligned} \tag{3.2.1}$$

where $\left\langle {a \atop k} \right\rangle_s = \frac{a^{[k,s]}}{k!} = \frac{a(a+s)\cdots(a+(k-1)s)}{k!}$. Here the parameters $a, b, n, \phi, z, s$ are such that for $k = 0, 1, \ldots;\ P(k) \geq 0$. If $s=-1$, $k = 0, 1, \ldots, \min(\frac{a}{1-z}, n)$. It is possible for $s$ to be negative when $\phi \geq 0$ provided

$$a+b+(n+\phi k)z+(n+\phi k-1)s) > 0.$$

Further, if $a + kz$ is a fractional number, $\left\langle {a+kz \atop k} \right\rangle$ is expressed in gamma function.

The UPM (3.2.1) can be written in the form

$$P(k) = \frac{n}{n+\phi k}\frac{\alpha}{\alpha+kt}\frac{\left\langle {\alpha+kt \atop k} \right\rangle \left\langle {\beta+(n+(\phi-1)k)t \atop n+(\phi-1)k} \right\rangle}{\left\langle {\alpha+\beta+(n+\phi k)t \atop n+\phi k} \right\rangle} \tag{3.2.2}$$

$$\text{where } \frac{a}{s} = \alpha,\ \frac{b}{s} = \beta,\ \frac{z}{s} = t,$$

$$\left\langle {\alpha \atop k} \right\rangle = \frac{\alpha^{[k]}}{k!} = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} \text{ and } \alpha^{[0]} = 1$$

and also as

$$P(k) = \frac{\alpha}{\alpha+kt}\frac{n}{n+\phi k}\binom{n+\phi k}{k}\frac{(\alpha+kt)^{[k]}(\beta+(n+(\phi-1)k)t)^{[n+(\phi-1)k]}}{(\alpha+\beta+(n+\phi k)t)^{[n+\phi k]}} \tag{3.2.3}$$

for $t = 0$ (3.2.3) reduces to GPE (Sen and Mishra [63]), with parameters $n, \phi, \alpha, \beta$.

### 3.2.1 Distributions as special cases

UPM (3.2.1) is not a proper discrete probability distribution for all values of $\phi, z$ and $s$, but it generates most of the well known discrete distributions as well as new distributions for different values of parameters $n, \phi, s, z, a$ and $b$. In the following a list of the well known discrete distributions (Das [29], Johnson, Kotz and Kemp [51], Nandi and Das [54], Sen and Mishra [63], Patil, Boswell, Joshi and Ratnaparkhi [59], Charalambides ([7], [8]) are presented which are particular cases of UPM.

Table 3.2: Special cases of unified probability model

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 1. | $\phi = 0$ | QED (Janardan [43]) | $\frac{a}{a+kz}\frac{\left\langle {a+kz \atop k} \right\rangle_s \left\langle {b+(n-k)z \atop n-k} \right\rangle_s}{\left\langle {a+b+nz \atop n} \right\rangle_s}$ | $0(1)n$ |
| 2. | $\phi = 0, z = 0$ | Polya-Eggenberger (PE) | $\frac{\left\langle {a \atop k} \right\rangle_s \left\langle {b \atop n-k} \right\rangle_s}{\left\langle {a+b \atop n} \right\rangle_s}$ | $0(1)n$ |
| 3. | $s = 1, \phi = 0$ | Quasi beta-binomial (QBB) | $\frac{a}{a+kz}\frac{\left\langle {a+kz \atop k} \right\rangle \left\langle {b+(n-k)z \atop n-k} \right\rangle}{\left\langle {a+b+nz \atop n} \right\rangle}$ | $0(1)n$ |

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 4. | $s=1, \phi=0$ | Quasi negative hypergeometric (Janardan [43]) | $\frac{a}{a+kz} \frac{\binom{a-1+k(z+1)}{k}\binom{b-1+(n-k)(z+1)}{n-k}}{\binom{a+b-1+n(+1)z}{n}}$ | $0(1)n$ |
| 5. | $s=1, \phi=0$, $z=0$ | Beta-Binomial (BB) (Ord [56], Patil & Joshi, [60]) | $\frac{\binom{a+k-1}{k}\binom{b+n-k-1}{n-k}}{\binom{a+b+n-1}{n}}$ | $0(1)n$ |
| 6. | $s=-1, \phi=0$ | Quasi hypergeometric (QH) (Nandi and Das [54], Janardan [43]) | $\frac{a}{a+kz} \frac{\binom{a+kz}{k}\binom{b+(n-k)z}{n-k}}{\binom{a+b+nz}{n}}$ | $\max(0, n-b)$ (1) $\min(n, a)$ |
| 7. | $\phi=0, s=0$ | Quasi Binomial (QB) (Consul and Mittal [11]), Janardan [43]) | $\binom{n}{k} \frac{a(a+kz)^{k-1}(b+(n-k)z)^{n-k}}{(a+b+nz)^n}$ | $0(1)n$ |
| 8. | $\phi=0, z=0$ $s=-1$ | Hypergeometric (HG) | $\frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}}$ | $0(1)\min(n, a)$ |
| 9. | $\phi=0, z=0$ Replace $s$ by $-s$ | Markov-Polya Survival Model (MPSM) (Janardan [46]) | $\frac{\binom{a}{k}_s\binom{b}{n-k}_s}{\binom{a+b}{n}_s}$ | $0(1)\min(n, a)$ |
| 10. | $\phi=0, s=0$, $z=0, a/(a+b)=p$ | Binomial | $\binom{n}{k}p^k(1-p)^{n-k}$ | $0(1)n$ |
| 11. | $\phi=0, z=0$, $a=b=s$ | Uniform or Discrete rectangular | $\frac{1}{n+1}$ | $0(1)n$ |

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 12. | $\phi = 1$ | QIPD (Janardan [43]) | $\binom{n+k-1}{k}\frac{a}{a+kz}$ $\frac{(a+kz)^{[k,s]}(b+nz)^{[n,s]}}{(a+b+(n+k)z)^{[n+k,s]}}$ | $0(1)\infty$ |
| 13. | $\phi = 1, s = -1$ | Quasi inverse hypergeometric (Janardan [43]) | $\binom{n+k-1}{k}\frac{a}{a+kz}$ $\frac{(a+kz)^{(k)}(b+nz)^{(n)}}{(a+b+(n+k)z)^{(n+k)}}$ | $0(1)\infty$ |
| 14. | $\phi = 1, z = 0$ | Inverse Polya-Eggenberger (IPE) | $\binom{n+k-1}{k}\frac{a^{[k,s]}b^{[n,s]}}{(a+b)^{[n+k,s]}}$ | $0(1)\infty$ |
| 15. | $\phi = 1, s = 0, b + nz = p, a + p = 1$ | Quasi negative Binomial (QNB) (Berg [3]) | $\binom{n+k-1}{k}\frac{p^n(1-p)(1-p+kz)^{k-1}}{(1+kz)^{n+k}}$ | $0(1)\infty$ |
| 16. | $\phi = 1, s = 0, z = 0, a/(a+b) = p$ | Negative Binomial (NB) | $\binom{n+k-1}{k}p^k(1-p)^n$ | $0(1)\infty$ |
| 16a. | $m = np/(1-p), p = a/(1+a)$ in 16. | A alternative form of NB used in Ecology (Evans [31]), Patil et al. [59]) | $\binom{m/a+k-1}{k}\left(\frac{a}{1+a}\right)^k\left(\frac{1}{1+a}\right)^{m/a}$ | $0(1)\infty$ |
| 17. | $\phi = 1, s = 0, n = 1, z = 0, a/b = P, Q = 1 + P$ | Geometric | $(P/Q)^kQ^{-1}$ | $0(1)\infty$ |
| 18. | $s = -1, \phi = 1, z = 0$ | Negative hypergeometric (NHG) | $\frac{n}{n+k}\frac{\binom{a}{k}\binom{b}{n}}{\binom{a+b}{n+k}}$ | $0(1)\min(0, a + b - n)$ |

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 19. | $\phi = 1, z = 0, a = b = s$ | Inverse factorial (IF) (Irwin [41]) | $\frac{n}{(n+k)(n+k+1)}$ | $0(1)\infty$ |
| 20. | $s = 1, \phi = 1, z = 0$ Or $s = -1, \phi = 1, z = 1$ | Beta-Pascal (BP) (Ord [56]) | $\frac{n}{n+k} \frac{\binom{a+k-1}{k}\binom{b+n-1}{n}}{\binom{a+b+n+k-1}{n+k}}$ | $0(1)\infty$ |
| 21. | $\phi = 2, s = 0, a/b = \alpha$ (Replacing $k$ by $k-n$), $z = 0$ | Haight (Haight [37]) | $\frac{n}{k}\binom{2k-n-1}{k-1}\frac{\alpha^{k-n}}{(1+\alpha)^{2k-n}}$ | $n(1)\infty$ |
| 22. | $z = 0, s = 0, a/b = P, Q = 1 + P$ | Negative binomial-negative binomial (Consul and Shenton [22]) | $\frac{n}{n+\phi k}\binom{n+\phi k}{k}\left(\frac{P}{Q}\right)^k Q^{-(n+(\phi-1)k)}$ | $0(1)\infty$ |
| 23. | $s = 0, z = 0, a/b = P, Q = 1+P$ (Replacing $n$ by $\phi - 1$ and then $k$ by $k-1$) | Takács (Takács [66]) | $\frac{\phi}{\phi k-1} \binom{\phi k-1}{k-1}\left(\frac{P}{Q}\right)^{k-1} Q^{-(\phi-1)k}$ | $1(1)\infty$ |
| 24. | $z = 0, s = 0, a/(a+b) = p$ | Generalized negative binomial (GNB) (Jain and Consul [42]), Consul and Gupta [17]) | $\frac{n}{n+\phi k}\binom{n+\phi k}{k}p^k (1-p)^{(n+(\phi-1)k)}$ | $0(1)\infty$ |

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 25. | $z = 0, s = 0, \phi = m, a/(a+b) = p$ (Replacing $n$ by $mn$ and $k$ by $k-n$) | Binomial-delta (BD) (Consul and Shenton [22]) | $\frac{n}{k}\binom{mk}{k-n} p^{k-n} (1-p)^{mk+n-k}$ | $n(1)\infty$ |
| 26. | $z = 0, s = 0, \phi = m, a/(a+b) = p$ (Replacing $n$ by $mn$ and $k$ by $k-n$) when $n = 1$ | Consul distribution (CD) (Consul and Shenton [24], Consul [13]) | $\frac{1}{k}\binom{mk}{k-1} p^{k-1} (1-p)^{mk+1-k}$ | $1(1)\infty$ |
| 27. | $z = 0, s = 0, a/b = P, Q = 1+P$ (Replacing $n$ by $(\phi-1)n$ and $k$ by $k-n$) | Negative binomial-delta (Consul and Shenton [22]) | $\frac{n}{k}\binom{\phi k-n-1}{k-n}\left(\frac{P}{Q}\right)^{k-n} Q^{-(\phi-1)k}$ | $n(1)\infty$ |
| 28. | $s = 0, \phi = 2, a/b = P, Q = 1+P, z = 0$ | Negative binomial | $\frac{n}{k}\binom{n+2k-1}{k-1}(P/Q)^k Q^{-n-k}$ | $0(1)\infty$ |
| 29. | $\phi = 0$, Replace $s$ by $-s$ | Generalized quasi hypergeometric (GQH) (Nandi and Das [54]) | $\frac{a}{a+kz}\frac{\binom{a+kz}{k}_s\binom{b+(n-k)z}{n-k}_s}{\binom{a+b+nz}{n}_s}$ | $\max(0, n-b)$ (1) $\min(n, a)$ |

### 3.2.2 Moments

Denoting the expression on the right hand side of (3.2.3) by $P(k; n, \nu, \alpha, \beta, t)$, where $\nu + 1 = \phi$, the $r$ th moment about origin for the model is defined as

$$\begin{aligned} M_r'(n, \nu, \alpha, \beta, t) &= \frac{n\alpha}{\alpha+\beta+t}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t(n+\nu)}{\alpha+t+\beta}\right)^m \\ &\sum_{\ell=0}^{m+1}\binom{m+1}{\ell}\left(\frac{\nu+1}{n+\nu}\right)^{\ell}\{M_{j+\ell}'(n+\nu, \nu, \alpha+t+1, \beta, t) + \\ &\frac{t}{\alpha+t+1}M_{j+\ell+1}'(n+\nu, \nu, \alpha+t+1, \beta, t)\} \end{aligned} \tag{3.2.4}$$

*Proof* :

$$\begin{aligned} M_r'(n, \nu, \alpha, \beta, t) &= \sum_{k\geq 0} k^r P(k; n, \nu, \alpha, \beta, t) \\ &= n\sum_{k\geq 1} k^{r-1}\binom{n+(\nu+1)k-1}{k-1}\frac{\alpha}{\alpha+kt}\frac{(\alpha+kt)^{[k]}(\beta+(n+\nu k)t)^{[n+\nu k]}}{(\alpha+\beta+(n+(\nu+1)k)t)^{[n+(\nu+1)k]}} \\ &= \frac{n\alpha}{\alpha+\beta+t}\sum_{k\geq 0}(1+k)^{r-1}(1+\frac{\nu+1}{n+\nu}k)(1+\frac{t}{\alpha+t+1}k) \\ &\left\{1+\frac{t(n+\nu)}{\alpha+\beta+t}(1+\frac{\nu+1}{n+\nu}k)\right\}^{-1} P(k; n+\nu, \nu, \alpha+t+1, \beta, t) \\ &= \frac{n\alpha}{\alpha+\beta+t}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k\geq 0}k^j(1+\frac{t}{\alpha+t+1}k)\sum_{m\geq 0}(-1)^m\left(\frac{t(n+\nu)}{\alpha+\beta+t}\right)^m \\ &\sum_{\ell=0}^{m+1}\binom{m+1}{\ell}\left(\frac{(\nu+1)k}{n+\nu}\right)^{\ell} P(k; n+\nu, \nu, \alpha+t+1, \beta, t) \\ &= \frac{n\alpha}{\alpha+\beta+t}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{m\geq 0}(-1)^m\left(\frac{t(n+\nu)}{\alpha+\beta+t}\right)^m\sum_{\ell=0}^{m+1}\binom{m+1}{\ell}\left(\frac{(\nu+1)}{n+\nu}\right)^{\ell} \\ &\left[M_{j+\ell}'(n+\nu, \nu, \alpha+t+1, \beta, t) + \frac{t}{\alpha+t+1}M_{j+\ell+1}'(n+\nu, \nu, \alpha+t+1, \beta, t)\right] \end{aligned}$$

For $t = 0$, (3.2.4) reduces to (Sen and Mishra [63])

$$\begin{aligned} M_r'(n, \nu, \alpha, \beta) &= \frac{n\alpha}{\alpha+\beta}\sum_{j=0}^{r-1}\binom{r-1}{j}\{M_j'(n+\nu, \nu, \alpha+1, \beta) + \frac{\nu+1}{n+\nu} \\ &M_{j+1}'(n+\nu, \nu, \alpha+1, \beta)\} \end{aligned} \tag{3.2.5}$$

### 3.2.3 Recurrence relations for mean and second order moment when $\nu = -1$

Putting $r = 1$ and $\nu = -1$ in (3.2.4) we get

$$M_1'(n,\alpha,\beta,t) = \frac{n\alpha}{\alpha+\beta+nt}\left\{1+\frac{t}{\alpha+t+1}M_1'(n-1,\alpha+t+1,\beta,t)\right\} \tag{3.2.6}$$

where $M_1'(n,\alpha,\beta,t) \equiv M_1'(n,-1,\alpha,\beta,t)$ is the mean of the QED (Janardan [43]).

Repeating successively, we get

$$M_1'(n,\alpha,\beta,t) = \frac{n\alpha}{\alpha+\beta+nt}\sum_{j=0}^{n-1}\frac{(n-1)^{(j)}}{(\alpha+\beta+nt+1)^{[j]}}t^j \tag{3.2.7}$$

This result is equivalent to the one obtained by Consul [11].

Putting $r = 2$ and $\nu = -1$ in (3.2.4) we get

$$\begin{aligned} M_2'(n,\alpha,\beta,t) &= M_1'(n,\alpha,\beta,t)+\frac{n\alpha}{\alpha+\beta+nt} \\ &\left\{M_1'(n-1,\alpha+t+1,\beta,t)+\frac{t}{\alpha+t+1}M_2'(n-1,\alpha+t+1,\beta,t)\right\} \end{aligned} \tag{3.2.8}$$

where $M_2'(n,\alpha,\beta,t) \equiv M_2'(n,-1,\alpha,\beta,t)$ is the second order moment of the QED (Janardan [43]).

Some interesting expressions for moments of $X$ following QED with $n,\alpha,\beta,t$ are presented below. Using the generalized quasi factorial distribution (Das [29]), it can be observed that

$$\begin{aligned} &\sum_{k\geq 0}\frac{\beta}{\beta+(n-k)t}\frac{\alpha+\beta+nt}{\alpha+\beta} \\ &\left\{\binom{n}{k}\frac{\alpha}{\alpha+kt}\frac{(\alpha+kt)^{[k]}(\beta+(n-k)t)^{[n-k]}}{(\alpha+\beta+nt)^{[n]}}\right\} = 1 \end{aligned} \tag{3.2.9}$$

But the quantity inside the second bracket is the probability function of $X$ following QED with $n,\alpha,\beta,t$. Therefore

$$\mathrm{E}\left[\frac{1}{\beta+(n-X)t}\right] = \frac{\alpha+\beta}{\beta(\alpha+\beta+nt)}$$

Also using a generalized Vandermonde convolution identity (Gould [34]), it can be seen that

$$\mathrm{E}\left[\frac{c+dX}{\beta+(n-x)t}\right] = \frac{c(\alpha+\beta)+n\alpha d}{\beta(\alpha+\beta+nt)} \quad \text{and}$$

$$\mathrm{E}\left[\frac{(c+dX)(u+v(n-X))}{\beta+(n-x)t}\right] = \frac{cu(\alpha+\beta)}{\beta(\alpha+\beta+nt)}+\frac{(du\alpha+cv\beta)n}{\beta(\alpha+\beta+nt)}+dv\alpha \left[\left\langle {\alpha+\beta+nt \atop n}\right\rangle\right]^{-1}\sum_{j=0}^{n-2}\left\langle {\alpha+\beta+nt+j+2 \atop n-j-2}\right\rangle t^j \quad (3.2.10)$$

Another result that can easily be obtained for QED is

$$\mathrm{E}\left[\alpha+X(t+1)\right] = \frac{\alpha}{\alpha-t}\frac{\alpha+\beta+n(t+1)-1}{n+1}\left[1-\frac{(\beta+(n+1)t)^{[n+1]}}{(\alpha+\beta+nt)^{[n+1]}}\right] \quad (3.2.11)$$

For the mean of the QIPD, we have the following recurrence relation

$$M_1'(n,a,b,z,s) = \frac{na}{b+nz+ns}\sum_{j=0}^{\infty}\frac{(n+1)^{(j)}}{(b+nz+ns)^{[j]}}(z+s)^j \quad (3.2.12)$$

In particular, for $s=0, z=0$ (i.e. for negative binomial) r.h.s. of (3.2.12) reduces to $\frac{na}{b}$ and for $s=-1, z=-1$ (beta Pascal distribution) r.h.s. of (3.2.12) becomes $\frac{na}{b}$.

### 3.2.4 Recurrence relation for probabilities

Following recurrence relations for the probabilities $P(k;n,\phi,\alpha,\beta,t)$ in equation (3.2.3) hold

$$I.\quad P(k+1;n,\phi,\alpha,\beta,t) = \frac{n}{k+1}\frac{n+\phi-1+k}{n+\phi-1}\frac{\alpha}{\alpha+t}\frac{\alpha+t+(t+1)k}{\alpha+\beta-1+(n+\phi+\phi k)(t+1)} P(k;n+\phi-1,\phi,\alpha+t,\beta,t)$$

$$II.\quad P(k;n+1,\phi,\alpha,\beta,t) = \frac{n+1}{n}\frac{n+\phi k}{n+(\phi-1)k+1}\frac{\beta+(n+(\phi-1)k)(t+1)+t}{\alpha+\beta+(n+\phi k)(t+1)+t} P(k;n,\phi,\alpha,\beta+t,t)$$

$$III.\quad P(k;n,\phi,\alpha+1,\beta,t) = \frac{\alpha+1}{\alpha}\frac{\alpha+kt+k}{\alpha+kt+1}\frac{\alpha+\beta+(n+\phi k)t}{\alpha+\beta+(n+\phi k)(t+1)}P(k;n,\phi,\alpha,\beta,t)$$

$$IV.\quad P(k;n,\phi,\alpha,\beta+1,t) = \frac{\beta+(n+(\phi-1)k)(t+1)}{\beta+(n+(\phi-1)k)t}\frac{\alpha+\beta+(n+\phi k)t}{\alpha+\beta+(n+\phi k)(t+1)} P(k;n,\phi,\alpha,\beta,t)$$

Recurrence relations for probabilities of various distributions occurring as particular cases of UPM (3.2.3) can be obtained from the above relations by choosing appropriate values for the parameters $n,\phi,\alpha,\beta,t$.

### 3.2.5 Estimation

In this section various steps involved in the ML estimation of the parameters of QED using iterative numerical method have been outlined. It is assumed that observed frequency in a random sample of size $N$ are $n_k$, $k = 0(1)m$ for different classes, i.e., $\sum_{k=0}^{m} n_k = N$, where $m$ is of course the largest value observed. Here, the parameter $n$ is estimated by $m$.

The pf of QED can be written as

$$p_k = \binom{n}{k} \frac{p \prod_{i=1}^{k-1} (p + k\phi + it) \prod_{i=0}^{n-k-1} (1 - p - (n-k)\phi + it)}{\prod_{i=0}^{n-1} (1 + n\phi + it)} \tag{3.2.13}$$

where $p = \frac{a}{a+b}$, $q = \frac{b}{a+b}$, $\phi = \frac{z}{a+b}$ and $t = \frac{s}{a+b}$.

The log likelihood function is given by

$$\begin{aligned} l = \log L \quad \propto \quad & N \log p + \sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \log(p + k\phi + it) + \sum_{i=0}^{n-k-1} (1 - p + (n-k)\phi + it) \right\} \\ - \quad & N \sum_{i=0}^{n-1} \log(1 + n\phi + it) \end{aligned} \tag{3.2.14}$$

The three likelihood equations obtained by partially differentiating $l$ with respect to $p, \phi$, and $t$ are

$$\begin{aligned} \frac{\partial l}{\partial p} &= \frac{N}{p} + \sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{1}{p + k\phi + it} - \sum_{i=0}^{n-k-1} \frac{1}{1 - p + (n-k)\phi + it} \right\} = 0 \\ \Rightarrow u_1 &= 0, \quad \text{Say} \end{aligned} \tag{3.2.15}$$

$$\begin{aligned} \frac{\partial l}{\partial \phi} &= \sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{k}{p + k\phi + it} + \sum_{i=0}^{n-k-1} \frac{n-k}{1 - p + (n-k)\phi + it} \right\} \\ &- nN \sum_{i=0}^{n-1} \frac{1}{1 + n\phi + it} = 0 \Rightarrow u_2 = 0, \quad \text{Say} \end{aligned} \tag{3.2.16}$$

$$\begin{aligned} \frac{\partial l}{\partial t} &= \sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{i}{p + k\phi + it} - \sum_{i=0}^{n-k-1} \frac{i}{1 - p + (n-k)\phi + it} \right\} \\ &- N \sum_{i=0}^{n-1} \frac{i}{1 + n\phi + it} = 0 \Rightarrow u_3 = 0, \quad \text{Say} \end{aligned} \tag{3.2.17}$$

It is not possible to solve these likelihood equations as above analytically even for small values of $m$. Hence we present here the numerical method of solving these equations by applying the Newton-

Rapson technique. Following are the partial derivatives of $u_1, u_2$ and $u_3$ required for implementing the algorithm.

$$\frac{\partial u_1}{\partial p} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{1}{(p+k\phi+it)^2} + \sum_{i=0}^{n-k-1} \frac{1}{(1-p+(n-k)\phi+it)^2} \right\} - \frac{N}{p^2} = d_{11}, \text{ Say} \tag{3.2.18}$$

$$\frac{\partial u_l}{\partial \phi} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{k}{(p+k\phi+it)^2} - \sum_{i=0}^{n-k-1} \frac{n-k}{(1-p+(n-k)\phi+it)^2} \right\} = d_{12}, \text{ Say} \tag{3.2.19}$$

$$\frac{\partial u_1}{\partial t} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{i}{(p+k\phi+it)^2} - \sum_{i=0}^{n-k-1} \frac{i}{(1-p+(n-k)\phi+it)^2} \right\} = d_{13}, \text{ Say} \tag{3.2.20}$$

$$\frac{\partial u_2}{\partial \phi} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{k^2}{(p+k\phi+it)^2} + \sum_{i=0}^{n-k-1} \frac{(n-k)^2}{(1-p+(n-k)\phi+it)^2} \right\} + n^2 N \sum_{i=0}^{n-1} \frac{1}{(1+n\phi+it)^2} = d_{22}, \text{ Say} \tag{3.2.21}$$

$$\frac{\partial u_2}{\partial t} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{ik}{(p+k\phi+it)^2} + \sum_{i=0}^{n-k-1} \frac{i(n-k)}{(1-p+(n-k)\phi+it)^2} \right\} + nN \sum_{i=0}^{n-1} \frac{i}{(1+n\phi+it)^2} = d_{23}, \text{ Say} \tag{3.2.22}$$

$$\frac{\partial u_3}{\partial t} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{i^2}{(p+k\phi+it)^2} - \sum_{i=0}^{n-k-1} \frac{i^2}{(1-p+(n-k)\phi+it)^2} \right\} + N \sum_{i=0}^{n-1} \frac{i^2}{(1+n\phi+it)^2} = d_{33}, \text{ Say} \tag{3.2.23}$$

It may be noted here that

$$\frac{\partial u_1}{\partial \phi} = \frac{\partial u_2}{\partial p}, \ \frac{\partial u_1}{\partial t} = \frac{\partial u_3}{\partial p}, \ \frac{\partial u_2}{\partial t} = \frac{\partial u_3}{\partial \phi}.$$

Now, the estimates of $p, \phi$ and $t$ can be obtained by generating the sequence of vectors $(p_i, \phi_i, t_i)$ using the recurrence relations

$$p_{i+1} = p_i + incp_i, \quad \phi_{i+1} = \phi_i + inc\phi_i \text{ and } t_{i+1} = t_i + inct_i$$

where $\underline{inc} = -D^{-1}\underline{u}$,

wherein $\underline{inc} = (incp_i \ \ inc\phi_i \ \ inct_i)'$, $\underline{u} = (u_1 \ \ u_2 \ \ u_3)'$ and $D = (d_{ij})_{3\times 3}$.

The iteration is stopped at the $r$th step if the distance between $r$th and the $(r+1)$th solution is less than a pre-assigned small positive number and $(p_r, \phi_r, t_r)$ is taken as the MLE of $(p, \phi, t)$.

### 3.2.6 Limiting cases :

1. If $a/s$, $b/s$ and $n/s$ are infinitely large quantities of the same order, as $t$ approaches zero and $\phi = 0$, then the UPM tends to a normal distribution.

2. The UPM with parameters $p(=\frac{a}{a+b}), q(=\frac{b}{a+b}), r(=\frac{s}{a+b}), t(=\frac{z}{a+b}), \phi = 0$ and $n$, tends to the generalized Poisson distribution with parameters $\lambda_1$ and $\lambda_2$ as $n \to \infty$, $p \to 0$ and $r \to 0$, $t \to 0$ such that $np = \lambda_1$, $nt = \lambda_2$ and $nr \to 0$.

3. If $a/s$ is a quantity of lower order and $n/s$ is of the same order as the quantity $b/s$, then the UPM tends to the Poisson distribution as $z$ and $\phi$ approaches zero.

## 3.3 Model II : *An urn model with five urns*

Let us consider five urns $U_1, U_2, U_3, U_4$ and $U_5$. $U_1$ is empty, $U_2$ contains $a$ white no black, $U_3$ no white $b$ black, $U_4$ and $U_5$ both contains $a$ white and $b$ black balls respectively. For given positive integers $n, \phi, z$ and integer $s$, a strategy is determined by selecting an integer $k, k \geq 0$. Once selected, $n$ white, $\phi k$ black balls to $U_1$, $kz$ black balls to $U_2$, $(n + (\phi - 1)k)z$ white balls to urn $U_3$, $kz$ white, $(n + (\phi - 1)k)z$ black balls to $U_4$ are added. The constitution of the urns are now shown as follows.

Table 3.3: Constitution of the urns

| | *Number of balls* | |
|---|---|---|
| *Urn* | white | black |
| $U_1$ | $n$ | $\phi k$ |
| $U_2$ | $a$ | $kz$ |
| $U_3$ | $(n + (\phi - 1)k)z$ | $b$ |
| $U_4$ | $a + kz$ | $b + (n + (\phi - 1)k)z$ |
| $U_5$ | $a$ | $b$ |

Now, a ball is drawn from the urn $U_1$, if it is white, then a ball is drawn from the $U_5$, if it is white, a ball is drawn from $U_3$ else from $U_2$. If the colours of the balls drawn in the last two trials are different, then $n+\phi k$ draws are be made from $U_4$ one by one with replacement, where after each draw the ball drawn is replaced with $s$ additional balls of the same colour before the next draw. Success is achieved, if exactly $k$ of these $n+\phi k$ balls are white.

Clearly, the probability of success is the joint probability of drawing a white from $U_1$ in the first trial, a white again from $U_5$ and black from $U_3$ or black from $U_5$ and a white from $U_3$ in second and third trials and then exactly $k$ white balls in $n+\phi k$ repeated trials from $U_4$ using sampling scheme stated above.

$$\begin{aligned} Pr(\text{Success} \mid \text{Strategy } k) &= P(k) = \frac{n}{n+\phi k}\left[\frac{a}{a+b}\,\frac{b}{b+(n+(\phi-1)k)z} + \frac{b}{a+b}\,\frac{a}{a+kz}\right] \\ &\quad \binom{n+\phi k}{k}\frac{(a+kz)^{[k,s]}\,(b+(n+(\phi-1)k)z)^{[n+(\phi-1)k,s]}}{(a+b+(n+\phi k)z)^{[n+\phi k,s]}} \\ &= \frac{n}{n+\phi k}\,\frac{a}{a+kz}\,\frac{b}{b+(n+(\phi-1)k)z}\,\frac{a+b+(n+\phi k)z}{a+b} \\ &\quad \frac{\left\langle {a+kz \atop k} \right\rangle_s \left\langle {b+(n+(\phi-1)k)z \atop n+(\phi-1)k} \right\rangle_s}{\left\langle {a+b+(n+\phi k)z \atop n+\phi k} \right\rangle_s}, \end{aligned} \tag{3.3.1}$$

where $\left\langle {a \atop k} \right\rangle_s = \frac{a^{[k,s]}}{k!} = \frac{a(a+s)\cdots(a+(k-1)s)}{k!}$. Here the parameters $a, b, n, \phi, z, s$ are such that for $k = 0, 1, \ldots;\ P(k) \geq 0$. If $s = -1,\ k = 0, 1, \ldots, \min(\frac{a}{1-z}, n)$. It is possible for $s$ to be negative when $\phi \geq 0$ provided $a + b + (n+\phi k)z + (n + \phi k - 1)s) > 0$. Further, if $a + kz$ is a fractional number, $\left\langle {a+kz \atop k} \right\rangle$ is expressed in gamma function.

The GPM (3.3.1) can be written in the form

$$\begin{aligned} P(k) &= \frac{n}{n+\phi k}\,\frac{\alpha}{\alpha+kt}\,\frac{\beta}{\beta+(n+(\phi-1)k)t}\,\frac{\alpha+\beta+(n+\phi k)t}{\alpha+\beta} \\ &\quad \frac{\left\langle {\alpha+kt \atop k} \right\rangle \left\langle {\beta+(n+(\phi-1)k)t \atop n+(\phi-1)k} \right\rangle}{\left\langle {\alpha+\beta+(n+\phi k)t \atop n+\phi k} \right\rangle} \end{aligned} \tag{3.3.2}$$

$$\text{where } \frac{a}{s} = \alpha,\ \frac{b}{s} = \beta,\ \frac{z}{s} = t,$$

$$\left\langle {\alpha \atop k} \right\rangle = \frac{\alpha^{[k]}}{k!} = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} \text{ and } \alpha^{[0]} = 1$$

and also as

$$P(k) = \frac{\alpha}{\alpha+kt}\frac{\beta}{\beta+(n+(\phi-1)k)t}\frac{\alpha+\beta+(n+\phi k)t}{\alpha+\beta}\frac{n}{n+\phi k} \binom{n+\phi k}{k}\frac{(\alpha+kt)^{[k]}(\beta+(n+(\phi-1)k)t)^{[n+(\phi-1)k]}}{(\alpha+\beta+(n+\phi k)t)^{[n+\phi k]}} \quad (3.3.3)$$

For $t=0$, the pf (3.3.3) reduces to GPE model of Sen and Mishra [63]) with parameters $n, \phi, \alpha, \beta$.

*Remark.* The same model can also be developed using four urns with a different strategy of defining success. (Janardan [44]).

### 3.3.1 Distributions as special cases

GPM (3.3.1) is not a proper discrete probability distribution for all values of $\phi, z$ and $s$, but it generates most of the well known discrete distributions as well as new distributions for different values of parameters $n, \phi, s, z, a$ and $b$. In the following, a list of the well known discrete distributions (Das [29], Johnson, Kotz and Kemp [51], Nandi and Das [54], Sen and Mishra [63], Patil, Boswell, Joshi and Ratnaparkhi [59], Charalambides ([7], [8]) are presented as particular cases of GPM.

Table 3.4: Distributions as special cases of GP model

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 1. | $\phi=0$ | GMPD (Janardan [44], Johnson et al. [51]) | $\frac{a}{a+kz}\frac{b}{b+(n-k)z}\frac{a+b+nz}{a+b} \frac{\binom{a+kz}{k}_s\binom{b+(n-k)z}{n-k}_s}{\binom{a+b+nz}{n}_s}$ | $0(1)n$ |
| 2. | $\phi=0, z=0$ | Polya-Eggenberger (PE) | $\frac{\binom{a}{k}_s\binom{b}{n-k}_s}{\binom{a+b}{n}_s}$ | $0(1)n$ |

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 3. | $s=1, \phi=0$ | Quasi beta-Binomial II (QBB II) | $\frac{a}{a+kz}\frac{b}{b+(n-k)z}\frac{a+b+nz}{a+b}\frac{\left\langle {a+kz \atop k} \right\rangle \left\langle {b+(n-k)z \atop n-k} \right\rangle}{\left\langle {a+b+nz \atop n} \right\rangle}$ | $0(1)n$ |
| 4. | $s=1, \phi=0$ | Mixed quasi negative hypergeometric (Janardan [44]) | $\frac{a}{a+kz}\frac{b}{b+(n-k)z}\frac{a+b+nz}{a+b}\frac{\binom{a-1+k(z+1)}{k}\binom{b-1+(n-k)(z+1)}{n-k}}{\binom{a+b-1+n(+1)z}{n}}$ | $0(1)n$ |
| 5. | $s=1, \phi=0$, $z=0$ or $s=-1, \phi=0, z=1$ | Beta-Binomial (BB) (Ord [56], Patil and Joshi [60]) | $\frac{\binom{a+k-1}{k}\binom{b+n-k-1}{n-k}}{\binom{a+b+n-1}{n}}$ | $0(1)n$ |
| 6. | $s=-1, \phi=0$ | Quasi hypergeometric II(QHII) ( Consul and Mittal [20], Nandi and Das [54], Janardan [44]) | $\frac{a}{a+kz}\frac{b}{b+(n-k)z}\frac{a+b+nz}{a+b}\frac{\binom{a+kz}{k}\binom{b+(n-k)z}{n-k}}{\binom{a+b+nz}{n}}$ | $\max(0, n-b)$ (1) $\min(n, a)$ |
| 7. | $\phi=0, s=0$ | Quasi Binomial II (QBII) (Consul and Mittal [20], Janardan [44]) | $\binom{n}{k}\frac{ab(a+kz)^{k-1}(b+(n-k)z)^{n-k-1}}{(a+b)(a+b+nz)^{n-1}}$ | $0(1)n$ |
| 8. | $\phi=0, z=0$ $s=-1$ | Hypergeometric (HG) | $\frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}}$ | $0(1)\min(n, a)$ |
| 9. | $\phi=0, z=0$ Replace $s$ by $-s$ | Markov-Polya Survival Model (MPSM) (Janardan [46]) | $\frac{\binom{a}{k}_s\binom{b}{n-k}_s}{\binom{a+b}{n}_s}$ | $0(1)\min(n, a)$ |

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 10. | $\phi = 0, s = 0, z = 0, a/(a+b) = p$ | Binomial | $\binom{n}{k}p^k(1-p)^{n-k}$ | $0(1)n$ |
| 11. | $\phi = 0, z = 0, a = b = s$ | Uniform or Discrete rectangular | $\frac{1}{n+1}$ | $0(1)n$ |
| 12. | $\phi = 1$ | GIMPD (Janardan [44]) | $\binom{n+k-1}{k}\frac{a}{a+kz}\frac{b}{b+nz}\frac{a+b+(n+k)z}{a+b}$ $\frac{(a+kz)^{[k,s]}(b+nz)^{[n,s]}}{(a+b+(n+k)z)^{[n+k,s]}}$ | $0(1)\infty$ |
| 13. | $\phi = 1, s = -1$ | Quasi inverse Polya-Eggenberger (Janardan [44]) | $\binom{n+k-1}{k}\frac{a}{a+kz}\frac{b}{b+nz}\frac{a+b+(n+k)z}{a+b}$ $\frac{(a+kz)^{(k)}(b+nz)^{(n)}}{(a+b+(n+k)z)^{(n+k)}}$ | $0(1)\infty$ |
| 14. | $\phi = 1, z = 0$ | Inverse Polya-Eggenberger (IPE) | $\binom{n+k-1}{k}\frac{a^{[k,s]}b^{[n,s]}}{(a+b)^{[n+k,s]}}$ | $0(1)\infty$ |
| 15. | $\phi = 1, s = 0, z = 0, a/(a+b) = p$ | Negative Binomial (NB) | $\binom{n+k-1}{k}p^k(1-p)^n$ | $0(1)\infty$ |
| 15a. | $m = np/(1-p), p = a/(1+a)$ in 16. | A alternative form of NB used in Ecology (Evans [31], Patil et al. [59]) | $\binom{m/a+k-1}{k}\left(\frac{a}{1+a}\right)^k\left(\frac{1}{1+a}\right)^{m/a}$ | $0(1)\infty$ |
| 16. | $\phi = 1, s = 0, n = 1, z = 0, a/b = P, Q = 1+P$ | Geometric | $(P/Q)^kQ^{-1}$ | $0(1)\infty$ |

| Class Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|
| 17. $s=-1, \phi=1, z=0$ | Negative hypergeometric (NHG) or distribution of no. of exccedence (Johnson et al. [51]) | $\frac{n}{n+k}\frac{\binom{a}{k}\binom{b}{n}}{\binom{a+b}{n+k}}$ or $\frac{\binom{-n}{k}\binom{-b+n-1}{a-k}}{\binom{-b-1}{a}}$ | $0(1)\min(a, a+b-n)$ |
| 18. $\phi=1, z=0, a=b=s$ | Inverse factorial (IF) (Irwin [41]) | $\frac{n}{(n+k)(n+k+1)}$ | $0(1)\infty$ |
| 19. $s=1, \phi=1, z=0$ Or $s=-1, \phi=1, z=1$ | Beta-Pascal (BP) (Ord [56], Johnson et al. [51]) | $\frac{n}{n+k}\frac{\binom{a+k-1}{k}\binom{b+n-1}{n}}{\binom{a+b+n+k-1}{n+k}}$ or $\frac{b}{a+b}\frac{\binom{a+k-1}{k}\binom{b+n-1}{b}}{\binom{a+b+n+k-1}{a+b}}$ | $0(1)\infty$ |
| 20. $\phi=2, s=0, a/b=\alpha$ (Replacing $k$ by $k-n$), $z=0$ | Haight (Haight [37]) | $\frac{n}{k}\binom{2k-n-1}{k-1}\frac{\alpha^{k-n}}{(1+\alpha)^{2k-n}}$ | $n(1)\infty$ |
| 21. $z=0$ | Generalized Polya-Eggenberger (GPE) (Sen and Mishra [63]) | $\frac{n}{n+\phi k}\frac{\left\langle{a \atop k}\right\rangle_s \left\langle{b \atop n+(\phi-1)k}\right\rangle_s}{\left\langle{a+b \atop n+\phi k}\right\rangle_s}$ | $0(1)\min(a,n)$ when $s=-1$ |

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 22. | $z=0, s=0$, $a/b=P, Q=1+P$ | Negative binomial-negative binomial (Consul and Shenton [22]) | $\frac{n}{n+\phi k}\binom{n+\phi k}{k}\left(\frac{P}{Q}\right)^k Q^{-(n+(\phi-1)k)}$ | $0(1)\infty$ |
| 23. | $s=0, z=0$, $a/b=P, Q=1+P$ (Replacing $n$ by $\phi-1$ and then $k$ by $k-1$) | Takács (Takács [66]) | $\frac{\phi}{\phi k-1}\binom{\phi k-1}{k-1}\left(\frac{P}{Q}\right)^{k-1} Q^{-(\phi-1)k}$ | $1(1)\infty$ |
| 24. | $z=0, s=0$, $a/(a+b)=p$ | Generalized negative binomial (GNB) (Jain and Consul [42], Consul and Gupta [17]) | $\frac{n}{n+\phi k}\binom{n+\phi k}{k}p^k (1-p)^{(n+(\phi-1)k)}$ | $0(1)\infty$ |
| 25. | $z=0, s=0$, $\phi=m, a/(a+b)=p$ (Replacing $n$ by $mn$ and $k$ by $k-n$) | Binomial-delta (BD) (Consul and Shenton [22]) | $\frac{n}{k}\binom{mk}{k-n}p^{k-n}(1-p)^{mk+n-k}$ | $n(1)\infty$ |
| 26. | $z=0, s=0$, $\phi=m, a/(a+b)=p$ (Replacing $n$ by $mn$ and $k$ by $k-n$) when $n=1$ | Consul distribution (CD) (Consul and Shenton [24], Consul [13]) | $\frac{1}{k}\binom{mk}{k-1}p^{k-1}(1-p)^{mk+1-k}$ | $1(1)\infty$ |

| Class | Parameters | Distribution | Mass Function | Range of $k$ |
|---|---|---|---|---|
| 27. | $z=0, s=0$, $a/b=P, Q=1+P$ (Replacing $n$ by $(\phi-1)n$ and $k$ by $k-n$) | Negative binomial-delta (Consul and Shenton [22]) | $\frac{n}{k}\binom{\phi k-n-1}{k-n}\left(\frac{P}{Q}\right)^{k-n} Q^{-(\phi-1)k}$ | $n(1)\infty$ |
| 28. | $s=0, \phi=2$, $a/b=P, Q=1+P, z=0$ | Negative binomial | $\frac{n}{k}\binom{n+2k-1}{k-1}(P/Q)^k Q^{-n-k}$ | $0(1)\infty$ |
| 29. | $\phi=0$, Replace $s$ by $-s$ | Generalized quasi hypergeometric (GQH II) (Nandi and Das [54]) | $\frac{a}{a+kz}\ \frac{b}{b+(n-k)z}\ \frac{a+b+nz}{a+b}\ \frac{\binom{a+kz}{k}_s\binom{b+(n-k)z}{n-k}_s}{\binom{a+b+nz}{n}_s}$ | $\max(0, n-b)$ (1) $\min(n, a)$ |

### 3.3.2 Moments

Denoting the expression on the right hand side of equation (3.3.3) by $P(k; n, \nu, \alpha, \beta, t)$, where $\nu+1=\phi$, the moment of the GPM model is defined as

$$\begin{aligned} M'_r(n,\nu,\alpha,\beta,t) &= \frac{n\alpha}{\alpha+\beta}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t(n+\nu)}{\alpha+t+1+\beta}\right)^m \\ &\sum_{\ell=0}^{m+1}\binom{m+1}{\ell}\left(\frac{\nu+1}{n+\nu}\right)^{\ell}\{M'_{j+\ell}(n+\nu,\nu,\alpha+t+1,\beta,t)+ \\ &\frac{t}{\alpha+t+1}M'_{j+\ell+1}(n+\nu,\nu,\alpha+t+1,\beta,t)\} \end{aligned} \tag{3.3.4}$$

*Proof.*

$$\begin{aligned} M'_r(n,\nu,\alpha,\beta,t) &= \sum_{k\geq 0}k^r P(k;n,\nu,\alpha,\beta,t) \\ &= \frac{n\alpha}{\alpha+\beta}\sum_{k\geq 0}(1+k)^{r-1}(1+\frac{\nu+1}{n+\nu}k)\left(\frac{\alpha+t+1+kt}{\alpha+t+1}\right) \end{aligned}$$

$$\left(\frac{\alpha+t+1+\beta}{\alpha+t+1+\beta+(n+\nu+(\nu+1)k)t}\right) P(k;n+\nu,\nu,\alpha+t+1,\beta,t)$$

$$= \frac{n\alpha}{\alpha+\beta}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k\geq 0}\left(1+\frac{\nu+1}{n+\nu}k\right)\left(1+\frac{t}{\alpha+t+1}k\right)$$
$$\left[\frac{\alpha+t+1+\beta}{\alpha+t+1+\beta+(n+\nu+(\nu+1)k)t}\right]k^j P(k;n+\nu,\nu,\alpha+t+1,\beta,t)$$

$$= \frac{n\alpha}{\alpha+\beta}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k\geq 0}\left(1+\frac{\nu+1}{n+\nu}k\right)\left(1+\frac{t}{\alpha+t+1}k\right)$$
$$\left[\frac{\alpha+t+1+\beta+(n+\nu+(\nu+1)k)t}{\alpha+t+1+\beta}\right]^{-1} k^j P(k;n+\nu,\nu,\alpha+t+1,\beta,t)$$

$$= \frac{n\alpha}{\alpha+\beta}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k\geq 0}\left(1+\frac{\nu+1}{n+\nu}k\right)\left(1+\frac{t}{\alpha+t+1}k\right)\sum_{m\geq 0}(-1)^m$$
$$\left[\frac{t(n+\nu)}{\alpha+t+1+\beta}\right]^m\left(1+\frac{\nu+1}{n+\nu}k\right)^m k^j P(k;n+\nu,\nu,\alpha+t+1,\beta,t)$$

$$= \frac{n\alpha}{\alpha+\beta}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{m\geq 0}(-1)^m\left[\frac{t(n+\nu)}{\alpha+t+1+\beta}\right]^m\sum_{k\geq 0}\left(1+\frac{t}{\alpha+t+1}k\right)$$
$$\sum_{\ell=0}^{m+1}\binom{m+1}{l}\left(\frac{\nu+1}{n+\nu}\right)^\ell k^{j+\ell} P(k;n+\nu,\nu,\alpha+t+1,\beta,t)$$

$$= \frac{n\alpha}{\alpha+\beta}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t(n+\nu)}{\alpha+t+1+\beta}\right)^m$$
$$\sum_{\ell=0}^{m+1}\binom{m+1}{\ell}\left(\frac{\nu+1}{n+\nu}\right)^\ell\{M'_{j+\ell}(n+\nu,\nu,\alpha+t+1,\beta,t)+$$
$$\frac{t}{\alpha+t+1}M'_{j+\ell+1}(n+\nu,\nu,\alpha+t+1,\beta,t)\}$$

Similar relation for the model (3.3.1) can be obtained as.

$$M'_r(n,\nu,a,b,z,s) = \frac{na}{a+b}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{m=0}^{\infty}(-1)^m\left(\frac{z(n+\nu)}{a+b+z+s}\right)^m$$
$$\sum_{\ell=0}^{m+1}\binom{m+1}{\ell}\left(\frac{\nu+1}{n+\nu}\right)^\ell\{M'_{j+\ell}(n+\nu,\nu,a+z+s,b,z.s)+$$
$$\frac{z}{a+z+s}M'_{j+\ell+1}(n+\nu,\nu,a+z+s,b,z,s)\} \tag{3.3.5}$$

In fact, the relation (3.3.5) reduces to the relation (3.3.4) for $s=1$ and $a=\alpha$, $b=\beta$ and $z=t$. For $z=0$, (3.3.5) reduces to

$$M'_r(n,\nu,a,b,s) = \frac{na}{a+b}\sum_{j=0}^{r-1}\binom{r-1}{j}\{M'_j(n+\nu,\nu,a+s,b,s)+$$

$$\frac{\nu+1}{n+\nu}M'_{j+1}(n+\nu,\nu,a+s,b,s)\} \quad (3.3.6)$$

When $\nu = 0$, relation (3.3.6) reduces to (Inverse Polya -Eggenberger distribution ( Johnson and Kotz [49])

$$M'_r(n,a,b,s) = \frac{na}{a+b}\sum_{j=0}^{r-1}\binom{r-1}{j}\{M'_j(n,a+s,b,s) + \frac{1}{n}M'_{j+1}(n,a+s,b,s)\} \quad (3.3.7)$$

Which for $r = 1$ becomes

$$M'_1(n,a,b,s) = \frac{na}{a+b}\left\{1 + \frac{1}{n}M'_1(n,a+s,b,s))\right\} \quad (3.3.8)$$

Putting $s = 0$ in relation (3.3.7) we get the following relation for negative binomial distribution (Johnson et al. [51], p.207)

$$\begin{aligned} M'_r(n,a,b) &= \frac{na}{a+b}\sum_{j=0}^{r-1}\binom{r-1}{j}\{M'_j(n,a,b) + \frac{1}{n}M'_{j+1}(n,a,b)\} \\ &= np\sum_{j=0}^{r-1}M'_j(n,a,b) + p\sum_{j=0}^{r-1}\binom{r-1}{j}M'_{j+1}(n,a,b) \end{aligned} \quad (3.3.9)$$

where $\frac{a}{a+b} = p$ while for $t = 0$, (3.3.4) reduces to (Sen and Mishra [63])

$$\begin{aligned} M'_r(n,\nu,\alpha,\beta) &= \frac{n\alpha}{\alpha+\beta}\sum_{j=0}^{r-1}\binom{r-1}{j}\{M'_j(n+\nu,\nu,\alpha+1,\beta) + \\ &\quad \frac{\nu+1}{n+\nu}M'_{j+1}(n+\nu,\nu,\alpha+1,\beta)\} \end{aligned} \quad (3.3.10)$$

It may be mentioned here that while for $z = 0$ relations (3.3.4) and (3.3.5) gives moment relations correctly for all the distributions belonging to the GPM with $z = 0$, unfortunately for $z \neq 0$ it seems to fail to give results when $\nu = 0$

### 3.3.3 Recurrence relations for mean and second order moment when $\nu = -1$

Putting $r = 1$ and $\nu = -1$ in (3.3.4) and (3.3.5) we get respectively

$$\begin{aligned} M'_1(n,\alpha,\beta,t) &= \frac{n\alpha}{\alpha+\beta}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t(n-1)}{\alpha+t+1+\beta}\right)^m\{M'_0(n-1,\alpha+t+1,\beta,t) + \\ &\quad \frac{t}{\alpha+t+1}M'_1(n-1,\alpha+t+1,\beta,t)\} \end{aligned} \quad (3.3.11)$$

where $M_1'(n, \alpha, \beta, t) \equiv M_1'(n, -1, \alpha, \beta, t)$ is the mean of the GMPD (Janardan [44]) and

$$M_1'(n, a, b, z, s) = \frac{na}{a+b} \sum_{m=0}^{\infty} (-1)^m \left(\frac{z(n-1)}{a+b+z+s}\right)^m \{M_0'(n-1, a+z+s, b, z, s) + \frac{z}{a+z+s} M_1'(n-1, a+z+s, b, z, s)\} \quad (3.3.12)$$

where $M_1'(n, a, b, z, s) \equiv M_1'(n, -1, a, b, z, s)$ is the mean of the GMPD (Janardan [44]).

Since $M_0'(n-1, \alpha+t+1, \beta, t) = 1$,

$$M_1'(n, \alpha, \beta, t) = \frac{n\alpha}{\alpha+\beta} \left\{1 + \left(\frac{t(n-1)}{\alpha+t+1+\beta}\right)\right\}^{-1} \{1 + \frac{t}{\alpha+t+1} M_1'(n-1, \alpha+t+1, \beta, t)\} \quad (3.3.13)$$

and $M_1'(1, \alpha+(n-1)(t+1), \beta, t) = \frac{\alpha+(n-1)(t+1)}{\alpha+\beta+(n-1)(t+1)}$, using (3.3.13) recursively we get (Janardan [44]

$$M_1'(n, \alpha, \beta, t) = \frac{n\alpha}{\alpha+\beta} \sum_{i=1}^{n} \frac{(n-1)^{(i-1)}(\alpha+\beta+(t+1)i)}{(\alpha+\beta+nt+1)^{[i]}} t^{i-1} = \frac{n\alpha}{\alpha+\beta} \quad (3.3.14)$$

The same result can be directly obtained using the fact that

$$\frac{\alpha}{\alpha+kt} \binom{n}{k} \frac{(\alpha+kt)^{[k]}(\beta+(n-k)t)^{[n-k]}}{(\alpha+\beta+nt)^{[n]}} \quad (3.3.15)$$

is a proper probability distribution for $k = 0, 1, \ldots, n$, referred to as quasi Polya distribution (QED) with parameters $n, \alpha, \beta, t$ (Consul [11], Janardan [43], Das [29]).

Also using a generalized Vandermonde convolution identity (Gould [34]), it can be seen that

$$\mathrm{E}(c+dX) = \frac{c(\alpha+\beta)+n\alpha d}{\alpha+\beta}$$

where $X$ follows GMPD with $(n, \alpha, \beta, t)$.

Putting $r = 2$ and $\nu = -1$ in (3.3.4), we get

$$\begin{aligned} M_2'(n, \alpha, \beta, t) &= \frac{n\alpha}{\alpha+\beta} \sum_{m=0}^{\infty} (-1)^m \left(\frac{t(n-1)}{\alpha+t+1+\beta}\right)^m \left\{1 + \frac{t}{\alpha+t+1} M_1'(n-1, \alpha+t+1, \beta, t) \right. \\ &\quad \left. + M_1'(n-1, \alpha+t+1, \beta, t) + \frac{t}{\alpha+t+1} M_2'(n-1, \alpha+t+1, \beta, t)\right\} \\ &= \frac{n\alpha}{\alpha+\beta} \left\{1 + \frac{(n-1)(\alpha+t+1)}{(\alpha+\beta+nt+1)}\right\} + \frac{n\alpha}{\alpha+\beta} \frac{\alpha+\beta+t+1}{\alpha+\beta+nt+1} \\ &\quad \frac{t}{\alpha+t+1} M_2'(n-1, \alpha+t+1, \beta, t) \end{aligned} \quad (3.3.16)$$

where $M_2'(n,\alpha,\beta,t) \equiv M_2'(n,-1,\alpha,\beta,t)$ is the second order moment of the GMPD (Janardan [44]).

The second order factorial moment can also be obtained in terms of the mean of quasi Polya distribution (QED) (Janardan [43]) given in (3.3.15) as follows

$$\mathrm{E}[X(X-1)] = \frac{n(n-1)\alpha}{\alpha+\beta}\frac{\alpha+nt+1}{\alpha+\beta+nt+1} - \frac{n(n-1)\alpha t}{(\alpha+\beta)(\alpha+\beta+nt+1)}\mathrm{E}[Y] \tag{3.3.17}$$

where $Y$ is distributed as QED with $(n-2,\beta,\alpha+2t+2,t)$ and $\mathrm{E}[Y]$ is given in (3.2.7).

Which reduces to $\mathrm{E}[X(X-1)]$ for PE when $t=0$ (Johnson and Kotz [49]).

A general formula for various first and second order moments for the variate $X$ distributed GMPD with $(n,\alpha,\beta,t)$ can be written using a generalized Vandermonde convolution identity (Gould [34]) as

$$\begin{aligned}\mathrm{E}[(c+dX)(u+v(n-X))] &= cu + (du\alpha + cv\beta)\frac{n}{\alpha+\beta} + dv\alpha\beta\frac{\alpha+\beta+nt}{\alpha+\beta} \\ &\left[\left\langle \begin{matrix}\alpha+\beta+nt\\ n\end{matrix}\right\rangle\right]^{-1}\sum_{j=0}^{n-2}\left\langle \begin{matrix}\alpha+\beta+nt+j+2\\ n-j-2\end{matrix}\right\rangle t^j\end{aligned} \tag{3.3.18}$$

By properly choosing the constants $c,d,u,v$ as below, various moments can be obtained from relation (3.3.18)

i) $\mathrm{E}[X^2]$ if $c=0, d=1, u=n, v=-1$.

ii) $\mathrm{E}[X(X-1)]$ if $c=-1, d=1, u=n, v=-1$.

iii) $\mathrm{E}[X(X+1)]$ for $c=1, d=1, u=n, v=-1$.

iv) $\mathrm{V}(X)$ if $c=-\frac{n\alpha}{\alpha+\beta}, d=1, u=\frac{n\beta}{\alpha+\beta}, v=-1$.

v) $\mathrm{E}[X]$ if $c=0, d=1, u=1, v=0$.

vi) $\mathrm{E}[n-X]$ if $c=1, d=0, u=0, v=1$.

Some of the formulas are listed below

$$\mathrm{E}[X^2] = \frac{n^2\alpha}{\alpha+\beta}\frac{\alpha}{\alpha+\beta}\sum_{j=0}^{n-2}\frac{n^{(j+2)}}{(\alpha+\beta+nt+1)^{[j+1]}}t^j \tag{3.3.19}$$

$$\mathrm{E}[X(X-1)] = \frac{n(n-1)\alpha}{\alpha+\beta}\frac{\alpha+nt+1}{\alpha+\beta+nt+1}\frac{\alpha}{\alpha+\beta}\sum_{j=0}^{n-2}\frac{n^{(j+2)}}{(\alpha+\beta+nt+1)^{[j+1]}}t^j \tag{3.3.20}$$

This result is equivalent to the (3.3.17)

$$\mathrm{V}[X] = \frac{n^2\alpha}{(\alpha+\beta)^2}\frac{\alpha\beta}{\alpha+\beta}\sum_{j=0}^{n-2}\frac{n^{(j+2)}}{(\alpha+\beta+nt+1)^{[j+1]}}t^j \qquad (3.3.21)$$

### 3.3.4 Recurrence relations for probabilities

Following the recurrence relations for the probabilities $P(k;n,\nu,\alpha,\beta,t)$ on the r.h.s. of (3.3.3) where $\nu+1=\phi$ holds.

1.
$$\begin{aligned}P(k+1;n,\nu,\alpha,\beta,t) &= \frac{n+\nu k}{(n+\nu k)^{[\nu+1]}}\frac{(n+(\nu+1)k)^{[\nu+2]}}{n+(\nu+1)(k+1)}\frac{\alpha+kt}{(\alpha+kt)^{[t]}}\\ &\frac{(\alpha+(t+1)k)^{[t+1]}}{\alpha+(k+1)t}\frac{\beta+(n+\nu k)t}{(\beta+(n+\nu k)t)^{[\nu t]}}\frac{(\beta+(n+\nu k)(t+1))^{[\nu(t+1)]}}{\beta+(n+\nu(k+1))t}\\ &\frac{\alpha+\beta+(n+(\nu+1)(k+1))t}{\alpha+\beta+(n+(\nu+1)k)t}\frac{(\alpha+\beta+(n+(\nu+1)k)t)^{[(\nu+1)t]}}{(\alpha+\beta+(n+(\nu+1)k)(t+1))^{[(\nu+1)(t+1)]}}\\ &\frac{1}{k+1}P(k;n,\nu,\alpha,\beta,t)\end{aligned} \qquad (3.3.22)$$

*Special cases of (3.3.22).* For

i) $t=0$,

$$\begin{aligned}GPE(k+1;n,\nu,\alpha,\beta) &= \frac{(n+(\nu+1)k)^{[\nu+1]}}{(n+\nu k+1)^{[\nu]}}\frac{(\beta+n+\nu k)^{[\nu]}}{(\alpha+\beta+n+(\nu+1)k)^{[\nu+1]}}\\ &\frac{\alpha+k}{k+1}GPE(k;n,\nu,\alpha,\beta)\end{aligned} \qquad (3.3.23)$$

ii) $t=0$, $\nu=-1$

$$PE(k+1;n,\alpha,\beta) = \frac{\alpha+k}{k+1}\frac{n-k}{\beta+n-k-1}PE(k;n,\alpha,\beta) \qquad (3.3.24)$$

iii) $t=0$, $\nu=0$

$$IPE(k+1;n,\alpha,\beta) = \frac{\alpha+k}{k+1}\frac{n+k}{\alpha+\beta+n+k}IPE(k;n,\alpha,\beta) \qquad (3.3.25)$$

iv) $t=0$, $\nu=-1$, $\alpha=-a, \beta=-b$ (See Johnson et al. [51], p.253)

$$HG(k+1;n,a,b) = \frac{n-k}{k+1}\frac{a-k}{b-n+k+1}HG(k;n,a,b) \qquad (3.3.26)$$

v) $t = 0,\ \nu = 0,\ \alpha = -a, \beta = -b$

$$NHG(k+1; n, a, b) = \frac{a-k}{k+1} \frac{n+k}{a+b-n-k} NHG(k; n, a, b) \tag{3.3.27}$$

2.
$$P(k; n+1, \nu, \alpha, \beta, t) = \frac{n+1}{n} \frac{n+(\nu+1)k}{(n+\nu k+1)} \frac{\beta+(n+\nu k)t}{(\beta+(n+\nu k)t)^{[t]}}$$
$$\frac{(\beta+(n+\nu k)(t+1))^{[(t+1)]}}{\beta+(n+\nu k)t+t} \frac{\alpha+\beta+(n+(\nu+1)k)t+t}{\alpha+\beta+(n+(\nu+1)k)t}$$
$$\frac{(\alpha+\beta+(n+(\nu+1)k)t)^{[t]}}{(\alpha+\beta+(n+(\nu+1)k)(t+1))^{[(t+1)]}} P(k; n, \nu, \alpha, \beta, t) \tag{3.3.28}$$

*Special cases of (3.3.28).* When

i) $t = 0$,

$$GPE(k; n+1, \nu, \alpha, \beta) = \frac{n+1}{n} \frac{(n+(\nu+1)k)}{(n+\nu k+1)} \frac{(\beta+n+\nu k)}{(\alpha+\beta+n+(\nu+1)k)} GPE(k; n, \nu, \alpha, \beta) \tag{3.3.29}$$

ii) $t = 0,\ \nu = -1$

$$PE(k; n+1, \alpha, \beta) = \frac{\beta+n-k}{\alpha+\beta+n} \frac{n+1}{n-k+1} PE(k; n, \alpha, \beta) \tag{3.3.30}$$

iii) $t = 0,\ \nu = 0$

$$IPE(k; n+1, \alpha, \beta) = \frac{n+k}{n} \frac{\beta+n}{\alpha+\beta+n+k} IPE(k; n, \alpha, \beta) \tag{3.3.31}$$

iv) $t = 0,\ \nu = -1,\ \alpha = -a, \beta = -b$ (See Johnson et al. [51], p.253)

$$HG(k; n+1, a, b) = \frac{n+1}{n-k+1} \frac{b-n+k}{a+b-n} HG(k; n, a, b) \tag{3.3.32}$$

v) $t = 0,\ \nu = 0,\ \alpha = -a, \beta = -b$

$$NHG(k; n+1, a, b) = \frac{n+k}{n} \frac{b-n}{a+b-n-k} NHG(k; n, a, b) \tag{3.3.33}$$

3.
$$P(k; n, \nu+1, \alpha, \beta, t) = \frac{(n+(\nu+1)k)^{[k]}}{(n+\nu k+1)^{[k]}} \frac{\beta+(n+\nu k)t}{(\beta+(n+\nu k)t)^{[kt]}}$$
$$\frac{(\beta+(n+\nu k)(t+1))^{[k(t+1)]}}{\beta+(n+(\nu+1)k)t} \frac{(\alpha+\beta+(n+(\nu+1)k)t)^{[kt]}}{\alpha+\beta+(n+(\nu+1)k)t}$$
$$\frac{(\alpha+\beta+(n+(\nu+1)k)t+kt)}{(\alpha+\beta+(n+(\nu+1)k)(t+1))^{[k(t+1)]}} P(k; n, \nu, \alpha, \beta, t) \tag{3.3.34}$$

*Special cases of (3.3.34).* If

i) $t = 0$,

$$GPE(k; n, \nu+1, \alpha, \beta) = \frac{(n+(\nu+1)k)^{[k]}}{(n+\nu k)^{[k]}} \frac{(\beta+n+\nu k)^{[k]}}{(\alpha+\beta+n+(\nu+1)k)^{[k]}} GPE(k; n, \nu, \alpha, \beta) \quad (3.3.35)$$

ii) $t = 0$, $\nu = -1$

$$IPE(k; n, \alpha, \beta) = \frac{n^{[k]}}{(n-k)^{[k]}} \frac{(\beta+n-k)^{[k]}}{(\alpha+\beta+n)^{[k]}} PE(k; n, \alpha, \beta) \quad (3.3.36)$$

which gives a relationship between probabilities of PE and IPE.

iii) $t = 0$, $\nu = 0$

$$GPE(k; n, 1, \alpha, \beta) = \frac{(n+k)^{[k]}}{(n+1)^{[k]}} \frac{(\beta+n)^{[k]}}{(\alpha+\beta+n+k)^{[k]}} IPE(k; n, \alpha, \beta) \quad (3.3.37)$$

4. $$P(k; n, \nu, \alpha+1, \beta, t) = \frac{\alpha+1}{\alpha} \frac{\alpha+\beta}{\alpha+\beta+1} \frac{\alpha+kt+k}{\alpha+kt+1} \frac{\alpha+\beta+(n+(\nu+1)k)t+1}{\alpha+\beta+(n+(\nu+1)k)(t+1)} P(k; n, \nu, \alpha, \beta, t) \quad (3.3.38)$$

*Special cases of (3.3.38).* In case

i) $t = 0$,

$$GPE(k; n, \nu, \alpha+1, \beta) = \frac{\alpha+k}{\alpha} \frac{\alpha+\beta}{\alpha+\beta+n+(\nu+1)k} GPE(k; n, \nu, \alpha, \beta) \quad (3.3.39)$$

ii) $t = 0$, $\nu = -1$

$$PE(k; n, \alpha+1, \beta) = \frac{\alpha+k}{\alpha} \frac{\alpha+\beta}{\alpha+\beta+n} PE(k; n, \alpha, \beta) \quad (3.3.40)$$

iii) $t = 0$, $\nu = 0$

$$IPE(k; n, \alpha+1, \beta) = \frac{\alpha+k}{\alpha} \frac{\alpha+\beta}{\alpha+\beta+n+k} IPE(k; n, \alpha, \beta) \quad (3.3.41)$$

iv) $t = 0$, $\nu = -1$, $\alpha = -a, \beta = -b$ (See Johnson et al. [51], p.253)

$$HG(k; n, a+1, b) = \frac{a+1}{a-k+1} \frac{a+b-n+1}{a+b+1} HG(k; n, a, b) \quad (3.3.42)$$

v) $t = 0,\ \nu = 0,\ \alpha = -a, \beta = -b$

$$NHG(k; n, a+1, b) = \frac{a+1}{a+k-1} \frac{a+b+1-n-k}{a+b+1} NHG(k; n, a, b) \tag{3.3.43}$$

$$5. \quad P(k; n, \nu, \alpha, \beta+1, t) = \frac{\beta+1}{\beta} \frac{\alpha+\beta}{\alpha+\beta+1} \frac{\beta+(n+\nu k)(t+1)}{\beta+(n+\nu k)t+1} \frac{\alpha+\beta+(n+(\nu+1)k)t+1}{\alpha+\beta+(n+(\nu+1)k)(t+1)} P(k; n, \nu, \alpha, \beta, t) \tag{3.3.44}$$

*Special cases of (3.3.44).* When

i) $t = 0$,

$$GPE(k; n, \nu, \alpha, \beta+1) = \frac{\alpha+\beta}{\beta} \frac{\beta+n+\nu k}{\alpha+\beta+n+(\nu+1)k} GPE(k; n, \nu, \alpha, \beta) \tag{3.3.45}$$

i) $t = 0,\ \nu = -1$

$$PE(k; n, \alpha, \beta+1) = \frac{\alpha+\beta}{\beta} \frac{\beta+n-k}{\alpha+\beta+n} PE(k; n, \alpha, \beta) \tag{3.3.46}$$

iii) $t = 0,\ \nu = 0$

$$IPE(k; n, \alpha, \beta+1) = \frac{\alpha+\beta}{\beta} \frac{\beta+n}{\alpha+\beta+n+k} IPE(k; n, \alpha, \beta) \tag{3.3.47}$$

iv) $t = 0,\ \nu = -1,\ \alpha = -a, \beta = -b$ (See Johnson et al. [51], p.253)

$$HG(k; n, a, b+1) = \frac{b+1}{a+b+1} \frac{a+b-n+1}{b-n+k+1} HG(k; n, a, b) \tag{3.3.48}$$

v) $t = 0,\ \nu = 0,\ \alpha = -a, \beta = -b$

$$NHG(k; n, a, b+1) = \frac{b+1}{a+b+1} \frac{a+b+1-n-k}{b+1-n} NHG(k; n, a, b) \tag{3.3.49}$$

$$6. \quad P(k; n, \nu, \alpha+1, \beta-1, t) = \frac{\beta-1}{\beta} \frac{\alpha+1}{\alpha} \frac{\alpha+kt+k}{\alpha+kt+1} \frac{\beta+(n+\nu k)t}{\beta+(n+\nu k)(t+1)-1} P(k; n, \nu, \alpha, \beta, t) \tag{3.3.50}$$

*Special cases of (3.3.50).* Let

i) $t = 0$,

$$GPE(k; n, \nu, \alpha+1, \beta-1) = \frac{\alpha+k}{\alpha} \frac{\beta-1}{\beta+n+\nu k-1} GPE(k; n, \nu, \alpha, \beta) \tag{3.3.51}$$

ii) $t = 0,\ \nu = -1$

$$PE(k; n, \alpha+1, \beta-1) = \frac{\alpha+k}{\alpha}\,\frac{\beta-1}{\beta+n-k-1} PE(k; n, \alpha, \beta) \tag{3.3.52}$$

iii) $t = 0,\ \nu = 0$

$$IPE(k; n, \alpha+1, \beta-1) = \frac{\alpha+k}{\alpha}\,\frac{\beta-1}{\beta+n-1} IPE(k; n, \alpha, \beta) \tag{3.3.53}$$

iv) $t = 0,\ \nu = -1,\ \alpha = -a, \beta = -b$ (See Johnson et al. [51], p.253)

$$HG(k; n, a+1, b-1) = \frac{a+1}{a+k-1}\,\frac{b-n+k}{b} HG(k; n, a, b) \tag{3.3.54}$$

v) $t = 0,\ \nu = 0,\ \alpha = -a, \beta = -b$

$$NHG(k; n, a+1, b-1) = \frac{a+1}{a+1-k}\,\frac{b-n}{b} NHG(k; n, a, b) \tag{3.3.55}$$

### 3.3.5 Estimation

Here the various steps involved in the ML estimation of the parameters of GMPD using iterative numerical scheme are presented. It is assumed that the observed frequencies in a random sample of size $N$ are $n_k$, $k = 0(1)m$ for different classes, i.e., $\sum_{k=0}^{m} n_k = N$, where $m$ is of course the largest value observed. Here the parameter $n$ is estimated by $m$.

The pf of GMPD can be written as

$$p_k = \binom{n}{k} \frac{p(1-p) \prod\limits_{i=1}^{k-1} (p + k\phi + it) \prod\limits_{i=1}^{n-k-1} (1 - p - (n-k)\phi + it)}{\prod\limits_{i=1}^{n-1} (1 + n\phi + it)} \tag{3.3.56}$$

where $p = \frac{a}{a+b}$, $q = \frac{b}{a+b}$, $\phi = \frac{z}{a+b}$ and $t = \frac{s}{a+b}$.

The log likelihood function is given by

$$\begin{aligned} l = \log L \ \propto\ & N \log p(1-p) + \sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \log(p + k\phi + it) + \sum_{i=1}^{n-k-1} (1 - p + (n-k)\phi + it) \right\} \\ & - N \sum_{i=0}^{n-1} \log(1 + n\phi + it) \end{aligned} \tag{3.3.57}$$

The three likelihood equations obtained by partially differentiating $l$ with respect to $p, \phi$, and $t$ are

$$\frac{\partial l}{\partial p} = N\frac{1-2p}{p(1-p)} + \sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{1}{p+k\phi+it} - \sum_{i=1}^{n-k-1} \frac{1}{1-p+(n-k)\phi+it} \right\} = 0$$

$$\Rightarrow u_1 = 0, \quad \text{Say} \tag{3.3.58}$$

$$\frac{\partial l}{\partial \phi} = \sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{k}{p+k\phi+it} + \sum_{i=0}^{n-k-1} \frac{n-k}{1-p+(n-k)\phi+it} \right\}$$

$$- \; nN\sum_{i=1}^{n-1} \frac{1}{1+n\phi+it} = 0 \quad \Rightarrow u_2 = 0, \quad \text{Say} \tag{3.3.59}$$

$$\frac{\partial l}{\partial t} = \sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{i}{p+k\phi+it} + \sum_{i=1}^{n-k-1} \frac{i}{1-p+(n-k)\phi+it} \right\}$$

$$- \; N\sum_{i=0}^{n-1} \frac{i}{1+n\phi+it} = 0 \quad \Rightarrow u_3 = 0, \quad \text{Say} \tag{3.3.60}$$

Analytical solution of the likelihood equations above even for small values of $m$ are not available. Hence we present here the numerical method of solving the equations by applying the Newton-Rapson technique.Following are the partial derivatives of$u_1, u_2$ and $u_3$ required for implementing the algorithm.

$$\frac{\partial u_1}{\partial p} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{1}{(p+k\phi+it)^2} + \sum_{i=1}^{n-k-1} \frac{1}{(1-p+(n-k)\phi+it)^2} \right\}$$

$$- \; N(\frac{1}{p^2} + \frac{1}{(1-p)^2}) = d_{11}, \quad \text{Say} \tag{3.3.61}$$

$$\frac{\partial u_l}{\partial \phi} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{k}{(p+k\phi+it)^2} - \sum_{i=1}^{n-k-1} \frac{n-k}{(1-p+(n-k)\phi+it)^2} \right\}$$

$$= d_{12}, \quad \text{Say} \tag{3.3.62}$$

$$\frac{\partial u_1}{\partial t} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{i}{(p+k\phi+it)^2} - \sum_{i=1}^{n-k-1} \frac{i}{(1-p+(n-k)\phi+it)^2} \right\}$$

$$= d_{13}, \quad \text{Say} \tag{3.3.63}$$

$$\frac{\partial u_2}{\partial \phi} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{k^2}{(p+k\phi+it)^2} + \sum_{i=1}^{n-k-1} \frac{(n-k)^2}{(1-p+(n-k)\phi+it)^2} \right\}$$

$$+ \; n^2N\sum_{i=0}^{n-1} \frac{1}{(1+n\phi+it)^2} = d_{22}, \quad \text{Say} \tag{3.3.64}$$

$$\frac{\partial u_2}{\partial t} = -\sum_{k=0}^{m} n_k \left\{ \sum_{i=1}^{k-1} \frac{ik}{(p+k\phi+it)^2} + \sum_{i=1}^{n-k-1} \frac{i(n-k)}{(1-p+(n-k)\phi+it)^2} \right\}$$

$$+ \quad nN\sum_{i=0}^{n-1}\frac{i}{(1+n\phi+it)^2} \quad = \quad d_{23}, \quad \text{Say} \tag{3.3.65}$$

$$\frac{\partial u_3}{\partial t} = -\sum_{k=0}^{m} n_k\left\{\sum_{i=1}^{k-1}\frac{i^2}{(p+k\phi+it)^2}+\sum_{i=1}^{n-k-1}\frac{i^2}{(1-p+(n-k)\phi+it)^2}\right\}$$

$$+ \quad N\sum_{i=0}^{n-1}\frac{i^2}{(1+n\phi+it)^2} \quad = \quad d_{33}, \quad \text{Say} \tag{3.3.66}$$

It may be noted here that

$$\frac{\partial u_1}{\partial \phi}=\frac{\partial u_2}{\partial p},\ \frac{\partial u_1}{\partial t}=\frac{\partial u_3}{\partial p},\ \frac{\partial u_2}{\partial t}=\frac{\partial u_3}{\partial \phi}$$

Now the estimate of $p, \phi$ and $p$ can be obtained by generating the sequence of vectors $(p_i, \phi_i, t_i)$ using the recurrence relations

$p_{i+1} = p_i + incp_i, \quad \phi_{i+1} = \phi_i + inc\phi_i \quad t_{i+1} = t_i + inct_i$

where $\underline{inc} = -D^{-1}\underline{u}$ wherein $\underline{inc} = (incp_i \;\; inc\phi_i \;\; inct_i)', \underline{u} = (u_1 \;\; u_2 \;\; u_3)'$ and $D = (d_{ij})_{3\times 3}$

The iteration is stopped at the $r$th step if the distance between $r$th and the $(r+1)$th solution is less than a pre-assigned small positive number and $(p_r, \phi_r, t_r)$ is taken as the MLE of $(p.\phi, t)$.

### 3.3.6 Limiting cases

1. If $a/s$, $b/s$ and $n/s$ are infinitely large quantities of the same order, as $t$ approaches zero and $\phi = 0$, then the GPM tends to a normal distribution.

2. The GPM with parameters $p(=\frac{a}{a+b}), q(=\frac{b}{a+b}), r(=\frac{s}{a+b}), t(=\frac{z}{a+b})$, $\phi = 0$ and $n$ tends to the generalized Poisson distribution (Consul and Jain [18]) with parameters $\lambda$ and $\psi$ as $n \to \infty$, $p \to 0$ and $r \to 0$, $t \to 0$ such that $np = \lambda_1$, $nt = \lambda_2$ and $nr \to 0$, where $\lambda = \frac{\lambda_1}{1+\lambda_2}$ and $\psi = \frac{\lambda_2}{1+\lambda_2}$.

3. The GPM with parameters $p(=\frac{a}{a+b+nz}), q(=\frac{b}{a+b+nz}), r(=\frac{s}{a+b+nz}), t(=\frac{z}{a+b+nz})$, $\phi = 0$ and $n$ tends to the generalized Poisson distribution (Consul and Jain [18]) with parameters $\lambda$ and $\psi$ as $n \to \infty$, $p \to 0$ and $r \to 0$, $t \to 0$ such that $np = \lambda$, $nt = \psi$ and $nr \to 0$.

4. If $a/s$ is a quantity of lower order and $n/s$ is of the same order as the quantity $b/s$, then the GPM tends to the Poisson distribution as $z, \phi$ and approaches zero.

5. If $a$, $b \to \infty$ with $\frac{a}{(a+b)} = p(o < p < 1)$ $n$ remaining constant then as $\phi$ and $z$ approaches zero, GPM tends to binomial distribution with $n, p$.

6. If $a$, $b \to \infty$ with $\frac{a}{(a+b)} \to \infty$ also $n \to \infty$ with $\frac{na}{(a+b)} \to \lambda(0 < \lambda < \infty)$ and $\frac{(a+b)}{ns} \to \mu(0 < \mu < \infty)$ then as $\phi$ and $z$ approaches zero, GPM tends to negative binomial distribution with $\lambda\mu$ and $(1+\mu)^{-1}$.

7. If $n \to \infty$ with $\frac{a+b}{n^2 s} \to \beta(0 < \beta < \infty)$ and $\frac{na}{(a+b)} \to \lambda(0 < \lambda < \infty)$ then as $\phi$ and $z$ approaches zero, GPM tends to Poisson distribution with $\lambda$.

8. If $n \to \infty$ with $\frac{a+b}{n^2 s} \to 0$, $\frac{(a+b)^2}{n^3 s^2} \to \infty$ and $\frac{na}{(a+b)} \to \lambda(0 < \lambda < \infty)$ then as $\phi$ and $z$ approaches zero, GPM tends to Poisson distribution with $\lambda$.

9. If $n \to \infty$ with $\frac{a+b}{n^2 s} \to \infty$, $\frac{a+b}{n^3 s} \to 0$ and $\frac{na}{(a+b)} \to \lambda(0 < \lambda < \infty)$ then as $\phi$ and $z$ approaches zero, GPM tends to Poisson distribution with $\lambda$.

10. If $n \to \infty$ with $\frac{a}{a+b} \to 0$, $\frac{s}{a+b} \to 0$ such that $\frac{na}{(a+b)} \to \theta_1, \frac{ns}{(a+b)} \to \theta_2$ finite nonzero then as $\phi$ and $z$ approaches zero GPM tends to negative binomial distribution with $\frac{\theta_1}{\theta_2}, \theta_2$.

11. When $a, a+b \to \infty$ with $\frac{a}{a+b} \to p(0 < p < 1)$ as z approaches zero GPM tends to negative binomial distribution. (Johnson,Kotz [48], p.157) when $\phi = 1$ and $s \to -1$.

12. For $s = 0, z = 0$, and $\frac{a}{a+b} = p$ is small GPM tends to GPD I with $\lambda$ and $\psi$ as $n$ and $\phi$ tends to $\infty$ such that $np = \lambda$ and $\phi = \psi$. (Consul [14], p.26).

## 3.4 A simple extension of the model

Instead of drawing a ball form $U_1$ and proceed if it is white, here $r$ draws are made from $U_1$ one by one without replacement before proceeding to the next stage if all $r$ balls are white. Rest of the steps being the same the probability of success under this setup is given by

$$
\begin{aligned}
Pr(\text{Success} \mid \text{Strategy}k) &= P(k) = \frac{n^{(r)}}{(n+\phi k)^{(r)}} \frac{a}{a+kz} \frac{b}{b+(n+(\phi-1)k)z} \frac{a+b+(n+\phi k)z}{a+b} \\
&\quad \binom{n+\phi k}{k} \frac{(a+kz)^{[k,s]}\,(b+(n+(\phi-1)k)z)^{[n+(\phi-1)k,s]}}{(a+b+(n+\phi k)z)^{[n+\phi k,s]}}, \qquad (3.4.1)
\end{aligned}
$$

with $r = 1$ (3.4.1) reduces to (3.3.1) and for $\phi = 0$ to GMPD. For $\phi = 1$ (3.4.1) transforms to

$$\binom{n+k-r}{k} \frac{a}{a+kz} \frac{b}{b+nz} \frac{a+b+(n+k)z}{a+b} \frac{(a+kz)^{[k,s]}(b+nz)^{[n,s]}}{(a+b+(n+k)z)^{[n+k,s]}} \tag{3.4.2}$$

which is the probability of $k$ whites preceding the $n$th black when $(r - 1)$ other blacks occurs in prefixed positions. This is a generalization of GIMPD (Janardan [44]).

## 3.5 Conclusion

Urn models helps in understanding complex physical systems by interpreting them mathematically and hence have tremendous importance in applications (Johnson and Kotz [49]). Urn models with pre-determined strategies have applications in many real life situations (Consul [11], Janardan [43] and Consul and Mittal [21]) ranging form Migration, War strategies, Ecology, Business, Industry, Agriculture, Medicine to Psychology, Sociology, Operation Research, etc. The Models described in this chapter are developed theoretically and is among the most wider urn models in terms of its coverage of number and variety of discrete probability distributions having diverse applications. Study of these models will certainly strengthen the base of unification of various distributions and lay foundation of a common frame work for future studies. These models can be used to develop general routines for computing probabilities for a large number of discrete distributions. Also using cumulative probabilities, one can simulate random samples through inversion method. GMPD is a symmetrical discrete distribution when $a = b$ or $p = q$ and takes variety of shapes.

# Chapter 4

# $\alpha$-Modified Binomial Distributions

## 4.1 Introduction

Here various aspects like pgf, moments, inter-relation among different pfs and limiting distributions of a class of weighted $\alpha$-modified binomial and related distributions are studied. Some applications of these distribution are considered.

## 4.2 A class of $\alpha$-modified binomial distributions

**Definition 1** *A discrete random variable $X$ is said to follow a $\alpha$-modified-binomial distribution with parameters $n, p, q, \phi$ if its probability function is given by*

$$Pr(X = k) = \binom{n}{k} \frac{(p + \alpha\phi)^k q^{n-k}}{(q + p + \alpha\phi)^n}, \quad k = 0(1)n; \phi \geq 0; p + \phi \geq 0; q > 0, \qquad (4.2.1)$$

*where on expansion $\alpha^i \equiv \alpha_i = i!$.*

For $p + q = 1$, the pf (4.2.1) reduces to $\alpha$-modified-binomial distribution of Berg and Jaworski [4] with $n, p, \phi$. While for $\phi = 0$ to common binomial distribution with $n, p$.

Clearly, if $X$ follows (4.2.1) $Y = n - X$, the complimentary variable has the pmf

$$Pr(Y = k) = \binom{n}{k} \frac{(p + \alpha\phi)^{n-k} q^k}{(q + p + \alpha\phi)^n}, \quad k = 0(1)n; \phi \geq 0; p + \phi \geq 0; q > 0. \tag{4.2.2}$$

This distribution is referred to as **complimentary-α-modified-binomial** distribution with parameters $(n, p, q, \phi)$

### 4.2.1 Factorial moments

Factorial moments of the complimentary variable $(n - X)$ is given by

$$\mathrm{E}[(n - X)^{(\mu)}] = \begin{cases} n^{(\mu)} q^{\mu} \frac{(q+p+\alpha\phi)^{n-\mu}}{(q+p+\alpha\phi)^n} & \text{if } \phi > 0 \\ n^{(\mu)} q^{\mu} & \text{if } \phi = 0 \end{cases} \tag{4.2.3}$$

It can be shown that

$$(q + p + \alpha\phi)^n = (q + p)^n + n\phi(q + p + \alpha\phi)^{n-1} \tag{4.2.4}$$

Applying the result recursively we get

$$(q + p + \alpha\phi)^n = \sum_{r=0}^{\mu-1} n^{(r)} \phi^r (q + p)^{n-r} + n^{(\mu)} \phi^{\mu} (q + p + \alpha\phi)^{n-\mu} \tag{4.2.5}$$

Hence from (4.2.3)

$$\mathrm{E}[(n - X)^{(\mu)}] = \frac{q^{\mu}}{\phi^{\mu}} \left[ 1 - \frac{\sum_{r=0}^{\mu-1} n^{(r)} \phi^r (q + p)^{n-r}}{(q + p + \alpha\phi)^n} \right] \tag{4.2.6}$$

When $\mu = 1$, the mean of (4.2.1) is given by

$$\mathrm{E}[(n - X)] = \frac{q}{\phi} \left[ 1 - \frac{(q + p)^{n-r}}{(q + p + \alpha\phi)^n} \right] \tag{4.2.7}$$

for $q + p = 1$, r.h.s. of (4.2.7) reduces to Berg and Jaworski [4]

$$\frac{q}{\phi} \left[ 1 - \frac{1}{(1 + \alpha\phi)^n} \right]$$

From (4.2.6)

$$\sum_{k=0}^{n} [q^{\mu} - (n - k)^{(\mu)} \phi^{\mu}] \binom{n}{k} (p + \alpha\phi)^k q^{n-k-\mu} = \sum_{r=0}^{\mu-1} n^{(r)} \phi^r (q + p)^{n-r} \tag{4.2.8}$$

From this relation, it is clear that the sum in the left hand side of (4.2.8) is equal to the sum of the first $\mu$ terms of the expansion $(q+p+\alpha\phi)^n$.

In fact

$$\sum_{k=0}^{n}[q-(n-k)\phi]\binom{n}{k}(p+\alpha\phi)^k q^{n-k-1} = (q+p)^n = 1 \tag{4.2.9}$$

$$\begin{aligned}\sum_{k=0}^{n}[q^2-(n-k)^{(2)}\phi^2]\binom{n}{k}(p+\alpha\phi)^k q^{n-k-2} &= (q+p)^n + n\phi(q+p)^{n-1} \\ &= 1+n\phi, \ \text{if}\, p+q=1 \end{aligned}\tag{4.2.10}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\sum_{k=0}^{n}\binom{n}{k}(p+\alpha\phi)^k q^{n-k} = (q+p+\alpha\phi)^n = (1+\alpha\phi)^n, \ \text{if}\, p+q=1 \tag{4.2.11}$$

Using the result above, we now introduce a class of weighted probability distribution of (4.2.1) as follows.

## 4.3 A class of weighted $\alpha$-modified binomial distributions

**Definition 2** *With $\alpha^{\imath} \equiv \alpha_{\imath}\cdot = i!$ the following pf represents a* ***class of discrete probability distributions***

$$Pr(X=k) = \binom{n}{k}\frac{(p+\alpha\phi)^k q^{n-k-\nu}\{q^\nu-(n-k)^{(\nu)}\phi^\nu\}}{\sum\limits_{r=0}^{\nu-1} n^{(r)}\phi^r(q+p)^{n-r}}, \tag{4.3.1}$$

*for* $k=0(1)n;\ \ 0\le\phi\le q/n;\ \ p+\phi\ge 0;\ \ q>0.$

### 4.3.1 Some special cases

For

i) $\nu = n+1$, the pf (4.3.1) reduces to the pf (4.2.1).

ii) $\nu=1, p+q=1$, the pf (4.3.1) transforms to the pf given by (Berg and Jaworski [4])

$$Pr(X=k) = \binom{n}{k}(p+\alpha\phi)^k q^{n-k-1}(q-(n-k)\phi), \quad k=0(1)n;\ \phi\le q/n, p+\phi\ge 0. \tag{4.3.2}$$

This distribution is referred to as **weighted-$\alpha$-modified-binomial** distribution with parameters $(n, p, \phi)$

### 4.3.2 Probability generating function

The probability generating function (pgf) of the complimentary random variable $(n - X)$ when $X$ follows $\alpha$-modified-binomial as stated in (4.3.1) is given by

$$G(s) = \frac{(qs + p + \alpha\phi)^n - n^{(\nu)}\phi^\nu s^\nu (qs + p + \alpha\phi)^{n-\nu}}{\sum_{r=0}^{\nu-1} n^{(r)}\phi^r (q + p)^{n-r}} \tag{4.3.3}$$

In particular, the pgf of $(n - X)$ when

(i) $X$ follows the distribution in (4.2.1) is

$$\frac{(qs + p + \alpha\phi)^n}{(q + p + \alpha\phi)^n} \tag{4.3.4}$$

(ii) $X$ follows the distribution in (4.3.2) is

$$(qs + p + \alpha\phi)^n - n\phi s(qs + p + \alpha\phi)^{n-1} \tag{4.3.5}$$

Using the following generalization of (4.2.5)

$$\begin{aligned} (qs + p + \alpha\phi)^n - n^{(\nu)}\phi^\nu s^\nu (qs + p + \alpha\phi)^{n-\nu} &= \sum_{r=0}^{\nu-1} n^{(r)}\phi^r (qs + p)^{n-r} \\ &+ (1 - s^\nu) n^{(\nu)}\phi^\nu (qs + p + \alpha\phi)^{n-\nu}, \end{aligned} \tag{4.3.6}$$

the pgf (4.3.3) can be alternatively written as

$$\frac{\sum_{r=0}^{\nu-1} n^{(r)}\phi^r (qs + p)^{n-r} + (1 - s^\nu) n^{(\nu)}\phi^\nu (qs + p + \alpha\phi)^{n-\nu}}{\sum_{r=0}^{\nu-1} n^{(r)}\phi^r (q + p)^{n-r}} \tag{4.3.7}$$

### 4.3.3 Factorial Moments

Denoting by $\mathrm{E}_\imath[n - X]$ the mathematical expectation of $(n - X)$ when $X$ follows (4.3.1) with $\nu = \imath$, it can be seen that

$$\mathrm{E}_\nu[n - X] \;=\; G'(1)$$

$$= \frac{[nq(q+p+\alpha\phi)^{n-1} - n^{(\nu)}\phi^{\nu}\{\nu(q+p+\alpha\phi)^{n-\nu} + q(n-\nu)(q+p+\alpha\phi)^{n-\nu-1}\}]}{\sum_{r=0}^{\nu-1} n^{(r)}\phi^{r}(q+p)^{n-r}} \tag{4.3.8}$$

In particular, with $\nu = 1$, we have E$[n-X]$ for (4.3.2) as

$$\mathrm{E}_1[n-X] = G'(1)\mid_{\nu=1} = n(q-\phi)(1+\alpha\phi)^{n-1} - n^{(2)}q\phi(1+\alpha\phi)^{n-2} \tag{4.3.9}$$

Using (4.2.4), (4.3.9) can be written as

$$\mathrm{E}_1[n-X] = -n\phi(1+\alpha\phi)^{n-1} + nq \tag{4.3.10}$$

Therefore,

$$\mathrm{E}_1[X] = n\phi(1+\alpha\phi)^{n-1} + np \tag{4.3.11}$$

For

i) $\nu = n+1$, we get

$$\mathrm{E}_{n+1}[n-X] = \frac{nq(q+p+\alpha\phi)^{n-1}}{(q+p+\alpha\phi)^{n}} \tag{4.3.12}$$

which is equal to the r.h.s of (4.2.7).

ii) $\nu = 2$,

$$\begin{aligned} \mathrm{E}_2[n-X] &= G'(1)\mid_{\nu=2} \\ &= \frac{\{nq(q+p+\alpha\phi)^{n-1} - 2n^{(2)}\phi^2(q+p+\alpha\phi)^{n-2} - n^{(3)}\phi^2 q(q+p+\alpha\phi)^{n-3}\}}{\{(q+p)^n + n\phi(q+p)^{n-1}\}} \\ &= (1+n\phi)^{-1}\{nq(1+\alpha\phi)^{n-1} - 2n^{(2)}\phi^2(1+\alpha\phi)^{n-2} - n^{(3)}\phi^2 q(1+\alpha\phi)^{n-3}\}, \\ &\qquad \text{if } p+q=1 \end{aligned} \tag{4.3.13}$$

Now

$$\begin{aligned} \mathrm{E}_{\nu}[n-X]^{(2)} &= G''(1) \\ &= [n^{(2)}q^2(q+p+\alpha\phi)^{n-2} - \phi^{\nu}\{\nu^{(2)}n^{(\nu)}(q+p+\alpha\phi)^{n-\nu} \\ &+ 2\nu n^{(\nu+1)}q(q+p+\alpha\phi)^{n-\nu-1} + n^{(\nu+2)}q^2(q+p+\alpha\phi)^{n-\nu-2}\}] \\ &\left(\sum_{r=0}^{\nu-1} n^{(r)}\phi^{r}(q+p)^{n-r}\right)^{-1} \end{aligned} \tag{4.3.14}$$

In particular, when

i) $\nu = 1$,

$$\begin{aligned} \mathrm{E}_1[n-X]^{(2)} &= n^{(2)}q(q+p+\alpha\phi)^{n-2}(q-2\phi) - n^{(3)}q^2\phi(q+p+\alpha\phi)^{n-3} \\ &= n^{(2)}q(1+\alpha\phi)^{n-2}(q-2\phi) - n^{(3)}q^2\phi(1+\alpha\phi)^{n-3} \end{aligned} \tag{4.3.15}$$

Using (4.2.4) it can be seen that

$$\mathrm{E}_1[n-X]^{(2)} = n^{(2)}q^2 - 2n^{(2)}\phi(1+\alpha\phi)^{n-2} \tag{4.3.16}$$

Hence,

$$\mathrm{E}_1[n-X]^2 = n[2+q(p+nq) - (\phi+2)(1+\alpha\phi)^{n-1}] \tag{4.3.17}$$

ii) $\nu = n+1$,

$$\begin{aligned} \mathrm{E}_{n+1}[n-X]^{(2)} &= \frac{n^{(2)}q^2(q+p+\alpha\phi)^{n-2}}{(q+p+\alpha\phi)^n} \qquad (4.3.18) \\ &= \frac{n^{(2)}q^2(1+\alpha\phi)^{n-2}}{(1+\alpha\phi)^n}, \quad q+p=1 \end{aligned}$$

Following relations hold provided $p+q=1$,

$$\mathrm{E}_1[n-X] = \left\{\mathrm{E}_{n+1}[n-X] - \frac{\phi}{q}\mathrm{E}_{n+1}[n-X]^2\right\}(1+\alpha\phi)^n \tag{4.3.19}$$

$$\mathrm{E}_2[n-X] = \left(1+\frac{\phi}{q}\right)\mathrm{E}_1[n-X] + \frac{\phi}{q}\mathrm{E}_1[n-X]^{(2)} + \frac{\phi^2}{q^2}(1+\alpha\phi)^n\{\mathrm{E}_{n+1}[n-X] + \mathrm{E}_{n+1}[n-X]^2\} \tag{4.3.20}$$

Mean, variance and other moments of different distributions belonging to the weighted $\alpha$-modified-binomial class can be derived using the results stated above along with the relationships between the ascending and descending factorials.

## 4.4 Generalized distributions

Let $Y$ be a discrete random variable with $\mathrm{E}[s^Y] = \sum_{r\geq 0} p_r s^r = G(s)$, where $p_r = Pr(Y=r)$. If $G(s)$ has the form

$$G(s) = \frac{(qg(s)+p+\alpha\phi)^n}{(q+p+\alpha\phi)^n}, \tag{4.4.1}$$

where the parameters $n, p, \phi$ obeys $q > 0, \phi \geq 0$ and $p + \phi > 0, n$ positive integer and $g(s)$ is a pgf, then the distribution of $X = n - Y$ is said to be a generalized $\alpha$-modified-binomial distribution generalized by the distribution whose pgf is $g(s)$. (Johnson and Kotz [48], p.202 and Johnson et al. [51], p.324 )

Clearly,

$$G(s) = \sum_{j=0}^{n} \binom{n}{j} \frac{(p + \alpha\phi)^{n-j} q^j [g(s)]^j}{(q + p + \alpha\phi)^n} \tag{4.4.2}$$

and $p_r =$ Coefficient of $s^r$ in $G(s)$.

In particular, if the distribution with the pgf $g$ depends on a parameter, say $z$, such that $[g(s \mid z)]^j = g(s \mid jz)$, then

$$p_r = \sum_{j=0}^{n} \binom{n}{j} \frac{(p + \alpha\phi)^{n-j} q^j}{(q + p + \alpha\phi)^n} \times \text{ Coefficient of } s^r \text{ in } g(s/jr) \tag{4.4.3}$$

**Some special cases**: For

I. $g(s) = e^{\lambda(s-1)}, \ \lambda \geq 0,$

$$Pr(Y = r) = \frac{\lambda^r}{r!} \sum_{j=0}^{n} j^r e^{-j\lambda} \binom{n}{j} \frac{(p + \alpha\phi)^{n-j} q^j}{(q + p + \alpha\phi)^n}, \quad r = 0, 1, \ldots \tag{4.4.4}$$

with $\phi = 0$ and $p + q = 1$, (4.4.4) reduces to Poisson($\theta$) $\bigwedge_{\theta/\lambda}$ Binomial($n, q$) (Johnson and Kotz [48], p.186 and Johnson et al. [51], p.333).

Alternatively, the pf (4.4.4) can be written as

$$Pr(Y = r) = \frac{\lambda^r}{r!} \sum_{l=0}^{r} \frac{\Delta^l 0^r}{l!} n^{(l)} (qe^{-\lambda})^l \frac{(qe^{-\lambda} + p + \alpha\phi)^{n-l}}{(q + p + \alpha\phi)^n} \tag{4.4.5}$$

The factorial moment for the distribution (4.4.4) is

$$\mathrm{E}[Y^{(k)}] = \lambda^k \ \mathrm{E}[j^k], \tag{4.4.6}$$

where $(n - j)$ follows $\alpha$-modified binomial distribution (4.2.1) with parameters $n, p, q$ and $\phi$.

Using the formula for $\mathrm{E}[j^k]$ in (4.2.3), it can be shown that

$$\mathrm{E}[Y^{(k)}] = \lambda^k \frac{(q + p + \alpha\phi + q\Delta)^n 0^k}{(q + p + \alpha\phi)^n}, \tag{4.4.7}$$

where $\Delta$ is the forward difference operator and in expansion $\Delta^j 0^N = \Delta^j x^N \mid_x = 0$ is known as the difference of zero. The values of $\frac{\Delta^j 0^N}{j!}$ for different values $N, j$ are tabulated in Table (34.2) of Johnson et al. [50] (p.6).

From (4.4.7) it can be observed that

$$E[Y] = nq\lambda \left\{ \frac{(q+p+\alpha\phi)^{n-1}}{(q+p+\alpha\phi)^n} \right\}, \tag{4.4.8}$$

$$E[Y^{(2)}] = \frac{\lambda^2}{(q+p+\alpha\phi)^n} \left\{ nq(q+p+\alpha\phi)^{n-1} + n^{(2)}q^2(q+p+\alpha\phi)^{n-2} \right\} \text{ and} \tag{4.4.9}$$

$$E[Y^2] = \frac{\lambda^2}{(q+p+\alpha\phi)^n} \left\{ nq(1+\lambda^{-1})(q+p+\alpha\phi)^{n-1} + n^{(2)}q^2(q+p+\alpha\phi)^{n-2} \right\} \tag{4.4.10}$$

For $\phi = 0, p+q = 1$, the above results reduces to (Johnson and Kotz [48], p.186; Johnson et al. [51], p.334).

$$\begin{aligned} E[Y] &= nq\lambda, \\ E[Y^2] &= \lambda^2 \left[ nq(1+\lambda^{-1}) + n^{(2)}q^2 \right] = \lambda^2 \left[ n^2q^2 + npq \right] + \lambda nq \\ \text{and } V[Y] &= \lambda^2 npq + \lambda nq \end{aligned} \tag{4.4.11}$$

The pgf of (4.4.4) is

$$G(s) = \frac{(qe^{\lambda(s-1)} + p + \alpha\phi)^n}{(q+p+\alpha\phi)^n} \tag{4.4.12}$$

Therefore

$$p_0 = \frac{(qe^{-\lambda} + p + \alpha\phi)^n}{(q+p+\alpha\phi)^n} \quad \text{and} \tag{4.4.13}$$

$$p_r = \frac{\lambda^r}{r!} \sum_{l \geq 0} \frac{(-\lambda)^l}{l!} \mu'_{r+l}, \tag{4.4.14}$$

where $\mu'_{r+l}$ is the $(r+l)$th order raw moment of the $\alpha$-modified -binomial distribution (4.2.2).

This results may be used for computation of probabilities.

II. $g(s) = (q' + p's)^{n'}$,

$$Pr(Y = r) = \sum_{j \geq r/n'}^{n} \binom{n}{j} \frac{(p+\alpha\phi)^{n-j} q^j}{(q+p+\alpha\phi)^n} \binom{n'j}{r} (p')^r (q')^{n'j-r}, \tag{4.4.15}$$

$$r = 0, 1, \ldots;\ q' = 1 - p'.$$

with $\phi = 0$ and $p + q = 1$, the pf (4.4.15) reduces to Binomial$(n', p') \wedge$ Binomial$(n, q)$ (Johnson and Kotz [48], p.194 and Johnson et al. [51], p.336).

$$\mathrm{E}[Y^{(k)}] = p'^k \mathrm{E}[n' j^{(k)}],$$

where $(n - j)$ follows $\alpha$-modified-binomial distribution (4.2.1) with $n, p, q$ and $\phi$.

When $n = 1$, the pf (4.4.15) reduces to

$$\begin{aligned} Pr(Y = r) &= \sum_{j=0}^{n} \binom{n}{j} \frac{(p+\alpha\phi)^{n-j} q^j}{(q+p+\alpha\phi)^n} \binom{j}{r} (p')^r (q')^{j-r}. \\ &= \binom{n}{r} \frac{(qp')^r (qq' + p + \alpha\phi)^{n-r}}{(q+p+\alpha\phi)^n}, \qquad (4.4.16) \\ & \qquad r = 0, 1, \ldots;\ q' = 1 - p'. \end{aligned}$$

is also a $\alpha$-modified-binomial of the form (4.2.2) with parameters $n, p, qp'$ and $\phi$.

III. $g(s) = (Q - Ps)^{-N}$,

$$\begin{aligned} Pr(Y = r) &= \sum_{j=0}^{n} \binom{n}{j} \frac{(p+\alpha\phi)^{n-j} q^j}{(q+p+\alpha\phi)^n} \binom{-Nj}{r} P^r Q^{Nj+r}, \qquad (4.4.17) \\ & \qquad r = 0, 1, \ldots;\ Q = 1 + P. \end{aligned}$$

with $\phi = 0$ and $p + q = 1$, the pf (4.4.17) reduces to Negative Binomial$(N, P) \wedge$ Binomial$(n, q)$ (Johnson and Kotz [48], p.200 and Johnson et al. [51], p.341).

IV. $g(s) = \beta \log(1 - \theta s)$, where $\beta = [\log(1 - \theta)]^{-1}$

$$\begin{aligned} Pr(Y = r) &= \sum_{j=0}^{n} \binom{n}{j} \frac{(p+\alpha\phi)^{n-j} q^j}{(q+p+\alpha\phi)^n} \frac{1}{[-\log(1-\theta)]^j} \frac{\theta^r j!}{r!} \mid S_r^{(j)} \mid, \qquad (4.4.18) \\ & \qquad r = 0, 1, \ldots \end{aligned}$$

where $S_r^{(j)} = \frac{1}{j!} \left[ \frac{d^j}{dx^j} \left( \prod_{t=1}^{r} (x - t + 1) \right) \right]_{x=0}$ the Stirling number of the first kind with arguments $j$ and $r$ (Johnson and Kotz [48], p.179).

For $\phi = 0$ and $p + q = 1$, the pf (4.4.18) becomes Binomial$(n, q) \vee$ logseries$(\theta)$ (Johnson and Kotz [48], p.203).

## 4.5 Some compound distributions

I. Complimentary-$\alpha$-modified-binomial$(N, p, \phi) \underset{N/n}{\wedge}$ Poisson $(\lambda)$

$$Pr(Y = r) = e^{-\lambda} \sum_{j \geq r/n}^{n} \binom{nj}{r} \frac{(p+\alpha\phi)^{nj-r} q^r}{(q+p+\alpha\phi)^{nj}} \frac{\lambda^j}{j!}, \quad r = 0, 1, \ldots \tag{4.5.1}$$

The pgf of (4.5.1) is given by

$G(s) = \exp[\lambda(g(s) - 1)]$, where $g(s)$ is given by (4.3.4).

Using the generating function yield

$$\mathrm{E}[Y] = nq\lambda \left\{ \frac{(q+p+\alpha\phi)^{n-1}}{(q+p+\alpha\phi)^n} \right\} \quad \text{and} \tag{4.5.2}$$

$$\mathrm{E}[Y^{(2)}] = \frac{nq\lambda}{(q+p+\alpha\phi)^n} \left\{ (n-1)q(q+p+\alpha\phi)^{n-2} + nq\lambda(q+p+\alpha\phi)^{n-1} \frac{(q+p+\alpha\phi)^{n-1}}{(q+p+\alpha\phi)^n} \right\} \tag{4.5.3}$$

For $\phi = 0$ and $p + q = 1$, the above results reduces to (Johnson and Kotz [48], p.191).

$$\mathrm{E}[Y] = nq\lambda \ , \ \ \mathrm{E}[Y^{(2)}] = n^{(2)} q^2 \lambda + n^2 q^2 \lambda^2 \ \text{ and } \ \mathrm{V}[Y] = n^2 q^2 \lambda + npq\lambda \tag{4.5.4}$$

II. Weighted-$\alpha$-modified-binomial$(N, p, \phi) \underset{N/n}{\wedge}$ Poission$(\lambda)$

$$Pr(Y = r) = \sum_{j \geq r/n}^{n} \binom{nj}{r} (p+\alpha\phi)^r q^{nj-r-1} [q - (nj - r)\phi] \frac{e^{-\lambda}\lambda^j}{j!}, \tag{4.5.5}$$

after some patch work

$$= \frac{e^{-\lambda(1-q^n)}}{r!} \frac{(p+\alpha\phi)^r}{q^r} \left[ \mu_{(r)}{}' - \frac{\phi}{q} \mu_{(r+1)}{}' \right], \quad r = 0, 1, \ldots \tag{4.5.6}$$

where $\mu_{(k)}{}'$ is the $k$th factorial moment of $nj$, where $j$ follows Poisson distribution with $\lambda q^n$.

$$\text{i.e.} \quad \mu'_{(k)} = \sum_{j \geq 0} nj^{(k)} \frac{e^{-\lambda q^n} (\lambda q^n)^j}{j!}$$

It can be seen that, $\mu_{(k)}{}' = \sum_{t=1}^{k} \sum_{i=t}^{k} S_i{}^{(k)} \left( \frac{\Delta^t O^i}{i!} \right) n^i \, (\lambda q^n)^t,$

where $S_i^{(k)}$'s are the Stirling number of the first kind. The pgf of (4.5.5) is given by $\exp[\lambda(g(s)-1)]$, where $g(s)$ is the pgf of Y having distribution (4.3.2).

Hence, $\mathrm{E}[Y] = \lambda\mathrm{E}[X]$, $\mathrm{E}[Y(Y-1)] = \lambda[\mathrm{E}(X(X-1)) + \mathrm{E}(Y)\mathrm{E}(X)]$, Using results of section §4.3.3 and $X^{(\nu)} = \sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j}(n-x)^{(j)}(n-\nu+1)^{[\nu-j]}$,

expressions for factorial moments can be obtained. In fact,

$$\mathrm{E}[Y] = \lambda[n\phi(1+\alpha\phi)^{(n-1)} + n(1-q)], \quad \text{and so on.}$$

## 4.6 Limiting distributions

I. For $p + q = c$ as $n \to \infty$; $p, \phi \to 0$; $np = \lambda$; and $n\phi = \psi$, the pf (4.3.1) tends to two parameter $\alpha$-modified-Poisson distribution given by

$$\frac{1}{x!}\left(\frac{\lambda}{c} + \frac{\psi}{c}\alpha\right)^x e^{-\frac{\lambda}{c}}\left(1 - \frac{\psi}{c}\right), \quad \lambda + \psi \geq 0, |\, \psi \,| < c. \tag{4.6.1}$$

which, for $c = 1$, reduces to two parameter $\alpha$-modified-Poisson (Berg [4]).

II. When $p + q = c$ as $n \to \infty$; $p, \phi \to 0$; $np = -\lambda$; and $n\phi = \lambda$, the pf (4.3.1) tends to

$$\frac{1}{x!}\left(\frac{\lambda}{c}\right)^x D_x e^{\frac{\lambda}{c}}\left(1 - \frac{\lambda}{c}\right), \quad x = 0, 1, \ldots; \quad 0 < \lambda < c \tag{4.6.2}$$

where $D_x = (-1+\alpha)^x$ is the displacement number. For $c = 1$ expression (4.6.2) reduces to one parameter $\alpha$-modified-Poisson (Berg and Jaworski [4]).

III. As $n \to \infty$ and $q + p = 0$, the pf (4.2.2) tends to Poisson distribution with $q/\phi$.

## 4.7 Some extensions of $\alpha$-modified binomial distributions

**Definition 3** *A discrete random variable X is said to follow a $\alpha$-**modified binomial distribution of order** j with parameters $n, p, \phi$, if its probability function is of the form*

$$Pr(X = k) = \binom{n}{k}\frac{\{p + \phi\alpha(j)\}^k q^{n-k}}{\{q + p + \phi\alpha(j)\}^n}, \quad x = 0(1)n \tag{4.7.1}$$

*where $\phi \geq 0, p + \phi \geq 0$. and on expansion $\alpha(j)^i = \binom{i+j-1}{i} i!$*

Clearly, the pf (4.7.1) reduces to $\alpha mb(n,p,q,\phi)$ for $j=1$. We write $X \sim \alpha mb_j(n,p,q,\phi)$.

### 4.7.1 Some properties

I. The $\nu$th factorial moment of $(n-X)$ when $X \sim \alpha mb_j(n,p,q,\phi)$ is

$$\mathrm{E}[(n-X)^{(\nu)}] = n^{(\nu)}q^{(\nu)}\left[\frac{\{q+p+\phi\alpha(j)\}^{n-\nu}}{\{q+p+\phi\alpha(j)\}^{n}}\right] \tag{4.7.2}$$

II. The **pgf** of the complimentary variable $(n-X)$ when $X \sim \alpha mb_j(n,p,q,\phi)$ is

$$\frac{\{qs+p+\alpha(j)\phi\}^n}{\{q+p+\alpha(j)\phi\}^n} \tag{4.7.3}$$

III. **Limiting distributions** :

(a) As $n \to \infty, p, \phi \to 0; np \to \lambda;$ and $n\phi \to \psi$, the pf (4.7.1) tends to

$$\frac{1}{x!}\left\{\frac{\lambda}{c}+\frac{\psi}{c}\alpha(j)\right\}^k \exp\left(-\frac{\lambda}{c}\right)(1-\psi)^j \tag{4.7.4}$$

(b) As $n \to \infty, p, \phi \to 0; np \to -\lambda;$ and $n\phi \to \lambda$, the pf (4.7.1) tends to

$$\frac{1}{x!}\left(\frac{\lambda}{c}\right)^k \{-1+\alpha(j)\}^k \exp\left(\frac{\lambda}{c}\right)\left(1-\frac{\lambda}{c}\right)^j \tag{4.7.5}$$

These distributions are referred to as $\alpha mp$ **of order** $j$ with two parameters $(\lambda,\psi)$ and one parameter $\lambda$ respectively. For $j=1$ and $c=1$ these reduces to $\alpha mp$ distribution of Berg and Jaworski [4]. However (4.7.4) is referred as $\alpha mp_j(\lambda,\psi)$ and (4.7.5) is referred as $\alpha mp_j(-\lambda,\lambda)$ when $c=1$.

IV. **Characterizations**:

(a) If $X \sim \alpha mp(\lambda,\psi)$ and $Y \sim$ Geometric($\psi$), then

$$Pr(X+Y=k) = \frac{1}{k!}(1-\psi)^2\exp(-\lambda)(\lambda+\psi\alpha(2))^k \tag{4.7.6}$$

In general, if $X \sim \alpha mp_{j-1}(\lambda,\psi)$ and $Y \sim$ Geometric($\psi$), then

$$Pr(X+Y=k) = \frac{1}{k!}(1-\psi)^j\exp(-\lambda)(\lambda+\psi\alpha(j))^k \tag{4.7.7}$$

This result is a direct generalisation of the fact that if $X \sim$ Poisson($\lambda$) and $Y \sim$Geometric($\psi$), then $X + Y \sim \alpha$mp($\lambda, \psi$) (Berg and Jaworski [4]).

(b) If $X \sim \alpha mp_i(\lambda_x, \psi)$ and $Y \sim \alpha mp_j(\lambda_y, \psi)$, then $X + Y \sim \alpha mp_{i+j}(\lambda_x + \lambda_y, \psi)$.

In general, if $X_k \sim \alpha mp_{i_k}(\lambda, \psi); k = 1(1)p$, then $X_1 + X_2 + \cdots + X_p \sim \alpha mp_i(\lambda, \psi)$, where $i = i_1 + i_2 + \cdots + i_p;\ \ \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_p$

In particular, if $Y_i \sim \alpha mp(-\lambda, \lambda); i = 1(1)p$, then $Y_1 + Y_2 + \cdots + Y_p \sim \alpha mp_p(-p\lambda, \lambda)$ with pf

$$\begin{aligned} Pr(Y = k) &= \frac{1}{k!}\,(1-\lambda)^p\, e^{p\lambda}\, \lambda^k(-p + \alpha(p))^k \\ &= \frac{1}{k!}\,(1-\lambda)^p\, e^{p\lambda}\, \lambda^k \left[\sum_{i=0}^{k} \binom{k}{i} (-p)^i \frac{(p+k-i-1)!}{(p-1)!}\right] \end{aligned} \tag{4.7.8}$$

[In Berg and Jaworski [4] the same result appeared to have wrongly printed in that there in the expression within the third bracket which they denoted by $A_{py}$ the term $(p-1)!$ is missing. Though their numerical results take care of the factor.]

(c) If $X \sim \alpha mp_i(\lambda_x, \psi)$ and $Y \sim \alpha mp_j(\lambda_y, \psi)$, then

$$Pr(X = r \mid X + Y = n) = \binom{n}{r} \frac{(\lambda_x + \alpha(i)\psi)^r (\lambda_y + \alpha(j)\psi)^{n-r}}{(\lambda_x + \lambda_y + \alpha(i+j)\psi)^n} \tag{4.7.9}$$

This distribution as **doubly α-modified-binomial distribution of the order** $(i, j)$.

(d) If

$$Pr(Y = r \mid X = k) = \binom{k}{r} \frac{\lambda_1^r (\lambda_2 + \alpha\psi)^{k-r}}{(\lambda_1 + \lambda_2 + \alpha\psi)^k} \tag{4.7.10}$$

and $X \sim \alpha mp(\lambda_1 + \lambda_2, \psi)$, then $Y \sim$ Poisson($\lambda_1$).

(e) If $X \sim$ Poisson($\lambda$) and $Y \sim$ Negative binomial($j, \psi$) and are independent, then $X + Y \sim \alpha mp_j(\lambda, \psi)$.

(f) If $X \sim$ Poisson($\lambda_x$) and $Y \sim \alpha mp(\lambda_y, \psi)$, then $X+Y \sim \alpha mp(\lambda_x + \lambda_y, \psi)$ and $X \mid (X+Y) \sim \alpha mb(\lambda_x, \lambda_y, \psi)$.

V. The **pgf** of (4.7.1) is given by

$$\left(\frac{1-\psi}{1-\psi s}\right)^j \exp(-\lambda(1-s)) \tag{4.7.11}$$

VI. The mean and variance of $\alpha mp_j$ with parameters $\lambda$ and $\psi$ are

$$\mathrm{E}[X] = \lambda + \frac{j\psi}{1-\psi}, \tag{4.7.12}$$

$$\mathrm{E}[X(X-1)] = \lambda^2 + \frac{2j\psi\lambda}{1-\psi} + \frac{j(j+1)\psi^2}{(1-\psi)^2} \tag{4.7.13}$$

$$\text{and}\quad \mathrm{V}[X] = \lambda + \frac{j\psi}{(1-\psi)^2} \tag{4.7.14}$$

VII. If $\mathrm{E}_p(X)$ and $\mathrm{E}_{mp}(X)$ stands for mathematical expectation of $X$, when $X \sim$ Poisson($\lambda$) and $\alpha mp(\lambda, \psi)$, then the following recurrence relation hold

$$(1-\psi^r)\mathrm{E}_{mp}X^{(r)} = (1-\psi)\sum_{i=0}^{r-1}\psi^i\lambda^{r-i}\mathrm{E}_p[(X+r)^{(i)}] + \psi^r\sum_{j=0}^{r-1}\binom{r}{j}r^{(r-j)}\mathrm{E}_{mp}X^{(j)} \tag{4.7.15}$$

VIII. If $X \sim \alpha mp_j(\lambda, \psi)$, then

$$\mathrm{E}(X^{(k)}) = \sum_{i=0}^{k}\binom{k}{i}\lambda^i j^{[k-i]}(\frac{\psi}{1-\psi})^{k-i} \tag{4.7.16}$$

This result reduces to equation (3.9) of Berg and Nowicki [6], (p.253) when $\lambda = n\psi$. In fact, for $\lambda = n\psi$, the pf (4.7.7) reduces to the class of distributions given in equation (3.1) of Berg and Nowicki [6], (p.250) .

IX. If $X \sim$ GPD III $(a; \lambda)$, then $X = X_1 + X_2 + \cdots + X_j$ has a probability distribution with pf

$$Pr(X = k) = \frac{1}{k!}\,(1-\lambda)^j e^{-(ja+\lambda k)}(ja + \alpha(j-1)\lambda + \lambda k)^k \tag{4.7.17}$$

[ See theorem (5.9.5) of page number 114 ]

for $ja = n\lambda$ (4.7.17) reduces to the class of distribution of Berg and Nowicki [6], eq no.(3.3). The moments of $\alpha$-modified GPD III of order $(j-1)$ (4.7.17) can be easily obtained from those of GPD III $(a, \lambda)$ as

$$\mathrm{E}(X) = \frac{ja(1-\lambda)+j\lambda}{(1-\lambda)^2} \quad\text{and}\quad \mathrm{V}(X) = j\left[\frac{\lambda^2+(1-a)\lambda+a}{(1-\lambda)^4}\right] \tag{4.7.18}$$

X. **Some new distributions.**

(a) If $X \sim$ Poisson($\lambda$), $Y \sim$ Generalized negative binomial(Johnson et al. [51], pp.142,230) with parameters $m, j$, and $\psi$, then $z = X + Y \sim$ **Generalized** $\alpha mp_j$ **of type I** having pf

$$Pr(Z = k) = \frac{1}{k!}(1-\psi)^j \exp(-\lambda)\{\lambda + \psi(1-\psi)^{m-1}a(j,m)\}^k \ , \tag{4.7.19}$$

where $[a(j,m)]^k = \binom{mk+j-1}{j-1}k!$ For $m = 1$, the pf (4.7.19) reduces to $\alpha mp_j$. Following formulas can be derived easily using the corresponding results of generalized negative binomial distribution (Johnson et al. [51]).

$$\begin{aligned} (i) \qquad \mathrm{E}(Z) &= \lambda + \frac{j\psi}{1-m\psi}\ , \\ \mathrm{V}(Z) &= \lambda + \frac{j\psi(1-\psi)}{(1-m\psi)^3} \text{ and} \\ \mathrm{E}(Z-\mathrm{E}(Z))^3 &= \lambda - \mu_2\left[\frac{3mj\psi(1-\psi)}{(1-m\psi)^2} + \frac{4(1-\psi)-1}{(1-m\psi)}\right] - \frac{2n\psi(1-\psi)^2}{(1-m\psi)^4} \end{aligned} \tag{4.7.20}$$

(ii) The pgf of (4.7.19) is given by

$$e^{\lambda(s-1)}\{1-\psi(1-s)\}^j \tag{4.7.21}$$

(b) If $X \sim$ Poission($\lambda$) and $Y \sim$ Ripp – Shuffle distribution ($\psi, j$) (Johnson et al. [51], p.234), then $Z = X + Y \sim$ **Generalized** $\alpha mp_j$ **of type II** with pf

$$\begin{aligned} Pr(Z=k) &= \frac{1}{k!}(1-\psi)^j \exp(-\lambda)(\lambda + \psi\alpha(j))^k \\ &+ \frac{1}{k!}\psi^j \exp(-\lambda)(\lambda + (1-\psi)\alpha(j))^k; \quad k = 0(1)j \end{aligned} \tag{4.7.22}$$

For large $j$ we have the following approximate moment formulas using the results of corresponding formulas of Ripp-Shuffle distribution.

$$\mathrm{E}(Z) = \lambda + \frac{j(1-\psi)}{\psi} \quad \text{and} \quad \mathrm{V}(Z) = \lambda + \frac{j(1-\psi)}{\psi^2} \tag{4.7.23}$$

(c) If $V_i \sim \alpha mp(\lambda_i, \psi_i)$; $i = 0, 1, 2$, then the joint distribution of $X = V_0 + V_1, Y = V_0 + V_2$ is termed as the bivariate $\alpha mp$ distribution having pf

$$Pr(X = x, Y = y) \ = \ e^{-(\lambda_0+\lambda_1+\lambda_2)}(1-\psi_0)(1-\psi_1)(1-\psi_2)$$

$$\sum_{i=0}^{min(x,y)} \frac{(\lambda_0+\alpha\psi_0)^i}{i!}\frac{(\lambda_1+\alpha\psi_1)^{x-i}}{(x-i)!}\frac{(\lambda_2+\alpha\psi_2)^{y-i}}{(y-i)!} \tag{4.7.24}$$

Obviously the marginal distribution of $X$ is

$$Pr(X=k)=e^{-(\lambda_0+\lambda_1)}(1-\psi_0)(1-\psi_1)\left\{\frac{(\lambda_0+\lambda_1+\alpha(\psi_0+\psi_1))^k}{k!}\right\} \tag{4.7.25}$$

and conditional distribution of $Y$ given $X$ is the convolution of a doubly $\alpha$-modified-binomial distribution and a $\alpha$-modified Poisson distribution.

XI. Using the fact that

$$\sum_{\boldsymbol{k}} \frac{n!}{k_p!} a_p^{n-k_p} \prod_{i=1}^{p-1} \frac{(a_i+\alpha\psi)^{k_i}}{k_i!} = (a+a_p+\alpha(p-1)\psi)^n, \tag{4.7.26}$$

where $\boldsymbol{k}=(k_1,k_2,\dots,k_p)$, $n=\sum_{i=1}^{p} k_i$; $a=a_1+a_2+\cdots+a_{p-1}$, a multivariate extension of (4.2.1) as

**Definition 4** *Multivariate $\alpha$-modified binomial distribution with parameters $(a_1, a_2, \cdots, a_p; \psi)$ with pf*

$$P\left(\bigcap_{i=1}^{p-1} X_i=k_i\right)=\binom{n}{\boldsymbol{k}}\frac{a_p^{n-k}\prod_{i=1}^{p-1}(a_i+\alpha\psi)^{k_i}}{(\sum_{i=1}^{p}a_i+\alpha(p-1)\psi)^n}, \tag{4.7.27}$$

*where $\alpha^k \equiv \alpha_k = k!$; $\alpha^k(p)\equiv\alpha_k(p)=(\underbrace{\alpha+\alpha+\cdots+\alpha}_{p\ terms})^k=\binom{k+p-1}{k}k!$ and $\binom{n}{\boldsymbol{k}}$ is the multinomial coefficient. Then we write $(x_1,x_2,\dots,x_{p-1})\sim mamb_p(a_1,a_2,\dots,a_p;\psi)$.*

For $p=2$, the pf (4.7.1) reduces to $amb(a_1,a_2;\psi)$.

## 4.8 Some applications

### I. Matching problem.

In classical matching problem the probability of exactly $r$ matches in $n$ is given by

$$\begin{aligned} p_r &= \frac{1}{r!}\sum_{i=0}^{n-r}(-1)^i/i!\,, \quad r=0,1,2,\dots,n. \\ &= \binom{n}{r}(-1+\alpha)^{n-r}1^r/(-1+1+\alpha)^n \end{aligned} \tag{4.8.1}$$

is a particular case of the pf (4.2.2) with $p = -1, q = 1, \phi = 1$.

Using (4.3.12) and (4.3.18) it can be seen that

$$\mathrm{E}[X] = 1\ ,\quad \mathrm{E}[X(X-1)] = 1;\ \text{ and }\ \mathrm{V}[X] = 1. \tag{4.8.2}$$

The above distribution also arises as a particular case of the factorial series distribution (Berg [2]).

As $n \to \infty$, the pf (4.8.1) tends to Poisson distribution with parameter 1.

II. **Extended Matching problem.**

Suppose that the probability that there are matches in $r$ places is given by $\frac{1}{n^{(r)}p^r}$ instead of $\frac{1}{n^{(r)}}$ as in classical case. Then the probability of no match in any position is $\sum_{k=0}^{n} \frac{(-p)^k}{k!}$ and probability of exactly $r$ matches is

$p_r = \binom{n}{r} \times$ Probability of $r$ matches in some position $\times$ Probability of no matches in the remaining $(n-r)$ positions.

$$\begin{aligned}\text{Clearly,}\quad p_r &= \binom{n}{r}\frac{1}{n^{(r)}}p^r\sum_{k=0}^{n-r}(-p)^k/k! \\ &= \frac{\binom{n}{r}(-p+\alpha)^{n-r}p^r}{(p-p+\alpha)^n},\quad 0 < p \le 1\end{aligned} \tag{4.8.3}$$

is a particular case of the pf (4.2.2) with $p = -p, q = p, \phi = 1$.

Using (4.3.12) and (4.3.18) it can be shown that

$$\mathrm{E}[X] = np\frac{\alpha^{n-1}}{\alpha^n} = p\ ,\quad \mathrm{E}[X(X-1)] = \frac{n(n-1)p^2\alpha^{n-2}}{\alpha^n} = p^2\ \text{ and }\ \mathrm{V}[X] = p. \tag{4.8.4}$$

As $n \to \infty$, the pf (4.8.3) tends to Poisson distribution with parameter $p$.

III. **Matching problem when $n$ is a Poisson r.v. with $\lambda$**

The probability of exactly $r$ matches by (4.5.1) is

$$p_r = \frac{e^{-\lambda}}{r!}\sum_{j \ge r/n}\frac{\lambda^j}{j!}\frac{(-1+\alpha)^{nj-r}}{(nj-r)!} \tag{4.8.5}$$

Which is a particular case of the pf (4.5.1). Therefore

$$\mathrm{E}[X] = \lambda\ ,\quad \mathrm{E}[X(X-1)] = \lambda + \lambda^2;\ \text{ and }\ \mathrm{V}[X] = 2\lambda. \tag{4.8.6}$$

IV. **Extended matching problem when $n$ is a Poisson r.v. with $\lambda$**

$$p_r = \frac{e^{-\lambda}}{r!}\sum_{j\geq r/n}\frac{\lambda^j}{j!}\frac{(-p+\alpha)^{nj-r}p^r}{(nj-r)!} \tag{4.8.7}$$

Which also is a particular case of the pf (4.5.1) with $p=-p, q=p$. Hence

$$\mathrm{E}[X]=\lambda p \quad \text{and} \quad \mathrm{V}[X]=\lambda p(1+p) \tag{4.8.8}$$

Clearly, (4.8.8) reduces to (4.8.6) when $p=1$.

V. **A problem of rumor.**

Suppose there are $(n+1)$ individuals in a region. A person narrates a rumor to a second person who in turn narrates it to a third person and so on. At each stage a recipient of the rumor is chosen at random from $n$ available persons excluding the narrator himself. Then

$p_r$ = Probability [ a rumor will be told $r$ times without being narrated to any person more than once | that the rumor will be told without narrating more than once ]

$$\begin{aligned} \text{Obviously,} \quad p_r &= \frac{n_{(r)}/n^r}{\sum\limits_{r=0}^{n} n_{(r)}/n^r} \\ &= \binom{n}{r}\frac{(\alpha/n)^r}{(1+\alpha/n)^n} \end{aligned} \tag{4.8.9}$$

is a particular case of the pf (4.2.1) with $p=0, q=1, \phi=1/n$.

Therefore by (4.3.12), it is observed that $\mathrm{E}[n-X]=n\frac{(1+\alpha/n)^{n-1}}{(1+\alpha/n)^n} = n\left[1-\frac{1}{(1+\alpha/n)^n}\right]$

VI. **Random mapping problem.**

For the random mapping model (Jaworski [47]) $(T;q)$ defined on a set of $n$ points such that $Pr[T(i)=i]=q',\ i=1(1)n$ and $Pr[T(i)=j]=\frac{1-q'}{n-1},\ i\neq j$. Consider the random variable $Y_n$ = number of cyclical vertices which are not loops. Berg and Jaworski [4] have shown that $\Pr[Y_n=k]$ is a special case of the pf (4.3.2) with $p=-Q, \phi=Q, Q=\frac{1-q'}{n-1}$. i.e.

$$Pr(Y_n=k)=\binom{n}{k}(-Q+\alpha Q)^k_\cdot(1+Q)^{n-k-1}(q'+kQ) \tag{4.8.10}$$

With $q' = 1/n$ i.e. $Q = 1/n$ (4.8.10) reduces to

$$Pr(Y_n = k) = \binom{n}{k} \frac{D_k(n+1)^{n-k-1}(k+1)}{n^n}. \tag{4.8.11}$$

Now by results of section §4.3.3.

$$\begin{aligned}
\mathrm{E}[Y_n] &= nQ\left[(1+\alpha Q)^{n-1} - 1\right], \\
\mathrm{E}[Y_n/q' = 1/n] &= (1+\alpha Q)^{n-1} - 1, \\
\mathrm{E}[(n-Y_n)^{(2)}] &= n^{(2)}(1+Q)^2 - 2n^{(2)}Q(1+\alpha Q)^{n-2} \text{ and} \\
\mathrm{E}[(n-Y_n)^{(2)}/q' = 1/n] &= (n-1)\left[\frac{(n+1)^2}{n} - 2(1+\frac{\alpha}{n})^{n-2}\right],
\end{aligned} \tag{4.8.12}$$

where of course $Q = \frac{1}{n}$.

VII. **Random mapping problem when $n$ is a Poisson r.v. with $\lambda$.**

If in the model in (VI) $n$ is assumed to be a r.v. having Poisson distribution with parameter $\lambda$, then denoting $Y_n'$ = number of cyclical vertices which are not loops, it can be shown by (4.5.5) that

$$Pr(Y_n' = k) = \frac{e^{-\lambda(1-(1+Q)^n)}}{k!} \frac{(-Q+\alpha Q)^k}{(1+Q)^k} \left[\mu_{(k)}{}' - \frac{Q}{1+Q}\mu_{(k+1)}{}'\right], \tag{4.8.13}$$

where $\mu_{(k)}{}'$ is the $k$th factorial moment of $nj$, where $j$ follows Poisson distribution with $\lambda(1+Q)^n$. In this case

$$\begin{aligned}
\mathrm{E}[Y_n'] &= \lambda \mathrm{E}[Y_n] = nQ\lambda\left[(1+\alpha Q)^{n-1} - 1\right], \\
\mathrm{E}[Y_n'/q' = 1/n] &= \lambda\left[(1+\alpha Q)^{n-1} - 1\right], \\
\text{and } \mathrm{E}[Y_n'^{(2)}] &= \lambda\left[\mathrm{E}(Y_n^2) + \mathrm{E}(Y_n)\mathrm{E}(Y_n')\right],
\end{aligned} \tag{4.8.14}$$

where $Y_n$ has probability distribution (4.8.10).

## 4.9 Some relation among the various probabilities

Denoting r.h.s. of the pfs (4.2.1),(4.3.1) and (4.3.2) by $\alpha$mb$(x; n, p, \phi)$, c$\alpha$mb$(x; n, p, \phi)$ and w$\alpha$mb$(x; n, p, \phi)$ we get the following relations

$$I.\ \alpha mb(x;n,p,\phi) = \frac{1}{(q+p+\alpha\phi)^n}[B(x;n,p) + n\phi(q+p+\alpha\phi)^{n-1} \alpha mb(x-1;n-1,p,\phi)]$$

$$II.\ \alpha mb(x;n,p,\phi) = \frac{1}{(q+p+\alpha\phi)^n}\sum_{i=0}^{x} x^{(i)}\left(\frac{\phi}{p}\right)^i B(x;n,p)$$

$$III.\ \alpha mb(x;n,p,\phi) = \frac{1}{(q+p+\alpha\phi)^n}B(x;n,p) + \phi\frac{n-x+1}{q}\alpha mb(x-1;n,p,\phi)$$

$$IV.\ B(x;n,p) = (q+p+\alpha\phi)^n \alpha mb(x;n,p,\phi) - n\phi(q+p+\alpha\phi)^{n-1} \alpha mb(x-1;n-1,p,\phi)$$

$$V.\ c\alpha mb(x;n,p,\phi,\nu) = \frac{1}{\sum_{r=0}^{\nu-1} n^{(r)}\phi^r(q+p)^{n-r}}[B(x;n,p) + \phi^\nu n^{(\nu)}B(x;n-\nu,p) + n\phi\left(\sum_{r=0}^{\nu-1}(n-1)^{(r)}\phi^r(q+p)^{n-1-r}\right) c\alpha mb(x-1;n-1,p,\phi,\nu)$$

$$VI.\ w\alpha mb(x;n,p,\phi) = b(x;n,p) - \frac{n\phi}{(q+p)}[b(x;n-1,p) - cw\alpha mb(x-1;n-1,p,\phi)]$$

$$VII.\ w\alpha mb(x;n,p,\phi,\nu) = \sum_{i=0}^{x}\frac{x^{(i)}\left(\frac{\phi}{p}\right)^i}{\sum_{r=0}^{\nu-1} n^{(r)}\phi^r(q+p)^{n-r}}[B(x;n,p) + \phi^\nu n^{(\nu)}B(x;n-\nu,p)]$$

$$VIII.\ w\alpha mb(x;n,p,\phi,\nu) = \sum_{i=0}^{x} x^{(i)}\left(\frac{\phi}{p}\right)^i[b(x;n,p) + n\phi b(x;n-1,p)]$$

$$IX.\ \frac{\alpha mb(x+1;n,p,\phi)}{\alpha mb(x;n,p,\phi)} = \frac{b(x+1;n,p)}{b(x;n,p)}\frac{\sum_{i=0}^{x+1}(x+1)^{(i)}\left(\frac{\phi}{p}\right)^i}{\sum_{i=0}^{x} x^{(i)}\left(\frac{\phi}{p}\right)^i}$$

$$= \frac{n-x}{x+1}\frac{p}{q}\left[\frac{(1+\alpha\frac{\phi}{p})^{x+1}}{(1+\alpha\frac{\phi}{p})^x}\right],$$

where $b(x;n,p) = \binom{n}{x}p^x q^{n-x}$ and $B(x;n,p) = (q+p)^n b(x;n,p)$.

# Chapter 5

# A Class of Weighted Generalized Poisson Distributions

## 5.1 Introduction

In this chapter various distributional properties of a class of weighted generalized Poisson distributions in general and that of some members of the class namely GPD II and GPD III in particular have been studied. The problem of parameter estimation by various methods including MLE has been discussed along with some data fittings.

## 5.2 A class of weighted generalized Poisson distributions

The pf of the GPD (Consul and Jain [18], [19]) is given by

$$Pr(X = k) = \frac{1}{k!}\, a\, (a + kz)^{k-1}\, e^{-(a+kz)} \tag{5.2.1}$$

Clearly,

$$\begin{aligned} \mathrm{E}\left[X^{(r+1)}\right] &= a\, e^{-(a+rz+z)} \sum_{i\geq 0} \frac{1}{k!}(a + rz + z + iz)^{i+r} e^{-(a+rz+z+iz)} \\ &= ae^{-(a+rz+z)} K(a + rz + z; r; z), \quad \text{say} \end{aligned} \tag{5.2.2}$$

where $K(a;s;z) = \sum_{i \geq 0} \frac{1}{i!}(a+iz)^{i+s}e^{-iz}$ and $X^{(r)} = X(X-1)\cdots(X-r+1)$.

Therefore the pf of the weighted GPD with weight function $X^{(r+1)}$ is given by

$$\frac{1}{(x-r-1)!}\frac{\{a+rz+z+(x-r-1)z\}^{x-r-1+r}e^{-(x-r-1)z}}{K(a+rz+z;r;z)} \tag{5.2.3}$$

making the transformation $Y = X - r - 1$; $b = a + rz + z$ The pf of $Y$ is obtained as

$$p_y = \frac{1}{y!}\frac{(b+yz)^{y+r}\ e^{-yz}}{K(b;r;z)}\ , \tag{5.2.4}$$

which is a class of weighted GPD derived from Consul's GPD. For $r = 0$ the size biased (Ord et al. [57], p.149, Johnson et al. [51], p.146) GPD distribution is obtained as

$$p_y = \frac{1}{y!}(1-z)(b+yz)^y e^{-(b+yz)} \tag{5.2.5}$$

Berg and Mutafchiev [5] mentioned about (5.2.5) and its convolution property. Nandi et al. [55] derived (5.2.4) by defining the exponential sums $K(a;s;z)$ (see appendix B) and referred it as a class of generalized Poisson distribution. Here it is shown that the above class of GPD is infact a family of weighted distributions of the GPD. This class (5.2.4) is therefore called a class of weighted generalized Poisson distribution (WGPD).

Denoting the weighted GP variate with parameters $b, z$ for given $r$ in (5.2.4) by WGPD$(b;r;z)$, it can be observed that the weighted GPD in (5.2.3) with weight function $X^{(r+1)}$ is $1+r+$WGPD$(b;r;z)$, where $b = a + rz + z$

In particular

I. $r = 0, z = 0$, size biased Poisson (Johnson et al. [51] p.146) is observed as 1+ Poisson $(a)$

II. $r = 0$, GPD III (Nandi et al. [55] can be seen as 1+WGPD$(a+z;0;z)$

III. $r = -1$, GPD in (5.2.1) as WGPD$(a;-1;z)$

IV. $r = -2$, gives GPD II (Nandi et al. [55] as $1-2+$WGPD$(a-z;-2;z)$

and many more for different choices of the parameter $r$. Following Nandi et al. [55] henceforth we refer (5.2.1) as GPD I.

## 5.3 Some Results on Moments

Moment properties of GPD I have been studied by Consul and Jain [19], Shoukri [64]. Gupta [36], Gupta and Singh [35] and Janardan [45] dealt with the moments of the restricted GPD I model (i.e. when $z = \alpha a$ in (5.2.1)). In this section, the moment properties of the class of WGPD have been discussed by deriving general formulas, recurrence relations and moments of some new distributions.

**Theorem 5.3.1** *If $X$ is* WGPD *(5.2.4) class with parameters $(a; s; z)$, then*

$$\mathrm{E}(a + Xz)^r = \frac{K(a; s + r; z)}{K(a; s; z)}$$

In particular,

$$\mathrm{E}(X) = \frac{1}{z}\left\{\frac{K(a; s+1; z)}{K(a; s; z)} - a\right\} \tag{5.3.1}$$

$$\mathrm{E}(X^2) = \frac{1}{z^2}\left\{\frac{K(a; s+2; z)}{K(a; s; z)} - 2az\mathrm{E}(X) - a^2\right\} \tag{5.3.2}$$

$$\mathrm{E}(X^3) = \frac{1}{z^3}\left\{\frac{K(a; s+3; z)}{K(a; s; z)} - 3az^2\mathrm{E}(X^2) - 3a^2z\mathrm{E}(X) - a^3\right\} \tag{5.3.3}$$

In general, we may write

$$\mathrm{E}(X^m) = \frac{1}{z^m}\left\{\frac{K(a; s+m; z)}{K(a; s; z)} - \sum_{i=0}^{m-1}\binom{m}{i} a^{m-i} z^i \mathrm{E}(X^i)\right\} \tag{5.3.4}$$

This relation can be used to determine the higher order moments starting with $\mathrm{E}(X)$ and values of exponential sums.

**Theorem 5.3.2**

$$\mu'_{r+1} = \frac{1}{1-z}\frac{d}{dt}\mu'_r(t)\mid_{t=1} + e^{-z}\mu'_1\mu'_r$$

*where $\mu'_r(t)$ is the $r$th moment about the origin for the* WGPD *class with parameters $(at; s; zt)$, i.e. $\mu'_r(1) = \mu'_r$ is the $r$th moment about the origin for the* WGPD *class with parameters $(a; s; z)$, and $\mu'_1$ is the mean of the distribution.*

For

I. $s = -1$, reduces to

$$\mu'_{r+1} = \frac{1}{1-z}\frac{d}{dt}\mu'_r(t) \mid_{t=1} + \frac{a}{1-z}\mu'_r \tag{5.3.5}$$

II. $s = 0$

$$\mu'_{r+1} = \frac{1}{1-z}\frac{d}{dt}\mu'_r(t) \mid_{t=1} + \frac{a(1-z)+z}{(1-z)^2}\mu'_r \tag{5.3.6}$$

III. $s = -2$

$$\mu'_{r+1} = \frac{1}{1-z}\frac{d}{dt}\mu'_r(t) \mid_{t=1} + \frac{a^2}{a(1-z)+z}\mu'_r \tag{5.3.7}$$

**Theorem 5.3.3** *If $X$ is* WGPD *class with parameters $(a;s;z)$, then*

$$\begin{aligned}\mu'_{(r)}(a;s;z) &= \frac{e^z}{K(a;s;z)}\sum_{k\geq 0} z^k e^{-kz}(a+rz+kz)K(a+(k+1)z;s;z) \\ & \quad \mu'_{(r-1)}(a+(k+1)z;s;z)\end{aligned} \tag{5.3.8}$$

This formula can be used to obtain the $i$th factorial moment about the origin when the form of the corresponding $(i-1)$th moment is available .

In particular, for

I. $s = -1$

$$\mu'_{(r)}(a;-1;z) = a\sum_{k\geq 0} z^k \frac{(a+rz+kz)}{(a+kz+z)}\mu'_{(r-1)}(a+(k+1)z;-1;z) \tag{5.3.9}$$

II. $s = 0$

$$\mu'_{(r)}(a;0;z) = \sum_{k\geq 0} z^k(a+rz+kz)\mu'_{(r-1)}(a+(k+1)z;0;s) \tag{5.3.10}$$

**Theorem 5.3.4** $\mu_{k+1} = \frac{k}{1-z}\frac{d}{dt}\mu(t) \mid_{t=1} \mu_{k-1} + \frac{1}{1-z}\frac{d}{dt}\mu_k(t) \mid_{t=1},$ *where*

$\mu_k(t) = \sum_x\{x-\mu(t)\}^k p_x(at;s;zt)$ *wherein* $\mu(t) = \sum_x x p_x(at;s;zt)$ *and*

$p_x(a;s;z)$ *is the pf of* WGPD$(a;s;z)$ *i.e.* $\mu = \mu(1)$*, the mean of* $X \sim$ WGPD*(a;s;z)*

The above result can be equivalently stated as

$$\mu_{k+1} = k\mu_2\,\mu_{k-1} + \frac{1}{1-z}\frac{d}{dt}\mu_k(t) \mid_{t=1}$$

For

I. $z = 0$, the following relation for Poisson distribution with parameter $a$ is obtained

$$\mu_{k+1} = ak\ \mu_{k-1} + \frac{d}{dt}\mu_k(t)\ |_{t=1} \tag{5.3.11}$$

II. $s = -1$, reduces to (Consul [14], p.51))

$$\mu_{k+1} = \frac{ak}{(1-z)^3}\ \mu_{k-1} + \frac{1}{1-z}\frac{d}{dt}\mu_k(t)\ |_{t=1} \tag{5.3.12}$$

III. $s = 0$

$$\mu_{k+1} = k\frac{z^2 + (1-a)z + a}{(1-z)^4}\ \mu_{k-1} + \frac{1}{1-z}\frac{d}{dt}\mu_k(t)\ |_{t=1} \tag{5.3.13}$$

IV. $s = -2$

$$\mu_{k+1} = k\frac{a^2(a-z)}{(1-z)(a+z-az)}\ \mu_{k-1} + \frac{1}{1-z}\frac{d}{dt}\mu_k(t)\ |_{t=1} \tag{5.3.14}$$

## 5.4 A relationship between central moments of GPD I and GPD III

Denoting the $r$th central moment of GPD I$(a; z)$ and GPD III$(a + z; z)$ by $\mu_r$ and ${\mu_r}^*$ respectively it can be easily seen that

$$\mu_r = \frac{a}{1-z}\sum_{j=0}^{r-1}\binom{r-1}{j}{\mu_r}^*$$

## 5.5 Inverse moments

Inverse integer moments of the class of WGPD are discussed here. These results are important in many problem of applied statistics. The corresponding results for GPD I (Consul and Shoukri [26]) are seen as particular cases.

**Theorem 5.5.1** *If X is* WGPD *class with parameters* $(a; s; z)$, *then*

$$E(a+Xz)^{-r} = \frac{K(a; s-r; z)}{K(a; s; z)} \tag{5.5.1}$$

In particular,

$$E\left(X+\frac{a}{z}\right)^{-1} = z\left\{\frac{K(a; s-1; z)}{K(a; s; z)}\right\} \tag{5.5.2}$$

$$E\left(X+\frac{a}{z}\right)^{-2} = z^2\left\{\frac{K(a; s-2; z)}{K(a; s; z)}\right\} \tag{5.5.3}$$

Now, for

I. $s=-1$, we get the corresponding GPD I(Consul [14], p.60) results.

$$E\left(X+\frac{a}{z}\right)^{-1} = \frac{z}{a} - \frac{z^2}{a+z} \tag{5.5.4}$$

$$E\left(X+\frac{a}{z}\right)^{-2} = \frac{z^2}{a^2} - \frac{z^3}{a(a+z)} - \frac{z^3}{(a+z)^2} + \frac{z^4}{(a+z)(a+2z)} \tag{5.5.5}$$

II. $s=0$

$$E\left(X+\frac{a}{z}\right)^{-1} = \frac{z(1-z)}{a} \tag{5.5.6}$$

$$E\left((X+\frac{a}{z}\right)^{-2} = \frac{z^2(1-z)}{a^2}\frac{a+z-az}{a+z} \tag{5.5.7}$$

III. $s=-2$

$$E\left(X+\frac{a}{z}\right)^{-1} = \frac{z}{a} - \frac{z^2 a}{(a+z)(a+2z)} - \frac{z^3(1-z)a}{(a+z)(a+2z)(a(1-z)+z)} \tag{5.5.8}$$

**Theorem 5.5.2** *If X is* WGPD *class with parameters* $(a; s; z)$, *then*

$$E(a+Xz)^{-r} = a^{-1}E(a+Xz)^{-(r-1)} - za^{-1}e^{-z}\frac{K(a+z; s; z)}{K(a; s; z)}E(a+z+Xz)^{-(r-1)} \tag{5.5.9}$$

*alternatively (5.5.9) can be written in the following different ways*

$$E(a+Xz)^{-r} = aE(a+Xz)^{-r-1} + ze^{-z}\frac{K(a+z; s; z)}{K(a; s; z)}E(a+z+Xz)^{-r-1} \tag{5.5.10}$$

*or*

$$E\left(X+\frac{a}{z}\right)^{-r} = \frac{a}{z}E\left(X+\frac{a}{z}\right)^{-r-1} + ze^{-z}\frac{K(a+z; s; z)}{K(a; s; z)}E\left(X+1+\frac{a}{z}\right)^{-r} \tag{5.5.11}$$

*or*

$$\mathrm{E}\left(X+\frac{a}{z}\right)^{-r} = \frac{z}{a}\mathrm{E}\left(X+\frac{a}{z}\right)^{-(r-1)} - \frac{z^2e^{-z}}{a}\frac{K(a+z;s;z)}{K(a;s;z)}\mathrm{E}\left(X+1+\frac{a}{z}\right)^{-(r-1)} \tag{5.5.12}$$

In particular, for

I. $s=-1$, it reduces to (Consul [14], p.62)

$$\mathrm{E}\left(X+\frac{a}{z}\right)^{-r} = za^{-1}\mathrm{E}\left(X+\frac{a}{z}\right)^{-(r-1)} - \frac{z^2}{a+z}\mathrm{E}\left(X+1+\frac{a}{z}\right)^{-(r-1)} \tag{5.5.13}$$

II. $s=0$

$$\mathrm{E}\left(X+\frac{a}{z}\right)^{-r} = za^{-1}\mathrm{E}\left(X+\frac{a}{z}\right)^{-(r-1)} - \frac{z^2}{a}\mathrm{E}\left(X+1+\frac{a}{z}\right)^{-(r-1)} \tag{5.5.14}$$

III. $s=1$

$$\mathrm{E}\left(X+\frac{a}{z}\right)^{-r} = za^{-1}\mathrm{E}\left(X+\frac{a}{z}\right)^{-(r-1)} - \frac{az^2[(a+z)(1-z)+z]}{(a+z)(a+2z)(a+z-az)}\mathrm{E}\left(X+1+\frac{a}{z}\right)^{-(r-1)} \tag{5.5.15}$$

**Theorem 5.5.3** *If $X$ is* WGPD *class with parameters $(a;s;z)$, then*

$$\mathrm{E}\left\{\frac{1}{(X+1)^{[r]}}\right\} = \frac{e^{rz}}{K(a;s;z)}\left\{K(a-rz;s-r;z) - \sum_{k\geq 0}^{r-1}\frac{e^{-kz}(a-rz+kz)^{k+s-r}}{k!}\right\} \tag{5.5.16}$$

In particular, for

1. $r=1$

$$\mathrm{E}\left\{\frac{1}{X+1}\right\} = \frac{e^z}{K(a;s;z)}\left\{K(a-z;s-1;z)-(a-z)^{s-1}\right\} \tag{5.5.17}$$

(a) $s=-1$, (Consul [14], p.60)

$$\mathrm{E}\left\{\frac{1}{X+1}\right\} = \frac{-z}{(a-z)} + \frac{a}{(a-z)^2}\{1-e^{-a+z}\} \tag{5.5.18}$$

(b) $s=0$

$$\mathrm{E}\left\{\frac{1}{X+1}\right\} = \frac{1-z}{a-z}\left(1-e^{-a+z}\right) \tag{5.5.19}$$

(c) $s = -2$

$$\mathrm{E}\left\{\frac{1}{X+1}\right\} = \Big\{(a-z)^3(1-2z+z^2)+(a-z)^2(4z+z^3-5z^2)+(a-z)(5z^2-2z^3) + 2z^3 - e^{-a+z}a^2(a+z)\Big\}\left\{(a-z)^3(a+z-az)\right\}^{-1} \quad (5.5.20)$$

2. $r = 2$

$$\mathrm{E}\left\{\frac{1}{(X+1)(X+2)}\right\} = \frac{e^{2z}}{K(a;s;z)}\Big\{K(a-2z;s-2;z) - ((a-2z)^{s-2}+e^{-z}(a-z)^{s-1})\Big\} \quad (5.5.21)$$

3. $r = 3$

$$\mathrm{E}[\frac{1}{(X+1)(X+2)(X+3)}] = \frac{e^{3z}}{K(a;s;z)}\Big[K(a-3z;s-3;z) - \Big\{(a-3z)^{s-3} + e^{-z}(a-2z)^{s-2} + \frac{e^{-2z}(a-z)^{s-1}}{2!}\Big\}\Big] \quad (5.5.22)$$

Given the values of $\mathrm{E}\left\{\frac{1}{(X+1)^{[r]}}\right\}$, it is possible to evaluate the values of $\mathrm{E}\left\{\frac{1}{(X+m)}\right\}$ for $m = 2, 3, \ldots,$ by using the following relations

$$\frac{1}{X+2} = \frac{1}{X+1} - \frac{1}{(X+1)^{[2]}}$$

$$\frac{1}{X+3} = \frac{1}{X+1} - \frac{2}{(X+1)^{[2]}} + \frac{2}{(X+1)^{[3]}}$$

$$\frac{1}{X+4} = \frac{1}{X+1} - \frac{3}{(X+1)^{[2]}} + \frac{6}{(X+1)^{[3]}} - \frac{6}{(X+1)^{[4]}}$$

In general on using the result

$$\frac{1}{(X+m)^{[r]}} = \frac{1}{(X+m-1)^{[r]}} - \frac{r}{(X+m-1)^{[r+1]}} \quad (5.5.23)$$

The following recurrence relation can be obtained.

**Theorem 5.5.4** *If $X$ is* WGPD *class with parameters $(a;s;z)$, then*

$$\mathrm{E}\left\{\frac{1}{(X+m)^{[r]}}\right\} = \mathrm{E}\left\{\frac{1}{(X+m-1)^{[r]}}\right\} - r\mathrm{E}\left\{\frac{1}{(X+m-1)^{[r+1]}}\right\} \quad (5.5.24)$$

**Theorem 5.5.5** *If* $X$ *is* WGPD *class with parameters* $(a;s;z)$*, then*

$$\begin{aligned} e^{-z}\frac{a-z-sz}{a-z}\frac{K(a;s;z)}{K(a-z;s;z)}\mathrm{E}_a(X+m+1)^{-r} &= -\left\{\frac{s}{a-z}-\frac{d}{da}\log K(a-z;s;z)\right\}\mathrm{E}_{a-z}(X+m)^{-r} \\ &+ \frac{d}{da}\mathrm{E}_{a-z}(X+m)^{-r} \end{aligned} \tag{5.5.25}$$

*where* $\mathrm{E}_t(\ )$ *denotes the mathematical expectation of* $X \sim$WGPD$(a;s;z)$ *when* $a=t$

In particular for

I. $s=-1$, reduces to (Consul [14], p.62)

$$\mathrm{E}_a(X+m+1)^{-r} = \mathrm{E}_{a-z}(X+m)^{-r} + \frac{d}{da}\mathrm{E}_{a-z}(X+m)^{-r} \tag{5.5.26}$$

II. $s=0$

$$\mathrm{E}_a(X+m+1)^{-r} = \mathrm{E}_{a-z}(X+m)^{-r} + \frac{d}{da}\mathrm{E}_{a-z}(X+m)^{-r} \tag{5.5.27}$$

III. $s=-2$

$$\begin{aligned} &\frac{(a-z)(a+z-az)}{a(a+z^2-az)}\mathrm{E}_a(X+m+1)^{-r} \\ = &-\left(1+\frac{1-z}{a-az+z^2}\right)\mathrm{E}_{a-z}(X+m)^{-r} + \frac{d}{da}\mathrm{E}_{a-z}(X+m)^{-r} \end{aligned} \tag{5.5.28}$$

## 5.6 Incomplete moments: Some relations

The incomplete moments are important as they can be used to derive expressions for mean deviation about mean and the generalized moments of the absolute deviations (Kamat [52]). Denoting incomplete moments of order $k$ on the right about origin by ${}_xM_k$ and on the left by ${}^xM^k$, we have for WGPD$(a;s;z)$

$$\begin{aligned} {}_xM_k(a;z) &= \sum_{j=x}^{\infty}\frac{j^k}{j!}\frac{(a+jz)^{j+s}e^{-jz}}{K(a;s;z)} && (5.6.1) \\ {}^xM^k(a;z) &= \sum_{j=0}^{x}\frac{j^k}{j!}\frac{(a+jz)^{j+s}e^{-jz}}{K(a;s;z)} && (5.6.2) \end{aligned}$$

Clearly, ${}_0M_k(a;z) = {}_\infty M_k(a;z) = \mathrm{E}(X^k)$

Hence, ${}_0M_0(a;z) = 1$; ${}_0M_1(a;z) = \mathrm{E}(X) = {}_1M_1(a;z); \ldots, {}_xM_0(a;z) = 1 - F_x(a;z) = Q_{x-1}(a;z)$

where $F_x(a;z)$ is the cumulative distribution function (cdf) of $X$ i.e.

$$F_x(a;z) = Pr(X \leq x) \text{ and } 1 - F_x(a;z) = Q_{x-1}(a;z)$$

### 5.6.1 Some relation for ${}_xM_k$

**Theorem 5.6.1**

$$\begin{aligned} {}_xM_k(a;z) &= e^{-z}\frac{K(a+z;s;z)}{K(a;s;z)}\left\{(a+z)\sum_{i=0}^{k-1}\binom{k-1}{i}{}_{x-1}M_i(a+z;z)\right. \\ &+ \left. z\sum_{i=0}^{k-1}{}_{x-1}M_{i+1}(a+z;z)\right\} \end{aligned} \tag{5.6.3}$$

In particular, for

I. $s = -1$, reduces to Consul [14], p.68)

$$\begin{aligned} {}_xM_k(a;z) &= a\sum_{i=0}^{k-1}\binom{k-1}{i}{}_{x-1}M_i(a+z;z) \\ &+ \frac{az}{a+z}\sum_{i=0}^{k-1}{}_{x-1}M_{i+1}(a+z;z) \end{aligned} \tag{5.6.4}$$

II. $s = 0$

$$\begin{aligned} {}_xM_k(a;z) &= (a+z)\sum_{i=0}^{k-1}\binom{k-1}{i}{}_{x-1}M_i(a+z;z) \\ &+ \frac{az}{a+z}\sum_{i=0}^{k-1}{}_{x-1}M_{i+1}(a+z;z) \end{aligned} \tag{5.6.5}$$

III. $s = -2$

$$\begin{aligned} {}_xM_k(a;z) &= \left\{\frac{a^2[(a+z)(1-z)+z]}{(a+2z)(a-az+z)}\right\}\sum_{i=0}^{k-1}\binom{k-1}{i}{}_{x-1}M_i(a+z;z) \\ &+ \left\{\frac{za^2[(a+z)(1-z)+z]}{(a+z)(a+2z)(a(1-z)+z)}\right\}\sum_{i=0}^{k-1}{}_{x-1}M_{i+1}(a+z;z) \end{aligned} \tag{5.6.6}$$

**Theorem 5.6.2**

$$
\begin{aligned}
{}_xM_k(a;z) &= e^{-z}\frac{K(a+z;s;z)}{K(a;s;z)}\left[\{(a+z)Q_{x-2}(a+z;z)+z_{x-1}M_k(a+z;z)\}\right. \\
&+ \left.\sum_{i=0}^{k-2}\left\{a\binom{k-1}{i+1}+z\binom{k}{i+1}\right\}{}_{x-1}M_i(a+z;z)\right]
\end{aligned} \tag{5.6.7}
$$

In particular, for

I. $s=-1$, reduces to Consul [14], p.68

$$
\begin{aligned}
{}_xM_k(a;z) &= \left\{aQ_{x-2}(a+z;z)+\frac{az}{a+z}{}_{x-1}M_k(a+z;z)\right\} \\
&+ \sum_{i=0}^{k-2}\left\{a\binom{k-1}{i+1}+\frac{az}{a+z}\binom{k-1}{i}\right\}{}_{x-1}M_{i+1}(a+z;z)
\end{aligned} \tag{5.6.8}
$$

II. $s=0$

$$
\begin{aligned}
{}_xM_k(a;z) &= \{(a+z)Q_{x-2}(a+z;z)+z_{x-1}M_k(a+z;z)\} \\
&+ \sum_{i=0}^{k-2}\left\{a\binom{k-1}{i+1}+z\binom{k}{i+1}\right\}{}_{x-1}M_{i+1}(a+z;z)
\end{aligned} \tag{5.6.9}
$$

III. $s=-2$

$$
\begin{aligned}
{}_xM_k(a;z) &= \{g(a,z)Q_{x-2}(a+z;z)+h(a,z)_{x-1}M_k(a+z;z)\} \\
&+ \sum_{i=0}^{k-2}\left\{g(a,z)\binom{k-1}{i+1}+h(a,z)\binom{k-1}{i}\right\}{}_{x-1}M_{i+1}(a+z;z)
\end{aligned} \tag{5.6.10}
$$

where

$$
g(a,z)=\frac{a^2\{(a+z)(1-z)+z\}}{(a+2z)(a(1-z)+z)}
$$

and

$$
h(a,z)=\frac{za^2\{(a+z)(1-z)+z\}}{(a+z)(a+2z)(a(1-z)+z)}
$$

**Theorem 5.6.3**

$$
\begin{aligned}
{}_xM_1(a;z) &= \sum_{i=0}^{x-2} z^i e^{-(i+1)z}(a+(i+1)z)\frac{K(a+(i+1)z;s;z)}{K(a;s;z)}Q_{x-2-i}(a+(i+1)z;z) \\
&+ z^{x-1}e^{-xz}\frac{K(a+xz;s+1;z)}{K(a;s;z)}
\end{aligned} \tag{5.6.11}
$$

For

I. $s = -1$, reduces to Consul [14], p.69

$$ {}_xM_1(a;z) = a\sum_{i=0}^{x-2} z^i Q_{x-2-i}(a+(i+1)z;z) + z^{x-1}\frac{a}{1-z} \tag{5.6.12}$$

II. $s = 0$

$$\begin{aligned} {}_xM_1(a;z) &= \sum_{i=0}^{x-2} z^i(a+(i+1)z)\ Q_{x-2-i}(a+(i+1)z;z) \\ &+ z^{x-1}\left(\frac{(a+xz)(1-z)+z^2}{(1-z)^2}\right) \end{aligned}\tag{5.6.13}$$

The following result is derived using the relation (5.6.3) when $k = 2$

**Theorem 5.6.4**

$$\begin{aligned} {}_xM_2(a;z) &= \sum_{i=0}^{x-2} z^i \frac{e^{-(i+1)z}K(a+(i+1)z;s;z)}{k(a;s;z)}\{(a+(i+1)z) \\ & Q_{x-2}(a+(i+1)z;z) + (a+(i+2)z)_{x-i-1}M_1(a+(i+1)z;z)\} \\ &+ z^{x-1}\frac{e^{-xz}k(a+(x-1)z;s;z)}{K(a;s;z)}\ {}_1M_2(a+(x-1)z;z) \end{aligned}\tag{5.6.14}$$

## 5.6.2 Some relation for ${}^xM^k$

Initial values are ${}^0M^k(a;z) = 0$; ${}^xM^0(a;z) = F_x(a;z)$ and ${}^1M^k(a;z) = p_1(a;s;z) = \frac{(a+z)^{s+1}e^{-z}}{K(a,s,z)}$

**Theorem 5.6.5**

$$\begin{aligned} {}^xM^k(a;z) &= e^{-z}\frac{k(a+z;s;z)}{K(a;s;z)}\sum_{i=0}^{k-1}\binom{k-1}{i}\{(a+z)\ {}^{x-i}M^i(a+z;z) \\ &+ z\ {}^{x-1}M^{i+1}(a+z;z)\} \end{aligned}\tag{5.6.15}$$

In particular, for

I. $s = -1$

$$ {}^xM^k(a;z) = \sum_{i=0}^{k-1}\{a\ {}^{x-i}M^i(a+z;z) + \frac{az}{a+z}\ {}^{x-1}M^{i+1}(a+z,z)\} \tag{5.6.16}$$

II. $s=0$

$$ {}^{x}M^{k}(a;z)=\sum_{\imath=0}^{k-1}\{(a+z)\ {}^{x-\imath}M^{\imath}(a+z;z)+z\ {}^{x-1}M^{\imath+1}(a+z;z)\} \tag{5.6.17}$$

III. $s=-2$

$$ {}^{x}M^{k}(a;z)=\sum_{\imath=0}^{k-1}\{(a+z)\ {}^{x-\imath}M^{\imath}(a+z;z)+z\ {}^{x-1}M^{\imath+1}(a+z;z)\} \tag{5.6.18}$$

**Theorem 5.6.6**

$$\begin{aligned} {}^{x}M^{1}(a;z) &= \sum_{\imath=0}^{x-2} z^{\imath}e^{-(\imath+1)}\frac{(a+(i+1)z)K(a+(i+1)z;s;z)}{K(a;s;z)} \\ & F_{x-\imath-1}(a+(i+1)z;z)+z^{x-1}e^{-xz}\frac{(a+(x-1)z+z)^{s+1}}{K(a;s;z)} \end{aligned} \tag{5.6.19}$$

$$\begin{aligned} {}^{x}M^{2}(a;z) &= \sum_{\imath=0}^{x-2} z^{\imath}e^{-(\imath+1)z}\frac{K(a+(\imath+1)z,s;z)}{K(a;s;z)}\{(a+(i+1)z)F_{x-\imath-1}(a+(\imath+1)z;z) \\ &+ (a+(\imath+2)z\ {}^{x-\imath-1}M^{1}\ (a+(i+1)z;z)\}+z^{x-1}\frac{e^{-xz}(a+xz)^{s+1}}{K(a;s;z)} \end{aligned} \tag{5.6.20}$$

In particular, for

I. $s=-1$, reduces to Consul [14] (p.70)

$$ {}^{x}M^{1}(a;z) = a\sum_{\imath=0}^{x-2} z^{\imath}F_{x-\imath-1}(a+(i+1)z;z)+az^{x-1}e^{-a-xz} \tag{5.6.21}$$

$$\begin{aligned} {}^{x}M^{2}(a;z) &= a\sum_{\imath=0}^{x-2} z^{\imath}\left\{F_{x-\imath-1}(a+(\imath+1)z;z)+\frac{a+(\imath+2)z}{a+(\imath+1)z}\ {}^{x-\imath-1}M^{1}\right. \\ & \left.(a+(\imath+1)z;z)\right\}+az^{x-1}e^{-a-xz} \end{aligned} \tag{5.6.22}$$

II. $s=0$

$$\begin{aligned} {}^{x}M^{1}(a;z) &= \sum_{\imath=0}^{x-2} z^{\imath}(a+(\imath+1)z)F_{x-\imath-1}(a+(i+1)z;z) \\ &+ z^{x-1}e^{-a-xz}(a+xz)(1-z) \end{aligned} \tag{5.6.23}$$

$$\begin{aligned} {}^{x}M^{2}(a;z) &= \sum_{\imath=0}^{x-2} z^{\imath}[(a+(i+1)z)F_{x-\imath-1}(a+(i+1)z;z)(a+(i+2)z)\ {}^{x-\imath-1}M^{1} \\ & (a+(i+1)z;z)]+z^{x-1}e^{-a-xz}(a+xz)(1-z) \end{aligned} \tag{5.6.24}$$

III. $s=-2$

$$ {}^{x}M^{1}(a;z) = \sum_{\imath=0}^{x-2} z^{\imath}\left\{\frac{[(a+(i+1)z)(1-z)+z]a^{2}(a+z)}{(a+(i+1)z)((a+(\imath+2)z)(a+z-az)}F_{x-\imath-1}(a+(i+1)z;z)\right\}$$

$$+ \quad \frac{z^{x-1}e^{-a-xz}a^2(a+z)}{(a+xz)(a+z-az)} \tag{5.6.25}$$

$$\begin{aligned} {}^{x}M^2(a;z) &= \sum_{i=0}^{x-2} z^i \left\{ \frac{[(a+(i+1)z)(1-z)+z]a^2(a+z)}{(a+(i+1)z)^2(a+(i+2)z)(a+z-az)}(a+(i+1)z)F_{x-i-1}(a+(i+1)z;z) \right. \\ &+ \left. (a+(i+2)z)\ {}^{x-i-1}M^1(a+(i+1)z;z) \right\} + z^{x-1}\frac{e^{-a-xz}a^2(a+z)}{(a+xz)(a+z-az)} \end{aligned} \tag{5.6.26}$$

### 5.6.3 A formula for doubly incomplete moment of order 1

**Theorem 5.6.7** *If X is* WGPD *class with parameters $(a;s;z)$, then the doubly incomplete moment of order 1 is given by*

$$\begin{aligned} \sum_{j=l}^{r} j\frac{(a+jz)^{j+s}e^{-jz}}{j!K(a;s;z)} &= {}_lM_1(a;z) - {}_{r+1}M_1(a;z) && (5.6.27) \\ &= {}^{r}M^1(a;z) - {}^{l-1}M^1(a;z) && (5.6.28) \end{aligned}$$

Putting $s=-1$ in the above formulas the results obtained by Consul [14] (p.70.) can be observed as particular cases.

## 5.7 Mean deviation about mean

**Theorem 5.7.1** *Let $\mu$ =mean and $(\mu)$ = integral part of mean of the distribution, then the mean deviation about mean for* WGPD$(a;s;z)$ *is given by*

$$\begin{aligned} \delta &= \mathrm{E}[\mid X-\mu \mid] \\ &= 2\left[\mu F_{(\mu)}(a;z) - \sum_{x=0}^{(\mu)} x\, P_x(a;s;z)\right] \\ &= 2\left[\mu F_{(\mu)}(a;z) - {}^{(\mu)}M^1(a;z)\right] \end{aligned} \tag{5.7.1}$$

In particular, for

I. $s=-1$ we have for GPD I

$$\begin{aligned} \delta &= 2\left[\mu F_{(\mu)}(a;z) - a\sum_{i=0}^{(\mu)-2} z^i F_{(\mu)-1-i}(a+(i+1)z;z) \right. \\ &- \left. a z^{(\mu)-1} e^{-a-(\mu)z}\right] \end{aligned} \tag{5.7.2}$$

II. $s = 0$

$$\begin{aligned}\delta &= 2\Bigg[\mu F_{(\mu)}(a;z) - a\sum_{i=0}^{(\mu)-2} z^i(a+(i+1)z)F_{(\mu)-1-i}(a+(i+1)z;z)\\ &- z^{(\mu)-1}e^{-a-(\mu)z}(a+(\mu)z)(1-z)\Bigg] \qquad (5.7.3)\end{aligned}$$

III. $s = -2$

$$\begin{aligned}\delta &= 2\Bigg[\mu F_{(\mu)}(a;z) - a\sum_{i=0}^{(\mu)-2} z^i \frac{[(a+(i+1)z)(1-z)+z]a^2(a+z)}{(a+(i+1)z)(a+(i+2)z)(a+z_a z)} F_{(\mu)-1-i}(a+(i+1)z;z)\\ &- \frac{a^2(a+z)z^{(\mu)-1}e^{-a-(\mu)z}}{(a+z-az)(a+(\mu)z)}\Bigg] \qquad (5.7.4)\end{aligned}$$

IV. $z = 0$, the corresponding result for Poisson distribution is observed as (Johnson and Kotz [48], p.91, and Johnson et al. [51], p.157)

$$\delta = 2\left\{\frac{e^{-\mu}\mu^{(\mu)+1}}{(\mu)!}\right\} \qquad (5.7.5)$$

## 5.8 Probability generating function of the class of WGPD

The probability generating function (pgf) of the class of WGPD is given by

$$G(u) = t^{-s}\frac{K(at;s;zt)}{K(a;s;z)} \qquad (5.8.1)$$

where $u = te^{z(1-t)}$.

For $s = -1$ the pgf of GPD I is obtained as $G(u) = e^{a(t-1)}$.

### Moments from pgf

In the following the expressions for first four moments about origin in terms of derivatives of the pgf are presented.

$$\begin{aligned} i)\ \frac{d}{dt}G(u)\mid_{t=1} &= (1-z)\mathrm{E}(X) && (5.8.2)\\ ii)\ \frac{d^2}{dt^2}G(u)\mid_{t=1} &= (1-z)^2\mathrm{E}(X^2) - \mathrm{E}(X) && (5.8.3)\\ iii)\ \frac{d^3}{dt^3}G(u)\mid_{t=1} &= (1-z)^3\mathrm{E}(X^3) - 3(1-z)\mathrm{E}(X^2) + 2\mathrm{E}(X) && (5.8.4)\end{aligned}$$

$$iv)\ \frac{d^4}{dt^4}G(u)\mid_{t=1} = (1-z)^4\mathrm{E}(X^4) - 6(1-z)^2\mathrm{E}(X^3) + (11-8z)\mathrm{E}(X) - 6\mathrm{E}(X) \tag{5.8.5}$$

## 5.9 Distribution of the sums

In the following the distributions of sums of independent WGP variates are discussed.

**Theorem 5.9.1** *If $X_i s$ are independent* WGPD *class with parameters $(a_i, s_i, z)$ , then $Y = X_1 + \cdots + X_j$ is distributed with pf*

$$Pr(Y = n) = \frac{e^{-nz}}{n!}\frac{B_n(a_1, a_2, \ldots, a_j; s_1, s_2, \ldots, s_j; z)}{\prod_{i=1}^{j} K(a_i; s_i; z)} \tag{5.9.1}$$

where

$$B_n(a_1, a_2, \ldots, a_j; s_1, s_2, \ldots, s_j; z) = \Sigma\binom{n}{\mathbf{k}}\prod_{i=1}^{j}(a_i + k_i z)^{k_i+s_i}$$

wherein $\mathbf{k} = (k_1, \ldots, k_j)$, $\binom{n}{\mathbf{k}}$ is the multinomial coefficient and the sum is over all non-negative integers $k_1, \ldots, k_j$ such that $\sum_{i=1}^{j} k_i = n$.

For $j = 2$, (5.9.1) reduces to

$$Pr(X_1 + X_2 = n) = \frac{e^{-nz}}{n!}\frac{B_n(a_1, a_2; s_1, s_2; z)}{K(a_1; s_1; z)K(a_2; s_2; z)} \tag{5.9.2}$$

The distributions (5.9.1) and (5.9.2) are referred to as $j$-**gpsum** and 2-**gpsum** distributions respectively.

### 5.9.1 Some important particular cases

Following are some important particular cases are derived from the above theorems.

**Theorem 5.9.2** *If $X_1$ and $X_2$ are two independent* GPD I *with parameters $(a_1, z)$ and $(a_2, z)$ respectively, then the sum $Y = X_1 + X_2$ is also a* GPD I *with parameters $a_1 + a_2, z)$ (Consul [14]).*

**Theorem 5.9.3** *If $X_1$ be a* GPD I *with $(a_1, z)$ and $X_2$ is a* GPD III *independent of $X_1$ with parameters $(a_2, z)$ respectively, then the sum $Y = X_1 + X_2$ is follows a* GPD III *with parameters $(a_1 + a_2, z)$.*

**Theorem 5.9.4** *If $X_1$ and $X_2$ are two independent* GPD III *with parameters $(a_1, z)$ and $(a_2, z)$ respectively, then the sum $Y = X_1 + X_2$ is a $\alpha$-modified* GPD III *of order one with parameters $(a_1 + a_2, z)$. The pf of the $\alpha$-modified* GPD III *of order one is given by*

$$Pr(X_1 + X_2 = n) = \frac{1}{n!}(1-z)^2(a_1 + a_2 + \alpha z + nz)^n e^{-(a_1+a_2+nz)} \tag{5.9.3}$$

*where $\alpha^k = \alpha_k \equiv k!$.*

Above theorem can be extended as follows:

**Theorem 5.9.5** *If $X_i s$ are independent* GPD III *with parameters $(a_i, z)$ , then the sum $Y = X_1 + \cdots + X_{j+1}$ is a $\alpha$-modified* GPD III *of order $j$ with parameters $(a_1 + \cdots + a_{j+1}, z)$. The pf of the $\alpha$-modified* GPD III *of order $j$ is given by*

$$Pr(Y = n) = \frac{1}{n!}(1-z)^{j+1}\left\{\sum_{i=1}^{j+1} a_i + \alpha(j)z + nz\right\}^n e^{-\left\{\sum_{i=1}^{j+1} a_i + nz\right\}} \tag{5.9.4}$$

*where $\alpha^k = \alpha_k \equiv k!$ and $\alpha^k(j) = \alpha_k(j) = (\underbrace{\alpha + \alpha + \cdots + \alpha}_{j})^k = \binom{k+j-1}{k}k!$*

*Remark* : The WGPD class is closed under convolution only for $s = -1$.

### 5.9.2 Two recurrence relations of the probabilities

It can be easily seen that (Riordan [62], p.24, eq. 32)

$$P_n(a_1, a_2, \ldots, a_j; s_1, s_2, \ldots, s_j; z) = \frac{e^{-z}}{n}\sum_{i=1}^{j}\frac{K(a_i + z; s_i + 1; z)}{K(a_i; s_i; z)} P_n{}^i(a_1, a_2, \ldots, a_j; s_1, s_2, \ldots, s_j; z) \tag{5.9.5}$$

Where $P_n(a_1, a_2, \ldots, a_j; s_1, s_2, \ldots, s_j; z)$ refers to (5.9.1) and

$$P_n{}^i(a_1, a_2, .., a_j; s_1, s_2, .., s_j; z) = P_n{}^i(a_1, a_2, .., a_i + z, .., a_j; s_1, s_2, .., s_i + 1, .., s_j; z)$$

The second relation is obtained using equation 33 of Riordan [62], p.24 as

$$P_n(a_1, a_2, \ldots, a_j; s_1, s_2, \ldots, s_j; z) = \sum_{k=0}^{n} \frac{K(a_1 + kz; s_1 - 1; z)}{K(a_1; s_1; z)} e^{-kz} P_{n-k}(a_1 + kz, a_2, \ldots, a_j; s_1 - 1, s_2, \ldots, s_j; z) \quad (5.9.6)$$

### 5.9.3 Factorial moments

Formula for the first two factorial moments can be derived using the first relation above as

$$\mathrm{E}[n] = e^{-z} \sum_{i=1}^{j} \frac{K(a_i + z; s_i + 1; z)}{K(a_i; s_i; z)} \quad (5.9.7)$$

$$\mathrm{E}[n^{(2)}] = e^{-2z} \sum_{i=1}^{j} \sum_{l=1}^{j} \frac{K(a_i + z; s_i + 1; z) K(a_l + (1 + \delta_{li})z; s_l + \delta_{li} + 1; z)}{K(a_i; s_i; z) K(a_l + \delta_{li} z; s_l + \delta_{li}; z)} \quad (5.9.8)$$

where for given $i$,

$$\delta_{li} = 0 \text{ if } l \neq i$$
$$= 1 \text{ if } l = i$$

## 5.10 Difference of two WGP Variates

**Theorem 5.10.1** *If $X_1$ ~WGPD$(a_1; s_2; z)$ and $X_2$ ~WGPD$(a_2; s_2; z)$ and are independent, then the probability distribution of $Y = X_1 - X_2$ is given by*

$$Pr(Y = d) = \frac{e^{dz}}{K(a_1; s_1; z)\ K(a_2; s_2; z)} \sum_{i \geq 0} e^{-2iz} \frac{(a_1 + (i + d)z)^{i+d+s_1} (a_2 + iz)^{i+s_2}}{i!\ (i + d)!} \quad (5.10.1)$$

I. For $z = 0$, (5.10.1) reduces to (Johnson et al. [51], p.191)

$$Pr(Y = d) = e^{-(a_1+a_2)} \left(\frac{a_1}{a_2}\right)^{d/2} I_d(2\sqrt{a_1 a_2}), \quad (5.10.2)$$

where

$$I_d(x) = \left(\frac{x}{2}\right)^d \sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^j}{j!\ \Gamma(d + j + 1)}$$

is the modified Bessel function of the first kind.(Johnson and Kotz [48], p.9; Johnson et al. [51], p.16).

II. When $X_1 \sim$GPD I $(a_1; z)$ and $X_2 \sim$GPD I $(a_2; z)$, then (5.10.1) reduces to (Consul [14], p.74)

$$Pr(Y = d) = a_1\ a_2 e^{-a_1-a_2-dz} \sum_{i\geq 0} \frac{e^{-2iz}}{i!\ (i+d)!}(a_1 + (i+d)z)^{i+d-1}(a_2 + iz)^{i-1} \tag{5.10.3}$$

III. If the variates are GPD III, then

$$Pr(Y = d) = (1-z)^2 e^{-a_1-a_2-dz} \sum_{i\geq 0} \frac{e^{-2iz}}{i!\ (i+d)!}(a_1 + (i+d)z)^{i+d}(a_2 + iz)^{i} \tag{5.10.4}$$

and

IV. For GPD II

$$\begin{aligned} Pr(Y = d) &= \left\{ \frac{(a_1\ a_2)^2(a_1+z)(a_2+z)e^{-a_1-a_2-dz}}{(a_1+z-a_1z)(a_2+z-a_2z)} \right\} \\ & \sum_{i\geq 0} \frac{e^{-2iz}}{i!\ (i+d)!}(a_1 + (i+d)z)^{i+d-2}(a_2 + iz)^{i-2} \end{aligned} \tag{5.10.5}$$

## 5.11 Distributions of sums of left truncated WGPD class of variates

Medhi [53], Consul [14] have derived the distributions of the sums of the left truncated generalized Poisson variates. In this section the corresponding results for the class of WGPD have been discussed.

Let $X_i,\ i = 1, 2, \ldots, n$ be $n$ independent left truncated WGPD variates with parameters $(a; z)$ for given value of $s$ with pf

$$Pr(X = x) = \frac{p_x(a; s; z)}{1 - F_{c-1}(a; s; z)};\quad x = c, c+1, \ldots$$

Here we seek the distribution of $Y = \sum_{i=1}^{n} X_i$ when $c = 1$ and $c = 2$ i.e. zero-truncated case and zero-one truncated case respectively.

### 5.11.1 Case I. Sum of zero-truncated WGPD variates

For, $c = 1$, the distribution of $Y$ is

$$Pr(Y = y) = \sum_t (-1)^{n-t} \binom{n}{t} \left( \frac{a^{-s(t+n)} e^{-yz}}{y!} \right) \left[ \frac{B_y(\overbrace{a, \dots, a}^{t\ terms}; \overbrace{s, \dots, s}^{t\ terms}; z)}{\{K(a; s; z) - a^s\}^n} \right] \tag{5.11.1}$$

where $B_y(a, \dots, a; s, \dots, s; z)$ is defined in theorem(5.9.1)

In particular, for

I. $z = 0$ (Johnson et al. [51], p.190) the distribution of the sum of $n$ i.i.d. zero-truncated Poisson variates is given by

$$Pr(Y = y) = \frac{n! a^y}{y!\ (e^a - 1)^n} S(y, n), \quad y = n, n+1, \dots \tag{5.11.2}$$

where $S(y, n)$ is the Stirling number of the 2nd kind. This distribution is also known as Stirling distribution of the second kind.

II. $s = -1$ (Consul [14], p.71)

$$Pr(Y = y) = \sum_t \binom{n}{t} (-1)^{n-t} \left\{ \frac{at(at + yz)^{y-1} e^{-yz}}{y!(e^a - 1)^n} \right\}, \ y = n, n+1, \dots \tag{5.11.3}$$

III. $s = 0$

$$Pr(Y = y) = \sum_t \binom{n}{t} (-1)^{n-t} \frac{\{at + yz + \alpha(t-1)z\}^y e^{-yz} (1 - z)^n}{y!\ (e^a - 1 + z)^n}, \ y = n, n+1, \dots \tag{5.11.4}$$

where $\alpha^k(j) = \binom{k+j-1}{k} k!$

Similarly it is possible to obtain the distributions for other values of $s$ using the results of $B_n(.)$

### 5.11.2 Case II. Sum of zero-one-truncated WGPD variates

For, $c = 2$, *the distribution of* $Y$ *is*

$$\begin{aligned} Pr(Y = y) &= \sum_{t=0}^{n} \sum_{u=0}^{n-t} (-1)^{n-u} \binom{n}{t} \binom{n-t}{u} \left\{ \frac{a^{s(n-u-t)} (a+z)^{t(s+1)} e^{-z(y+t)}}{y!} \right\} \\ & \left[ \frac{B_y(\overbrace{a, \dots, a}^{u\ terms}; \overbrace{s, \dots, s}^{u\ terms}; z)}{\{K(a; s; z) - a^s - e^{-z}(a+z)^{s+1}\}^n} \right] \end{aligned} \tag{5.11.5}$$

Alternatively,

$$Pr(Y=y) = \sum_{t=0}^{n}(-1)^t\binom{n}{t}\left(\frac{e^{-z}(a+z)^{s+1}}{a^s}\right)^t\left\{\sum_{u=0}^{n-t}\binom{n-t}{u}(-1)^u\left(\frac{a^{-s(u-t)}e^{-zy}}{y!}\right)\right\}$$
$$\left[\frac{B_y(\overbrace{a,a,\ldots,a}^{(n-t-u)\ terms};\overbrace{s,s,\ldots,s}^{(n-t-u)\ terms};z)}{\{K(a;s;z)-a^s-e^{-z}(a+z)^{s+1}\}^n}\right] \tag{5.11.6}$$

In particular, for

I. $z=0$, the pf of the sum of $n$ iid zero-one truncated Poisson variates is given by

$$Pr(Y=y)=\frac{n!\ a^y}{y!\ (e^a-1-a)^n}\sum_{t\geq 0}^{n-1}(-1)^t\ \frac{a^t}{t!}\ S(y,n-t) \tag{5.11.7}$$

II. $s=-1$ , the pf the sum of $n$ iid zero-one truncated GPD I variates is given by (Consul [14], p.73)

$$Pr(Y=y) = \left(\frac{n!\ a^y\ e^{-yz}}{y!\ (e^a-1-ae^{-z})^n}\right)\sum_{t+0}^{n-1}(-1)^t\binom{y}{k}$$
$$\sum_{k=0}^{y-t-1}\binom{y-t-1}{t}\left(\frac{z(y-t)}{a}\right)^k S(y-t-k,n-t)\ , \tag{5.11.8}$$

where $S(y-t-k,n-t)=\frac{1}{(n-t)!}\sum_{u=0}^{n-t-1}(-1)^u\binom{n-t-1}{u}(n-t-u)^{y-t-k}$

## 5.12 Some known and new generalized Poisson distributions

Probability functions, pgfs, and the first two moments of some members of the class of WGPD (5.2.4) with parameters $(a;s;z)$ are listed below.

i) $s=-3$, the pf is

$$P_k(a;z)=\frac{1}{k!}\ \frac{a^3(a+z)^2(a+2z)(a+kz)^{k-3}e^{-(a+kz)}}{a^3(z^2-2z+1)+a^2(z^3-5z^2+4z)+a(-2z^3+5z^2)+2z^3} \tag{5.12.1}$$

The pgf is

$$G(u)=\frac{g_3(at,zt)}{g_3(a,z)}\ e^{a(t-1)} \tag{5.12.2}$$

where $g_3(a,z)=z^3(a^2-2a+2)+z^2(a^3+5a-5a^2)+z(-2a^3+4a^2)+a^3$ The mean and the variance are

$$E(X) = \frac{a^3(-z^2-(a-2)z+a)}{z^3(a^2-2a+2)+z^2(a^3+5a-5a^2)+z(-2a^3+4a^2)+a^3} \tag{5.12.3}$$

$$\mathrm{V}(X) = a^3 \frac{(4-2a)z^4 - (-12a+5a^2)z^3 - (-13a^2+4a^3)z^2 - (a^4-6a^3)z + a^4}{z^3(a^2-2a+2) + z^2(a^3+5a-5a^2) + z(-2a^3+4a^2) + a^3} \quad (5.12.4)$$

ii) $s=-2$, (GPD II) the pf become

$$P_k(a;z) = \frac{a^2(a+z)}{a(1-z)+z} \frac{(a+kz)^{k-2}}{k!} e^{-(a+kz)} \quad (5.12.5)$$

The pgf is

$$G(u) = \frac{a(1-zt)+z}{a(1-z)+z} e^{a(t-1)} \quad (5.12.6)$$

Mean and variance of are

$$\mathrm{E}[X] = \frac{a^2}{a(1-z)+z} \quad \text{and} \quad \mathrm{V}(X) = \frac{a^2(a+z)}{(1-z)(a-z+az)^2} \quad (5.12.7)$$

iii) $s=-1$, (GPD I) The pgf is given by

$$G(u) = e^{a(t-1)} \quad (5.12.8)$$

Mean and variance of GPD I are

$$\mathrm{E}[X] = \frac{a}{(1-z)} \quad \text{and} \quad \mathrm{V}(X) = \frac{a}{(1-z)^3} \quad (5.12.9)$$

iv) $s=0$, (GPD III) the pf is

$$p_k(a;z) = \frac{(1-z)(a+kz)^k}{k!} e^{-(a+kz)} \quad (5.12.10)$$

The pgf is

$$G(u) = \frac{1-z}{1-zt} e^{a(t-1)} \quad (5.12.11)$$

Mean and variance are

$$\mathrm{E}[X] = \frac{a(1-z)+z}{(1-z)^2} \quad \text{and} \quad \mathrm{V}(X) = \frac{z^2+(1-a)z+a}{(1-z)^4} \quad (5.12.12)$$

v) $s=1$, the pf become

$$P_k(a;z) = \frac{(1-z)^3}{a(1-z)+z^2} \frac{(a+kz)^{k+1}}{k!} e^{-(a+kz)} \quad (5.12.13)$$

The pgf is

$$G(u) = \frac{1}{t}\left(\frac{1-z}{1-zt}\right)^3 \frac{g_1(at,zt)}{g_1(a,z)}\, e^{a(t-1)} \tag{5.12.14}$$

where

$$g_1(a,z) = a(1-z) + z^2$$

$$\mathrm{E}(X) = \frac{(2-a)z^3 - (a-a^2-1)z^2 - (2a^2-2a)z + a^2}{(a-az+z^2)(1-z)^2} \tag{5.12.15}$$

$$\mathrm{V}(X) = \frac{2z^6 + (4-5a)z^5 + (4a^2-a)z^4 + (5a-6a^2-a^3)z^3 + (a+3a^3)z^2 + (2a^2-3a^3)z + a^3}{(a-az+z^2)(1-z)^4} \tag{5.12.16}$$

vi) $s = 2$, we get the pf as

$$P_k(a;z) = \frac{(1-z)^5}{a^2(z^2-2z+1) + a(-3z^3+3z^2) + 2z^4 + z^3} \frac{(a+kz)^{k+2}}{k!} e^{-(a+kz)} \tag{5.12.17}$$

The pgf is

$$G(u) = \frac{1}{t^2}\left(\frac{1-z}{1-zt}\right)^5 \frac{g_2(at,zt)}{g_2(a,z)}\, e^{a(t-1)} \tag{5.12.18}$$

where

$$g_2(a,z) = 2z^4 + (1-3a)z^3 + (a^2+3a)z^2 - 2a^2z + a^2$$

The mean and variance are

$$\mathrm{E}(X) = \frac{f_1(a,z)}{(2z^4 + (-3a+1)z^3 + (a^2+3a)z^2 - 2a^2z + a^2)(1-z)^2} \tag{5.12.19}$$

where

$$\begin{aligned} f_1(a,z) \;&=\; (6a+2a)z^5 - (8a-3a^2-8)z^4 - (-7a+3a^2-1+a^3)z^3 \\ &\quad +(3a-3a^2+3a^3)z^2 + (3a^2-3a^3)z + a^3 \end{aligned}$$

$$\mathrm{V}(X) = \frac{f_2(a,z)}{(2z^4 + (-3a+1)z^3 + (a^2+3a)z^2 - 2a^2z + a^2)(1-z)^2} \tag{5.12.20}$$

where

$$
\begin{aligned}
f_2(a,z) &= 12z^{10} + (-40a+44)z^9 + (-60a+26+51a^2)z^8 + (8+60a-23a^2-31a^3)z^7 \\
&+ (9a^4+40a+67a^3-88a^2)z^6 + (42a^2-18a^3-33a^4-a^5)z^5 \\
&+ (-38a^3+42a^4+5a^5+17a^2)z^4 + (17a^3-18a^4+a^2-10a^5)z^3 \\
&+ (3a^3-3a^4+10a^5)z^2 + (-5a^5+3a^4)z + a^5
\end{aligned}
$$

*Remark* : In all the pgfs $G(u)$ above $t = ue^{z(1-t)}$.

## 5.13 Skewness and Kurtosis

In this section the expressions for the third and fourth central moments, fourth cumulant, skewness and kurtosis for GPD II and GPD III are furnished. Some numerical results are also presented.

### 5.13.1 GPD II

$$
\begin{aligned}
\mu_3 &= -a^2\left[\frac{2az^3+(2a^2-a-1)z^2-(a^2+2a)z-a^2}{((a-1)z-a)^3(z-1)3}\right] \\
\mu_4 &= -\frac{a^2}{(z-1)^5(a(z-1)-z)^4}[3a^2z^5+(5a^2-10a+2)z^4 \\
&+ (1+10a+14a^3-28a^2-3a^4)z^3+(9a^4-27a^3+3a+17a^2)z^2 \\
&+ (3a^2+12a^3-9a^4)z+a^3+3a^4] \\
k_4 &= -\frac{a^2}{(z-1)^5(a(z-1)-z)^4}[6a^2z^5+(6a^3-4a^2-10a+2)z^4+(10a-19a^2-4a^3+1)z^3 \\
&+ (3a+14a^2-9a^3)z^2+(3a^2+6a^3)z+a^3] \\
\beta_1 &= -\frac{(2az^2-az-z-a)^2}{(a+z)(z-1)^3a^2} \\
\beta_2 &= -\frac{1}{(a+z)(z-1)^3a^2}[3a^2z^4+(5a^2-10a+2-3a^3)z^3+(1+8a+9a^3-18a^2)z^2 \\
&+ (9a^2-9a^3+2a)z+a^2+3a^3]
\end{aligned}
$$

### 5.13.2 GPD III

$$
\begin{aligned}
\mu_3 &= \frac{1}{(1-z)^6}[2z^3 + (5-2a)z^2 + (1+a)z + a] \\
\mu_4 &= \frac{1}{(1-z)^8}[9z^4 + (32+2a)z^3 + (18+3a^2-2a)z^2 \\
&+ (1-6a^2+13a)z + 3a^2 + a] \\
k_4 &= \frac{1}{(1-z)^8}[6z^4 + (26-6a)z^3 + (15-2a)z^2 + (1+7a)z + a] \\
\beta_1 &= \frac{1}{(z^2+(1-a)z+a)^3}[4z^6 + (20-8a)z^5 + (4+4a+(5-2a)^2)z^4 \\
&+ \{4a+2(1+a)(5-2a)\}z^3 + \{2a(5-2a)+(1+a)^2\}z^2 + 2a(1+a)z + a^2] \\
\beta_2 &= \frac{1}{\{z^2+(1-a)z+a\}^2}[9z^4 + (32-12a)z^3 + (18+3a^2-3a)z^2 \\
&+ (1-6a^2+13a)z + 3a^2 + a]
\end{aligned}
$$

It is easy to observe that for large values of $a$, $\beta_1 \to 0$ and $\beta_2 \to 3$ for both the distributions.

A table of numerical values to see the behaviour of the coefficients $\beta_1$ and $\beta_2$ for $z = .1$ for different values of $a$ is presented below.

Table 5.1: Values of $\beta_1$ and $\beta_2$ for GPD II and GPD III

| | GPD II | | GPD III | |
|---|---|---|---|---|
| a | $\beta_1$ | $\beta_2$ | $\beta_1$ | $\beta_2$ |
| .1 | 29.6740 | 40.0820 | 8.4500 | 14.100 |
| 1 | 1.7364 | 5.2305 | 1.4732 | 4.9122 |
| 3 | 0.5485 | 3.7071 | 0.5186 | 3.6710 |
| 5 | 0.3254 | 3.4199 | 0.3146 | 3.4069 |
| 6 | 0.2705 | 3.3490 | 0.2629 | 3.3399 |
| 8 | 0.2021 | 3.2609 | 0.1979 | 3.2558 |
| 9 | 0.1795 | 3.2319 | 0.1761 | 3.2276 |
| 12 | 0.1343 | 3.1734 | 0.1324 | 3.1711 |
| 15 | 0.1073 | 3.1358 | 0.1061 | 3.1317 |
| 20 | 0.0803 | 3.1037 | 0.0797 | 3.1029 |
| 30 | 0.0535 | 3.0691 | 0.0532 | 3.0687 |

## 5.14 Estimation

Consul [14] discussed the estimation of the two parameters of GPD I using (i) mean and zero class frequency, (ii) maximum likelihood method (Consul and Shoukri [25], Consul and Famoye [16]) (iii) method of moments (Consul and Jain [18]) and presented some data fittings. In this section the problem of estimation of the two parameters of GPD II and GPD III using different methods have been discussed. Here, $\bar{x}$ = sample mean, $m_i'$ = $i$th sample moment about origin. It is assumed that the observed frequencies in a random sample of size $n$ are $n_k = 0, 1, \ldots, m$ for the different classes such that $\sum_{k=0}^{m} n_k = n$, where $m$ is of course the largest value observed.

### 5.14.1 GPD II

The pf of the GPD II is given by

$$p_k = \frac{a^2(a+z)}{a(1-z)+z} \frac{(a+kz)^{k-2}}{k!} e^{-(a+kz)} \tag{5.14.1}$$

I. **By proportion of zeros and mean**

Here the estimate of $a$ is first obtained by solving

$$p_0 e^a (a-1) - \bar{x} + 1 = 0 \tag{5.14.2}$$

numerically and then $z$ is estimated by substituting the value of $a$ in

$$z = \frac{a(p_0 e^a - 1)}{p_0 e^a (a-1) + 1} \tag{5.14.3}$$

II. **Moments (Nandi et al. [55])**

$$\hat{z} = 1 - (1-a)\frac{m_1'}{m_2'}$$

$$\hat{a} = \left[\frac{m_1'(m_2' - m_1')}{m_2'}\right]^{1/2}$$

III. **MLE**

The log-likelihood function is given by,

$$l = \log L \;\propto\; 2n\log a + n\log(a+z) - n\log(a+z-az) - na - nz\bar{x} + \sum_{k=0}^{m}(k-2)n_k \log(a+kz) \tag{5.14.4}$$

The two likelihood equations are

$$\frac{\partial l}{\partial a} = \frac{2n}{a} + \frac{n}{a+z} - \frac{n(1-z)}{a+z-az} - n + \sum_{k=0}^{m}\frac{n_k(k-2)}{a+kz} = 0 \Rightarrow g = 0\,,\ \text{say} \tag{5.14.5}$$

$$\frac{\partial l}{\partial z} = \frac{n}{a+z} - \frac{n(1-z)}{a+z-az} - n\bar{x} + \sum_{k=0}^{m}\frac{(k-2)kn_k}{a+kz} = 0 \Rightarrow h = 0\,,\ \text{say} \tag{5.14.6}$$

as it is not easy to solve this equations analytically, MLE of $a$ and $z$ are obtained by solving (5.14.5) and (5.14.6) numerically with Newton-Rapson technique. Following partial derivatives are required for applying the method.

$$\frac{\partial g}{\partial a} = -\frac{2n}{a^2} - \frac{n}{(a+z)^2} + \frac{n(1-z)^2}{(a+z-az)^2} - \sum_{k=0}^{m}\frac{n_k(k-2)}{(a+kz)^2} = d_{11},\ \text{say} \tag{5.14.7}$$

$$\frac{\partial g}{\partial z} = -\frac{n}{(a+z)^2} + \frac{n(1-z)(1-a)}{(a+z-az)^2} + \frac{n}{a+z-az} - \sum_{k=0}^{m}\frac{k(k-2)n_k}{(a+kz)^2} = d_{12},\ \text{say} \tag{5.14.8}$$

$$\frac{\partial h}{\partial z} = -\frac{n}{(a+z)^2} + \frac{n(1-a)^2}{(a+z-az)^2} - \sum_{k=0}^{m}\frac{n_k(k-2)k^2}{(a+kz)^2} = d_{22},\ \text{say} \tag{5.14.9}$$

It may be noted that

$$\frac{\partial h}{\partial a} = \frac{\partial g}{\partial z}$$

Now the ML estimates of $a, z$ are obtained by generating a sequence of pairs $(a_i, z_i); i = 1, 2, \ldots$ using the recurrence relations $a_{i=1} = a_i + inca_i$ and $z_{i+1} = z_i + incz_i$ for $i = 0, 1, \ldots$ starting with an initial pair of $(a_0, z_0)$.

where $\underline{inc} = -D^{-1}\underline{u}$, wherein $\underline{inc} = (inca_i \quad incz_i)'$, $\underline{u} = (g \quad h)'$ and $D = (d_{ij})$ evaluated at the point $a = a_i,\ z = z_i$.

The iteration is stopped at the $r$th step if the distance between the $r$th and the $(r+1)$th solution is less than a preassigned small positive number and $(a_r, z_r)$ is taken as the MLE of $(a, z)$.

### 5.14.2 GPD III

The pf of GPD III is given by

$$p_k = \frac{1}{k!}(1-z)(a+kz)^k e^{-(a+kz)} \tag{5.14.10}$$

I. **Proportion of zeros and ones**

Here first the estimates of $a$ is obtained by solving

$$(p_0 e^a)(1+a-p_0 e^a)e^{-(1+a-p_0 e^a)} - p_1 = 0 \tag{5.14.11}$$

numerically and then the estimate of $z$ calculated from the equation

$$\hat{z} = 1 - p_0 e^a \tag{5.14.12}$$

II. **Proportion of zeros and mean**

Here too the estimate of $a$ is obtained by numerically solving

$$(1-p_0 e^a)e^{-2a} + ae^{-a}p_0 - \bar{x}p_0^2 = 0 \tag{5.14.13}$$

Then $z$ is estimated as in the last case.

III. **Moments**

Moment estimators are obtained by equating the sample mean and variance with the corresponding population expressions. Here the estimate of $z$ is first obtained by numerically solving

$$m_2 z^4 - 4m_2 z^3 + (6m_2 - m_1' - 1)z^2 + (2m_1' - 4m_2)z + m_2 - m_1' = 0 \tag{5.14.14}$$

then $a$ is estimated from

$$\hat{a} = (1-\hat{z})m_1' - \frac{\hat{z}}{1-\hat{z}} \tag{5.14.15}$$

Where $m_2$ = sample variance.

IV. **MLE**

The log-likelihood function is given by

$$l = \log L \quad \propto \quad n\log(1-z) - na - nz\bar{x} + \sum_{k=0}^{m} kn_k \log(a+kz) \tag{5.14.16}$$

The two likelihood equations are

$$\frac{\partial l}{\partial a} = -n + \sum_{k=0}^{m} \frac{n_k k}{a+kz} = 0 \tag{5.14.17}$$

$$\frac{\partial l}{\partial z} = \frac{n}{1-z} + n\bar{x} - \sum_{k=0}^{m} \frac{k^2 n_k}{a+kz} = 0 \tag{5.14.18}$$

multiplying (5.14.17) by $a$ and (5.14.18) by $z$ and then subtracting we get

$$\hat{a} = -\frac{z}{1-z} + \bar{x}(1-z) \text{ and} \tag{5.14.19}$$

$$\sum_{k=0}^{m} n_k \left( \frac{k^2}{(k-\bar{x}-\frac{1}{1-z})z+\bar{x}} - \frac{1}{1-z} \right) - n\bar{x} = 0 \Rightarrow g = 0 \text{ say} \tag{5.14.20}$$

The estimate of $z$ is first obtained by generating a sequence of $z_i; i = 0, 1, \ldots$ from the recurrence relation

$$z_{i+1} = z_i - \frac{g(z_i)}{g'(z_i)}, \text{ for } i = 0, 1, \ldots$$

. The iteration is stopped at the $r$th step if for some positive integer value $r$, the value of $\mid z_{r+1} - z_r \mid < \psi$, a preassigned arbitrary small positive number. Then $z_r$ is taken as the ML estimate of $z$, where

$$g'(z) = \sum_{k=1}^{m} n_k k^2 \left[ \frac{\bar{x} + \frac{z}{(1-z)^2} + \frac{1}{1-z} - k}{[\bar{x}(1-z) - \frac{z}{1-z} + kz]^2} \right] - \frac{n}{(1-z)^2} \tag{5.14.21}$$

Estimate of $a$ is then obtain using (5.14.19).

## 5.15 Data fitting

In this section fitting of GPD I, GPD II, and GPD III to four sets of data taken using maximum likelihood method of estimation are presented.

Example 1. Data of Lucy Whiteker, Biometrika, 1914, 10, p.36 about the distribution of the number of days according to the number of deaths of women per day over 85 published in Times during 1910-12 is taken from Yule and Kendall [67], used by Nandi et al. [55].

Table 5.2: Distribution of the number of days according to the number of deaths of women per day over 85 published Times during 1910-12

| No. of deaths per day | Observed frequency | Poisson | GPD I | GPD II | GPD III |
|---|---|---|---|---|---|
| 0 | 364 | 336.2495 | 364.7279 | 364.7098 | 364.7474 |
| 1 | 376 | 397.3020 | 374.6163 | 374.6493 | 374.5805 |
| 2 | 218 | 234.7200 | 216.4483 | 216.4466 | 216.4503 |
| 3 | 89 | 92.4460 | 92.8999 | 92.8874 | 92.9135 |
| 4 | 33 | 27.3078 | 33.0449 | 33.0401 | 33.0502 |
| 5 | 13 | 6.4532 | 10.3158 | 10.3165 | 10.3152 |
| 6 | 2 | 1.2708 | 2.9254 | 2.9271 | 2.9236 |
| 7 | 1 | 0.2509 | 1.0212 | 1.0231 | 1.0193 |
| Total | 1096 | 1096 | 1096 | 1096 | 1096 |
| $\hat{a}$ | | 1.1816 | 1.1003 | 1.1694 | 1.0311 |
| $\hat{z}$ | | | 0.0688 | 0.0708 | 0.0668 |
| $\chi^2$ | | 14.0136 | 0.3934 | 0.3960 | 0.3956 |
| d.f. | | 4 | 3 | 3 | 3 |

All the three models are almost equally good as shown by the value of the $\chi^2$ . Of course GPD I is slightly better than the rest. All of them are far better than the Poisson distribution. Using method of moments it has been observed that GPD II gives the minimum value of $\chi^2$.

Example 2. Data of Adelstien [1] about the accident proneness experienced by Shunters during 1937-1942, used by Consul [14].

Table 5.3: Comparison of the observed and expected frequencies of accidents of 122 experienced shunting men over 6 years (1937-42)

| No. of deaths per day | Observed frequency | GPD I | GPD II | GPD III |
|---|---|---|---|---|
| 0 | 40 | 39.9796 | 39.9674 | 42.9373 |
| 1 | 39 | 39.4855 | 39.5049 | 39.6524 |
| 2 | 26 | 23.7589 | 23.7608 | 22.6314 |
| 3 | 8 | 11.3162 | 11.3098 | 10.3244 |
| 4 | 6 | 4.7028 | 4.6988 | 4.1374 |
| 5 | 2 | 1.7894 | 1.7886 | 1.5253 |
| $\geq 6$ | 1 | 0.9676 | 0.967 | 0.7919 |
| Total | 122 | 122 | 122 | 122 |
| $\hat{a}$ | | 1.1156 | 1.2391 | 0.9211 |
| $\hat{z}$ | | 0.1219 | 0.1218 | 0.1159 |
| $\chi^2$ | | 1.5684 | 1.5676 | 2.2400 |
| d.f. | | 3 | 3 | 2 |

As measured by $\chi^2$ all the models are almost equally good. Here GPD II gives better fit than the rest of the models.

Example 3. Data of Adelstien [1] about the accident proneness experienced by Shunters during 1943-1947, used by Consul [14].

Table 5.4: Observed and expected frequencies of accidents of 122 experienced shunters over 5 years (1943-47)

| No. of deaths per day | Observed frequency | GPD I | GPD II | GPD III |
|---|---|---|---|---|
| 0 | 50 | 50.9108 | 50.9130 | 50.9073 |
| 1 | 43 | 40.4033 | 40.4082 | 40.3084 |
| 2 | 17 | 19.5701 | 19.5634 | 19.5782 |
| 3 | 9 | 7.5135 | 7.5096 | 7.5178 |
| 4 | 2 | 2.5196 | 2.5198 | 2.5192 |
| 5 | 0 | 0.7743 | 0.7755 | 0.7726 |
| $\geq 6$ | 1 | 0.3086 | 0.3105 | 0.3064 |
| Total | 122 | 122 | 122 | 122 |
| $\hat{a}$ | | 0.8740 | 0.9702 | 0.7781 |
| $\hat{z}$ | | 0.0964 | 0.1014 | 0.0914 |
| $\chi^2$ | | 0.9154 | 0.9161 | 0.9150 |
| d.f. | | 2 | 2 | 2 |

Here GPD III is found to be better than the rest of the models.

Example 4. Data of Adelstien [1] about the accident proneness experienced by Shunters during 1937-1947, used by Consul [14].

Table 5.5: Distributions of accidents of 122 experienced shunters over 11 years (1937-47)

| No of deaths per day | Observed frequency | GPD I | GPD II | GPD III |
|---|---|---|---|---|
| 0 | 21 | 20 2944 | 20 2526 | 20 3433 |
| 1 | 31 | 30 3925 | 30 4117 | 30 3696 |
| 2 | 26 | 27 3357 | 27 3684 | 27 2977 |
| 3 | 19 | 19 2510 | 19 2637 | 19 2364 |
| 4 | 7 | 11 7297 | 11 7260 | 11 7340 |
| 5 | 9 | 6 4997 | 6 4914 | 6 5093 |
| $\geq 6$ | 9 | 6 4971 | 6 4800 | 6 5097 |
| Total | 122 | 122 | 122 | 122 |
| $\hat{a}$ | | 1 7937 | 1 9843 | 1 6016 |
| $\hat{z}$ | | 0 1804 | 0 1881 | 0 1728 |
| $\chi^2$ | | 3 9383 | 3 9652 | 3 9145 |
| d f | | 4 | 4 | 4 |

From the chi-square value it is clear that the GPD III is better than the other models.

## 5.16 Characterization of the class of WGPD

**Theorem 5.16.1** *If $X_1$ and $X_2$ are two independent* WGP *variates with parameters $(a_1; s_1; z)$ and $(a_2; s_2; z)$ respectively, then the conditional distribution of $X_1$ given $X_1 + X_2 = n$ is the class of weighted quasi binomial distributions (2.2.5) with parameters$(a_1, a_2; s_1, s_2; z)$, when $a_1 + a_2 + nz = 1$.*

*Proof.*

$$
\begin{aligned}
Pr(X_1 = k \mid X_1 + X_2 = n) &= \frac{Pr(X_1 = k)Pr(X_2 = n-k)}{Pr(X_1 + X_2 = n)} \\
&= \frac{\binom{n}{k}(a_1 + kz)^{k+s_1}(a_2 + (n-k)z)^{n-k+s_2}}{B_n(a_1, a_2; s_1, s_2; z)} \qquad (5.16.1)
\end{aligned}
$$

This theorem is a generalization of the result that if $X_1$ and $X_2$ are two independent GPD I variates with parameters $(a_1; z)$ and $(a_2; z)$ respectively, the conditional probability of $X_1$ given $X_1 + X_2 = n$ is a QBD II with pf (2.6.11) in page (33). (Consul [12]).

*In general , for $X_i$, $i = 1(1)m$, independent random variables the following theorem can be established.*

**Theorem 5.16.2** *If $X_i$, $i = 1(1)m$, are $m$ independent WGP variates with parameters $(a_i; s_i; z)$ $i = 1(1)m$, then the conditional distribution of $X_1$ given $X_1 + X_2 + \cdots + X_m = n$ is a class of quasi multinomial distributions (Das [28]).*

*Proof.*

$$
\begin{aligned}
P\left(X_1 = k \mid \sum_{i=1}^{m} X_i = n\right) &= \frac{\prod_{i=1}^{m} Pr(X_i = k_i)}{P\left(\sum_{i=1}^{m} X_i = n\right)} \\
&= \frac{\binom{n}{k}\prod_{i=1}^{m}(a_i + k_i z)^{k+s_i}}{B_n(a_1, \ldots, a_m; s_1, \ldots, s_m; z)} \qquad (5.16.2)
\end{aligned}
$$

**Theorem 5.16.3** *If $X_1$ and $X_2$ are two independent discrete rv s whose sum $Y$ is a 2-gpsum distribution with parameters $(n; a_1, a_2; s_1, s_2; z)$ defined by (5.9.2), then $X_1$ and $X_2$ must each be a class of* GP *variates.*

*Proof.* It can be seen that

$$\begin{aligned}
t^n B_n(a_1, a_2; s_1, s_2; z) &= \frac{B_n(a_1 t, a_2 t; s_1, s_2; zt)}{t^{s_1+s_2}} \\
K(a_1; s_1; z)k(a_2; s_2; z) &= \sum_{i\geq 0} \frac{e^{-iz}}{i!} B_i(a_1, a_2; s_1, s_2; z) \\
S(a_1, a_2; s_1, s_2; z) &= \sum_{n\geq 0} t^n \frac{B_n(a_1, a_2; s_1, s_2; z)}{n!} \\
&= v^{-(s_1+s_2)} K(a_1 v; s_1; zv) K(a_2 v; s_2; zv) \qquad (5.16.3)
\end{aligned}$$

where $v = e^{zv}$.

Therefore the pgf of the 2-gpsum distribution can be obtained as

$$\begin{aligned}
&= \frac{e^{z(s_1+s_2)} t^{-(s_1+s_2)} S(a_1 e^{-z} t, a_2 e^{-z} t; s_1, s_2; z e^{-z} t)}{K(a_1; s_1; z) K(a_2; s_2; z)} \\
&= u^{-s_1} \frac{K(a_1 u; s_1; zu)}{K(a_1; s_1; z)} u^{-s_2} \frac{K(a_2 u; s_2; zu)}{K(a_2; s_2; z)} \\
&= g_1(t) g_2(t) \text{ say} \qquad (5.16.4)
\end{aligned}$$

where

$$t = u e^{z(1-u)}$$

Clearly $g_1$ and $g_2$ are pgf of WGPDs, hence by uniqueness theorem of the pgfs $X_1$ and $X_2$ must follow WGPD with parameters $(a_1; s_1; z)$ and $(a_2; s_2; z)$ respectively. For $s_1 = s_2 = -1$, the corresponding result for GPD I (Consul [10]) can be obtained as particular case.

**Theorem 5.16.4** *If rv $W$ assumes only non-negative integer values with pf (5.9.2) with parameters $(a_1\theta, a_2\theta; s_1, s_2; z\theta)$ is sub-divided into two parts $X$ and $Y$ such that $Pr(X = k, Y = n - k \mid W = n)$ is a class of* QBD *with parameters $(n, a_1, a_2, s_1, s_2, z)$. Then the rv's $X$ and $Y$ are independent and have* WGP *distribution with parameters $(a_1\theta; s_1; z\theta)$ and $(a_2\theta; s_2; z\theta)$ respectively.*

*Proof.*

$$Pr(X = k, Y = n - k) = Pr(X = k, Y = n - k \mid W = n) Pr(W = n)$$

$$= \frac{\binom{n}{k}(a_1+kz)^{k+s_1}(a_2+(n-k)z)^{n-k+s_2}}{B_n(a_1,a_2;s_1,s_2;z)}$$
$$\frac{e^{-nz}}{n!}\frac{B_n(a_1\theta,a_2\theta;s_1,s_2;z\theta)}{K(a_1;s_1;z)K(a_2;s_2;z)}$$
$$= \frac{(a_1\theta+kz\theta)^{k+s_1}e^{-kz\theta}}{k!K(a_1\theta;s;z\theta)}\ \frac{(a_2+(n-k)z\theta)^{k+s_2}e^{-(n-k)z\theta}}{(n-k)!K(a_2\theta;s_2;z\theta)}$$

Hence $X$ and $Y$ are independent and have class of WGPD with parameters $(a_1\theta; s_1; z\theta)$ and $(a_2\theta; s_2; z\theta)$ respectively.

Putting $s_1 = s_2 = -1$, Consul's result for GPD I (Consul [10]) is seen as particular case.

**Theorem 5.16.5** *If $X$ and $Y$ are two independent rv's defined on a set of all non-negative integers such that $Pr(X = k \mid X + Y = n)$ is* QBD *(2.2.5) with parameters $(n, a_n, b_n, s, t, z)$, then*

*i) $a_n$ is independent of $n$ and is equal to $a$ for all $n$.*

*ii) $X$ and $Y$ must have class of* WGP *distribution with parameters $(av, s, zv)$ and $(bv, t, zv)$ respectively, where $v > 0$ is an arbitrary number such that $zv \leq 1$.*

*Proof.* Let the pfs of $X$ and $Y$ be denoted by $f(x)$ and $g(y)$.Science $X$ and $Y$ are independent

$$Pr(X=k \mid X+Y=n) = \frac{Pr(X=k)Pr(Y=n-k)}{Pr(X+Y=n)} = \frac{f(k)g(n-k)}{\sum_{k=0}^{n} f(k)g(n-k)} \tag{5.16.5}$$

$$f(k)g(n-k) = Pr(X=k \mid X+Y=n)\sum_{k=0}^{n} f(k)g(n-k)$$

Therefore

$$\frac{f(k)g(n-k)}{f(k-1)g(n-k-1)} = \frac{(n-k-1)(a_n+k)z)^{k+s}(b_n+(n-k)z)^{n-k+t}}{k(a_n+kz-z)^{k+s-1}(b_n+z+(n-k)z)^{n-k+t+1}} \tag{5.16.6}$$

Now replacing $k$ and $n$ by $k+1$ and $n+1$ in (5.16.5) and then dividing by (5.16.6) and noting that the lhs of the resulting expression is independents of $n$ we get

$$\frac{f(k+1)f(k-1)}{f(k)^2} = \frac{k}{k+1}\frac{\{a+(k+1)z\}^{k+1+s}(a-z+kz)^{k+s+1}}{(a+kz)^{2k+2s}} \tag{5.16.7}$$

Putting $k = 1, 2, \dots, n-1$ in (5.16.7) and multiplying we get

$$\frac{f(n)}{f(n-1)} = \frac{f(1)}{f(0)} \frac{a^s}{(a+z)^{s+1}} \frac{(a+nz)^{n+s}}{n(a+nz-z)^{n+s-1}} \tag{5.16.8}$$

Let

$$u = \frac{f(1)}{f(0)} \frac{a^s}{(a+z)^{s+1}}$$

Therefore

$$f(n) = \frac{u^n}{n!} \frac{(a+nz)^{n+s}}{a^s} f(0) \tag{5.16.9}$$

But $\sum f(n) = 1$ implies

$$f(0) = \frac{a^s}{v^{-s} K(av; s; zv)}$$

Where $v = ue^{zv}$. Hence

$$f(n) = \frac{1}{n!} \frac{(av + nzv)^{n+s} e^{-nzv}}{K(av; s; zv)}$$

Therefore $X$ has a class of WGP distribution with $(av; s; zv)$.

Putting $k = 1$ in (5.16.6) and proceeding as above it can be shown that $Y$ also follows the WGPD with $(bv; t; zv)$

Similar characterisation for GPD I obtained by (Consul [10]) can be derived from the above theorem by taking $s_1 = S - 2 = -1$.

**Theorem 5.16.6** *If $X$ and $Y$ are two independent non-negative integer valued rv's such that*

$$\begin{aligned} i)\ Pr(Y = 0 \mid X + Y = n) &= \left(\frac{a_1 + a_2}{a_1}\right)^s \frac{(a_1 + nz)^{n+s}}{(a_1 + a_2 + nz)^{n+s}} \\ ii)\ Pr(Y = 1 \mid X + Y = n) &= n\left(\frac{a_1 + a_2}{a_1}\right)^s \frac{(a_1 + nz - z)^{n+s-1}}{(a_1 + a_2 + nz)^{n+s}} \end{aligned}$$

where $a_1, a_2 > 0; 0 \le z \le 1$, then $X$ and $Y$ are class of WGPD with parameters $(a_1 v; s; zv)$ and $(a_2 v; s; zv)$ respectively,where $v$ is arbitrary number and $0 < v < 1$.

*Proof.* Suppose that

$Pr(X = x) = f(x)$ and $Pr(Y = y) = g(y)$ then by (i)

$$\frac{f(n)g(0)}{\sum_{i=0}^{n} f(i)g(n-i)} = \left(\frac{a_1+a_2}{a_1}\right)^s \frac{(a_1+nz)^{n+s}}{(a_1+a_2+nz)^{n+s}} \tag{5.16.10}$$

by (ii)

$$\frac{f(n)g(1)}{\sum_{i=0}^{n} f(i)g(n-i)} = n\left(\frac{a_1+a_2}{a_1a_2}\right)^s \frac{(a_1+nz-z)^{n+s-1}}{(a_1+a_2+nz)^{n+s}} \tag{5.16.11}$$

dividing (5.16.10) by (5.16.11) and repeating the recurrence and remembering that

$$\sum_{n\geq 0} f(n) = 1$$

we get

$$f(n) = \frac{1}{n!}\frac{(a_1v+nzv)^{n+s}e^{-nzv}}{K(a_1v;s;zv)} \quad ;n = 0,1,\ldots$$

As a particular case this theorem provides corresponding characterisation theorem of GPD I (Consul [12]).

## 5.17 Models leading to GPD III

**Theorem 5.17.1** *If the mean $m(a,z)$ for the probability distribution $p_x(a;z)$ is increased by changing the parameter $a$ to $a+\delta z$ in such away that*

$$\frac{dp_0(a;z)}{da} = -p_0(a;z) \tag{5.17.1}$$

*and*

$$\frac{dp_x(a;z)}{da} = -p_x(a;z) + p_{x-1}(a+z;z) \tag{5.17.2}$$

*for all integral values of $x > 0$ with the initial conditions $p_0(0,z) = 1-z$ and $p_x(0;z) = (1-z)\frac{e^{-xz}(xz)^x}{x!}$ for $x > 0$, then*

$$p_x(a;z) = \frac{(1-z)}{x!}(a+xz)^x e^{-(a+xz)} \quad a > 0;\ \ 0 < z < 1 \tag{5.17.3}$$

**Theorem 5.17.2** *Let $g(s)=$ pgf of the number of customers arriving for service at a server, $X=$ number of customers already waiting for service before the beginning of the service, $f(s)=$ pgf of $X$. Then under the assumption that the service time for each customer served in any busy period of the server, will have a Lagrangian probability distribution given by (Consul and Shenton [23])*

$$Pr(Y=y)=\frac{1}{y!}\frac{d^{y-1}}{ds^{y-1}}\left[g(s)^y\frac{d}{ds}f(s)\right]_{s=0} \tag{5.17.4}$$

*Taking*

$$f(s)=\frac{1-z}{1-zs}e^{a(s-1)}$$

*and* $g(s)=e^{z(s-1)}$ *it can be shown that*

$$Pr(Y=y)=\frac{1-z}{y!}(a+yz)^y e^{-(a+yz)} \quad a>0;\ \ 0<z<1 \tag{5.17.5}$$

**Theorem 5.17.3** *Under steady state conditions the probability distribution of a first order kinetic energy process having forward and backward rate*

$\frac{1}{(a+kz)^k}$ *and* $\frac{ke^z}{(a+kz)^k}$ *respectively, is given by*

$$\frac{1-z}{k!}(a+kz)^k e^{-(a+kz)} \quad a>0;\ \ 0<z<1 \tag{5.17.6}$$

All these theorem can be proved following Consul [14].

## 5.18 Limiting Distribution

**Theorem 5.18.1** *As $n\to\infty$ and $na_1=\lambda, nz=\psi$ the class of weighted quasi binomial distributions (*WQBD*) (2.2.5) with parameters $(n,a_1,a_2;s_1,s_2;z)$ tends to the class of* WGPD *with parameters $(\lambda;s_1;\psi)$.*

Proof: Since

$$\begin{aligned}&\lim_{n\to\infty} n^{s_1}\binom{n}{k}(a_1+kz)^{k+s_1}(1-a_1-kz)^{n-k+s_2}\\ =\ &\frac{1}{k!}(\lambda+k\psi)^{k+s_1}e^{-(\lambda+k\psi)}\end{aligned} \tag{5.18.1}$$

Hence

$$\lim_{n\to\infty} n^{s_1} B_n(a_1, a_2; s_1, s_2; z) \to K(\lambda; s_1; z).$$

*Remark* : Limiting distribution is independent of the choice of $s_2$. i.e. all the members of the QBD class with fixed $s_1$ approaches to WGPD($\lambda; s_1; \psi$) for all choice of $s_2$. Thus QBD I (i.e. when $s_1 = -1, s_2 = 0$) (Consul [11]), QBD II (i.e. $s_1 = -1, s_2 = -1$) (Consul [20]) both tends to GPD I with $(\lambda; \psi)$ which is the result obtained in Consul [14] (p.27).

**Theorem 5.18.2** *If* $X \sim$ GPD *III with* $a; z$ $\mathrm{E}(X) = \mu, \mathrm{V}(X) = \sigma^2$, *then as* $a \to \infty, U = \frac{X-\mu}{\sigma}$ *approaches to the standard normal distribution.*

Proof: The cumulant generating function (cgf) of the standardized rv $U$ is given by

$$\frac{-\mu}{\sigma}t + \frac{k_1}{\sigma}t + \frac{k_2}{\sigma^2}\frac{t^2}{2!} + \frac{k_3}{\sigma^3}\frac{t^3}{3!} + \cdots \tag{5.18.2}$$

where $k_r; r = 1, 2, \ldots$ are the successive cumulants of GPD III. Putting these values in the above expression and taking limit as $n \to \infty$ the cgf tends to $\frac{t^2}{2}$, which is the cgf of the standard normal distribution. Hence the result.

# Chapter 6

# A class of Generalized Multivariate Generalized Poisson Distributions

## 6.1 Introduction

A class of generalized multivariate generalized Poisson distributions is proposed by defining a class of multivariate identities. Some known distributions are obtained as particular cases of this class. Moment properties of these distribution are studied. Parameter estimation for two bivariate distributions are discussed.

## 6.2 Some multivariate exponential identities

**Definition 5** *For all* $i = 0, 1, 2, \ldots, n; b_i > 0; \mid z_i \mid < 1;$ *and* $s_i$ *integer values, a class of multivariate exponential sums is defined as*

$$M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) = \sum_{x_1 \geq 0} \cdots \sum_{x_n \geq 0} \sum_{j \geq 0}^{\min} (b_0 + jz_0)^{j+s_0} \frac{e^{-jz_0}}{j!}$$
$$\prod_{i=1}^{n} \left[ \frac{(b_i + (x_i - j)z_i)^{x_i - j + s_i} e^{-(x_i - j)z_i}}{(x_i - j)!} \right], \tag{6.2.1}$$

*where* $b_i + (x_i - j)z_i > 0$ *for all* $x_i \geq 0$ *and min* = *minimum* $\{x_1, \cdots, x_n\}$

### 6.2.1 Some special cases

(i) $s_0 = s_1 = \cdots = s_n = -1$

$$M = \left(\prod_{i=0}^{n} b_i\right)^{-1} \exp\left(\sum_{i=0}^{n} b_i\right)$$

(ii) $s_0 = 0, s_1 = \cdots = s_n = -1$

$$M = \left((1 - z_0)\prod_{i=1}^{n} b_i\right)^{-1} \exp\left(\sum_{i=0}^{n} b_i\right)$$

(iii) $s_0 = s_1 = 0, s_2 = \cdots = s_n = -1$

$$M = \left((1 - z_0)(1 - z_1)\prod_{i=2}^{n} b_i\right)^{-1} \exp\left(\sum_{i=0}^{n} b_i\right)$$

(iv) $s_0 = s_1 = \cdots = s_{r-1} = 0, s_r = \cdots = s_n = -1$

$$M = \left(\prod_{j=0}^{r-1}(1 - z_j)\prod_{i=r}^{n} b_i\right)^{-1} \exp\left(\sum_{i=0}^{n} b_i\right)$$

(v) $s_0 = -2, s_1 = \cdots = s_n = -1$

$$M = \left(b_0(b_0 + z)\prod_{i=0}^{n} b_i\right)^{-1} \{b_0 + z_0(1 - b_0)\} \exp\left(\sum_{i=0}^{n} b_i\right)$$

(vi) $s_k = -2, s_i = -1; i = 0, 1, \ldots, n; i \neq k$

$$M = \left(b_k(b_k + z)\prod_{i=0}^{n} b_i\right)^{-1} \{b_k + z_k(1 - b_k)\} \exp\left(\sum_{i=0}^{n} b_i\right)$$

(vii) $s_k = -2, s_l = 0, s_i = -1;\ i = 0, 1, \ldots, n; i \neq ki \neq l$

$$M = \left[\exp\left(\sum_{i=0}^{n} b_i\right)\{b_k + z_k(1 - b_k)\}\right] \left(b_k(1 - z_l)(b_k + z_k) \prod_{i=0, i\neq l}^{n} b_i\right)^{-1}$$

more such identities can be deduced using the recurrence relations stated below.

### 6.2.2 Some recurrence relations.

$$\text{I.} \quad M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) = b_0 M(b_0, b_1, \ldots, b_n; s_0 - 1, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) + z_0 e^{-z_0} M(b_0 + z_0, b_1, \ldots, b_n; s_0 - 1, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) \quad (6.2.2)$$

Proof: Separating $(b_0 + jz_0)$ from (6.2.1).

$$II. \quad M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) = b_1 M(b_0, b_1, \ldots, b_n; s_0, s_1 - 1, \ldots, s_n; z_0, z_1, \ldots, z_n) + z_1 e^{-z_1} M(b_0, b_1 + z_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) \tag{6.2.3}$$

Proof: Separating $(b_1 + (x_1 - j)z_1)$ in (6.2.1)

$$III. \quad M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) = b_k M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_{k-1}, s_k - 1, s_{k+1}, \ldots, s_n; z_0, z_1, \ldots, z_n) + z_k e^{-z_k} M(b_0, b_1, \ldots, b_{k-1}, b_k + z_k, b_{k+1}, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) \tag{6.2.4}$$

Proof: By separating $(b_k + (x_k - j)z_k)$.

Alternatively (6.2.4) can be written as

$$IV. \quad M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_{k-1}, s_k - 1, s_k + 1, \ldots, s_n; z_0, z_1, \ldots, z_n) = \frac{1}{b_k} M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) - z_k e^{-z_k} M(b_0, b_1, \ldots, b_{k-1}, b_k + z, b_{k+1}, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) \tag{6.2.5}$$

$$V. \quad M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) = \sum_{\alpha \geq o} (b_k + \alpha z_k) {z_k}^{\alpha} e^{-\alpha z_k} M(b_0, b_1, \ldots, b_{k-1}, b_k + \alpha z_k, b_{k+1}, \ldots, b_n; s_0, s_1, \ldots, s_{k-1}, s_k - 1, s_{k+1}, \ldots, s_n; z_0, z_1, \ldots, z_n) \tag{6.2.6}$$

Proof : By repeated application of (6.2.5).

*Remark* : For $z_0 = z_1 =, \ldots, z_n = z$, the relation (6.2.1) reduces to a class of exponential sums given by

$$\mathrm{E}(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n) = \sum_{x_1 \geq 0} \cdots \sum_{x_n \geq 0} \sum_{j \geq 0}^{\min} (b_0 + jz)^{j + s_0} \frac{e^{-jz}}{j!} \prod_{i=1}^{n} \left[ \frac{(b_i + (x_i - j)z)^{x_i - j + s_i} e^{-(x_i - j)z}}{(x_i - j)!} \right] \tag{6.2.7}$$

Special cases for different values of $s_0, s_1, \ldots, s_n$ and recurrence relations for the above sums can be easily deduced by using the corresponding results of (6.2.1).

## 6.3 A class of generalized multivariate generalized poission distributions

**Definition 6** *A random vector $X = (X_1, X_2, \ldots, X_n)$ is said to follow a* GMGP *distribution with parameters $b_0, b_1, \ldots, b_n; z_0, z_1, \ldots, z_n$; for given $s_0, s_1, \ldots, s_n$, if its probability function is of the form*

$$[M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n)]^{-1} \sum_{j \geq 0}^{min} (b_0 + jz_0)^{j+s_0} \frac{e^{-jz_0}}{j!}$$
$$\prod_{i=1}^{n} \left[ \frac{(b_i + (x_i - j)z_i)^{x_i - j + s_i} e^{-(x_i - j)z_i}}{(x_i - j)!} \right] \tag{6.3.1}$$

*where $M(.)$ is defined in (6.2.1) and $min$ = minimum $\{x_1, x_2, \ldots, x_n\}$.*

**Theorem 6.3.1** *If $X_i = V_0 + V_i; i = 1, \ldots, n$ where $V_i$ follows* WGPD$(b_i, s_i, z_i)$ *for $i = 0(1)n$, then the joint pf of $X_1, X_2, \ldots, X_n$ is given by (6.3.1)*

Particular cases :

I. For $s_0 = s_1 = \ldots = s_n = -1$, GMGPD I, the multivariate extension of the bivariate generalized Poisson distribution (Johnson et al. [50], p.133) is obtained with probability function

$$\sum_{j \geq 0}^{min} \left[ b_0 (b_0 + jz_0)^{j-1} \frac{e^{-(b_0 + jz_0)}}{j!} \prod_{i=1}^{n} \left\{ \frac{b_i (b_i + (x_i - j)z_i)^{x_i - j - 1} e^{-(b_i + (x_i - j)z_i)}}{(x_i - j)!} \right\} \right] \tag{6.3.2}$$

which is the joint distribution of $X_1, X_2, \ldots, X_n$ where $X_i = V_0 + V_i; i = 1, \ldots, n$ and $V_i$ follows GPD I (Consul and Jain [18]) with $b_i, z_i$ and are mutually independent.

II. Further, for $z_0 = z_1 =, \ldots, z_n = z$, the pf (6.3.2) reduces to a class of multivariate generalized Poisson (Das [28]) and when $z = 0$, to multivariate Poisson with pf

$$Pr(\bigcap_{i=1}^{n} X_i = x_i) = \exp(-\sum_{i=1}^{n} b_i) \sum_{i=0}^{\min\{x_1, .., x_n\}} \frac{b_0{}^{i}}{j!} \prod_{i=1}^{n} \frac{b_i{}^{x_i - j}}{(x_i - j)!} \tag{6.3.3}$$

which is a multivariate extension of the class of bivariate Poisson distributions (Holgate [40]).

Some distributions of the GMGPD class by choosing various values of the $s_i$' s will now be derived.

## 6.4 Some new multivariate generalized Poisson distributions

(i) $s_0 = 0, s_1 = \ldots, s_n = -1$ (GMGPD II)

$$e^{-\sum b_i} \sum_{j\geq 0}^{\min} \left[ \left\{ (1-z_0)(b_0+jz_0)^j \frac{e^{-jz_0}}{j!} \right\} \prod_{i=1}^{n} \left\{ \frac{b_i(b_i+(x_i-j)z_i)^{x_i-j-1} e^{-(x_i-j)z_i}}{(x_i-j)!} \right\} \right] \tag{6.4.1}$$

(ii) $s_0 = s_1 = 0, s_2 = \cdots = s_n = -1$ (GMGPD III)

$$e^{-\sum b_i} \sum_{j\geq 0}^{\min} \left[ \left\{ (1-z_0)(b_0+jz_0)^j \frac{e^{-jz_0}}{j!} \right\} \left\{ \frac{(1-z_1)\{b_1+(x_1-j)z_1\}^{x_1-j} e^{(x_i-j)z_i}}{(x_i-j)!} \right\} \prod_{i=2}^{n} \left\{ \frac{b_i(b_i+(x_i-j)z_i)^{x_i-j-1} e^{-(x_i-j)z_i}}{(x_i-j)!} \right\} \right] \tag{6.4.2}$$

(iii) $s_0 = s_1 = \cdots = s_{r-1} = 0, s_r = \cdots = s_n = -1$ (GMGPD VI)

$$e^{-\sum b_i} \sum_{j\geq 0}^{\min} \left[ \left\{ (1-z_0)(b_0+jz_0)^j \frac{e^{-jz_0}}{j!} \right\} \prod_{k=1}^{r-1} \left\{ \frac{(1-z_k)(b_k+(x_k-j)z_k)^{x_k-j} e^{-(x_k-j)z_k}}{(x_k-j)!} \right\} \prod_{i=r}^{n} \left\{ \frac{b_i(b_i+(x_i-j)z_i)^{x_i-j-1} e^{-(x_i-j)z_i}}{(x_i-j)!} \right\} \right] \tag{6.4.3}$$

(iv) $s_0 = -2, s_1 = \ldots = s_n = -1$ (GMGPD IV)

$$e^{-\sum b_i} \sum_{j\geq 0}^{\min} \left[ \left\{ \frac{b_0^2(b_0+z_0)}{b_0+z_0(1-b_0)} (b_0+jz_0)^{j-2} \frac{e^{-jz_0}}{j!} \right\} \prod_{i=1}^{n} \left\{ \frac{b_i(b_i+(x_i-j)z_i)^{x_i-j-1} e^{-(x_i-j)z_i}}{(x_i-j)!} \right\} \right] \tag{6.4.4}$$

(v) $s_k = -2, s_i = -1; i \neq k = 0(1)n$ (GMGPD V)

$$e^{-\sum b_i} \sum_{j\geq 0}^{\min} \left[ \left\{ (b_0+jz_0)^{j-2} \frac{e^{-jz_0}}{j!} \right\} \left\{ \frac{b_k^2(b_k+z_k)}{b_k+z_k(1-b_k)} \frac{(b_k+(x_k-j)z_k)^{x_k-j-2} e^{-(x_k-j)z_k}}{(x_k-j)!} \right\} \prod_{i\neq k=1}^{n} \left\{ \frac{b_i(b_i+(x_i-j)z_i)^{x_i-j-1} e^{-(x_i-j)z_i}}{(x_i-j)!} \right\} \right] \tag{6.4.5}$$

(vi) $s_k = -2, s_l = 0, s_0 = -1; i \neq k \neq l = 0(1)n$ (GMGPD VII)

$$e^{-\sum b_i} \sum_{j\geq 0}^{\min} \left[ \left\{ (b_0+jz_0)^{j-2} \frac{e^{-jz_0}}{j!} \right\} \left\{ \frac{b_k^2(b_k+z_k)}{b_k+z_k(1-b_k)} \frac{(b_k+(x_k-j)z_k)^{x_k-j-2} e^{-(x_k-j)z_k}}{(x_k-j)!} \right\} \left\{ \frac{(1-z_l)(b_l+(x_l-j)z_l)^{x_l-j} e^{-(x_l-j)z_l}}{(x_l-j)!} \right\} \prod_{i\neq k\neq l=1}^{n} \left\{ \frac{b_i(b_i+(x_i-j)z_i)^{x_i-j-1} e^{-(x_i-j)z_i}}{(x_i-j)!} \right\} \right] \tag{6.4.6}$$

Corresponding Bivariate distributions can be obtained putting $n = 2$ in the above results. In all the above expressions *min* stands for $\min\{x_1, x_2, \ldots, x_n\}$

## 6.5 Marginal and conditional distributons

**Theorem 6.5.1** *The marginal distribution of any subset* $X_{d_1}, \ldots, X_{d_s}$ *of* $X_1, \ldots, X_n$ *following* GMGPD *(6.3.1) also follows* GMGPD *with parameters* $b_0, b_{d_1}, \ldots, b_{d_s}; s_0, s_{d_1}, \ldots, s_{d_s}$ *and* $z_0, z_{d_1}, \ldots, z_{d_s}$

Proof:

$$Pr\left(\bigcap_{i=1}^{s} X_{d_i} = x_{d_i}\right) = \sum_{\{x_l\},(l \neq d_1,\cdots,d_s)} Pr\left(\bigcap_{i=1}^{s} X_i = x_i\right) \tag{6.5.1}$$

$$= [M(b_0, b_1, \ldots, b_n; s_0, s_1, \ldots, s_n; z_0, z_1, \ldots, z_n)]^{-1} \sum_{j\geq 0}^{\min} (b_0 + jz_0)^{j+s_0} \frac{e^{-jz_0}}{j!} \prod_{l=1}^{s} \left[\frac{(b_{d_l} + (x_{d_l} - j)z_{d_l})^{x_{d_l}-j+s_{d_l}} e^{-(x_{d_l}-j)z_{d_l}}}{(x_{d_l} - j)!}\right] \sum_{\{x_l\},(l\neq d_1,\cdots,d_s)} \left[\prod_{l=1,(l\neq d_1,\cdots,d_s)}^{n} \left[\frac{(b_l + (x_l - j)z_l)^{x_l-j+s_l} e^{-(x_l-j)z_l}}{(x_l - j)!}\right]\right] \tag{6.5.2}$$

Corollary: The marginal distribution of $X_{d_1}$ is given by

$$\frac{e^{-(x_{d_1} z_{d_1})}}{x_{d_1}!} \frac{B'_{x_{d_1}}(b_0, b_{d_1}; s_0, s_{d_1}; z_0, z_{d_1})}{M(b_0, b_{d_1}; s_0, s_{d_1}; z_0, z_{d_1})} \tag{6.5.3}$$

where

$$B'_{x_{d_1}}(b_0, b_{d_1}; s_0, s_{d_1}; z_0, z_{d_1}) = \sum_{j=0}^{x_{d_1}} \binom{x_{d_1}}{j} (b_0 + jz_0)^{j+s_0} (b_{d_1} + (x_{d_1} - j)z_{d_1})^{x_{d_1}-j+s_{d_1}} e^{-(z_0-z_{d_1})j} \tag{6.5.4}$$

which for $z_0 = z_{d_1}$ reduces to Abel's generalization of binomial identity. (Riordan [62]) In particular when $z_0 = z_{d_1} = z$ then

$$\frac{e^{-x_{d_1} z}}{x_{d_1}!} \frac{B_{x_{d_1}}(b_0, b_{d_1}; s_0, s_{d_1}; z)}{M(b_0, b_{d_1}; s_0, s_{d_1}; z, z)} \tag{6.5.5}$$

for $s_0 = s_{d_1} = -1$

$$\frac{e^{-(b_0+b_{d_1}+x_{d_1} z)}(b_0 + b_{d_1} + x_{d_1} z)^{x_{d_1}-1}}{x_{d_1}!} \tag{6.5.6}$$

which is the probability function of GPD I (Consul and Jain [18]) with parameters $b_0 + b_{d_1}; z$.

**Theorem 6.5.2** *The conditional distribution of $X_1, \ldots, X_n$ given $X_i = 0$ is $n-1$ variate multiple generalized Poisson distribution (Das [28]) with $(b_k, s_k, z_k); k = 1, \ldots, n; k \neq i$ having pf*

$$\prod_{k=1}^{n} \left\{ \frac{1}{x_k!} \frac{(b_k + x_k z_k)^{x_k + s_k} e^{-x_k z_k}}{K(b_k; s_k; z_k)} \right\} \tag{6.5.7}$$

**Theorem 6.5.3** *For generalized bivariate* GPD *(i.e.* GMGPD *with $n = 2$) the conditional distribution of $X_1$ given $X_2$ is given by*

$$\sum_{j \geq 0} \left\{ \frac{\binom{x_2}{j}(b_0 + jz_0)^{j+s_0}(b_2 + (x_2 - j)z_2)^{x_2 - j + s_2} e^{-j(z_0 - z_2)}}{B'_{x_2}(b_0, b_2; s_0, s_2; z_0, z_2)} \right\}$$
$$\left\{ \frac{1}{(x_1 - j)!} \frac{(b_1 + (x_1 - j)z_1)^{x_1 - j + s_1} e^{-(x_1 - j)z_1}}{K(b_1; s_1; z_1)} \right\} \tag{6.5.8}$$

*When $z_0 = z_2 = z$ (6.5.8) is clearly the convolution of weighted quasi binomial distribution (2.2.5) and* WGPD *distribution (5.2.4).*

Converse of the theorem also holds.

**Theorem 6.5.4** $\mathrm{E}[X_1 \mid X_2] = \frac{e^{-z_1} K(b_1 + z_1; s_1 + 1; z_1)}{K(b_1; s_1; z_1)} + x_2 \frac{B'_{x_2 - 1}(b_0 + z_0, b_2; s_0 + 1, s_2; z_0, z_2)}{B'_{x_2}(b_0, b_2; s_0, s_2; z_0, z_2)}$

For different values of $s_0, s_1, s_2$ the regression function of $X_1$ on $X_2$ can be derived from theorem (6.5.4) using values of the sums $K(.)$ and $B'(.)$.

e.g. when (i) $z_0 = z_1 = z_2 = z$ and $s_0 = s_1 = s_2 = -1$, the pf (6.5.4) reduces to (Johnson et al. p.134)

$$\mathrm{E}[X_1 \mid X_2] = \frac{b_1}{(1 - z)} + x_2 \frac{b_0}{(b_0 + b_2)} \tag{6.5.9}$$

(ii) $z_0 = z_1 = z_2 = z$ and $s_0 = 0, s_1 = s_2 = -1$, we have

$$\begin{aligned} \mathrm{E}[X_1 \mid X_2] &= \frac{b_1}{(1 - z)} + \frac{x_2}{(b_0 + b_2 + x_2 z)^{x_2}} \\ &\quad \sum_{k=0}^{x_2 - 1} (b_0 + (k+1)z) z^k (x_2 - 1)^{(k)} (b_0 + b_2 + x_2 z)^{x_2 - k - 1} \end{aligned} \tag{6.5.10}$$

(iii) $z_0 = z_1 = z_2 = z$ and $s_0 = 0, s_1 = 0, s_2 = -1$

$$\begin{aligned} \mathrm{E}[X_1 \mid X_2] &= (b_1+z)(1-z)^2 + z^2(1-z) + \frac{x_2}{(b_0+b_2+x_2z)^{x_2}} \\ & \sum_{k=0}^{x_2-1} (x_2-1)^{(k)} z^k (b_0+(k+1)z)(b_0+b_2+x_2z)^{x_2-k-1} \end{aligned} \tag{6.5.11}$$

(iv) $z_0 = z_1 = z_2 = z$ and $s_0 = s_1 = s_2 = 0$

$$\begin{aligned} \mathrm{E}[X_1 \mid X_2] &= (1-z)\{(b_1+z)(1-z)+z^2)\} + \frac{x_2}{\sum_{i=0}^{x_2} x_2^{(i)} z^i (b_0+b_2+x_2z)^{x_2-i}} \\ & \left[(b_0+z)\sum_{j=0}^{x_2-1}(j+1)(x_2-1)^{(j)}z^j(b_0+b_2+x_2z)^{x_2-j-1} + \frac{1}{2}\right. \\ & \left.\sum_{j=0}^{x_2-1} j(j+1)(x_2-1)^{(j)} z^{j+1}(b_0+b_2+x_2z)^{x_2-j-1}\right] \end{aligned} \tag{6.5.12}$$

For $z = 0$, relations (ii), (iii) and (iv) all reduces to relation (i) as expected.

## 6.6 Mean and Dispersion

**Theorem 6.6.1** *If a random vector* $X = (X_1, X_2, \ldots, X_n) \sim$ GMGP *distribution with parameters* $b_0, b_1, \ldots, b_n; z_0, z_1, \ldots, z_n$; *for given* $s_0, s_1, \ldots, s_n$, *then*

$$\mathrm{E}[X_i] = e^{-z_0}a_{01} + e^{-z_i}a_{i1} \tag{6.6.1}$$

$$\mathrm{V}[X_i] = e^{-2z_0}\left(a_{02} - a_{01}^2 + e^{z_0}a_{01}\right) + e^{-2z_i}\left(a_{i2} - a_{i1}^2 + e^{z_i}a_{i1}\right) \tag{6.6.2}$$

$$\text{and } \mathrm{Cov}[X_i, X_j] = e^{-2z_0}\left(a_{02} - a_{01}^2 + e^{z_0}a_{01}\right) \tag{6.6.3}$$

*where*

$$a_{ij} = \frac{K(b_i + jz_i; s_i + j; z_i)}{K(b_i; s_i; z_i)}$$

Note: For obvious reason it can be seen that the correlation between any two variables is always positive.

Using the results described above below formulas for mean and dispersion matrix for different distributions of the GMGPD class are listed below.

*Case I*: For $s_0 = s_1 = \cdots == s_n = -1$

$$\mu_i = \frac{b_0}{1-z_0} + \frac{b_i}{1-z_i}; \quad \sigma_i^2 = \frac{b_0}{(1-z_0)^3} + \frac{b_i}{(1-z_i)^3}; \quad \sigma_{ij} = \frac{b_0}{(1-z_0)^3}$$

*Case II*: In the above case if further $z_0 = z_1 = \cdots = z_n = z$, then (Das [28])

$$\mu_i = \frac{b_0+b_i}{1-z}; \quad \sigma_i^2 = \frac{b_0+b_i}{(1-z)^3}; \quad \sigma_{ij} = \frac{b_0}{(1-z)^3}$$

*Case III*: Suppose $s_0 = 0, s_1 = s_2 = \cdots = s_n = -1$, then

$$\begin{aligned}
\mu_i &= \frac{b_0(1-z_0)+z_0^2}{(1-z_0)^2} + \frac{b_i}{1-z_i} \\
\sigma_i^2 &= \frac{b_0+(1-b_0)z_0+z_0^2}{(1-z_0)^4} + \frac{b_i}{(1-z_i)^3} \\
\sigma_{ij} &= \frac{b_0+(1-b_0)z_0+z_0^2}{(1-z_0)^4}
\end{aligned}$$

*Case IV*: If $s_0 = s_1 = 0, s_2 = s_3 = \cdots = s_n = -1$, then

$$\begin{aligned}
\mu_1 &= \frac{b_0(1-z_0)+z_0^2}{(1-z_0)^2} + \frac{b_1(1-z_1)+z_1^2}{(1-z_1)^2} \\
\mu_i &= \frac{b_0(1-z_0)+z_0^2}{(1-z_0)^2} + \frac{b_i}{1-z_i}, \quad i = 2(1)n \\
\sigma_1^2 &= \frac{b_0+(1-b_0)z_0+z_0^2}{(1-z_0)^4} + \frac{b_1+(1-b_1)z_1+z_1^2}{(1-z_i)^2} \\
\sigma_i^2 &= \frac{b_0+(1-b_0)z_0+z_0^2}{(1-z_0)^4} + \frac{b_i}{(1-z_i)^3}, \quad i = 2(1)n \\
\sigma_{ij} &= \frac{b_0+(1-b_0)z_0+z_0^2}{(1-z_0)^4}
\end{aligned}$$

*Case V*: In case $s_0 = -2, s_1 = s_2 = \cdots = s_n = -1$, then

$$\begin{aligned}
\mu_i &= \frac{b_0^2}{b_0+z_0-b_0z_0} + \frac{b_i}{1-z_i} \\
\sigma_i^2 &= \frac{b_0^2(b_0+z_0)}{(1-z_0)(b_0+z_0-b_0z_0)^2} + \frac{b_i}{(1-z_i)^3} \\
\sigma_{ij} &= \frac{b_0^2(b_0+z_0)}{(1-z_0)(b_0+z_0-b_0z_0)^2}
\end{aligned}$$

*Case VI*: For $s_k = -2, s_i = -1, i \neq k = 0, 1, \ldots, n$

$$\mu_k = \frac{b_0}{1-z_0} + \frac{b_k^2}{b_k+z_k-b_kz_k}$$

$$\mu_i = \frac{b_0}{1-z_0} + \frac{b_i}{1-z_i}$$
$$\sigma_k^2 = \frac{b_0}{(1-z_0)^3} + \frac{b_k^2(b_k+z_k)}{(1-z_k)(b_k+z_k-b_kz_k)^2}$$
$$\sigma_i^2 = \frac{b_0}{(1-z_0)^3} + \frac{b_i}{(1-z_i)^3}$$
$$\sigma_{ij} = \frac{b_0}{(1-z_0)^3} \quad i \neq j;\ i \neq k;\ i,j = 1(1)n$$

## 6.7 Some formula for mixed moments

**Theorem 6.7.1** *If exists, the mixed descending factorial moment* $\mu'_{(r_1,r_2,\ldots,r_n)}$ *of the random vector* $(X_1, X_2, \ldots, X_n)$ *following* GMGPD *(6.3.1) can be expressed as*

$$[M(b_0,\ldots,b_n;s_0,\ldots,s_n;z_0,\ldots,z_n)]^{-1}\left[\sum_{j\geq 0}\left\{\prod_{i=1}^{n}[e^{-z_i}K(b_i;s_i;z_i)+j]^{\underline{r_i}}\right\}\frac{(b_0+jz_0)^{j+s_0}e^{-jz_0}}{j!}\right], \tag{6.7.1}$$

*Where* $(a+b)^{\underline{r}} = \sum\binom{r}{i}a^i b^{(r-i)}$*, wherein expansion* $\{e^{-z}K(a;s;z)\}^i = e^{-iz}K(a+iz;s+i;z)$ *and* $j^i = j^{(i)}$

**Theorem 6.7.2** $\mathrm{E}[\bigcap_{i=1}^{p} X_i] = \mathrm{E}(V_0^p) + \sum_{l=1}^{p}\mathrm{E}(V_0^{p-l}) \sum_{i_1<i_2<\cdots<i_l} \mathrm{E}(\bigcap_{j=1}^{l} V_{i_j})$

where $V_i$ is WGPD with $b_i, s_i, z_i$ for $i = 0(1)n$ as in (5.2.4)

## 6.8 Estimation

In this section, the problem of estimation of the parameters of GBGPD I and GBGPD IV using method based on the observed proportion of double zeros and the first two sample moments have been studied.

(i) GBGPD I

Famoye and Consul [32] discussed a method of estimating the six parameters $b_0, b_1, b_2; z_0, z_1, z_2$

based on observed proportion of double zeros($p_{00}$) and the observed sample first and second moments(Johnson et al. [50], p.134).

(ii) GBGPD II : Here the parameters are obtained from the following formulas:

$$\widehat{b_0} = (1-\widehat{z_0})^3 m_{11} - \frac{\widehat{z_0}^2}{(1-\widehat{z_0})} \tag{6.8.1}$$

$$\widehat{b_1} = (1-\widehat{z_1})^3 [m_{20} - m_{11})] \tag{6.8.2}$$

$$\widehat{b_2} = (1-\widehat{z_2})^3 [m_{02} - m_{11}] \tag{6.8.3}$$

$$\widehat{z_1} = 1 - \left[\frac{m'_{10} - m'_{01} + (1-\widehat{z_2})^2 [m_{02} - m_{11}]}{m_{20} - m_{11}}\right]^{\frac{1}{2}} \tag{6.8.4}$$

Where $z_0, z_2$ satisfy the following equations.

$$\begin{aligned}(1-\widehat{z_0})^4 &- (m'_{01} - (1-\widehat{z_2}))^2 (m_{02} \\ &- m_{11})(1-\widehat{z_0})^2 - \widehat{z_0} = 0\end{aligned} \tag{6.8.5}$$

$$\begin{aligned}\log\left(\frac{1-\widehat{z_0}}{p_{00}}\right) &- (1-\widehat{z_0})^3 m_{11} + \frac{\widehat{z_0^2}}{1-z_0} \\ &- \frac{(m'_{10} - m'_{20} + (1-\widehat{z_2})^2 (m_{20} - m_{11}))^{\frac{3}{2}}}{\sqrt{m_{20} - m_{11}}} \\ &- (1-\widehat{z_2})^3 (m_{02} - m_{11}) = 0\end{aligned} \tag{6.8.6}$$

(ii) GBGPD IV : Here the parameters are estimated using the following equations

$$\widehat{b_1} = (1-\widehat{z_1})^3 [m_{20} - m_{11}] \tag{6.8.7}$$

$$\widehat{b_2} = (1-\widehat{z_2})^3 [m_{02} - m_{11}] \tag{6.8.8}$$

$$\widehat{z_1} = 1 - \left\{\frac{m'_{10} - m'_{01} + (1-z_2)^2 (m_{02} - m_{11})}{m_{20} - m_{11}}\right\}^{\frac{1}{2}} \tag{6.8.9}$$

Where $\widehat{b_0}, \widehat{z_0}$ and $\widehat{z_2}$ satisfy the following equations.

$$\begin{aligned}&\widehat{b_0}^3 + \{(1-\widehat{z_0})^3 m_{11} - \widehat{z_0}\}\widehat{b_0}^2 \\ &+2\widehat{z_0}(1-\widehat{z_0})^2 m_{11}\widehat{b_0} + \widehat{z_0}^2(1-\widehat{z_0})m_{11} = 0\end{aligned} \tag{6.8.10}$$

$$(1-\hat{z}_0)(1-\hat{z}_2)(\hat{b}_0+\hat{z}_0)m'_{01} - (1-\hat{z}_0)(1-\hat{z}_2)^3(\hat{b}_0+\hat{z}_0)$$

$$\{m_{02}-m_{11}\} - (1-\hat{z}_2)\{\hat{b}_0(1-\hat{z}_0)+\hat{z}_0\}m_{11} = 0 \tag{6.8.11}$$

$$\log p_{00} - 2\log\hat{b}_0 - \log(\hat{b}_0+\hat{z}_0) + \log\{\hat{b}_0(1-\hat{z}_0)+\hat{z}_0\}$$

$$+\hat{b}_0 + (1-z_0)^3\{m_{20}-m_{11}\} - 3\log(1-\hat{z}_2)$$

$$-\log\{m_{20}-m_{11}\} - \log\{m_{02}-m_{11}\}$$

$$+\{m_{20}-m_{11}\}\left\{\frac{m'_{10}-m'_{01}}{m_{20}-m_{11}} - (1-\hat{z}_2)^2\right\}^{\frac{3}{2}}$$

$$-3\log\left\{\frac{m'_{10}-m'_{01}}{m_{20}-m_{11}} - (1-\hat{z}_2)^2\right\}^{\frac{1}{2}} = 0 \tag{6.8.12}$$

where, $m_{11}$ = Sample covariance $(X_1, X_2)$, $m'_{10}$ = Sample mean $X_1$, $m'_{01}$ = Sample mean $X_2$, $m_{20}$ = Sample variance $X_1$, and $m_{02}$ = Sample variance $X_2$.

# Appendix A

Some important relations and identities related to a class of Abel's generalizations of binomial identities and two results of multinomial Abel identities (Riordan [62]) which are used repeatedly in deriving many results of the present work are presented here.

The general form of the class of the Able sums is given by

$$B_n(a_1, a_2; s, t; z) = \sum_{k=0}^{n} \binom{n}{k} (a_1 + kz)^{k+s}(a_2 + (n-k)z)^{n-k+t} \tag{A.1.1}$$

The parameter $z$ will not be disposed of as it is important in the context of the present work.

For $z = 1$, (A.1.1) reduces to $A_n(a_1, a_2; s, t)$ of Riordan [62]. Some of the important new results along with the known ones are listed below. More results can be derived using the recurrence relations (Riordan [62]). Though the process is straight forward, sometimes may be quite messy.

$$\begin{aligned} B_n(a_1, a_2; -4, 0; z) &= a_1^{-4}(a_1 + a_2 + nz)^n - nza_1^{-1}(a + z)^{-1}[a_1^{-2} + a_1^{-1}(a_1 + z)^{-1} \\ &+ (a_1 + z)^{-2}](a_1 + a_2 + nz)^{n-1} + n^{(2)}z^2a_1^{-1}(a_1 + z)^{-1}(a_1 + 2z)^{-1}[a_1{}^{-1} \\ &+ (a_1 + 2z)^{-2} + (a_1 + z)^{-1}(a_1 + 2z)^{-1}](a_1 + a_2 + nz)^{n-2} \\ &+ n^{(3)}z^3a_1^{-1}(a_1 + z)^{-1}(a_1 + 2z)^{-1}(a_1 + 3z)^{-1}(a_1 + a_2 + nz)^{n-3} \end{aligned} \tag{A.1.2}$$

$$\begin{aligned} B_n(a_1, a_2; -3, 0; z) &= a_1^{-3}(a_1 + z)^{-2}(a_1 + 2z)^{-1}[(a_1 + z)^2(a_1 + 2z)(a_1 + a_2 + nz)^n \\ &- nza_1(a_1 + 2z)(2a_1 + z)(a_1 + a_2 + nz)^{n-1} + n(n-1)a_1^2z^2(a_1 + z) \\ &\quad (a_1 + a_2 + nz)^{n-2}] \end{aligned} \tag{A.1.3}$$

$$B_n(a_1,a_2;-2,0;z) = a_1^{-2}(a_1+z)^{-1}[(a_1+z)(a_1+a_2+nz)^n - nza_1(a_1+a_2+nz)^{n-1}] \quad \text{(A.1.4)}$$

$$B_n(a_1,a_2;-1,0;z) = a_1^{-1}(a_1+a_2+nz)^n \quad \text{(A.1.5)}$$

$$B_n(a_1,a_2;0,0;z) = (a_1+a_2+nz+z\alpha)^n \quad \text{(A.1.6)}$$

$$B_n(a_1,a_2;1,0;z) = (a_1+a_2+nz+z\alpha+z\beta'(a_1))^n \quad \text{(A.1.7)}$$

$$B_n(a_1,a_2;2,0;z) = (a_1+a_2+nz+z\alpha+z\beta'(a_1;2))^n + (a_1+a_2+nz+z\alpha(2)+z\gamma'(a_1))^n \quad \text{(A.1.8)}$$

$$\begin{aligned} B_n(a_1,a_2;3,0;z) &= (a_1+a_2+nz+z\alpha+z\beta'(a_1;3))^n \\ &+ 3(a_1+a_2+nz+z\alpha(2)+z\beta'(a_1)+z\gamma'(a_1))^n \\ &+ (a_1+a_2+nz+z\alpha(2)+z\beta'(0)+z\gamma'(a_1))^n \\ &+ (a_1+a_2+nz+z\alpha(3)+z\psi'(a_1))^n \end{aligned} \quad \text{(A.1.9)}$$

$$\begin{aligned} B_n(a_1,a_2;-3,-1;z) &= \frac{a_1+a_2}{a_1^3a_2}(a_1+a_2+nz)^{n-1} - \frac{nz(2a_1+z)(a_1+a_2+z)}{a_1^2(a_1+z)^2a_2} \\ &(a_1+a_2+nz)^{n-2} + \frac{n(n-1)z^2(a_1+a_2+2z)}{a_1(a_1+z)(a_1+2z)a_2} \\ &(a_1+a_2+nz)^{n-3} \end{aligned} \quad \text{(A.1.10)}$$

$$\begin{aligned} B_n(a_1,a_2;-2,-1;z) &= \frac{a_1+a_2}{a_1^2a_2}(a_1+a_2+nz)^{n-1} - \frac{nz(a_1+a_2+z)}{(a_1+z)a_2a_1} \\ &(a_1+a_2+nz)^{n-2} \end{aligned} \quad \text{(A.1.11)}$$

$$B_n(a_1,a_2;-1,-1;z) = \frac{a_1+a_2}{a_1a_2}(a_1+a_2+nz)^{n-1} \quad \text{(A.1.12)}$$

$$\begin{aligned} B_n(a_1,a_2;-2,1;z) &= \frac{1}{a_1^2}(a_1+a_2+nz+z\beta'(a_2))^n \\ &- \frac{nz}{a_1(a_1+z)}(a_1+a_2+nz+z\beta'(a_2))^{n-1} \end{aligned} \quad \text{(A.1.13)}$$

$$B_n(a_1,a_2;-1,1;z) = a_1^{-1}(a_1+a_2+nz+z\beta'(a_2))^n \quad \text{(A.1.14)}$$

$$B_n(a_1,a_2;1,1;z) = (a_1+a_2+nz+z\alpha+z\beta'(a_1)+z\beta'(a_2))^n \quad \text{(A.1.15)}$$

$$B_n(a_1,a_2;-1,2;z) = a_1^{-1}[(a_1+a_2+nz+z\beta'(a_2;2))^n$$

$$(a_1 + a_2 + nz + z\alpha + z\gamma'(a_2))^n] \quad \text{(A.1.16)}$$

$$B_n(a_1, a_2; 1, 2; z) = (a_1 + a_2 + nz + z\alpha + z\beta'(a_1) + z\beta'(a_2; 2))^n \quad \text{(A.1.17)}$$

$$\begin{aligned} B_n(a_1, a_2; 2, 2; z) &= [(a_1 + a_2 + nz + \alpha z + z\beta'(a_1; 2))^n \\ &\quad (a_1 + a_2 + nz + z\alpha(2) + z\gamma'(a_1) + z\beta'(a_2; 2))^n \\ &+ (a_1 + a_2 + nz + z\alpha(2) + z\beta'(a_1; 2) + z\gamma'(a_2))^n \\ &+ (a_1 + a_2 + nz + z\alpha(3) + z\gamma'(a_1) + z\gamma'(a_2))^n] \end{aligned} \quad \text{(A.1.18)}$$

$$\begin{aligned} B_n(a_1, a_2; -2, -2; z) &= \frac{a_1 + a_2}{a_1^2 a_2^2}(a_1 + a_2 + nz)^{n-1} - \frac{nz(a_1 + a_2 + z)}{a_1^2 a_2^2 (a_1 + z)(a_2 + z)}(a_1 + a_2 + nz)^{n-2} \\ &\quad \{a_2(a_1 + z) + a_1(a_2 + z)\} + \frac{n^{(2)} z^2 (a_1 + a_2 + 2z)}{a_1 a_2 (a_1 + z)(a_2 + z)} \\ &\quad (a_1 + a_2 + nz)^{n-3} \end{aligned} \quad \text{(A.1.19)}$$

**Multinomial Abel identities:**

The general form of the multinomial extension of Abel sum (A.1.1) is given by

$$B_n(a_1, \ldots, a_m; s_1, \ldots, s_m; z) = \sum \frac{n!}{n_1! \cdots n_m!} \prod_{i=1}^{m} (a_i + n_i z)^{n_i + s_i} \quad \text{(A.1.20)}$$

where the sum is over all positive integers $n_1, \ldots, n_m$ such that $\sum_{i=1}^{m} n_i = n$

For $z = 1$ (A.1.20) reduces to $A_n(a_1, \ldots, a_m; s_1, \ldots, s_m;)$ of Riordan(1968).

Two important identities:

$$B_n(a_1, \ldots, a_m; -1, \cdots, -1) = \left(\prod_{i=1}^{m} a_i\right)^{-1} \left(\sum_{i=1}^{m} a_i\right) \left(\sum_{i=1}^{m} a_i + nz\right)^{n-1} \quad \text{(A.1.21)}$$

$$B_n(a_1, \ldots, a_m; 0, \cdots, 0) = \left\{\sum_{i=1}^{m} a_i + nz + z\alpha(m-1)\right\}^n \quad \text{(A.1.22)}$$

where

$$\alpha_k \equiv \alpha^k = k!$$

$$\alpha^k(j) \equiv \alpha_k(j) = (\underbrace{\alpha + \cdots + \alpha}_{j \text{ terms}})^k = \binom{k + j - 1}{k} k!$$

$$\beta'_k(x) \equiv \beta'^k(x) = k!\,(x + kz)$$

$$\gamma'_k(x) \equiv \gamma'^k(x) = (kz)\ k!\ (x+kz)$$

$$\psi'_k(x) \equiv \psi'^k(x) = (kz)\ (kz)\ k!\ (x+kz)$$

$$\beta'^k(x;2) = [\beta'(x) + \beta'(x)]^k, \text{etc.}$$

Putting $z = 1$, the corresponding results for $A_n(a_1, a_2; s, t)$ tabulated in page 23 of Riordan [62] can be obtained. An important relation for $A_n(x, y; s, t)$

$$\begin{aligned} A_n(x,y;s,t) &= (x-1)\sum_{k=0}^{n}\binom{n}{k}\alpha^k A_{n-k}(x+k,y;s-1,t) \\ &+ \sum_{k=0}^{n}\binom{n}{k}\alpha^k(2)A_{n-k}(x+k,y;s-1,t) \end{aligned} \tag{A.1.23}$$

This relation is useful in converting results of $A_n(x, y; s, t)$ only involving the umbral $\alpha$.

For example:

$$\begin{aligned} A_n(x,y;2,0) &= (x-1)^{(2)}(x+y+n+\alpha(3))^n + [2(x-1)+(x-3)](x+y+n+\alpha(4))^n \\ &+ 3(x+y+n+\alpha(5))^n \end{aligned}$$

Expansions of the following type can be developed easily which will help to further simplify the expressions of Abel identities.

$$\{\alpha + \beta'(x)\}^\nu = \nu!\ \left\{(\nu+1)x + z\frac{\nu(\nu+1)}{2}\right\}$$

# Appendix B

As in appendix A here too some results related to a class of exponential sums (Nandi et al. [55]) are listed as these are used in arriving some results specially related to class of weighted generalized Poisson distributions discussed in the present work.

The general form of the exponential sum is given by

$$K(a,s,z) = \sum_{k\geq 0} \frac{1}{k} e^{-kz}(a+kz)^{k+s} \tag{B.1.1}$$

subject to the simultaneous realization of the constrains $a + kz > 0$, for all $k$ and

$$0 < \left(z + \frac{a}{k+1}\right)\left(1 + \frac{z}{a+kz}\right)^{k+s} e^{-z} < 1$$

for all sufficiently large $k$ where $s$ is an integer.

*Recurrence relations of* $K(a;s;z)$

$$a)\ K(a;s-1;z) = \frac{1}{a}\left[K(a;s;z) - ze^{-z}K(a+z;s;z)\right] \tag{B.1.2}$$

$$b)\ K(a;s;z) = \sum_{\nu=0}^{\infty} z^{\nu} e^{-\nu z}(a_1 + \nu z)K(a+\nu z, a; s-1; z) \tag{B.1.3}$$

$$c)\ K(a;s-1;z) = \frac{1}{s}\left[\frac{d}{da}K(a;s;z) - e^{-z}K(a+z;s;z)\right] \tag{B.1.4}$$

Some important results derived using the recurrence relations are presented below

$$K(a,-1,z) = \frac{e^{a}}{a} \tag{B.1.5}$$

$$K(a,-2,z) = \frac{e^{a}\{a(1-z)+z\}}{a^{2}(a+z)} \tag{B.1.6}$$

$$K(a,-3,z) = \frac{e^a}{a^3}\left\{\frac{(-2a+2+a^2)z^3+(5a-5a^2+a^3)z^2+(4a^2-2a^3)z+a^3}{(a+z)^2(a+2z)}\right\} \quad \text{(B.1.7)}$$

$$\begin{aligned} K(a,-4,z) &= -\frac{e^a}{a^4(a+z)^3(a+2z)^2(a+3z)}\Big\{(2a^3-6a^2+12a-12)z^6 \\ &+ (5a^4-26a^3+52a^2-52a)z^5+(4a^5-35a^4+91a^3-91a^2)z^4 \\ &+ (a^6-18a^5+70a^4-82a^3)z^3+(-3a^6+24a^5-40a^4)z^2 \\ &+ (3a^6-10a^5)z-a^6\Big\} \end{aligned} \quad \text{(B.1.8)}$$

$$\begin{aligned} K(a,-5,z) &= \frac{e^5}{a^5(a+z)^4(a+2z)^3(a+3z)^2(a+4z)}\{(12a^4-48a^3+144a^2-228a+228)z^{10} \\ &+ (52a^5-308a^4+924a^3-1848a^2+1848a)z^9+(91a^6-754a^5+2602a^4 \\ &- 5204a^3+5204a^2)z^8+(82a^7-938a^6+3959a^5-8458a^4+8458a^3)z^7 \\ &+ (40a^8-648a^7+3493a^6-8489a^5+8777a^4)z^6+(10a^9-250a^8+1827a^7 \\ &- 5376a^6+6027a^5)z^5+(a^{10}-50a^9+555a^8-2143a^7+2835a^6)z^4 \\ &+ (-4a^{10}+90a^9-520a^8+882a^7)z^3+(6a^{10}-70a^9+175a^8)z^2 \\ &+ (-4a^{10}+20a^9)z+a^{10}\} \end{aligned} \quad \text{(B.1.9)}$$

$$K(a,0,z) = \frac{e^a}{(1-z)} \quad \text{(B.1.10)}$$

$$K(a,1,z) = \frac{e^a}{(1-z)}\left\{\frac{a(1-z)+z^2}{(1-z)^2}\right\} \quad \text{(B.1.11)}$$

$$K(a,2,z) = \frac{e^a}{(1-z)^5}\{2z^4+(1-3a)z^3+(a^2+3a)z^2-2a^2z+a^2\} \quad \text{(B.1.12)}$$

$$\begin{aligned} K(a,3,z) &= \frac{e^a}{(1-z)^7}\{6z^6+(8-11a)z^5+(6a^2+7a+1)z^4 \\ &+ (-a^3-12a^2+4a)z^3+(3a^3+6a^2)z^2-3a^3z+a^3\} \end{aligned} \quad \text{(B.1.13)}$$

$$\begin{aligned} K(a,4,z) &= \frac{e^a}{(1-z)^9}\{24z^8+(58-5a)z^7+(35a^2+22)z^6 \\ &+ (-10a^3-60a^2+45a+1)z^5+(a^4+30a^3+15a^2+5a)z^4 \\ &+ (-4a^4-30a^3+10a^2)z^3+(6a^4+10a^3)z^2-4a^4z+a^4\} \end{aligned} \quad \text{(B.1.14)}$$

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# Abbreviations

QBD Quasi binomial distribution

QPD Quasi Polya distribution

QIPD Quasi inverse Polya distribution

GMPD Generalized Markov Polya Distribution

GIMPD Generalized inverse Markov Polya distribution

GPD Generalized Poisson distribution

WQBD Weighted QBD

WGPD Weghted GPD

GPM Generalized probability model

UPM Unified probability model

pgf probability generating function

cgf cumulant generating function

pf probability function

PE Polya Eggenberger

IPE Inverse Polya Eggenberger

QBB quasi beta binomial

BB beta binomial

QH quasi hypergeometric

HG hypergeometric

MPSM Markov Polya survival model

QNB quasi negative binomial

NB negative binomial

NHG negative hypergeometric

IF inverse factorial

BP beta Pascal

GNB generalized negative binomial

BD binomial delta

CD Consul distribution

GQH generalized quasi hypergeometric

GPE generalized Polya Eggenberger

MLE maximum likelihood estimate

$\alpha$mb $\alpha$-modified binomial

$\alpha$mp $\alpha$-modified Poisson

$\alpha mp_j$ $\alpha$-modified Poisson of order $j$

c$\alpha$mb class of $\alpha$-modified binomial

w$\alpha$mp weighted $\alpha$-modified Poisson

GMGPD generalized multivariate GPD

GBGPD generalized bivariate GPD

# List of papers

## Papers published:

1. "Moments of a class of quasi binomial distributions" *Proceedings of the annual technical session,* Assam Science Society, 1999.

2. "An Unified discrete probability model" *Journal of the Assam Science Society,* Vol. 1, No. 1, pp. 15-25, 2000.

## Papers communicated:

1. "A generalized probability model" communicated to *Bulletin of Calcutta Statistical Association.*

## Papers being communicated:

1. "Some aspects of a class of $\alpha$-modified binomial distributions".

2. "Some results on a class of weighted generalized Poisson distributions".

3. "A Class of generalized multivariate generalized Poisson distributions".